



Lecture Handouts

on

**VECTORS AND CLASSICAL MECHANICS
(MTH-622)**

**Virtual University of Pakistan
Pakistan**

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

Contents

Module No. 1.....	1
Introduction to the Course and Mathematics	1
Module No. 2.....	2
Scalar and Vector Fields	2
Module No. 3.....	4
The Operator Del and Gradient of Function.....	4
Module No. 4.....	6
Properties of the Gradient.....	6
Module No. 5.....	8
Directional Derivative.....	8
Module No. 6.....	10
Theorem related Directional Derivative	10
Module No. 7.....	11
Example of the Directional Derivative	11
Module No. 8.....	12
Related Problem 1 of the Directional Derivative	12
Module No. 9.....	13
Related problem 2 of Directional Derivative	13
Module No. 10.....	14
Related Problem 3 of the Directional Derivative	14
Module No. 11.....	16
Geometrical Interpretation of Gradient	16
Module No. 12.....	17
Theorem Related to Gradient	17
Module No. 13.....	19
Related Problem 1; Gradient	19
Module No. 14.....	20
Related Problem 2; Gradient	20
Module No. 15.....	21
Related Problem 3; Gradient	21
Module No. 16.....	23
Divergence of a Vector Point Function	23

Module No. 17.....	25
Properties of the Divergence	25
Module No. 18.....	27
Laplacian	27
Module No. 19.....	29
Example of Divergence	29
Module No. 20.....	30
Related Problem 1: Divergence	30
Module No. 21.....	33
Related Problem 2: Divergence	33
Module No. 22.....	35
Related Problem 3: Laplacian	35
Module No. 23.....	37
Curl of a vector Point Function	37
Module No. 24.....	39
Properties of the Curl	39
Module No. 25.....	42
Example of Curl	42
Module No. 26.....	43
Related Problem 1: Curl	43
Module No. 27.....	45
Related Problem 2: Curl	45
Module No. 28.....	48
Related Problem 3: Curl	48
Module No. 29.....	50
Vector Identities	50
Module No. 30.....	54
Line Integral	54
Module No. 31.....	56
Other forms and General Properties of line Integrals.....	56
Module No. 32.....	58
Example of Line Integral.....	58
Module No. 33.....	60

Example of Line Integral	60
Module No. 34	62
Line Integral Dependent on Path	62
Module No. 35	64
Independence of Path	64
Module No. 36	66
Theorems on line Integral	66
Module No. 37	70
Selected Example/Problem 1	70
Module No. 38	72
Selected Example/Problem 2: Line Integrals	72
Module No. 39	73
Selected Example/Problem 3: Line Integrals	73
Module No. 40	75
Selected Example/Problem 4: Line Integral	75
Module No. 41	77
Surface Integral	77
Module No. 42	79
Evaluation of the Surface Integral	79
Module No. 43	81
Example to the previous topic: Surface Integrals	81
Module No. 44	84
Further Example on Surface Integral	84
Module No. 45	86
Selected Example/Problem 1: Surface Integrals	86
Module No. 46	88
Selected Example/Problem 2: Surface Integral	88
Module No. 47	91
Selected Example/Problem 3: Surface Integral	91
Module No. 48	94
Volume Integral	94
Module No. 49	96
Example on Volume Integral	96

Module No. 50.....	98
Selected Example/Problem 1: Volume Integral	98

Module No. 1

Introduction to the Course and Mathematics

Introduction to Mathematics

“Mathematics is the branch of science which deals with the study of relations and patterns, and means to represent and communicate them.”

Introduction to the Course

There are two main portions of this course:

- Vectors
- Classical Mechanics

Module No. 2

Scalar and Vector Fields

Scalar Point Function

If to each point (x, y, z) of a region R in space there corresponds a scalar $\varphi(x, y, z)$, then φ is called a scalar point function in R .

Scalar Field

Scalar field is a function define on space whose value at each point is a scalar quantity
The set of all values of scalar point function φ in R together forms a Scalar field.

Examples of Scalar Fields

1. The temperature $T(x, y, z)$ within a body A is a scalar point function because there exist only one temperature at each point of A .
2. The pressure and potential due to gravity of the air in the earth's atmosphere define scalar field.

Vector Point Function

If to each point (x, y, z) of a region R in space there exist a unique vector $\vec{A}(x, y, z)$, then $\vec{A}$ is called a vector point function in R .

Vector Field

A function of a space whose value at each point is a vector quantity is called vector field.
Mathematically, we can write it as

$$\vec{A} = \vec{A}(x, y, z) = A_1(x, y, z) + A_2(x, y, z) + A_3(x, y, z)$$

The set of all values of $\vec{A}$ in R constitute a vector field.

Examples of Vector Field

1. $\vec{A} = \vec{A}(x, y, z) = xy\hat{i} - 2yz^3\hat{j} + y^2z\hat{k}$ defines a vector point function and hence is a vector field.
2. The motion of a moving fluid at any time define vector field.
3. The set of tangent vector of a curve C and the set of normal vectors of a surface S are examples of vector field.

Module No. 3

The Operator Del and Gradient of Function

The Del Operator

The vector differential operator del, symbolize as ∇ , is defined by

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

The symbol ∇ called **del** or **nabla** is used to symbolize del operator. It is only applied as defined derivative on one-dimensional function, and for more dimensions it may be applied as partial derivative on the function. The del operator is not a particular operator but when we applied it on a scalar point function or a vector point function; this may be known as the gradient, the divergence, and the curl.

Gradient: $\text{grad}(f) = \nabla \phi$

Curl: $\text{curl } \vec{A} = \nabla \times \vec{A}$

Divergence: $\text{div}(\vec{A}) = \nabla \cdot \vec{A}$

We will discuss here the first application of del operator as gradient.

Gradient Function

Let $\phi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. Then we can apply del operator on ϕ in order to obtain gradient of scalar function ϕ of $\text{grad}(\phi)$ written as $\nabla \phi$ is defined by

$$\begin{aligned}\nabla\varphi &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\varphi \\ &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\end{aligned}$$

it is to be noted that $\nabla\varphi$ defined a vector field. Also $\nabla\varphi = 0$ if and only if φ is constant.

Module No. 4

Properties of the Gradient

If φ and ψ are Scalar point functions as well as differentiable on a specific domain and C is a constant then we can show that the gradient holds the following properties.

- i. $\nabla(C\varphi) = C\nabla\varphi$
- ii. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$
- iii. $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$
- iv. $\nabla\left(\frac{\varphi}{\psi}\right) = \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}, \psi \neq 0$

Proof:

i. $\nabla(C\varphi) = C\nabla\varphi$

$$\begin{aligned}
 \text{L.H.S} &= \nabla(C\varphi) = \frac{\partial(C\varphi)}{\partial x}\hat{i} + \frac{\partial(C\varphi)}{\partial y}\hat{j} + \frac{\partial(C\varphi)}{\partial z}\hat{k} \\
 &= C\frac{\partial\varphi}{\partial x}\hat{i} + C\frac{\partial\varphi}{\partial y}\hat{j} + C\frac{\partial\varphi}{\partial z}\hat{k} \\
 &= C\left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) \\
 &= C\nabla\varphi = \text{R.H.S}
 \end{aligned}$$

ii. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

$$\begin{aligned}
 \text{L.H.S} &= \nabla(\varphi + \psi) = \frac{\partial(\varphi + \psi)}{\partial x}\hat{i} + \frac{\partial(\varphi + \psi)}{\partial y}\hat{j} + \frac{\partial(\varphi + \psi)}{\partial z}\hat{k} \\
 &= \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) + \left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}\right) \\
 &= \nabla\varphi + \nabla\psi = \text{R.H.S}
 \end{aligned}$$

iii. $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$

$$\text{L.H.S} = \nabla(\varphi\psi) = \frac{\partial(\varphi\psi)}{\partial x}\hat{i} + \frac{\partial(\varphi\psi)}{\partial y}\hat{j} + \frac{\partial(\varphi\psi)}{\partial z}\hat{k}$$

$$\begin{aligned}
&= \left(\varphi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \varphi}{\partial x} \right) \hat{i} + \left(\varphi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \varphi}{\partial y} \right) \hat{j} + \left(\varphi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \varphi}{\partial z} \right) \hat{k} \\
&= \varphi \left(\frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j} + \frac{\partial \psi}{\partial z} \hat{k} \right) + \psi \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \\
&= \varphi \nabla \psi + \psi \nabla \varphi = R.H.S
\end{aligned}$$

iv. $\nabla \left(\frac{\varphi}{\psi} \right) = \frac{\psi \nabla \varphi - \varphi \nabla \psi}{\psi^2}$

$$\begin{aligned}
L.H.S &= \nabla \left(\frac{\varphi}{\psi} \right) = \nabla \left(\varphi \frac{1}{\psi} \right) = \varphi \nabla \left(\frac{1}{\psi} \right) + \psi \nabla \left(\frac{1}{\varphi} \right) \\
&= \varphi \left(\frac{-1}{\psi^2} \right) \nabla \psi + \psi \nabla \left(\frac{1}{\varphi} \right) \\
&= \frac{\psi \nabla \varphi - \varphi \nabla \psi}{\psi^2} = R.H.S
\end{aligned}$$

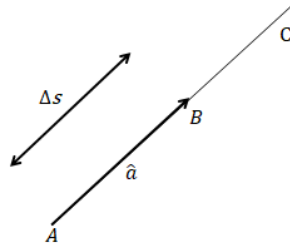
Module No. 5

Directional Derivative

The procedure to determine the derivative in a specific direction, other than the coordinate axes (x, y, z) is called **directional derivative**.

Let $\varphi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. The first partial derivatives of $\varphi(x, y, z)$ are the rate of change of φ in the direction of coordinate axes (x, y, z) . It is a restricted way to calculate the rate change is given function. Maybe one ought to need the derivative in a specific direction. Therefore the idea of directional derivative introduced.

To define the directional derivative we choose a point $A(x, y, z)$ in space and a direction at P , given by a unit vector $\hat{a}$. Let C be the ray drawn from P in the direction of $\hat{a}$, and let $A(x + \Delta x, y + \Delta y, z + \Delta z)$ denoted by B be a neighboring point on C , whose distance from P is Δs as shown in figure,



The value of given scalar point function is $\varphi(x, y, z)$ and $\varphi(+\Delta x, y + \Delta y, z + \Delta z)$ at P and P' respectively.

