

MTH641-Functional Analysis

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Topic:001

Introduction

Functional Analysis itself a very applied type of course. It has its applications in differential equations, ordinary differential equations, numerical analysis, approximation theory, calculus of variations, so there are number of fields in which functional analysis is used.

Course Outcome:

To be able to understand basic concepts, principles, and methods of Functional Analysis and its applications.

Prerequisites:

An undergraduate course of linear algebra and ordinary calculus is sufficient as a prerequisites.

Topic:002

Course Outline

Topics:

- Metric Space
- Normed Space

- Banach Space
- Further Properties of Normed Spaces
- Finite Dimensional Normed Spaces and Subspaces
- Compactness and Finite Dimension
- Linear Operators
- Bounded and Continuous Linear Operators
- Linear Functional
- Linear Operators and Functionals on Finite Dimensional Spaces
- Normed Spaces of Operators
- Dual space
- Inner Product Space
- Hilbert Space
- Orthogonal Complements and Direct Sums
- Orthonormal Sets and Sequences
- Representation of Functional on Hilbert Spaces
- Hilbert Adjoint Operators
- Self Adjoint
- Unitary and Normal Operators

Topic:003

Recommended Books

Book 1: Introductory Functional Analysis with Applications by Erwin Kreyszig

Book 2: Introduction to Functional Analysis by Angus E.Taylor

Book 3: Functional Analysis, sobolev spaces and Partial Differential Equations by Haim Brezis

Book 4: Elements of Functional Analysis by Robert Zimmer

Topic:004

Metric Space

Metric space is a very fundamental tool in functional analysis.

In Functional Analysis, we shall study more general spaces and functions defined on them.

Metric Space: Let $x, y \in \mathbb{R}$ $d(x,y) = |x-y|$

Example: Let $3, -1 \in \mathbb{R}$ $d(3,-1) = d(-1,3) = 4$

Example: Let $-3.5, 4.8 \in \mathbb{R}$ $d(-3.5,4.8) = 8.3$

Topic:005

Metric Space

Definition: A metric space is a pair (X,d) where X is a set and d is a metric on X (or distance function on X), that is a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

- **M1:** d is real valued, finite and non negative.
- **M2:** $d(x,y) = 0$ iff $x = y$
- **M3:** $d(x,y) = d(y,x)$ (Symmetry)
- **M4:** $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle Inequality)

If $x_1, x_2, x_3, \dots, x_n \in \mathbb{R}$ then,

- $d(x_1, x_n) \leq d(x_1, x_2) + d(x_2, x_3) + \dots + d(x_{n-1}, x_n)$
(generalized triangle inequality)

Topic:006

Subspace

Definition: A subspace (Y, d) of (X, d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$. Thus the metric on Y is the restriction:

- $\hat{d} = d|_{Y \times Y}$
- $\hat{d}$ is called the metric induced on Y by d .

Topic:007

Examples (Metric Space)

Example 1:

Let $X = \text{Real line } \mathbb{R}$. Let $d(x, y)$ is defined as:

$$d(x, y) = |x - y|$$

Now, we have to check whether this set with metric d form a metric space or not?

For this, we have to check all the four properties or axioms of metric space which we have seen in the definition of metric space.

M1,M2 and M3 are simple and obvious. Lets check M4:

$$\begin{aligned}d(x,y) &= |x - y| \\&= |x - z + z - y| \\&\leq |x - z| + |z - y| \\&= d(x,z) + d(z,y)\end{aligned}$$

Hence, Real Line with metric $d(x,y) = |x - y|$ is a metric space.

Example 2: Show that Euclidean Plane $\mathbb{R}^2$ is a metric space.

Set $X = \mathbb{R}^2 = \{(x,y) | x, y \in \mathbb{R}\}$ with metric

$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ is a metric space.

Also

Example 3: $X = \mathbb{R}^2 = \{(x,y) | x, y \in \mathbb{R}\}$ with metric $d' = |x_2 - x_1| + |y_2 - y_1|$ is a metric space.

Topic:008

Examples (Metric Space)

Example: Three dimensional Euclidean Space $\mathbb{R}^3$.

$X = \mathbb{R}^3 = \{(x,y,z) | x,y,z \in \mathbb{R}\}$ with

metric $d(u, v) = \sqrt{(\partial_1 - \delta_1)^2 + (\partial_2 - \delta_2)^2 + (\partial_3 - \delta_3)^2}$
where $u = (\partial_1, \partial_2, \partial_3)$ and $v = (\delta_1, \delta_2, \delta_3)$ forms a metric space.

Example: Euclidean space $\mathbb{R}^n$.

$X = \mathbb{R}^n = \{(x,y,z,\dots,n) | x,y,z, \dots, n \in \mathbb{R}\}$ with

metric $d(u, v) = \sqrt{(\partial_1 - \delta_1)^2 + (\partial_2 - \delta_2)^2 + \dots + (\partial_n - \delta_n)^2}$
where $u = (\partial_1, \partial_2, \dots, \partial_n)$ and $v = (\delta_1, \delta_2, \dots, \delta_n)$ forms a
metric space.

Example: Unitary Space $\mathbb{C}^n$.

$X = \mathbb{C}^n = \{(a,b,c,\dots,n) | a,b,c,\dots,n \in \mathbb{C}\}$

Recall:

$$z = a+ib \quad |z| = \sqrt{a^2 + b^2}$$

$$z = a+ib, \quad w = c+id$$

$$|z-w| = \sqrt{(a-c)^2 + (b-d)^2}$$

$$d(u,v) = \sqrt{|\partial_1 - \delta_1|^2 + |\partial_2 - \delta_2|^2 + \dots + |\partial_n - \delta_n|^2}$$

$(\mathbb{C}^n, d)$ is a metric space.

