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MTH641 Assignment 1 Solution

Fall 2026

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MTH641: Functional Analysis

Assignment No. 1 — Fall 2026 | Total Marks: 20

Coverage: Modules 1–30. Due Date: 15 November 2026.

Name: VU Learning Hub

Question 1: Metric Spaces and Open Sets [5 Marks]

Let X = ℝ, and define d(x, y) = min{1, |x − y|} for x, y in ℝ.

(a) Prove that d is a metric on R. [3 Marks]

First, d(x, y) is never negative. |x − y| is never negative and 1 is positive, so the smaller of the two is never negative.

Next, d(x, y) = 0 exactly when x = y. If x = y, then |x − y| = 0, so min{1, 0} = 0. If d(x, y) = 0, then min{1, |x − y|} = 0. Since 1 is not 0, we must have |x − y| = 0, so x = y.

Also, d(x, y) = d(y, x), because |x − y| = |y − x|.

For the triangle inequality, first prove a lemma. For a, b ≥ 0, min{1, a} + min{1, b} ≥ min{1, a + b}. If a + b ≤ 1, then a ≤ 1 and b ≤ 1, so both sides equal a + b. If a + b > 1, the right side is 1. The left side is at least 1: if a ≥ 1, then min{1, a} = 1; if a < 1 and b ≥ 1, then min{1, b} = 1; if both are less than 1, the left side is a + b, which is more than 1.

Now |x − z| ≤ |x − y| + |y − z| by the triangle inequality for |.|. The function min{1, t} grows when t grows, so min{1, |x − z|} ≤ min{1, |x − y| + |y − z|}. By the lemma this is ≤ min{1, |x − y|} + min{1, |y − z|}. That is d(x, z) ≤ d(x, y) + d(y, z).

All four rules hold, so d is a metric on R.

(b) Determine the open balls B_d(0, 1/2) and B_d(0, 1). [1 Mark]

B_d(0, r) is the set of x with d(0, x) < r, that is min{1, |x|} < r. For r ≤ 1, min{1, |x|} < r exactly when |x| < r. Hence:

Open ball	Set
B_d(0, 1/2)	(−1/2, 1/2)
B_d(0, 1)	(−1, 1)

(c) Is the set A = (0, 1) open in (R, d)? [1 Mark]

Yes, A is open. Take any x in (0, 1). Let r = min{x, 1 − x}/2. Then r > 0 and

r < 1. Since r < 1, B_d(x, r) = (x − r, x + r). Also r < x and r < 1 − x, so (x − r, x + r) sits inside (0, 1). Hence every point of A has a d-ball around it inside A, so A is open.

Question 2: Continuous Mappings [5 Marks]

Let (X, d_X) and (Y, d_Y) be metric spaces, and let f : X → Y.

(a) Prove that if f is continuous, then $f^{-1}(V)$ is open in X for every open set V ⊆ Y. [3 Marks]

Take any x in $f^{-1}(V)$. Then f(x) is in V. Since V is open, there is an ε > 0 such that the ball B(f(x), ε) sits inside V. Since f is continuous at x, there is a δ > 0 such that f maps the ball B(x, δ) inside B(f(x), ε). Hence B(x, δ) sits inside $f^{-1}(V)$. So every point of $f^{-1}(V)$ has a ball around it inside $f^{-1}(V)$. Hence $f^{-1}(V)$ is open in X.

(b) The discrete metric and the two identity mappings. [2 Marks]

The discrete metric is δ(x, y) = 0 if x = y, and δ(x, y) = 1 if x ≠ y.

I is continuous. In the discrete space every set is open. For any x, the ball B_δ(x, 1/2) is just {x}, because δ(x, y) < 1/2 only when y = x. So every singleton {x} is open, and every set is a union of singletons, so every set is open. Hence for every open V in (R, |.|), $I^{-1}(V) = V$ is open in (R, δ). So I is continuous.