Then the limit

$$\lim_{\Delta s \rightarrow 0} \frac{\Delta \varphi}{\Delta s} = \lim_{\Delta s \rightarrow 0} \frac{\varphi(P') - \varphi(P)}{\Delta s}$$

if it exists, is called the directional derivative of φ at P in the direction of $\hat{a}$ and is denoted by $\frac{\partial \varphi}{\partial s}$.

Obviously,

$$\begin{aligned}
\frac{\partial \varphi}{\partial s} &= \frac{\partial \varphi}{\partial x} \frac{dx}{ds} + \frac{\partial \varphi}{\partial y} \frac{dy}{ds} + \frac{\partial \varphi}{\partial z} \frac{dz}{ds} \\
&= \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k} \right) \\
&= \nabla \varphi \cdot \frac{d\vec{r}}{ds} = \nabla \varphi \cdot \hat{a}
\end{aligned} \tag{1}$$

Since $\hat{a}$ is a unit vector, directional derivative of φ (i.e. $\frac{\partial \varphi}{\partial s}$) is the component of $\nabla \varphi$ in the direction of this unit vector.

From equation (1), we have the operator equivalence,

$$\frac{\partial}{\partial s} = \nabla \cdot \hat{a}$$

This means that the operator $\nabla \cdot \hat{a}$ applied to the scalar function φ differentiates it w.r.t the distance s in the direction of $\hat{a}$.

Deductions

In particular, if we assume $\hat{a}$ has the direction of the positive x-axis, the $\hat{a} = \hat{i}$ then equation (1) will become $\nabla \varphi \cdot \hat{i}$

$$\begin{aligned}
&= \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \cdot \hat{i} \\
\nabla \varphi \cdot \hat{i} &= \frac{\partial \varphi}{\partial x}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nabla \varphi \cdot \hat{j} &= \frac{\partial \varphi}{\partial y}, \\
\nabla \varphi \cdot \hat{k} &= \frac{\partial \varphi}{\partial z}
\end{aligned}$$

Module No. 6

Theorem related Directional Derivative

Statement

Show that the maximum value of the directional derivative of $\varphi(x, y, z)$ is equal to the magnitude of $\varphi(x, y, z)$ is equal to the magnitude of $\nabla\varphi$ (i.e. $|\nabla\varphi|$) and it takes place in the direction of $\nabla\varphi$.

Proof

We know that

$$\begin{aligned}\frac{\partial\varphi}{\partial s} &= \nabla\varphi \cdot \hat{a} \\ &= |\nabla\varphi| |\hat{a}| \cos\theta\end{aligned}$$

where θ is the angle between $\nabla\varphi$ and $\hat{a}$. Since $-1 \leq \cos\theta \leq 1$, therefore $\frac{\partial\varphi}{\partial s}$ is maximum when $\cos\theta = 1$ or $\theta = 0^\circ$ i.e. when the direction of $\hat{a}$ is the direction of $\nabla\varphi$ and $\max\left(\frac{\partial\varphi}{\partial s}\right) = |\nabla\varphi|$.

Thus the maximum value of directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude $|\nabla\varphi|$.

It is important to be note that directional derivative $\frac{\partial\varphi}{\partial s}$ is zero, when $\theta = 90^\circ$ i.e when $\nabla\varphi$ and $\hat{a}$ are orthogonal to each other.

Module No. 7

Example of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi \cdot \hat{a}$$

$$\nabla\varphi = \nabla(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\varphi \text{ at } (1, -2, -1) = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

then the required directional unit vector is

$$\begin{aligned}\nabla\varphi \cdot \hat{a} &= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) \\ &= \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}\end{aligned}$$

Since this is positive, φ is increasing in this direction.

Module No. 8

Related Problem 1 of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi \cdot \hat{a}$$

$$\nabla\varphi = \nabla(4xz^3 - 3x^2y^2z)$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 6x^2y^2)\hat{k}$$

$$\nabla\varphi \text{ at } (2, -1, 2) = 8\hat{i} + 24\hat{j} + 84\hat{k}$$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

then the required directional unit vector is

$$\begin{aligned}\nabla\varphi \cdot \hat{a} &= (8\hat{i} + 24\hat{j} + 84\hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \\ &= \frac{16}{7} - \frac{144}{7} + \frac{504}{7} = \frac{376}{7}\end{aligned}$$

Since this is positive, φ is increasing in this direction.

Module No. 9

Related problem 2 of Directional Derivative

Problem Statement

- In what direction from the point $(2, 1, -1)$ is the directional derivative of $\varphi = x^2yz^3$ a maximum?
- What is the magnitude of this maximum?

Solution

First of all, we calculate gradient of $\varphi = x^2yz^3$

$$\nabla\varphi = \nabla(x^2yz^3) = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (3x^2yz^2)\hat{k}$$

$\nabla\varphi$ at $(2, 1, -1)$ is $-4\hat{i} - 4\hat{j} + 12\hat{k}$

using the result that directional derivative is maximum in the direction of $\nabla\varphi$ and its maximum magnitude is $|\nabla\varphi|$

The directional derivative is a maximum in the direction $\nabla\varphi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

the magnitude of this maximum is $|\nabla\varphi| = |(-4)^2 + (-4)^2 + (12)^2| = \sqrt{176} = 4\sqrt{11}$

Module No. 10

Related Problem 3 of the Directional Derivative

Problem Statement:

Find the values of the constants a, b, c so that the directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Solution:

Since $\varphi = axy^2 + byz + cz^2x^3$, therefore

$$\begin{aligned}\nabla\varphi &= \nabla(axy^2 + byz + cz^2x^3) \\ &= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}\end{aligned}$$

At the point $(1, 2, -1)$, the value of this gradient is

$$= (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$$

We know that the maximum directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude of $\nabla\varphi$. The maximum directional derivative will be parallel to the z-axis if

$$4a + 3c = 0 \tag{1}$$

$$4a - b = 0 \tag{2}$$

Therefore

$$\nabla\varphi = (2b - 2c)\hat{k}$$

since the magnitude of this maximum directional derivative is 64, there fore

$$2b - 2c = 64 \tag{3}$$

Solving equation (1), (2) and (3), we find

$$a = 6, b = 24, c = -8$$

which is the required solution.

Module No. 11

Geometrical Interpretation of Gradient

Geometrically, gradient of a scalar function represents a normal vector to the surface.

$\nabla\varphi|_{(x,y,z)}$ represents the normal vector of the surface at (x, y, z) .

Example

If $\varphi = x^2yz$

$$\nabla\varphi = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

At $(1,1,1)$

$$\nabla\varphi|_{(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$$

which is normal to the surface $\varphi = x^2yz$

Module No. 12

Theorem Related to Gradient

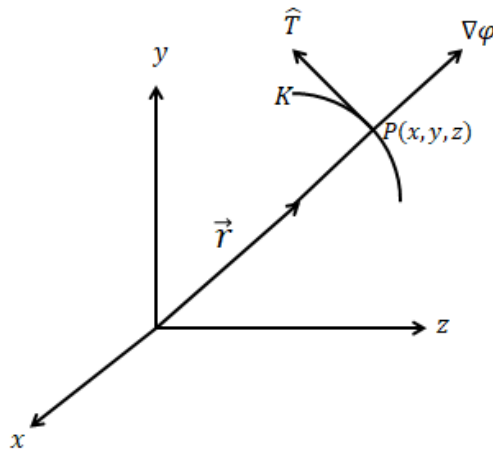
Theorem

Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$, where C is a constant.

Proof

Let the parametric equation of the curve K be $x = x(s)$, $y = y(s)$, $z = z(s)$.

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the point $P(x, y, z)$, then $\hat{T} = \frac{dr}{ds}$ is unit tangent vector to the curve K at P as shown in figure.



Since the curve K lies on the level surface, together, the condition of any point on the curve must satisfy equation (1), and so $[x(s), y(s), z(s)] = C$.

Differentiating above equation w.r.t 's' using the chain rule,

$$\frac{\partial\phi}{\partial x} \frac{dx}{ds} + \frac{\partial\phi}{\partial y} \frac{dy}{ds} + \frac{\partial\phi}{\partial z} \frac{dz}{ds} = 0$$

or

$$= \left(\frac{\partial\phi}{\partial x} \hat{i} + \frac{\partial\phi}{\partial y} \hat{j} + \frac{\partial\phi}{\partial z} \hat{k} \right) \cdot \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k} \right) = 0$$

or

$$\nabla\phi \cdot \frac{dr}{ds} = 0 \text{ or } \nabla\phi \cdot \hat{T} = 0$$

Which implies that $\nabla\varphi$ is a vector perpendicular to the unit tangent vector $\hat{T}$ and therefore on the surface $\varphi(x, y, z) = C$.

Module No. 13

Related Problem 1; Gradient

Problem Statement

If $\vec{A} = 2x^2i - 3yzj + xz^2k$ and $\varphi = 2z - x^3y$

Find

- i. $\vec{A} \cdot \nabla \varphi$
- ii. $\vec{A} \times \nabla \varphi$

Solution

Since $\varphi = 2z - x^3y$ then

$$\begin{aligned}\nabla \varphi &= \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \\ &= \frac{\partial (2z - x^3y)}{\partial x} \hat{i} + \frac{\partial (2z - x^3y)}{\partial y} \hat{j} + \frac{\partial (2z - x^3y)}{\partial z} \hat{k} \\ \nabla \varphi &= -3x^2yi - x^3j + 2k\end{aligned}$$

$\nabla \varphi$ at $(1, -1, 1)$

$$\nabla \varphi = 3i - j + 2k$$

Given vector $\vec{A} = 2x^2i - 3yzj + xz^2k$ at $(1, -1, 1)$

$$\vec{A} = 2i + 3j + k$$

- i. $\vec{A} \cdot \nabla \varphi = (2i + 3j + k) \cdot (3i - j + 2k)$
 $(2)(3) - (3)(1) + (1)(2) = 5$
- ii. $\vec{A} \times \nabla \varphi = (2i + 3j + k) \times (3i - j + 2k)$

$$\begin{aligned}&= \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix} \\ &= (6 - (-1))i - (4 - 3)j + (-2 - 9)k \\ &= 7i - j - 11k\end{aligned}$$