Complex Plane: If $n = 1$ then, $\mathbb{C}^n$ plane become complex plane $\mathbb{C}$ with metric $d(u,v) = |u-v|$ where u and v are complex numbers.

Topic:009

Examples (Metric Space)

Example: Sequence Space l^∞ .

As a set X we take the set of all bounded sequences of Complex numbers.

$x \in X$ then $x = (\partial 1, \partial 2, \partial 3, \dots)$, $x = (\partial i)$, $i = 1, 2, 3, \dots$

If the sequence is bounded, then

$|\partial i| \leq c_x$, c_x depends on sequence.

$d(x, y) = \text{Sup} |\partial i - \delta i|$, $y = \delta i$, $x, y \in X$

(X, d) forms a metric space.

Topic:010

Examples (Metric Space)

Example: Function space $C[a, b]$.

As a set X we take the set of all real valued functions $x, y, \dots$

Which are functions of an independent real variable t , and are defined and continuous on a given closed interval $J = [a, b]$.

$x, y \in C[a, b]$ $x : x(t)$, $y : y(t)$, real valued, continuous.

$x : [a, b] \rightarrow \mathbb{R}$, $y : [a, b] \rightarrow \mathbb{R}$

$d(x, y) = \max_{t \in J} |x(t) - y(t)|$

This set X with metric d forms a metric space.

Example: Consider a set X with metric defined as:

$d(x, x) = 0$, $d(x, y) = 1$

forms a metric space. This metric space is called **discrete metric space**.

Topic:011

Examples (Metric Space)

Example: Sequence Space s .

This space consists of the set of all (bounded or unbounded) sequence of complex numbers. The metric d is defined by:

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \text{ where } x = (\xi_j) \text{ and } y = (\eta_j).$$

We will check M_1 , M_2 , M_3 and M_4 .

M_1, M_2 and M_3 are obvious and easy to check. Let us check M_4 or triangular inequality.

$$\text{Let } |\xi_j - \eta_j| = t$$

$$\text{Then, } f(t) = \frac{t}{1+t}, \quad f'(t) = \frac{1}{(1+t)^2}$$

Derivative of the function is positive means that sequence is increasing sequence.

For example if we take the element $|a+b| \leq |a| + |b|$. If this function is increasing then,

$$f(|a+b|) \leq f(|a| + |b|)$$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a| + |b|}{1+|a| + |b|} = \frac{|a|}{1+|a| + |b|} + \frac{|b|}{1+|a| + |b|}$$

If we remove $|a|$ and $|b|$ in the following way, we get:

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

using this we proceed towards triangular inequality.

Consider, $a = \xi_i - \alpha_i$, $b = \alpha_i - \eta_i$; $z = (\alpha_i)$

Replacing the values:

$$\frac{|\xi_i - \alpha_i + \alpha_i - \eta_i|}{1+|\xi_i - \alpha_i + \alpha_i - \eta_i|} \leq \frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

$$\sum \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \sum \frac{1}{2^i} \frac{|\xi_i - \alpha_i|}{1+|\xi_i - \alpha_i|} + \sum \frac{1}{2^i} \frac{|\alpha_i - \eta_i|}{1+|\alpha_i - \eta_i|}$$

$$d(x,y) \leq d(x,z) + d(z,y) \quad \textbf{Proved!}$$

Hence, sequence space s with this metric d forms a metric space.

Topic:012

Examples (Metric Space)

Example: Space l^p .

Let $p \geq 1$ be a fixed real number.

By definition,

each element in the space l^p is a sequence

$x = (\xi_i) = (\xi_1, \xi_2, \dots)$ of numbers such that $|\xi_1|^p + |\xi_2|^p + \dots$ converges. Thus:

$$\sum_{j=1}^{\infty} |\xi_j|^p < \infty$$

And the metric is defined by:

$$d(x,y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^p)^{1/p}$$

where $y = (\eta_j)$ and $\sum_{j=1}^{\infty} |\eta_j|^p < \infty$ forms a metric space.

If the elements of sequences are from real numbers, then this is called real space l^p .

If the entries of sequences are from complex numbers, then this is called complex space l^p .

But the conditions remains same for both cases.

Example: The case $p = 2$.

The Hilbert sequence space l^2 with the metric defined by:

$$d(x,y) = (\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2)^{1/2}$$

$$d(x,y) = \sqrt{\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2}$$

forms a metric space.

Topic:013

Open set , Closed set

Real line: In real line, we get open set instead of open ball.

e.g open set of 2 and 5 is written as (2,5). Boundaries are not included.

Similarly closed set is written as [2,5]. Boundaries are included.

In $\mathbf{R}^2$, we have an open ball, having some center say x_0 and radius r . In **open ball** boundary is not included.

i.e $d(x_0, x) < r$ (all points x inside boundary)

$d(x_0, x) = r$ (all points x on boundary)

$d(x_0, x) > r$ (all points x outside boundary)

In **Closed ball**, boundary is also included.

i.e $d(x_0, x) \leq r$

Similarly, we have sphere in $\mathbf{R}^3$.

Ball and Sphere:

$B(x_0 ; r) = \{x \in X \mid d(x, x_0) < r\}$ (Open Ball)

$\tilde{B}(x_0 ; r) = \{x \in X \mid d(x, x_0) \leq r\}$ (Closed Ball)

$S(x_0 ; r) = \{x \in X \mid d(x, x_0) = r\}$ (Sphere)

In all three cases, x_0 is the center and r is the radius.