J is not continuous. The set {0} is open in (R, δ), because every set is open there. But $J^{-1}(\{0\}) = \{0\}$ is not open in (R, |.|), because every ball (0 − ε, 0 + ε) holds points other than 0. So J is not continuous. In sequence form: x_n = 1/n tends to 0 in |.|, but δ(J(x_n), J(0)) = 1 for every n, so J(x_n) does not tend to J(0).

Question 3: Sequence Spaces and Separability [5 Marks]

l^2 is the set of real sequences x = (x_k) with $\sum |x_k|^2 < \infty$, and $d(x, y) = (\sum |x_k - y_k|^2)^{1/2}$, the sum taken from k = 1 to ∞. D is the set of sequences with rational coordinates and only finitely many nonzero terms.

(a) Show that D is countable. [1 Mark]

For each N, let D_N be the set of sequences in D whose terms after the N-th are 0. D_N matches Q^N, the set of N-tuples of rationals, which is countable. D is the union of D_1, D_2, D_3, ... A countable union of countable sets is countable. So D is countable.

(b) Prove that D is dense in l^2. Hence, conclude that l^2 is separable. [3 Marks]

Take x in l^2 and $\varepsilon > 0$. The tail of the series tends to 0, so pick N such that the sum of $|x_k|^2$ over $k > N$ is less than $\varepsilon^2/2$. For each $k \leq N$, pick a rational q_k with $|x_k - q_k|^2 < \varepsilon^2/(2N)$. This is possible because Q is dense in R. Let $y = (q_1, ..., q_N, 0, 0, ...)$. Then y is in D, and $d(x, y)^2$ = the sum of $|x_k - q_k|^2$ over $k \leq N$ plus the sum of $|x_k|^2$ over $k > N$, which is less than $N \cdot \varepsilon^2/(2N) + \varepsilon^2/2 = \varepsilon^2$. So $d(x, y) < \varepsilon$. Every ball around x holds a point of D, so D is dense in l^2. A countable dense set exists, so l^2 is separable.

(c) Show that (e_n) is bounded but is not a Cauchy sequence. [1 Mark]

Each e_n has norm 1: the distance from e_n to 0 is 1. So the whole sequence sits inside the ball of radius 1 around 0, and (e_n) is bounded. For $n \neq m$, $d(e_n, e_m) = \sqrt{2}$. Take $\varepsilon = 1$. For any N, pick $n, m > N$ with $n \neq m$. Then $d(e_n, e_m) = \sqrt{2} > 1 = \varepsilon$. So (e_n) is not a Cauchy sequence.

Question 4: Convergence and Completeness [5 Marks]

Let (X, d) be a complete metric space, and let Y be a nonempty subset of X, equipped with the induced metric.

(a) Prove that if Y is closed in X, then Y is complete. [2 Marks]

Let (y_n) be a Cauchy sequence in Y. The metric on Y is the same as on X, so (y_n) is Cauchy in X. X is complete, so y_n tends to some x in X. Y is closed, so it holds all its limit points, and x is in Y. Hence (y_n) converges to a point of Y. So Y is complete.

(b) Prove that if Y is complete, then Y is closed in X. [2 Marks]

Let y_n in Y tend to x in X. A convergent sequence is Cauchy, so (y_n) is Cauchy in Y. Y is complete, so y_n tends to some y in Y. Limits in a metric space are unique, so $x = y$. Hence x is in Y. So Y holds all its limit points, and Y is closed in X.

(c) Give a Cauchy sequence in Y = (0, 1) that does not converge to any point of Y. [1 Mark]

Take Y = (0, 1) with the usual metric, and $x_n = 1/(n + 1)$. Each x_n is in Y. The sequence tends to 0 in R, so it is Cauchy in R, and hence Cauchy in Y. But 0 is not in Y. Any limit of (x_n) in Y would also be a limit in R, and limits are unique, so the only possible limit is 0, which is not in Y. Hence (x_n) has no limit in Y, so Y is not complete.