Module No. 14

Related Problem 2; Gradient

Statement

Show that

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

Proof

We know that

$$\begin{aligned}\nabla f(r) &= \frac{\partial}{\partial x} f(r)\hat{i} + \frac{\partial}{\partial y} f(r)\hat{j} + \frac{\partial}{\partial z} f(r)\hat{k} \\ &= \frac{\partial f}{\partial x} \frac{\partial r}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \frac{\partial r}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \frac{\partial r}{\partial z} \hat{k} \\ &= \frac{df}{dx} \frac{\partial r}{\partial x} \hat{i} + \frac{df}{dy} \frac{\partial r}{\partial y} \hat{j} + \frac{df}{dz} \frac{\partial r}{\partial z} \hat{k}\end{aligned}\quad (1)$$

since $r = \sqrt{x^2 + y^2 + z^2}$, it follows that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

putting these values in (1), we get

$$\begin{aligned}&= f'(r) \frac{x}{r} \hat{i} + f'(r) \frac{y}{r} \hat{j} + f'(r) \frac{z}{r} \hat{k} \\ &= \frac{f'(r)}{r} (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \frac{f'(r)\vec{r}}{r}\end{aligned}$$

Hence Proved

Module No. 15

Related Problem 3; Gradient

Statement

If $\nabla\varphi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$,

Find

$\varphi(x, y, z)$ if $\varphi(1, -2, 2) = 4$

Solution

Given equation is

$$\nabla\varphi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k} \quad (1)$$

We know that

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \quad (2)$$

therefore by comparing equation(1) and (2), we get

$$\frac{\partial\varphi}{\partial x} = 2xyz^3 \quad (3)$$

$$\frac{\partial\varphi}{\partial y} = x^2z^3 \quad (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2yz^2 \quad (5)$$

Integrating Equation (3) w.r.t x , keeping y and z constants

$$\varphi = x^2yz^3 + f(y, z) \quad (6)$$

Similarly, from equation (4) and (5), we get

$$\varphi = x^2yz^3 + g(x, z) \quad (7)$$

$$\varphi = x^2yz^3 + h(x, y) \quad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$f(y, z) = g(x, z) = h(x, y) = C$$

where C is an arbitrary constant.

Thus

$$\varphi = x^2yz^3 + C \quad (9)$$

using $\varphi(1, -2, 2) = 4$ in equation (9) we get

$$C = 20$$

hence from equation (9), we obtained

$$\varphi = x^2yz^3 + 20$$

Which is the required solution of φ .

Module No. 16

Divergence of a Vector Point Function

Definition

Divergence is a vector operator, when we applied this operator to the quantity of a vector field, it produces a scalar field.

Let $\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. V defines a differentiable vector field). Then the divergence of V , written $\nabla \cdot V$ or $\text{div } V$, is defined by

$$\begin{aligned}\nabla \cdot \vec{V} &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (V_1\hat{i} + V_2\hat{j} + V_3\hat{k}) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\end{aligned}$$

Some Deductions

It is important to be noted that $\nabla \cdot \vec{V}$ is a scalar quantity. Also $\nabla \cdot \vec{V} \neq \vec{V} \cdot \nabla$

If $\vec{V}$ is a constant vector, then $\nabla \cdot \vec{V} = 0$.

if $\nabla \cdot \vec{V} = 0$ everywhere in some region of R , Then $\vec{V}$ is called **solenoid vector point function** in the region.

We can consider an example of solenoid to understand the concept.

Example

Determine the constant a so that the vector $\vec{V} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

Solution

$$\begin{aligned}\nabla \cdot V &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot ((x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}) \\ &= \frac{\partial(x + 3y)}{\partial x} + \frac{\partial(y - 2z)}{\partial y} + \frac{\partial(x + az)}{\partial z}\end{aligned}$$

$$= 1 + 1 + a = 2 + a$$

$$a = -2$$

Hence if we substitute $a = -2$ in given vector field then given $\vec{V}$ will become solenoidal.

Module No. 17

Properties of the Divergence

Statement

If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and φ is differentiable scalar point function, then prove that

- i. $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
- ii. $\nabla \cdot (\varphi \vec{A}) = \varphi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)$

Proof

Let

$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

i. $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

$$L.H.S = (\vec{A} + \vec{B}) = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

hence

$$\begin{aligned} \nabla \cdot (\vec{A} + \vec{B}) &= \frac{\partial}{\partial x} (A_1 + B_1)\hat{i} + \frac{\partial}{\partial y} (A_2 + B_2)\hat{j} + \frac{\partial}{\partial z} (A_3 + B_3)\hat{k} \\ &= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) \\ &= \nabla \cdot \vec{A} + \nabla \cdot \vec{B} = R.H.S \end{aligned}$$

ii. $\nabla \cdot (\varphi \vec{A}) = \varphi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)$

$$L.H.S = \nabla \cdot (\varphi \vec{A})$$

$$\text{We have } \varphi \vec{A} = \varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k}$$

Hence

$$\begin{aligned} \nabla \cdot (\varphi \vec{A}) &= \frac{\partial}{\partial x} (\varphi A_1)\hat{i} + \frac{\partial}{\partial y} (\varphi A_2)\hat{j} + \frac{\partial}{\partial z} (\varphi A_3)\hat{k} \\ &= \varphi \frac{\partial A_1}{\partial x} + A_1 \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial A_2}{\partial y} + A_2 \frac{\partial \varphi}{\partial y} + \varphi \frac{\partial A_3}{\partial z} + A_3 \frac{\partial \varphi}{\partial z} \end{aligned}$$

$$\begin{aligned}
&= \varphi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + (A_1 \frac{\partial \varphi}{\partial x} + A_2 \frac{\partial \varphi}{\partial y} + A_3 \frac{\partial \varphi}{\partial z}) \\
&= \varphi(\nabla \cdot \vec{A}) + \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \\
&\nabla \cdot (\varphi \vec{A}) = \varphi(\nabla \cdot \vec{A}) + (\nabla \varphi) \cdot \vec{A} = \varphi(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)
\end{aligned}$$

$$\nabla \cdot (\varphi \vec{A}) = \varphi(\nabla \cdot \vec{A}) \text{ if } \varphi \text{ is constant.}$$

Module No. 18

Laplacian

The Laplacian or Laplace operator is a second-order differential operator given by the divergence of the gradient of a given function defined over a space R . It is usually denoted by $\nabla \cdot \nabla$, or ∇^2 or Δ .

Thus if $\vec{u}$ is a twice differentiable function, then the Laplacian of $\vec{u}$ is defined by

$$\Delta \vec{u} = \nabla^2 \vec{u} = \nabla \cdot \nabla \vec{u}$$

In cartesian coordinate system, the Laplacian is given by the sum of second order partial derivatives of the function w.r.t each independent variable. Laplace is a Second order differential operator which is obtained by taking the divergence of gradient of any scalar point function.

It is denoted as

$$\begin{aligned} \nabla \cdot \nabla &= \nabla^2 = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

It is a scalar operator.

In one and two dimension, the Laplace operator reduces to

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

If $\varphi(x, y, z)$ is a scalar point function, then the divergence of gradient of φ written as $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$ is called the Laplacian of φ and the equation $\nabla^2 \varphi = 0$ is called Laplace's equation.

If a scalar function φ satisfies the Laplace equation $\nabla^2 \varphi = 0$ in a Cartesian region R , then φ is said to be a harmonic function in the region R .

Mathematically, we can write it as

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

We can also express the Laplace operator in polar coordinate (r, θ) notation.

Two dimensional Laplace operators can be expressed as

$$\begin{aligned}\nabla^2\varphi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right) \\ &= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\end{aligned}$$

The Laplacian occurs in differential equation that describes many physical phenomena, such as Diffusion equation for heat and fluid flow, gravitational potentials and quantum mechanics.

Module No. 19

Example of Divergence

Statement

If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \vec{A}$ (or div A) at the point $(1, -1, 1)$.

Solution

As we know del operator ∇ is

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Since given vector point function $\vec{A}$ is

$$\begin{aligned}\vec{A} &= x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k} \\ \nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}) \\ &= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z} \\ &= 2xz - 6y^2z^2 + xy^2\end{aligned}$$

As we calculated the expression $\nabla \cdot \vec{A}$, we can easily determine its value at $(1, -1, 1)$.

$$\begin{aligned}\nabla \cdot \vec{A} \text{ at } (1, -1, 1) &= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 \\ &= 2 - 6 + 1 = -3 \\ \nabla \cdot \vec{A} \text{ at } (1, -1, 1) &= -3\end{aligned}$$

which is the required Result.

Module No. 20

Related Problem 1: Divergence

Statement

If $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$ and $\varphi = 3x^2 - yz$,

find

- i. $\nabla \cdot \vec{A}$
- ii. $\vec{A} \cdot \nabla \varphi$
- iii. $\nabla \cdot (\varphi \vec{A})$
- iv. $\nabla \cdot (\nabla \varphi)$

at the point $(1, -1, 1)$.

Solution

Since $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$

also

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

then

i. $\nabla \cdot \vec{A}$

$$\begin{aligned}\nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \\ &= 3yz^2 + 6xy^2\hat{j} - x^2y\end{aligned}$$

$\nabla \cdot \vec{A}$ at point $(1, -1, 1)$ is

$$\begin{aligned}&= 3(-1)(1)^2 + 6(1)(-1)^2 - (1)^2(-1) \\ &= -3 + 6 - 1 = 2\end{aligned}$$

ii. $\vec{A} \cdot \nabla \varphi$

Since $\varphi = 3x^2 - yz$

then

$$\begin{aligned}
\nabla\varphi &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \\
&= \frac{\partial(3x^2 - yz)}{\partial x}\hat{i} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{j} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k} \\
&= 6x\hat{i} - z\hat{j} - y\hat{k}
\end{aligned}$$

Now,

$$\begin{aligned}
\vec{A} \cdot \nabla\varphi &= (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \cdot (6x\hat{i} - z\hat{j} - y\hat{k}) \\
&= (18x^2yz^2 - 2xy^3z + x^2y^2z)
\end{aligned}$$

$\vec{A} \cdot \nabla\varphi$ at point $(1, -1, 1)$ is

$$\begin{aligned}
&= [18(1)^2(-1)(1)^2 - 2(1)(-1)^3(1) + (1)^2(-1)^2(1)] \\
&= -18 + 2 + 1 = -15
\end{aligned}$$

iii. $\nabla \cdot (\varphi\vec{A})$

then

$$\begin{aligned}
\varphi\vec{A} &= (3x^2 - yz)(3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \\
&= (9x^3yz^2 - 3xy^2z^3)\hat{i} + (6x^3y^3 - 2xy^4z)\hat{j} + (3x^4yz - x^2y^2z^2)\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
\nabla \cdot (\varphi\vec{A}) &= \nabla \cdot ((9x^3yz^2 - 3xy^2z^3)\hat{i} + (6x^3y^3 - 2xy^4z)\hat{j} + (3x^4yz - x^2y^2z^2)\hat{k}) \\
&= \frac{\partial(9x^3yz^2 - 3xy^2z^3)}{\partial x} + \frac{\partial(6x^3y^3 - 2xy^4z)}{\partial y} + \frac{\partial(3x^4yz - x^2y^2z^2)}{\partial z} \\
&= 27x^2yz^2 - 3y^2z^3 + 18x^3y^2 - 8xy^3z + 3x^4y - 2x^4yz
\end{aligned}$$