Warning:

Till now we are just keeping general ideas in our mind while defining all the above terms. First we see $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R}^3$ and then move on towards general form. The problem is that why we can't remain on $\mathbf{R}^3$ or $\mathbf{R}^n$. Have you ever seen that the sphere in $\mathbf{R}^3$ is empty? No. But if we talk in metric space, we have a sphere that is empty. So it is not always true that we talk only on $\mathbf{R}^2$ or $\mathbf{R}^3$. Conceptual wise it's good but not true for all cases.

e.g Discrete Metric Space

In discrete metric space we have $d(x,x) = 0$ and $d(x,y) = 1$

If we use the above concepts of open and closed balls, if we subtract open and closed balls, we will get boundary.

But, in discrete case, if we have a sphere, inside the sphere in this case is no value because distance of same points is 0. All the points having distance ≤ 1 are at the boundary. And when we subtract, we have nothing inside. In this case the sphere is an empty set. Because we fixed the distance that is 1. But in $\mathbb{R}^2$, $\mathbb{R}^3$ we can't have empty sphere.

Open Set , Closed Set:

A subset M of a metric space X is said to be open if it contains a ball about each of its points.

A subset K of a metric space X is said to be closed if it's complement (in X) is open. That is, $K^c = X - K$ is open.

Topic:014

Open set , Closed set (Examples)

Example (Real Line)

Consider an open interval $(-1, 6)$ on real line. We will get open intervals on all the points from $(-1,6)$ that totally contained in $(-1,6)$.

If we consider closed interval $[-1,6]$, we will not get open intervals on the boundary points.

If we consider $\mathbb{R}^2$, we have open ball. We will get open balls on all the points from the interval because boundary is not included. But in Closed intervals, we will not have open balls on boundary points.

In this topic, we see:

Examples of open and closed intervals and balls.

The idea is same i.e when we talk about closed it means boundary is closed and we can't get open ball on the boundary points.

On the other hand, in open intervals or balls boundary is not included and we get open ball on every point from the interval that contains in that interval.

Topic:015

Open set , Closed set

Neighborhood of a point: An open ball $B(x_0 ; \varepsilon)$ of radius ε is often called an ε -Neighborhood of x_0 . By a neighborhood of x_0 we mean any subset of X which contain an ε -Neighborhood of x_0 .

Interior Point: We call x_0 an interior point of a set $M \subset X$ if M is a neighborhood of x_0 . The interior of M is the set of all interior points of M . $\text{Int}(M)$ is largest open set contained in M .

Topic:016

Topological Spaces

We have learnt about interior point and neighborhood. Based on this concept we are going to define topological spaces. We will see that every topological space is also a metric space.

Definition (Topological Space)

Let τ be the collection of all open subsets of X . Then (X, τ) is said to be a topological space if it satisfies following properties:

- (T1) $\emptyset \in \tau$ and $X \in \tau$.
- (T2) The union of any members of τ is a member of τ .
- (T3) The intersection of finitely many members of τ is a member of τ .

Proof of these properties:

τ is a collection of all open subsets of X .

T1: $\emptyset$ is always open. X is a open set. So, $\emptyset$ and $X \in \tau$.

T2: Let $U =$ union of open subsets.

Let $x \in U$. There exists at least one open subset containing x

Let that one set is M . Since M is open.

Hence, M contains an open ball B about x .

Since $M \in U$, so $B \subset U$

$\Rightarrow U$ is open

T3: We have to prove that the intersection of open intervals is also open interval.

Consider $y \in \bigcap_{i=1}^m M_i$

$\Rightarrow y \in M_i \quad \forall i = 1, 2, \dots, m$

We will find a ball in all M_i centered at y . Then we will take the smallest radius of all the balls.

Let $\varepsilon = \min \{ \varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \}$. Then we will take a ball $B(y, \varepsilon)$ that contain in all M_i , because ε is minimum.

$\Rightarrow B(y, \varepsilon) \subset M_i \quad \forall i$

$\Rightarrow B(y, \varepsilon) \in \bigcap_{i=1}^m M_i$

$\Rightarrow$ Intersection is also open.

Hence, all the three axioms has proved!

A metric space is a topological space.

Topic: 017

Continuous Mappings

Our next stage is about Continuous Mapping.

Recall the definition of limit of continuous function studied in the Calculus.

In this course we are generalizing the things. We are not specific about any set or sequence. We are talking about sets and sequences in general form. In mappings, we are not talking about curves, here we are talking about open balls.

Continuous mapping:

Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. Then a mapping $X \rightarrow Y$ is said to be continuous at a point $x_0 \in X$, if for every $\varepsilon > 0$ there is a $\delta > 0$ such that:

$$\tilde{d}(Tx, Tx_0) < \varepsilon \text{ for all } x \text{ satisfying } d(x, x_0) < \delta.$$

Geometrically:

Topic: 018

Continuous Mappings (Theorem)

Now we will see a theorem that is another definition of mapping.

Statement: A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X .

Proof: Suppose that mapping T of a metric space X into a metric space Y is continuous. We have to prove that the inverse image of any open subset of Y is an open subset of X .

Let $S \subset Y$ be an open subset. Let S_0 be the inverse image of S .

Now we just have to prove that S_0 is open.

Here we have two cases:

1. $S_0 = \emptyset$, then it is open.
2. $S_0 \neq \emptyset$, let $x_0 \in S_0$ then there is $y = Tx_0$

Since we have S is open, $\exists$ ε -neighborhood of y_0 .

Since T is continuous, then according to the definition of continuous mapping,

$\exists$ δ – neighborhood of x_0 say N_0 . Which is mapped into N .