$\nabla \cdot (\varphi\vec{A})$ at point $(1, -1, 1)$ is

$$\nabla \cdot (\varphi\vec{A}) = -27 - 3 + 18 + 8 - 3 + 2 = -5$$

iv. $\nabla \cdot (\nabla\varphi)$

$$\begin{aligned}
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \\
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(\frac{\partial(3x^2 - yz)}{\partial x}\hat{i} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{j} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k} \right)
\end{aligned}$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (6x\hat{i} - z\hat{j} - y\hat{k}) = 6$$

Module No. 21

Related Problem 2: Divergence

Problem Statement

Given

$$\varphi = 2x^3y^2z^4.$$

To find

- i. $\nabla \cdot \nabla \varphi$ (or $\text{div grad} \varphi$).
- ii. Show that $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution

i. $\nabla \cdot \nabla \varphi$

As we know the grad of φ

$$\begin{aligned}\nabla \varphi &= \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \\ &= \frac{\partial(2x^3y^2z^4)}{\partial x} \hat{i} + \frac{\partial(2x^3y^2z^4)}{\partial y} \hat{j} + \frac{\partial(2x^3y^2z^4)}{\partial z} \hat{k} \\ &= 6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}\end{aligned}$$

Then the divergence of the grad of φ

$$\begin{aligned}\nabla \cdot \nabla \varphi &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial z} \right) \\ &= \frac{\partial(6x^2y^2z^4)}{\partial x} + \frac{\partial(4x^3yz^4)}{\partial y} + \frac{\partial(8x^3y^2z^3)}{\partial z} \\ &= 12xy^2z^4 + 4x^3yz^4 + 24x^3y^2z^2\end{aligned}$$

ii. $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$

$$\text{L.H.S} = \nabla \cdot \nabla \varphi$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right)$$

$$\begin{aligned} &= \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial z} \right) \\ &= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi \\ &= \nabla^2 \varphi \end{aligned}$$

Hence proved.

Module No. 22

Related Problem 3: Laplacian

Problem Statement

Show that

- i. $\nabla \cdot (r^3 \vec{r})$
- ii. $\nabla \cdot \left[\vec{r} \nabla \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$
- iii. $\nabla \cdot \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right] = -\frac{2\vec{r}}{r^3}$

where $\vec{r}$ is position vector.

Solution

First of all we will derive a general relation for $\nabla \cdot [\vec{r} f(r)]$ then we will substitute our given values to evaluate our required result.

$$\nabla \cdot [\vec{r} f(r)] = f(r)(\nabla \cdot \vec{r}) + \vec{r} \cdot (\nabla f(r))$$

$$\therefore \nabla \cdot \vec{r} = 3$$

We have

$$= 3f(r) + \vec{r} f'(r) \hat{r} = 3f(r) + \vec{r} \cdot \frac{f'(r) \vec{r}}{r}$$

$$\therefore \hat{r} = \frac{\vec{r}}{r}$$

$$= 3f(r) + \frac{f'(r)}{r} \vec{r} \cdot \vec{r}$$

$$\therefore \vec{r} \cdot \vec{r} = r^2$$

$$\nabla \cdot [\vec{r} f(r)] = 3f(r) + r f'(r)$$

Now setting $f(r) = r^n$ in above relation, we obtain

$$\nabla \cdot (\vec{r} r^n) = 3r^n + r(nr^{n-1})$$

$$\nabla \cdot (\vec{r} r^n) = 3r^n + (nr^n)$$

$$\nabla \cdot (r^n \vec{r}) = (3 + n)r^n$$

- i. We have $\nabla \cdot (r^3 \vec{r})$ to evaluate, it is of the form $\nabla \cdot [\vec{r} f(r)]$, where $\square(r) = r^3$

Put $n = 3$ in above relation, we will obtain the required result

$$\nabla \cdot (r^3 \vec{r}) = (3 + 3)r^3 = 6r^3$$

Is the required solution.

$$\text{ii. } L.H.S = \nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right]$$

We can also write it as

$$= \nabla \cdot [r(\nabla r^{-3})] = \nabla \cdot [r(-3r^{-5} \vec{r})]$$

where we used the result

$$\nabla f(r) = \frac{f'(r) \vec{r}}{r}$$

We have $\nabla \cdot [r(-3r^{-5} \vec{r})]$ to evaluate, it is of the form $\nabla \cdot [\vec{r} f(r)]$, where $f(r) = r^{-4} \vec{r}$ thus,

$$\nabla \cdot \left[\vec{r} \nabla \left(\frac{1}{r^3} \right) \right] = -3 \nabla \cdot (r^{-4} \vec{r}) = -3(-4 + 3)r^{-4} = \frac{3}{r^4} = R.H.S$$

is the required solution.

$$\text{iii. } L.H.S = \nabla \cdot \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right]$$

frist we evaluate $\nabla \cdot \left(\frac{\vec{r}}{r} \right)$ using formula $\nabla \cdot [\vec{r} f(r)]$, where $f(r) = r^{-1} \vec{r}$

$$\nabla \cdot (r^n \vec{r}) = (3 + n)r^n$$

put $n = 1$, we have

$$\nabla \cdot (r^{-1} \vec{r}) = (3 - 1)r^{-1} = \frac{2}{r}$$

therefore

$$= \nabla \cdot \left(\frac{2}{r} \right) = -2r^{-3} \vec{r} = -\frac{2\vec{r}}{r^3} = R.H.S$$

Module No. 23

Curl of a vector Point Function

Introduction

The infinitesimal rotation of a vector field is described by the curl. The curl of a specific point is shown by a vector at every point of the vector field. The attributes of this vector (length and direction) characterize the rotation at that point.

The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.

Definition

If $\vec{V}(x, y, z)$ is a differentiable vector field in a certain region of space, then the curl or rotation of V , written $\nabla \times \vec{V}$, curl V or rot V , is defined by

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}).$$

Where V_1, V_2, V_3 are the components of vector field along x, y and z-axis.

We can write this expression in matrix form

$$\begin{aligned} \nabla \times \vec{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \hat{k} \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \hat{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z} \right) \hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \hat{k} \end{aligned}$$

A vector field whose curl is zero is called irrotational.

Identities of Curl

Some identities of curl are stated below

A gradient has zero curl:

$$\nabla \times \nabla \phi = 0.$$

A curl has zero divergence:

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Note: $\nabla \cdot (\nabla \times \vec{V}) = (\nabla \times \nabla) \cdot \vec{V} = 0$ because $\nabla \times \nabla = 0$.

Module No. 24

Properties of the Curl

If $\vec{A}$ and $\vec{B}$ are differentiable vector functions, and φ is differentiable scalar functions of position (x, y, z) , then

- i. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$ or $\text{curl } (\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$
- ii. $\nabla \times (\varphi \vec{A}) = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}$
- iii. $\nabla \times (\nabla \varphi) = 0$ The curl of the gradient of φ is zero.
- iv. $\nabla \cdot (\nabla \times \vec{A}) = 0$ The divergence of the curl of $\vec{A}$ is zero.

Proof

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

i. **$\text{curl } (\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$**

L.H. $S = \text{curl } (\vec{A} + \vec{B})$

$$\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

Hence by using the definition of curl

$$\begin{aligned} \nabla \times (\vec{A} + \vec{B}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= \nabla \times \vec{A} + \nabla \times \vec{B} \end{aligned}$$

Hence proved

ii. **$\nabla \times (\varphi \vec{A}) = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}$**

L.H. $S = \nabla \times (\varphi \vec{A})$

$$\varphi \vec{A} = \varphi (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = \varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k},$$

Then by using the definition of curl

$$\begin{aligned}
\nabla \times (\varphi \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi A_1 & \varphi A_2 & \varphi A_3 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y} (\varphi A_3) - \frac{\partial}{\partial z} (\varphi A_2) \right) \hat{i} - \left(\frac{\partial}{\partial x} (\varphi A_3) - \frac{\partial}{\partial z} (\varphi A_1) \right) \hat{j} + \left(\frac{\partial}{\partial x} (\varphi A_2) - \frac{\partial}{\partial y} (\varphi A_1) \right) \hat{k} \\
&= \left[\varphi \frac{\partial(A_3)}{\partial y} + A_3 \frac{\partial(\varphi)}{\partial y} - \varphi \frac{\partial(A_2)}{\partial z} - A_2 \frac{\partial(\varphi)}{\partial z} \right] \hat{i} - \left[\varphi \frac{\partial(A_3)}{\partial x} + A_3 \frac{\partial(\varphi)}{\partial x} - \varphi \frac{\partial(A_1)}{\partial z} - A_1 \frac{\partial(\varphi)}{\partial z} \right] \hat{j} \\
&\quad + \left[\varphi \frac{\partial(A_2)}{\partial x} + A_2 \frac{\partial(\varphi)}{\partial x} - \varphi \frac{\partial(A_1)}{\partial y} - A_1 \frac{\partial(\varphi)}{\partial y} \right] \hat{k} \\
&= \varphi \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \right] + \left[(A_3 \frac{\partial \varphi}{\partial y} - A_2 \frac{\partial \varphi}{\partial z}) \hat{i} + (A_3 \frac{\partial \varphi}{\partial x} - A_1 \frac{\partial \varphi}{\partial z}) \hat{j} + (A_2 \frac{\partial \varphi}{\partial x} - A_1 \frac{\partial \varphi}{\partial y}) \hat{k} \right] \\
&= \varphi (\nabla \times \vec{A}) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}
\end{aligned}$$

Hence Proved

iii. $\nabla \times (\nabla \varphi) = \mathbf{0}$

$$\begin{aligned}
\text{L.H.S.} = \nabla \times (\nabla \varphi) &= \nabla \times \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) \hat{i} - \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) \hat{j} + \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) \hat{k} \quad (1)
\end{aligned}$$

We assume that φ has continuous second order partial derivative so the order of differentiation can be neglected.

i.e.

$$\frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi}{\partial z \partial y}, \frac{\partial^2 \varphi}{\partial z \partial x} = \frac{\partial^2 \varphi}{\partial x \partial z}, \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$$

$\Rightarrow$ equation (1) must be equal to zero

Hence the expression.

iv. $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\text{L.H.S.} = \nabla \cdot (\nabla \times \vec{A})$$

Since

$$\begin{aligned}\nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k}\end{aligned}$$

$$\begin{aligned}\text{Hence } \nabla \cdot (\nabla \times \vec{A}) &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\ &= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} \right) - \left(\frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z} \right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} \right) \\ &= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0\end{aligned}$$

Assuming that $\vec{A}$ has continuous second order derivative.

Hence the theorem.

Module No. 25

Example of Curl

Statement

If $\vec{A} = xz^3 \hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$,

Find

$\nabla \times \vec{A}$ (or curl $\vec{A}$) at the point $(1, -1, 1)$.

Solution

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times (xz^3 \hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k})$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z} \right) \hat{i} + \left(\frac{\partial(2yz^4)}{\partial x} - \frac{\partial(xz^3)}{\partial z} \right) \hat{j} + \left(\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y} \right) \hat{k}$$

$$\nabla \times \vec{A} = (2z^4 + 2x^2yz)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

Now we calculate the value of $\nabla \times \vec{A}$ at $(1, -1, 1)$

$$\nabla \times \vec{A} = (2 - 2)\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\nabla \times \vec{A} = 3\hat{j} + 4\hat{k}$$

is the required solution.

Module No. 26

Related Problem 1: Curl

Problem Statement

If $A = 2yz \, i - x^2y \, j + xz^2 \, k$ and $\varphi = 2x^2yz^3$,

Find

- i. $(A \times \nabla)\varphi$
- ii. $A \times (\nabla\varphi)$

Also show whether they are identical or not.

Solution

Since $A = 2yz \, i - x^2y \, j + xz^2 \, k$ and $\varphi = 2x^2yz^3$

- i. $(A \times \nabla)\varphi$

$$(A \times \nabla)\varphi = \left[(2yz \, i - x^2y \, j + xz^2 \, k) \times \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \right] \varphi$$

$$(A \times \nabla)\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \varphi$$

$$= \left[\left(-x^2y \frac{\partial}{\partial z} - xz^2 \frac{\partial}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial}{\partial z} - xz^2 \frac{\partial}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial}{\partial y} + x^2y \frac{\partial}{\partial x} \right) \hat{k} \right] \varphi$$

Now substituting φ

$$= \left[\left(-x^2y \frac{\partial \varphi}{\partial z} - xz^2 \frac{\partial \varphi}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial \varphi}{\partial z} - xz^2 \frac{\partial \varphi}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial \varphi}{\partial y} + x^2y \frac{\partial \varphi}{\partial x} \right) \hat{k} \right]$$

Since $\varphi = 2x^2yz^3$, substituting value

$$= \left[\left(-x^2y \frac{\partial(2x^2yz^3)}{\partial z} - xz^2 \frac{\partial(2x^2yz^3)}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial(2x^2yz^3)}{\partial z} - xz^2 \frac{\partial(2x^2yz^3)}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial(2x^2yz^3)}{\partial y} + x^2y \frac{\partial(2x^2yz^3)}{\partial x} \right) \hat{k} \right]$$

$$\begin{aligned}
&= \left[\{-x^2y(6x^2yz^2) - xz^2(2x^2z^3)\}\hat{i} - \{2yz(6x^2yz^2) - xz^2(4xyz^3)\}\hat{j} \right. \\
&\quad \left. + \{2yz(2x^2z^3) + x^2y(4xyz^3)\}\hat{k} \right] \\
(A \times \nabla)\varphi &= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} - (12x^2y^2z^3 - 4x^2yz^5)\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k} \\
(1)
\end{aligned}$$

ii. $A \times (\nabla\varphi)$

Here $A = 2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}$ and $\varphi = 2x^2yz^3$

$$\begin{aligned}
\nabla\varphi &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \\
&= \frac{\partial(2x^2yz^3)}{\partial x}\hat{i} + \frac{\partial(2x^2yz^3)}{\partial y}\hat{j} + \frac{\partial(2x^2yz^3)}{\partial z}\hat{k} \\
\nabla\varphi &= 4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
A \times (\nabla\varphi) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix} \\
&= [(-x^2y)(6x^2yz^2) - (xz^2)(2x^2z^3)]\hat{i} - [(2yz)(6x^2yz^2) - (xz^2)(4xyz^3)]\hat{j} \\
&\quad + ((2yz)(2x^2z^3) - (-x^2y)(4xyz^3))\hat{k}
\end{aligned}$$

$$\begin{aligned}
A \times (\nabla\varphi) &= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} - (12x^2y^2z^3 - 4x^2yz^5)\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k} \\
(2)
\end{aligned}$$

From equation (1) and (2), we conclude that

$$A \times (\nabla\varphi) = (A \times \nabla)\varphi$$

Module No. 27

Related Problem 2: Curl

Problem Statement

A vector V is called irrotational if $\text{curl} v = 0$.

- i. Find constants a, b, c so that

$$\vec{A} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational.

- ii. Show that $\vec{A}$ can be expressed as the gradient of a scalar function.

Solution

Since give vector field is

$$\vec{A} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

- i. Also for a vector to be irrotational, we have expression

$$\text{curl} \vec{A} = \nabla \times \vec{A} = 0$$

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z) \right) \hat{i} - \left(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az) \right) \hat{j} \\ &\quad + \left(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az) \right) \hat{k} \\ &= (c - 1)\hat{i} - (4 - a)\hat{j} + (b - 2)\hat{k} \end{aligned}$$

or

$$= (c - 1)\hat{i} + (a - 4)\hat{j} + (b - 2)\hat{k}$$

According to the given condition

$$(c - 1)\hat{i} + (a - 4)\hat{j} + (b - 2)\hat{k} = 0$$

$$\Rightarrow c - 1 = 0 \Rightarrow c = 1$$

$$\Rightarrow a - 4 = 0 \Rightarrow a = 4$$

$$\Rightarrow b - 2 = 0 \Rightarrow b = 2$$

and thus

$$\vec{A} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x + y + 2z)\hat{k} \quad (1)$$

is irrotational vector field.

ii. Assume that

$$\vec{A} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \quad (2)$$

By comparing equation (1) and (2), we obtain

$$\frac{\partial\varphi}{\partial x} = x + 2y + 4z \quad (3)$$

$$\frac{\partial\varphi}{\partial y} = 2x - 3y - z \quad (4)$$

$$\frac{\partial\varphi}{\partial z} = 4x + y + 2z \quad (5)$$

Integrating equation (3) w.r.t x , keeping y and z constants,

$$\varphi = \frac{x^2}{2} + 2xy + 4xz + f(y, z) \quad (6)$$

$$\varphi = 2xy - \frac{3y^2}{2} - yz + g(x, z) \quad (7)$$

$$\varphi = 4xz - yz + z^2 + h(x, y) \quad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$f(y, z) = -\frac{3y^2}{2} + z^2$$

$$g(x, z) = \frac{x^2}{2} + z^2$$

$$h(x, y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\varphi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{constant}$$

Note that we can also add any constant to φ . In general if $\nabla \times \vec{A} = 0$, then we can find φ so that $\vec{A} = \nabla\varphi$. A vector field $\vec{A}$ which can be derived from a scalar field φ so that $\vec{A} = \nabla\varphi$ is called a conservative vector field and φ is called the scalar potential.

Note that conversely if $\vec{A} = \nabla\varphi$, then $\nabla \times \vec{A} = 0$

Module No. 28

Related Problem 3: Curl

Problem Statement

If $v = \omega \times r$, prove $\omega = \frac{1}{2} \text{curl } v$ where ω is a constant vector.

Solution

$$\begin{aligned}
 \text{curl } v &= \nabla \times v = \nabla \times (\omega \times r) \\
 &= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}] \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_3 x - \omega_1 z) \right) \hat{i} - \left(\frac{\partial}{\partial x} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_2 z - \omega_3 y) \right) \hat{j} \\
 &\quad + \left(\frac{\partial}{\partial x} (\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y} (\omega_2 z - \omega_3 y) \right) \hat{k} \\
 &= (\omega_1 + \omega_1)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (\omega_3 - (-\omega_3))\hat{k} \\
 &= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) \\
 \Rightarrow \nabla \times (\omega \times r) &= 2\vec{\omega} \\
 \Rightarrow \nabla \times v &= 2\vec{\omega} \\
 \Rightarrow \vec{\omega} &= \frac{1}{2} \nabla \times v \\
 \Rightarrow \vec{\omega} &= \frac{1}{2} \text{curl } v
 \end{aligned}$$

This problem indicates that the curl of a vector field is linked with rotational properties of the field. We might say that if $\text{curl } \vec{A} = 0$, then there would be no rotation and the field is called irrotational field. A field which is not irrotational is sometimes called a vortex field.

Module No. 29

Vector Identities

Problem Statement

Evaluate $\nabla \cdot (\vec{A} \times \vec{r})$ if $\nabla \times \vec{A} = 0$.

Solution

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{aligned}\vec{A} \times \vec{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} \\ &= (A_2z - A_3y)\hat{i} - (A_1z - A_3x)\hat{j} + (A_1y - A_2x)\hat{k}\end{aligned}$$

And

$$\begin{aligned}\nabla \cdot (\vec{A} \times \vec{r}) &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left((A_2z - A_3y)\hat{i} - (A_1z - A_3x)\hat{j} + (A_1y - A_2x)\hat{k} \right) \\ &= \frac{\partial}{\partial x}(A_2z - A_3y) - \frac{\partial}{\partial y}(A_1z - A_3x) + \frac{\partial}{\partial z}(A_1y - A_2x) \\ &= z \frac{\partial A_2}{\partial x} - y \frac{\partial A_3}{\partial x} - z \frac{\partial A_1}{\partial y} + x \frac{\partial A_3}{\partial y} + y \frac{\partial A_1}{\partial z} - x \frac{\partial A_2}{\partial z} \\ &= x \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + y \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + z \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\ &= (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \left(\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\hat{k} \right) \\ &= \vec{r} \cdot (\nabla \times \vec{A}) = \vec{r} \cdot \text{curl } \vec{A}\end{aligned}$$

If $\nabla \times \vec{A} = 0$ this reduces to zero.