Since for $N \subset S$ we have $N_0 \subset S_0$.

$\Rightarrow S_0$ is open as $x_0 \in S_0$ and has δ -neighborhood.

We have done one sided proof.

Conversely, now we have to prove that T is continuous.

For every $x_0 \in X$ and ε -neighborhood N of Tx_0 , the inverse image N_0 of N is open. Since N is open and N_0 contains x_0 .

$\Rightarrow N_0$ also contains a δ – neighborhood of x_0 (being open) which is mapped into N because N_0 is mapped into N .

$\Rightarrow$ by definition, T is continuous at x_0 .

Since x_0 was any arbitrary element.

$\Rightarrow$ it is true for all $x_0 \in X$.

Proved!

Topic: 019

Accumulation Point (Limit Point)

Definition: If M is a subset of a metric space X then x_0 is a limit point of M if it is the limit of an eventually non-constant sequence (a_i) of points of M . x_0 may or may not belong to M .

Now, we will define it in the language of Metric space.

Let M be a subset of a metric space X . Then a point x_0 of X (which may or may not be a point of M) is called an **accumulation point (Limit point)** of M . If every neighborhood of x_0 contains at least one point $y \in M$ distinct from x_0 .

Example: Let Suppose, we have $\mathbb{R}$ and distance function

$d(x,y) = |x - y|$, and we know it forms a metric space.

Consider, we are taking $M = (0,1)$

If we check 0:

$0 \notin M = (0, 1)$ (we can make any sequence $(\frac{1}{n})$ whose limit is 0)

Hence 0 is the limit point of M .

Similarly, we can make sequence $(1 + \frac{1}{n})$ whose limit is 1)

So, 1 is also a limit point of M .

If we take $Z \subset \mathbb{R}$, Z has no limit point because Z is in the form of integers and in integers we find no sequence.

e.g any sequence in Z converging to any integer is eventually constant. (we require non-constant sequence)

Example: $\mathbb{R}^2$: $d(\bar{x}, \bar{y}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

We have open disk defined by $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$

The points on the boundary of this disk are limit or accumulation points.

Closure of M:

The set consisting of points of M and the accumulation points of M is called the closure of M and is denoted by $\bar{M}$.

e.g $M = (0,1)$ its limit points are: $\{0,1\}$

then, $\bar{M} = (0,1) \cup \{0,1\} = [0,1]$

Topic: 020

Dense Set , Separable Space

Dense Set:

A subset M of a metric space X is said to be dense in X if $\bar{M} = X$.

Example:

The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$.

Let $x \in \mathbb{R}$, x can be written as:

$$x = n + r \quad n \in \mathbb{Z}, \quad 0 \leq r < 1$$

Hence $r = 0.r_1r_2r_3\dots$

We say $x_k = n + 0.r_1r_2r_3\dots r_k$

$\Rightarrow$ each x_k is a rational number (because we are fixing its fraction, when we fix fractional part then it will give rational number).

If we take the limit:

$$\lim_{k \rightarrow \infty} x_k = x \quad (x \text{ is irrational number})$$

When we apply limit to rational number, it gave irrational number that is its limit point)

So, we find a limit point (irrational number) correspond to every rational number.

$$\Rightarrow \overline{\mathbb{Q}} = \mathbb{R}$$

Separable Space:

A metric space X is said to be separable if it has a countable subset which is dense in X .

In next lecture, we will see examples of separable space.

Topic: 021

Examples (Separable Spaces)

Example: The real line $\mathbb{R}$.

In real line, we have $(\mathbb{R}, d)$ with $d(x,y) = |x-y|$

As we know: $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q}$ is countable also: $\overline{\mathbb{Q}} = \mathbb{R}$

$\Rightarrow \mathbb{R}$ is separable.

In general if we take $(\mathbb{R}^n, d)$ with

$d(\underline{x}, \underline{y}) = \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2}$ where $\underline{x}$ and $\underline{y}$ are vectors.

$\mathbb{Q}^n = \{(c_1, c_2, \dots, c_n) | c_i \in \mathbb{Q}\}$ is dense in $\mathbb{R}^n$. As $\mathbb{Q}$ is countable then $\mathbb{Q}^n$ is also countable.

$\Rightarrow \mathbb{R}^n$ is separable.

Example: The Complex Plane $\mathbb{C}$.

Elements are of the form: $\{a+ib | a, b \in \mathbb{R}\}$ and $i = \sqrt{-1}$

We can say that it is same as ordered pair form: (a,b)

Consider $\mathbb{Q}^2 = \{(c_1, c_2) | c_1, c_2 \in \mathbb{Q}\}$ is dense and countable in $\mathbb{C}$.

$\Rightarrow \mathbb{C}$ is separable.

Example: Discrete Metric Space.

In this case, dense set is the same set X itself. There exists no limiting point other than this.

If the set X is countable then this discrete metric space is countable. Otherwise, it is not.

Topic: 022

Examples (Separable Spaces)

Example: Space l^p .

The space l^p with $1 \leq p < +\infty$ is separable.

We have to find a countable subset which is dense in l^p .

l^p is a space consisting of bounded or unbounded sequences such that:

$$\sum_{i=1}^{\infty} |\xi_i|^p < \infty \quad \text{i.e. convergent}$$

$$d(x,y) = (\sum_{i=1}^{\infty} |\xi_i - \mu_i|^p)^{1/p}$$

Let M be the set of all sequences of the form

$y = (\eta_1, \eta_2, \dots, \eta_n, 0, 0, \dots)$ n is positive integer and this η_i are rational numbers.

$\Rightarrow M$ is countable.

We need to prove that M is dense in l^p . i.e. $\bar{M} = l^p$

Let $x = (\xi_j) \in l^p$ be arbitrary sequence. We need to show that $\exists y \in M$ such that $d(x,y) < \varepsilon$ (we actually want to prove that these points are very close to each other i.e. limiting point)

Now, $x = (\xi_j) \in l^p$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_j| < \infty \quad (\text{Convergent})$$

$$\Rightarrow \sum_{j=1}^n |\xi_j| + \sum_{j=n+1}^{\infty} |\xi_j| < \infty$$

Then, for every $\varepsilon > 0$, there is n (dependent on ε)

$$\Rightarrow \sum_{j=n+1}^{\infty} |\xi_j| < \frac{\varepsilon^p}{2}$$

As we know rational numbers are dense in $\mathbb{R}$. As we have:

$$x = (\xi_j) = \{ \xi_1, \xi_2, \dots, \xi_n, \xi_{n+1}, \dots \}$$

and we choose $y \in M$, $y = (\eta_1, \eta_2, \dots, \eta_n, 0, 0, \dots)$

$$\sum_{j=1}^n |\xi_j - \eta_j| < \frac{\varepsilon^p}{2}$$

$$\begin{aligned} \text{Let } d(x, y)^p &= \sum_{j=1}^{\infty} |\xi_j - \eta_j|^p \\ &= \sum_{j=1}^n |\xi_j - \eta_j|^p + \sum_{j=n+1}^{\infty} |\xi_j - \eta_j|^p \end{aligned}$$

$$\text{Here, } \sum_{j=1}^n |\xi_j - \eta_j|^p < \frac{\varepsilon^p}{2}, \quad \sum_{j=n+1}^{\infty} |\xi_j - \eta_j|^p < \frac{\varepsilon^p}{2}$$

$$d(x, y)^p < \varepsilon^p \Rightarrow d(x, y) < \varepsilon$$

Hence, we found a point y that belongs to M and is a limit point of x .

$\Rightarrow M$ is dense in l^p .

$\Rightarrow l^p$ is separable.

Topic: 023

Sequences

We are going to start a new thing i.e Sequences. We will convert the sequences that we have studied to the language of metric

space. Basically, sequence is that function whose domain is natural numbers. In this lecture, we will take sequences from metric space, we will define convergence and limit of that sequence.

Convergence of Sequence: A sequence (x_n) in a metric space $X = (X, d)$ is said to converge or to be convergent if there is an $x \in X$ such that:

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0$$

x is called the limit of x_n and we write

$$\lim_{n \rightarrow \infty} x_n = x$$

Or simply, $x_n \rightarrow x$

Examples: Let $x_n = \frac{1}{n}$, $n = 1, 2, 3, 4, \dots$

The limit point of this sequence is 0 as it converges to 0.

Let $x_n = (-1)^n$ is not convergent as it gives 1 when n is even and gives -1 when n is odd.

Let $x_n = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{if otherwise} \end{cases}$ not convergent.

We say that (x_n) converges to x or has the limit x .

If (x_n) is not convergent, it is said to be divergent.

Now we will see sequences in the metric space from:

Let (x_n) converges to x . In the language of metric space there is an $\varepsilon > 0$ being given there is $N = N(\varepsilon)$ such that all x_n with $n > N$ lie in the ε -neighborhood $B(x; \varepsilon)$ of x .

This is a translation in the language of metric space.

Topic: 024

Bounded Sequences

We call a non empty subset $M \subset X$ a bounded set if its diameter $\delta(M) = \sup_{x, y \in M} d(x, y)$ is finite.

A sequence (X_n) in X is bounded sequence if the corresponding point set is a bounded subset of X .

Topic: 025

Lemma (Bounded Sequences)

Lemma: Let $X = (X, d)$ be a metric space. Then,

- (a) A convergent sequence in X is bounded and its limit is unique.
- (b) If $x_n \rightarrow x$ and $y_n \rightarrow y$, then $d(x_n, y_n) \rightarrow d(x, y)$.

Proof:

- (a) We have a convergent sequence, we have to prove that this sequence is bounded.

For every $\varepsilon > 0 \exists N = N(\varepsilon)$ such that for all x_n with $n > N$ lie in the ε -neighborhood $B(x; \varepsilon)$ of x . Using this, we

have to prove that $d(x_n, x) < \text{some number}$.

Since, it is true for all ε , we choose $\varepsilon = 1 \exists N$ such that

$$d(x_n, x) < 1 \quad \forall n > N \quad (1)$$

Suppose $d(x_1, x), d(x_2, x), \dots, d(x_N, x)$

$$\text{Max } (d(x_1, x), d(x_2, x), \dots, d(x_N, x)) = d$$

$$d(x_n, x) < d \quad \forall n < N \quad (2)$$

by (1) and (2), $d(x_n, x) < d + 1 \quad \forall n$

Uniqueness:

Let $x_n \rightarrow x$, $x_n \rightarrow z$

We need to prove that $x = z$

$$d(x, z) \geq 0 \quad (1)$$

on the other hand, using triangular inequality:

$$d(x, z) \leq d(x, x_n) + d(x_n, z)$$

As x_n converges to x and z , it means

$$d(x, x_n) \rightarrow 0, \quad d(x_n, z) \rightarrow 0$$

$$d(x, z) \leq 0 \quad (2)$$

by (1) and (2)

$$d(x, z) = 0 \quad \text{Hence, } x = z$$

(b) If $x_n \rightarrow x$, $y_n \rightarrow y$ then, $d(x_n, y_n) \rightarrow d(x, y)$