Statement

Prove that

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Proof

Let $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ be a vector point function

Then

$$\text{L.H.S} = \nabla \times (\nabla \times \vec{A})$$

Since

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \end{aligned}$$

$$\begin{aligned} \text{Hence } \nabla \times (\nabla \times \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) & \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \right] \hat{i} - \left[\frac{\partial}{\partial x} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \right] \hat{j} \\ &\quad + \left[\frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \right] \hat{k} \\ &= \left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} \\ &\quad + \left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \\ &= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} \right) \hat{k} + \left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) \hat{i} \\ &\quad + \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} \right) \hat{j} + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2} \right) \hat{k} \\
&\quad + \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) \hat{i} + \left(\frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} \right) \hat{j} \\
&\quad + \left(\frac{\partial^2 A_3}{\partial z^2} + \frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \\
&= - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&\quad + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&\quad = -\nabla^2 \vec{A} + \nabla \cdot \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&\quad = -\nabla^2 \vec{A} + \nabla \cdot (\nabla \cdot \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}
\end{aligned}$$

Hence the result.

Problem Statement

Find $\text{curl}(\vec{r}f(r))$ where $f(r)$ is differentiable.

Solution

Let $\vec{r} = xi + yj + zk$ be the given vector

then

$$\begin{aligned}
\text{curl}(\vec{r}f(r)) &= \nabla \times (\vec{r}f(r)) \\
&= \nabla \times (xf(r)\hat{i} + yf(r)\hat{j} + zf(r)\hat{k}) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y} zf(r) - \frac{\partial}{\partial z} yf(r) \right] \hat{i} + \left[\frac{\partial}{\partial z} xf(r) - \frac{\partial}{\partial x} zf(r) \right] \hat{j} + \left[\frac{\partial}{\partial x} yf(r) - \frac{\partial}{\partial y} xf(r) \right] \hat{k} \\
&= \left[z \frac{\partial f(r)}{\partial y} - y \frac{\partial f(r)}{\partial z} \right] \hat{i} + \left[x \frac{\partial f(r)}{\partial z} - z \frac{\partial f(r)}{\partial x} \right] \hat{j} \\
&\quad + \left[y \frac{\partial f(r)}{\partial x} - x \frac{\partial f(r)}{\partial y} \right] \hat{k} \tag{1}
\end{aligned}$$

As we know $r = \sqrt{x^2 + y^2 + z^2}$, therefore

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = f'(r) \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{f'(r)x}{r}$$

Similarly,

$$\frac{\partial f(r)}{\partial y} = \frac{f'(r)y}{r}$$

And

$$\frac{\partial f(r)}{\partial z} = \frac{f'(r)z}{r}$$

By substituting these values in equation (1), we obtain

$$= \left[z \frac{f'(r)y}{r} - y \frac{f'(r)z}{r} \right] i + \left[x \frac{f'(r)z}{r} - z \frac{f'(r)x}{r} \right] j + \left[y \frac{f'(r)x}{r} - x \frac{f'(r)y}{r} \right] \hat{k} = 0$$

Hence the result.

Module No. 30

Line Integral

An integral where the function is evaluated along a curve is called line integral. It is also named as path integral, curve integral, and curvilinear integral; contour integral as well. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve. A line integral is a natural generalization of definite integral. Line integral can be transformed into double integral and surface integrals and vice versa.

The symbolic form of line integral is

$$\int_C \vec{A} \cdot d\vec{r}$$

Where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ is called the differential displacement vector. The integrals that involves differential displacement vector $d\vec{r}$ are called line integrals.

Line integral can also be expressed as

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C (A_1dx + A_2dy + A_3dz) \end{aligned}$$

The line integral $\int_C \vec{A} \cdot d\vec{r}$ is sometimes called a scalar line integral of a vector field $\vec{A}$.

If C is a closed curve which we shall assume a simple closed curve (a curve which does not intersect itself anywhere), the line integral around C is denoted by

$$\oint_C \vec{A} \cdot d\vec{r} = \oint_C (A_1dx + A_2dy + A_3dz)$$

Applications

Some applications of line integrals are:

- If the vector field to be integrated $\vec{A}$ is the force $\vec{F}$ on a particle move along C, then this line integral show the work done by the force.

- ii. In fluid mechanics, if the vector field to be integrated $\vec{A}$ represents the velocity of some fluid then the line integral is called the circulation of $\vec{A}$ about C .
- iii. In general, we can say that any integral which is to be calculated along a curve is called a line integral.

Module No. 31

Other forms and General Properties of line Integrals

Other forms of Line integrals

Generally, we define line integral by the vector function $\vec{A}$ by

$$\int_C \vec{A} \cdot d\vec{r}$$

As we studied previously, that the function to be integrated can be either scalar point function or vector point function.

Here, we will define line integral using scalar function,

$$\begin{aligned} \int_C \phi d\vec{r} &= \int_C \phi (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C (\phi dx\hat{i} + \phi dy\hat{j} + \phi dz\hat{k}) \\ \int_C \phi \cdot d\vec{r} &= \hat{i} \int_C \phi dx + \hat{j} \int_C \phi dy + \hat{k} \int_C \phi dz \end{aligned}$$

Also the general line integral indicates the dot product (“.”) between the given vector field and the differential displacement vector. However we can also express it as a cross product (×) of both the vector according to our requirement.

$$\int_C \vec{A} \times d\vec{r} = \hat{i} \int_C (A_2 dz - A_3 dy) + \hat{j} \int_C (A_3 dx - A_1 dz) + \hat{k} \int_C (A_1 dy - A_2 dx)$$

Thus any integral that involves differential displacement vector $d\vec{r}$ are called line integrals.

General Properties of Line Integral

The following are the properties of the line integrals that are useful in computational subjects and application

- i. $\int_C K \vec{A} \cdot d\vec{r} = K \int_C \vec{A} \cdot d\vec{r}$ (K is any real constant)
- ii. $\int_C (\vec{A} + \vec{B}) \cdot d\vec{r} = \int_C \vec{A} \cdot d\vec{r} + \int_C \vec{B} \cdot d\vec{r}$

$$\text{iii. } \int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r}$$

Where the path C is subdivided into two arcs C_1 and C_2 that has the same orientations as C . If the sense of orientation along C is reversed, the value of the integral is multiplied by ‘-1’.

iv. If C is piece-wise smooth, consisting of smooth curves $C_1, C_2, \dots, C_n$ then the line integral of $\vec{A}$ over C is defined as the sum of the line integrals of $\vec{A}$ over each of the smooth curve making up C :

$$\int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} + \dots + \int_{C_n} \vec{A} \cdot d\vec{r}$$

In the sum, the orientation along C must be maintained over curves $C_1, C_2, \dots, C_n$. That is the initial point of C_j is the terminal point of C_{j-1} .

Module No. 32

Example of Line Integral

Problem Statement

If $\varphi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$.

Evaluate the line integrals

- i. $\int_C \varphi \cdot d\vec{r}$
- ii. $\int_C \vec{F} \times d\vec{r}$

Solution

i. $\int_C \varphi \cdot d\vec{r}$

By substituting the given values of x, y, z, we obtain

$$\varphi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9$$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \vec{r} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

Also $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$$

As we studied the relation

$$\begin{aligned} \int_C \varphi \cdot d\vec{r} &= \hat{i} \int_C \varphi dx + \hat{j} \int_C \varphi dy + \hat{k} \int_C \varphi dz \\ \Rightarrow \int_C \varphi d\vec{r} &= \int_0^1 4t^9(2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) dt = \hat{i} \int_0^1 8t^{10} dt + \hat{j} \int_0^1 8t^9 dt + \hat{k} \int_0^1 12t^{11} dt \\ &= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k} \end{aligned}$$

Hence the required solution.

ii. $\int_C \vec{F} \times d\vec{r}$

Since $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, first of all we will calculate the value of F using parametric equations of x,y,z

$$\vec{F} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$$

Also we calculated, $d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

Then

$$\begin{aligned}\vec{F} \times d\vec{r} &= (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = (-3t^5 - 2t^4)\hat{i} - (6t^5 - 2t^5)\hat{j} + (4t^3 + 2t^4)\hat{k}\end{aligned}$$

Using the expression

$$\int_C \vec{A} \times d\vec{r} = \hat{i} \int_C (A_2 dz - A_3 dy) + \hat{j} \int_C (A_3 dx - A_1 dz) + \hat{k} \int_C (A_1 dy - A_2 dx)$$

Substituting values, we get

$$\begin{aligned}\int_C \vec{F} \times d\vec{r} &= \hat{i} \int_0^1 (-3t^5 - 2t^4)dt - \hat{j} \int_0^1 (6t^5 - 2t^5)dt + \hat{k} \int_0^1 (4t^3 + 2t^4)dt \\ &= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}\end{aligned}$$

Module No. 33

Example of Line Integral

Problem statement

Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

Solution

Since $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$,

Substituting values $x = t^2 + 1, y = 2t^2, z = t^3$, in $\vec{F}$.

$$\vec{F} = 3(t^2 + 1)(2t^2)\hat{i} - 5(t^3)\hat{j} + 10(t^2 + 1)\hat{k}$$

$$\vec{F} = (6t^4 + 6t^2)\hat{i} - 5(t^3)\hat{j} + (10t^2 + 10)\hat{k}$$

Also, $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow d\vec{r} = (2t\hat{i} + 4t\hat{j} + 3t^2\hat{k})dt$$

Total work done $= \int_C \vec{F} \cdot d\vec{r} =$

$$= \int_1^2 [(6t^4 + 6t^2)\hat{i} - 5(t^3)\hat{j} + (10t^2 + 10)\hat{k}] \cdot (2t\hat{i} + 4t\hat{j} + 3t^2\hat{k})dt$$

$$= \int_1^2 2t(6t^4 + 6t^2) - 4t(5t^3) + 3t^2(10t^2 + 10)dt$$

$$= \int_1^2 (12t^5 + 12t^3) - (20t^4) + (30t^4 + 30t^2)dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= \int_1^2 12t^5 dt + \int_1^2 10t^4 dt + \int_1^2 12t^3 dt + \int_1^2 30t^2 dt$$

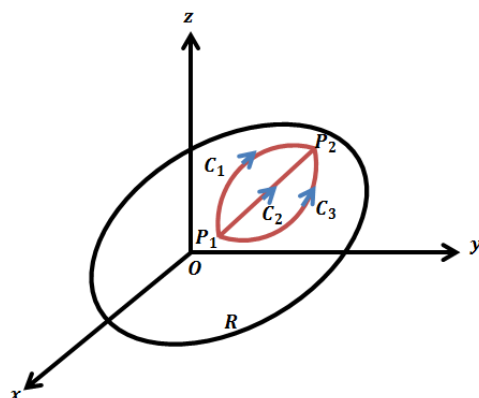
$$= \left| \frac{12t^6}{6} \right|_1^2 + \left| \frac{10t^5}{5} \right|_1^2 + \left| \frac{12t^4}{4} \right|_1^2 + \left| \frac{30t^3}{3} \right|_1^2$$

$$\begin{aligned} &= 2[(2)^6 - (1)^6] + 2[(2)^5 - (1)^5] + 3[(2)^4 - (1)^4] + 10[(2)^3 - (1)^3] \\ &= 303 \end{aligned}$$

Module No. 34

Line Integral Dependent on Path

The value of line integrals $\int_C \vec{A} \cdot d\vec{r}$ generally depends not only on the end points P_1 and P_2 of the curve C but also the geometric shape of the path C i.e. if we integrate from point P_1 to P_2 along different paths, we in general, obtain different values of the integral.