Using triangular inequality:

$$\begin{aligned}d(x_n, y_n) &\leq d(x_n, x) + d(x, y) + d(y, y_n) \\d(x_n, y_n) - d(x, y) &\leq d(x_n, x) + d(y_n, y)\end{aligned}\tag{3}$$

as $x_n \rightarrow x$, $y_n \rightarrow y$ then

$$\begin{aligned}d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\d(x, y) - d(x_n, y_n) &\leq d(x_n, x) + d(y_n, y) \\d(x_n, y_n) - d(x, y) &\geq - (d(x_n, x) + d(y_n, y))\end{aligned}\tag{4}$$

As $|x| \leq a \iff -a \leq x \leq a$

Using this, (3) and (4) implies:

$$\begin{aligned}|d(x_n, y_n) - d(x, y)| &\leq d(x_n, x) + d(y_n, y) \\ \text{as } x_n \rightarrow x, y_n \rightarrow y \text{ then } d(x_n, x) + d(y_n, y) &\rightarrow 0 + 0 \\ \implies |d(x_n, y_n) - d(x, y)| &\rightarrow 0 \\ \implies d(x_n, y_n) &\rightarrow d(x, y) \qquad \text{Proved!}\end{aligned}$$

Topic: 026

Cauchy Sequence

We are now moving to another property of metric spaces, that is completeness. Completeness is not just to satisfy four axioms or

metric space.

Definition: A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy (or fundamental) if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n > N$.

Equalent Notation:

$$d(x_m, x_n) \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Example: We take $X = (0, 1]$, with the usual metric defined by $d(x, y) = |x - y|$, and the sequence (x_n) , where $x_n = \frac{1}{n}$ and $n = 1, 2, \dots$. This is a Cauchy sequence, because let's assume we have a small value ε and then we have:

$$\left| \frac{1}{m} - \frac{1}{n} \right| \leq \frac{1}{m} + \frac{1}{n}$$

When $m, n \rightarrow \infty$, $\frac{1}{m}$ and $\frac{1}{n} \rightarrow 0$

$$\Rightarrow \left| \frac{1}{m} - \frac{1}{n} \right| \rightarrow 0$$

Complete Space: The space X is said to be complete if every Cauchy sequence in X converges. That is, has a limit which is an element of X .

For example: We take $X = (0, 1]$, with the usual metric defined by $d(x, y) = |x - y|$, and the sequence (x_n) , where $x_n = \frac{1}{n}$ and $n = 1, 2, \dots$. This is a Cauchy sequence, but it does not converge, because the point 0 is not point of X . It means this space is not complete.

Topic: 027

Theorem (Convergent Sequence)

Every convergent sequence in a metric space is a Cauchy sequence.

Proof: If $x_n \rightarrow x$, then for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n > N$.

We have to prove that (x_n) is Cauchy sequence.

Hence by the triangle inequality we obtain for $m, n > N$

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow d(x_m, x_n) < \varepsilon \quad \text{where } m, n > N$$

$$\Rightarrow (x_n) \text{ is Cauchy sequence.} \quad \text{Proved!}$$

Converse is not true. Example:

$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $(0,1]$. But it is not convergent in $(0,1]$.

Example related to completeness:

The metric space Q with $d(x,y) = |x - y|$ is not complete. i.e we need at least one Cauchy sequence that does not converge in this space.

The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$; $n = 1, 2, 3, \dots$

This is a Cauchy sequence in $\mathbb{Q}$ and converges to 'e' in $\mathbb{R}$
and $e \notin \mathbb{Q}$.

Hence, $\mathbb{Q}$ is not complete.

Topic: 028

Theorem (Closure, Closed set)

Theorem: Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure as defined before. Then:

- (a) $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \rightarrow x$.
- (b) M is closed if and only if the situation $x_n \in M, x_n \rightarrow x$ implies that $x \in M$.

Proof:

(a) Let $x \in \bar{M}$ where $\bar{M}$ is the collection of M and its limit points. We have to prove that there is a sequence (x_n) in M such that $x_n \rightarrow x$.

We have two choices:

$$x \in M, (x_1, x_2, \dots) = (x_n) = (x, x, \dots) \quad x_n \rightarrow x \in M$$

$x \notin M$ (x is a limit point)

$\Rightarrow$ for every $n = 1, 2, \dots$ the ball $B(x; \frac{1}{n})$ contains an $x_n \in M$ other than x .

Now as $n \rightarrow \infty \Rightarrow \frac{1}{n} \rightarrow 0 \Rightarrow x_n \rightarrow x$

Conversely: If there is a sequence (x_n) in M such that $x_n \rightarrow x$. We have to prove that $x \in \bar{M}$.

Now we have two cases, either $x \in M$ or every neighborhood of x contains points $x_n \neq x$.

$\Rightarrow x$ is a limit point. $\Rightarrow x \in \bar{M}$.

(b) Let $x_n \in M$, $x_n \rightarrow x$ implies that $x \in M$. We have to prove that M is closed. i.e $M = \bar{M}$.

To prove $M = \bar{M}$ we have to show that $M \subset \bar{M}$ and $\bar{M} \subset M$.

$M \subset \bar{M}$ is obvious because $\bar{M}$ contains M and all its limit points.

Now we have to show that $\bar{M} \subset M$.

Let $x \in \bar{M}$ (a) $\Rightarrow x_n \rightarrow x \Rightarrow x \in M$.

$\Rightarrow \bar{M} \subset M$.

Hence, $M = \bar{M}$.

Topic: 029

Theorem (Complete Subspace)

Theorem:

A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X .

As this condition is if and only if so vice versa. From previous theorem we have

Theorem: Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure as defined before. Then:

- (a) $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \rightarrow x$.
- (b) M is closed if and only if the situation $x_n \in M, x_n \rightarrow x$ implies that $x \in M$.

Proof: Let M is a subset of a complete metric space (X, d) . We have to prove that M is complete.

M is complete if and only if M is closed. M is closed if and only if $M = \bar{M}$.

Ultimately, we have:

M is complete if and only if $M = \bar{M}$.

Consider M is complete, we need to show that $M = \bar{M}$.

We will show that $M \subset \bar{M}$ and $\bar{M} \subset M$. It is obvious that $M \subset \bar{M}$.

To prove $\bar{M} \subset M$, let $x \in \bar{M}$ (using (a) of previous theorem)

there is a sequence (x_n) in M such that $x_n \rightarrow x$.

This sequence belongs to M and M is a subspace of (X, d) .

Hence this sequence is also in X .

Since every convergent sequence in a metric space is Cauchy.

$\Rightarrow (x_n)$ is Cauchy.

Since M is complete $\Rightarrow (x_n)$ converges in M .

$$\Rightarrow x_n \rightarrow x \in M \Rightarrow \overline{M} \subset M \Rightarrow M = \overline{M}$$

Hence, M is closed.

Conversely: Now, let M is closed i.e $M = \overline{M}$

We need to show that M is complete. i. e we need to show that every Cauchy sequence in M converges in M .

Let (x_n) be a Cauchy sequence in M such that $x_n \rightarrow x$. we have to show that this x belongs to M .

Using (a) of previous theorem: $x \in \overline{M}$

as we have $M = \overline{M} \Rightarrow x \in M$.

Since, (x_n) is an arbitrary sequence.

Hence, this is true for all Cauchy sequences in M .

Proved!

Topic: 030

Theorem (Continuous mapping)

Theorem: A mapping $T: X \rightarrow Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ **if and only if** $x_n \rightarrow x_0$ implies $Tx_n \rightarrow Tx_0$.

Proof: Suppose T is continuous, we will prove if $x_n \rightarrow x_0$

implies $Tx_n \rightarrow Tx_0$.

Since T is continuous, implies that for every $\varepsilon > 0$ there exist $\delta > 0$ such that $d(x, x_0) < \delta$ implies $\tilde{d}(Tx, Tx_0) < \varepsilon$.

Let $x_n \rightarrow x_0$, $\exists N$ such that for all $n > N$ we have $d(x_n, x_0) < \delta$.

Now applying definition of continuity to this definition of convergence.

That gives us $\tilde{d}(Tx_n, Tx_0) < \varepsilon \quad \forall n > N$

By definition, $Tx_n \rightarrow Tx_0$

Conversely: Let $x_n \rightarrow x_0$ implies $Tx_n \rightarrow Tx_0$. We need to show that T is continuous.

We will prove this by contradiction.

To prove T is continuous, suppose it is not true. Then there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is $x \neq x_0$ such that

$$d(x_n, x_0) < \delta \text{ but } \tilde{d}(Tx_n, Tx_0) \geq \varepsilon$$

in particular let $\delta = \frac{1}{n}$ then there is some x_n such that

$$d(x_n, x_0) < \frac{1}{n} \implies x_n \rightarrow x_0$$

but if we implement $d(x_n, x_0) < \delta$ then we get $\tilde{d}(Tx_n, Tx_0) \geq \varepsilon$

$$\implies Tx_n \not\rightarrow Tx_0$$

Contradiction occurs. Hence T is continuous.

Topic: 031

Examples (Completeness)

Real and Complex number spaces both are complete spaces.

Rational number space is not complete.

We will prove completeness of $\mathbb{R}$ (Real number space)

To prove this we will use the following results.

Lemma a:

Every Cauchy sequence in a metric space is bounded.

This result is satisfied for all metric spaces.

Lemma b:

If a Cauchy sequence has a subsequence that converges to $\bar{x}$, then the sequence converges to $\bar{x}$.

This condition is satisfied for Cauchy sequences only not for every sequence.

We will use these two Lemmas to prove that $\mathbb{R}$ is complete Space.

Proposition:

Every sequence of real numbers has a monotone subsequence.

Proof: Suppose the sequence $\{x_n\}$ has no monotone increasing subsequence. We will show that it has monotone decreasing

Subsequence. So the sequence x_n must have a first term say x_{n_1} such that all subsequent terms are smaller.

If $n > n_1 \Rightarrow x_n < x_{n_1}$

Otherwise $\{x_n\}$ would have a monotone increasing subsequence. Similarly, the remaining sequence $\{x_{n+1}, x_{n+2}, \dots\}$ must have first term.

Let first term of remaining sequence is $x_{n_2} < x_{n_1}$.

This process will continue as:

$x_{n_1} > x_{n_2} > x_{n_3} > x_{n_4} \dots$ and this is a monotone decreasing subsequence.

Now we will prove our main theorem. i.e

Theorem: R is a complete metric space.

i.e every Cauchy sequence in R converges.

Proof: Let $\{x_n\}$ be a Cauchy sequence.

Lemma a $\Rightarrow \{x_n\}$ is bounded.

$\Rightarrow$ every subsequence of $\{x_n\}$ is bounded.

Also $\{x_n\}$ has a monotone subsequence.

Now we will use monotone convergence theorem.

Theorem: If a sequence (a_n) is monotone and bounded then the sequence is convergent.

$\Rightarrow$ Subsequence $\{x_n\}$ is convergent.

$\Rightarrow R$ is Complete.