We will illustrate this concept by the following example

If $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C

- The straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$, and then to $(1,1,1)$.
- The straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution

- Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y = 0, z = 0, dy = 0, dz = 0$ while x varies from 0 to 1. Then

the integral over this part of the path is

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (A_1 dx + A_2 dy + A_3 dz) \\ &= \int_0^1 3x^2 + 6(0) dx - 14y(0)dy + 20x(0)^2 = \int_0^1 3x^2 dx = 1 \end{aligned}$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x = 1, z = 0, dx = 0, dz = 0$ while y varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6y \, dx - 14y(0)dy + 20x(0)^2 = \int_0^1 3 + 6y \, dx = 0$$

Along the straight line from $(1,1,0)$ to $(1,1,1)$ $x = 1, y = 1, dx = 0, dy = 0$ while z varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6(1)(0) - 14(1)z(0) + 20(1)z^2 \, dz = \int_0^1 20z^2 \, dz = \frac{20}{3}$$

By adding all the results, we have

$$\int_0^1 \vec{A} \cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

- ii. The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric form by $x = t, y = t, z = t$. Then

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k} \cdot d\vec{r} \\ &= \int_0^1 (3t^2 + 6t) - 14t^2 + 20t^3 \, dt = \int_0^1 6t - 11t^2 + 20t^3 \, dt = \frac{13}{3} \end{aligned}$$

In example, we discuss two cases here. In each case the end points of the curve were same but we chose different path to obtain results. The results obtained by both the cases are different. Hence we conclude that the line integral not only depends on the end points of the curve but also on the path taken by the particle.

Module No. 35

Independence of Path

Definition

The line integral $\int_C \vec{V} \cdot d\vec{r}$ is said to be independent of the path C in a given region R , if the value of the line integral $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ is the same for all paths C joining any two points P_1 and P_2 in R .

Theorem

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ to be independent of the path joining any two points P_1 and P_2 (i.e. $\vec{V}$ to be conservative) in a given region is that $\oint_C \vec{V} \cdot d\vec{r} = 0$ for all closed path C in the region.

Proof

Let C be any simple closed curve, and let P_1 and P_2 be any two points on C as shown.

Then since by the supposition, the integral is independent of path (i.e. $\vec{V}$ to be conservative), we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = \int_{P_1 B P_2} \vec{V} \cdot d\vec{r}$$

Reversing the direction of integration in the integral on the right, we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = - \int_{P_2 B P_1} \vec{V} \cdot d\vec{r}$$

or

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} + \int_{P_2 B P_1} \vec{V} \cdot d\vec{r} = 0$$

or

$$\oint_C \vec{V} \cdot d\vec{r} = 0$$

Conversely, if $\oint_C \vec{V} \cdot d\vec{r} = 0$,

then

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} + \int_{P_2 B P_1} \vec{V} \cdot d\vec{r} = 0$$

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = - \int_{P_2 B P_1} \vec{V} \cdot d\vec{r}$$

Again changing the direction in the integral, we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = \int_{P_1 B P_2} \vec{V} \cdot d\vec{r}$$

Which shows the line integral is independent of the path joining P_1 and P_2 as required.

Module No. 36

Theorems on line Integral

Scalar Potential Function

A scalar potential function φ is a single-valued function for which there exists a continuous vector field $\vec{V}$ in a simply connected region R that satisfies the relation $\vec{V} = \nabla\varphi$.

Theorem 1 Statement

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ to be independent of the path joining any two points $P_1(x, y, z)$ and $P_2(x, y, z)$ (i.e. $\vec{V}$ to be conservative) is that there exist a scalar function φ such that $\vec{V} = \nabla\varphi$, where φ is single valued and has continuous partial derivatives.

Proof

Let $\vec{V} = \nabla\varphi$, then

$$\begin{aligned} \int_C \vec{V} \cdot d\vec{r} &= \int_{P_1}^{P_2} \vec{V} \cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\varphi \cdot d\vec{r} \\ &= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x} \hat{i} + \frac{\partial\varphi}{\partial y} \hat{j} + \frac{\partial\varphi}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz \right) \\ &= \int_{P_1}^{P_2} d\varphi = \varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) \end{aligned}$$

Thus the line integrals only depends only on points P_1 and P_2 and not on the path joining them i.e. $\vec{V}$ is conservative.

Conversely, let $\int_C \vec{V} \cdot d\vec{r}$ be the independent of the path C joining any two points, We choose these points as a fixed point $P_1 \equiv (x_1, y_1, z_1)$ and a variable point $P_2 \equiv (x, y, z)$, so that the result is a function of only of the coordinates (x, y, z) of the variable end points. Then

$$\varphi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{V} \cdot d\vec{r} = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{V} \cdot \frac{d\vec{r}}{ds} ds$$

By taking derivative, we have

$$\frac{d\varphi}{ds} = \vec{V} \cdot \frac{d\vec{r}}{ds} \quad (1)$$

But

$$\frac{d\varphi}{ds} = \frac{\partial \varphi}{\partial s} = \nabla \varphi \cdot \frac{d\vec{r}}{ds} \quad (2)$$

From (1) and (2) we have

$$\vec{V} \cdot \frac{d\vec{r}}{ds} = \nabla \varphi \cdot \frac{d\vec{r}}{ds} = (\vec{V} - \nabla \varphi) \cdot \frac{d\vec{r}}{ds} = 0$$

Since $\frac{d\vec{r}}{ds}$ is a unit tangent vector, therefore $\frac{d\vec{r}}{ds} \neq 0$

$$\Rightarrow \vec{V} - \nabla \varphi = 0$$

Or

$$\vec{V} = \nabla \varphi$$

Hence Proved.

Theorem 2 Statement

Prove that a necessary and sufficient condition that a vector field $\vec{V}$ be conservative is that $\nabla \times \vec{V} = 0$ (i.e. $\vec{V}$ is irrotational).

Proof

If $\vec{V}$ is conservative field then by the previous theorem, we have $\vec{V} = \nabla \varphi$.

Thus

$$\nabla \times \vec{V} = \nabla \times \nabla \varphi = 0$$

Conversely, if $\nabla \times \vec{V} = 0$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

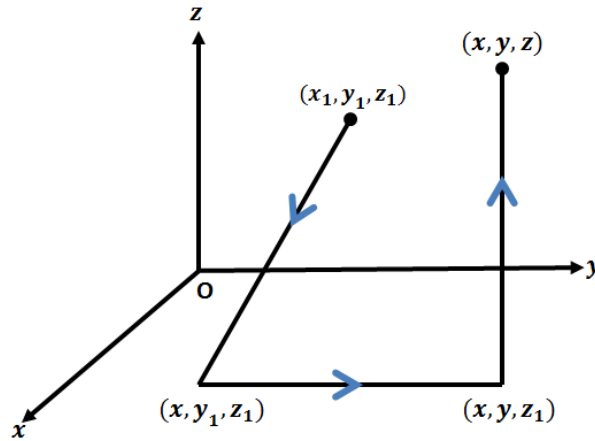
and thus

$$\frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y}$$

we must prove that $\vec{V} = \nabla\phi$ follows as a consequence of this,

Now,

$$\int_C \vec{V} \cdot d\vec{r} = \int_C (V_1(x, y, z)dx + V_2(x, y, z)dy + V_3(x, y, z)dz)$$



where C is path joining (x_1, y_1, z_1) and (x_2, y_2, z_3) . Let us choose as a particular path, the straight line segment from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z) and $\phi(x, y, z)$ the value of the integral along this particular path. Then omitting the integrand, we have

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y_1, z_1)} [] + \int_{(x, y_1, z_1)}^{(x, y, z_1)} [] + \int_{(x, y, z_1)}^{(x, y, z)} [] \quad (1)$$

- i. Along the straight line (x_1, y_1, z_1) to (x, y_1, z_1) , $y = \text{constant} = y_1, z = \text{constant} = z_1$, so that $dy = 0, dz = 0$, while x varies from x_1 to x .
- ii. Along the straight line (x, y_1, z_1) to (x, y, z_1) , $x = \text{constant}, z = \text{constant}$, so that $dx = 0, dz = 0$, while y varies from y_1 to y .
- iii. Along the straight line (x, y, z_1) to (x, y, z) , $x = \text{constant}, y = \text{constant}$, so that $dx = 0, dy = 0$, while z varies from z_1 to z .

Thus we can write equation (1) as

$$\phi(x, y, z) = \int_{x_1}^x V_1(x, y_1, z_1)dx + \int_{y_1}^y V_2(x, y, z_1)dy + \int_{z_1}^z V_3(x, y, z)dz$$

it follows that

$$\begin{aligned}
\frac{\partial \varphi}{\partial z} &= V_3(x, y, z) \\
\frac{\partial \varphi}{\partial y} &= V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_3}{\partial y}(x, y, z) dz \\
&= V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_2}{\partial z}(x, y, z) dz = V_2(x, y, z_1) + |V_2(x, y, z)|_{z_1}^z = V_2(x, y, z) \\
\frac{\partial \varphi}{\partial x} &= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_2}{\partial x}(x, y, z_1) dy + \int_{z_1}^z \frac{\partial V_3}{\partial x}(x, y, z) dz \\
&= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_1}{\partial y}(x, y, z_1) dy + \int_{z_1}^z \frac{\partial V_1}{\partial z}(x, y, z) dz \\
&= V_1(x, y_1, z_1) + |V_1(x, y, z_1)|_{y_1}^y + |V_1(x, y, z_1)|_{z_1}^z = V_1(x, y, z) \\
&= V_1(x, y_1, z_1) + V_1(x, y, z_1) - V_1(x, y_1, z_1) + V_1(x, y, z) - V_1(x, y, z_1) \\
\frac{\partial \varphi}{\partial x} &= V_1(x, y, z)
\end{aligned}$$

Then we have,

$$\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k} = \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} = \nabla \varphi$$

hence $\vec{V}$ is conservative i.e $\vec{V} = \nabla \varphi$.