Topic: 032

Examples (Completeness)

Example: The Euclidean space R^n is complete.

Proof: We have R^n . The elements in R^n are n-tuples.

Let $x, y \in R^n$ then $x = (a_1, a_2, a_3, \dots, a_n)$, $y = (b_1, b_2, b_3, \dots, b_n)$

$$d(x, y) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \dots + (a_n - b_n)^2}$$

Let $\{x_m\}, \{x_r\}$ be Cauchy sequences in R^n .

We have to prove this sequence to be convergent.

Let $x_m = (a_1^{(m)}, a_2^{(m)}, \dots, a_n^{(m)})$ and

$x_r = (a_1^{(r)}, a_2^{(r)}, \dots, a_n^{(r)})$ be two sequences in R^n .

$$d(x_m, x_r) = \sqrt{(a_1^{(m)} - a_1^{(r)})^2 + \dots + (a_n^{(m)} - a_n^{(r)})^2} < \varepsilon$$

$$\forall \quad m, r > N$$

Due to the definition of Cauchy sequence.

Taking square on both sides, we get:

$$(a_1^{(m)} - a_1^{(r)})^2 + \dots + (a_n^{(m)} - a_n^{(r)})^2 < \varepsilon^2$$

We can say that every term $(a_j^{(m)} - a_j^{(r)})^2 < \varepsilon^2 \quad \forall m, r > N$

$$\Rightarrow |a_j^{(m)} - a_j^{(r)}| < \varepsilon \quad \forall m, r > N$$

$\Rightarrow$ for a fixed j $(a_j^{(1)}, a_j^{(2)}, \dots)$ is a Cauchy sequence in $\mathbb{R}$.

$\Rightarrow$ it is convergent in $\mathbb{R}$ because $\mathbb{R}$ is a complete metric space.

$$\Rightarrow a_j^{(m)} \rightarrow a_j, \text{ as } m \rightarrow \infty \quad a_j \in \mathbb{R} \quad j = 1, 2, 3, \dots, n$$

$$x = (a_1, a_2, a_3, \dots, a_n) \in \mathbb{R}^n$$

$$\Rightarrow d(x_m, x) \leq \varepsilon$$

$\Rightarrow x$ is a limit of x_m . Since (x_m) was general element.

$\Rightarrow$ Every sequence will converge in $\mathbb{R}^n$.

$\Rightarrow \mathbb{R}^n$ is complete.

Topic: 033

Examples (Completeness)

Example: The space l^∞ is complete.

Let (x_m) be a Cauchy sequence in l^∞ . The elements in l^∞ are all sequences of the form:

$$x = (a_1, a_2, \dots) \text{ such that } |a_j| < c_x$$

$$y = (b_1, b_2, \dots) \text{ such that } |b_j| < c_y$$

$$d(x, y) = \sup_{j \in \mathbb{N}} |a_j - b_j|$$

$$x_m = (a_1^{(m)}, a_2^{(m)}, \dots)$$

$$x_1 = (a_1^{(1)}, a_2^{(1)}, \dots) \quad , \quad x_2 = (a_1^{(2)}, a_2^{(2)}, \dots)$$

Since it is a cauchy sequence then by definition:

For any $\varepsilon > 0 \quad \exists N > 0$ such that for all $m, n > N$

$$d(x_m, x_n) = \sup_{j \in \mathbb{N}} |a_j^{(m)} - a_j^{(n)}| < \varepsilon$$

So if $\sup < \varepsilon$, for a fixed $j \quad |a_j^{(m)} - a_j^{(n)}| < \varepsilon \quad m, n \geq N$

$\Rightarrow$ for every fixed j the sequence $(a_j^{(1)}, a_j^{(2)}, \dots)$ is cauchy sequence of real numbers $\mathbb{R}$.

Since $\mathbb{R}$ is complete, $a_j^{(m)}$ is convergent in $\mathbb{R}$.

$$a_j^{(m)} \rightarrow a_j \in \mathbb{R} \text{ as } m \rightarrow \infty \quad j=1,2,3,\dots$$

For these infinite limits $a_1, a_2, \dots$ we define $x = (a_1, a_2, \dots)$
 $a_i \in \mathbb{R}$

We need to prove that $x = (a_1, a_2, \dots) \in l^\infty$ and $x_m \rightarrow x$.

$$|a_j^{(m)} - a_j^{(n)}| < \varepsilon$$

$$\Rightarrow |a_j^{(m)} - a_j| < \varepsilon \quad \text{as } n \rightarrow \infty$$

First we will prove $x_m \rightarrow x$.

$$d(x_m, x) = \sup |a_j^{(m)} - a_j| < \varepsilon \quad \text{as } n \rightarrow \infty$$

Hence, $x_m \rightarrow x$.

Now, we need to prove that $x = (a_1, a_2, \dots) \in l^\infty$. For this we have to prove that x is a bounded sequence.

Since $x_m = a_j^{(m)} \in l^\infty$

$|a_j^{(m)}| < km \quad \forall j$ because, it's dependent on cauchy sequence m .

$$|a_j| = |a_j - a_j^{(m)} + a_j^{(m)}|$$

$$|a_j| \leq |a_j - a_j^{(m)}| + |a_j^{(m)}|$$

$$|a_j| < \varepsilon + km$$

$\Rightarrow a_j$ is bounded.

Hence, $x = |a_j| \in l^\infty$

Topic: 034

Examples (Completeness)

Example: The function space $C[a,b]$ is complete. Here $[a,b]$ is any closed interval on $\mathbb{R}$.

Let (x_m) be a Cauchy sequence in $C[a,b]$.

Let $[a,b]=J$

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