Module No. 37

Selected Example/Problem 1

Problem Statement

If $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t - 1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\hat{k}$,

Evaluate

- i. $\int_0^2 \vec{A} \cdot \vec{B} dt$
- ii. $\int_0^2 \vec{A} \times \vec{B} dt$

Solution

i. $\int_0^2 \vec{A} \cdot \vec{B} dt$

Since $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t - 1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\hat{k}$

So,

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (t\hat{i} - t^2\hat{j} + (t - 1)\hat{k}) \cdot (2t^2\hat{i} + 0\hat{j} + 6t\hat{k}) \\ &= 2t^3 + 6t(t - 1)\end{aligned}$$

Now

$$\begin{aligned}\int_0^2 \vec{A} \cdot \vec{B} dt &= \int_0^2 [2t^3 + 6t(t - 1)] dt \\ &= \int_0^2 2t^3 dt + \int_0^2 6t^2 dt - \int_0^2 6t dt \\ &= \left[\frac{t^4}{2} + 2t^3 - 3t^2 \right]_0^2 \\ &= 8 + 16 - 12 = 12\end{aligned}$$

Hence

$$\int_0^2 \vec{A} \cdot \vec{B} dt = 12$$

is required solution.

ii. $\int_0^2 \vec{A} \times \vec{B} dt$

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} \\ &= [(-t^2)(6t) - (t-1)(0)]\hat{i} - [(t)(6t) - (t-1)(2t^2)]\hat{j} + [(t)(0) - (-t^2)(2t^2)]\hat{k} \\ &= -6t^3\hat{i} - (6t^2 + 2t^2 - 2t^3)\hat{j} + 2t^4\hat{k} \\ &= -6t^3\hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4\hat{k}\end{aligned}$$

Now,

$$\begin{aligned}\int_0^2 \vec{A} \times \vec{B} dt &= \int_0^2 (-6t^3\hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4\hat{k}) dt \\ &= -\hat{i} \int_0^2 6t^3 dt - \hat{j} \int_0^2 (8t^2 - 2t^3) dt + \hat{k} \int_0^2 2t^4 dt \\ &= \left| -\hat{i} \frac{3}{2} t^4 - \hat{j} \left(\frac{8}{3} t^3 - \frac{t^4}{2} \right) + \hat{k} \frac{2}{5} t^5 \right|_0^2 \\ &= -\hat{i} \frac{3(16)}{2} - \hat{j} \left(\frac{8(8)}{3} - \frac{16}{2} \right) + \hat{k} \frac{2(32)}{5} \\ &= -\hat{i} 24 - \hat{j} \frac{80}{6} + \hat{k} \frac{64}{5} \\ \int_0^2 \vec{A} \times \vec{B} dt &= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}\end{aligned}$$

is the required result.

Module No. 38

Selected Example/Problem 2: Line Integrals

Problem Statement

If $F = 3xy\hat{i} - y^2\hat{j}$, evaluate work done by the force on the curve C in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

Solution

As we know total work done on a curve is

$$\int_C \vec{F} \cdot d\vec{r}$$

Also,

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

By substituting

$$\begin{aligned} &= \int_0^1 (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_0^1 (3xy \, dx - y^2 \, dy) \end{aligned}$$

As $y = 2x^2$ is given, this implies $dy = 4x \, dx$, now substituting these values in above integral, we obtain

$$\begin{aligned} &= \int_0^1 (3x(2x^2) \, dx - (2x^2)^2 4x \, dx) \\ &= \int_0^1 6x^3 \, dx - \int_0^1 16x^5 \, dx \\ &= \left[\frac{3}{2}x^4 - \frac{8}{3}x^6 \right]_0^1 \\ &= \frac{3}{2} - \frac{8}{3} = \frac{-7}{6} \end{aligned}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been 7/6 instead of $-7/6$.

Module No. 39

Selected Example/Problem 3: Line Integrals

Problem Statement

Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

Solution

In the plane $z = 0$, $\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j}$ and $d\vec{r} = dx\hat{i} + dy\hat{j}$

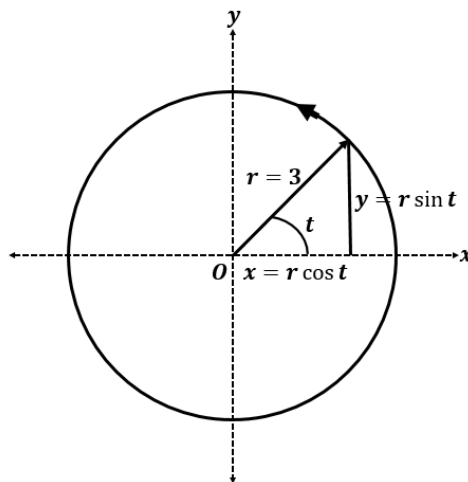
So, that the work done on the curve is

$$\begin{aligned} \text{Total work done} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C ((2x - y)\hat{i} + (x + y)\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_C ((2x - y)dx + (x + y)dy) \end{aligned}$$

As radius of the circle is $r = 3$, we choose the parametric equations of the circle as

$$x = 3 \cos t \Rightarrow dx = -3 \sin t \, dt$$

$$y = 3 \sin t \Rightarrow dy = 3 \cos t \, dt$$



where t varies from 0 to 2π . Then the line integral

$$\begin{aligned}
 &= \int_0^{2\pi} ((2(3 \cos t) - 3 \sin t)(-3 \sin t) dt) + (3 \cos t + 3 \sin t)(3 \cos t) dt \\
 &= \int_0^{2\pi} (6 \cos t - 3 \sin t)(-3 \sin t) + 9(\cos t)^2 + 9 \sin t \cos t \, dt \\
 &= \int_0^{2\pi} -18 \cos t \sin t + 9 \sin^2 t + 9 \cos^2 t + 9 \sin t \cos t \, dt \\
 &= \int_0^{2\pi} 9 - 9 \cos t \sin t \, dt \\
 &= \left[9t - \frac{9}{2} \sin^2 t \right]_0^{2\pi} \\
 &= 18\pi
 \end{aligned}$$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the positive direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π .

Module No. 40

Selected Example/Problem 4: Line Integral

Problem Statement

Given a force vector $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$.

- Show that $\vec{F}$ a conservative force field
- Find the scalar potential
- Find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

Solution

- We derived a necessary and sufficient condition that a force will be conservative is that

$$\text{curl}\vec{F} = \nabla \times \vec{F} = 0.$$

Now

$$\begin{aligned}\text{curl}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2) \right) \hat{i} - \left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3) \right) \hat{j} \\ &\quad + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3) \right) \hat{k} \\ &= 0\hat{i} - (3z^2 - 3z^2)\hat{j} + (2x - 2x)\hat{k} = 0 \\ \text{curl}\vec{F} &= 0\end{aligned}$$

Hence $\vec{F}$ is conservative.

- As $\vec{F}$ is conservative, so $\vec{F} = \nabla\varphi$

and

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

this implies=

$$\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

By comparing, we get

$$\frac{\partial\varphi}{\partial x} = 2xy + z^3 \quad (1)$$

$$\frac{\partial\varphi}{\partial y} = x^2 \quad (2)$$

$$\frac{\partial\varphi}{\partial z} = 3xz^2 \quad (3)$$

By integrating equation (1), (2) and (3), we get

$$\varphi = x^2y + xz^3 + f(y, z)$$

$$\varphi = x^2y + g(x, z)$$

$$\varphi = xz^3 + h(x, y)$$

These equation will agree if we choose

$$f(y, z) = 0, g(x, z) = xz^3 \text{ and } h(x, y) = x^2y$$

so that $\varphi = x^2y + xz^3$ to which may be added any constant.

iii. From part (ii), we have

$$\varphi = x^2y + xz^3 + K$$

Where K is any constant,

$$\text{Then work done} = \varphi(P_2) - \varphi(P_1)$$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) \quad (4)$$

$$\varphi(x_2, y_2, z_2) = \varphi(3, 1, 4) = (3)^2(1) + 3(4)^3 = 9 + 192 = 201$$

$$\varphi(x_1, y_1, z_1) = \varphi(1, -2, 1) = (1)^2(-2) + 1(1)^3 = -2 + 1 = -1$$

Now substituting these values in equation (4)

$$\text{Work done} = \varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) = 201 - (-1) = 202$$

202 is the required work done.

Module No. 41

Surface Integral

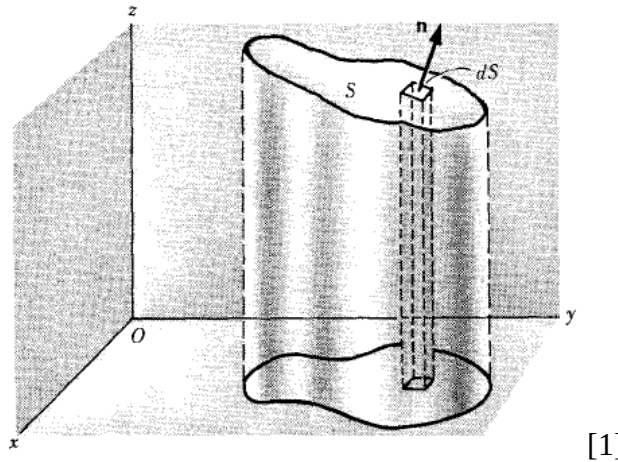
Definition

Integrals that involves the differential surface elements $d\vec{S}$ are called surface integrals.

Mathematically, we can represents it as

$$\iint_S \vec{A} \cdot d\vec{S}$$

Let S be a two-sided surface (open or closed), and $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a defined and continuous vector function over the surface S . Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\hat{n}$ to any point of the positive side of S is called a positive or outward drawn unit normal.



[1]

Associate with the differential of surface area dS a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of n . Then $d\vec{S} = \hat{n} dS$.

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_S \vec{A} \cdot d\vec{S}$$

The surface integral of a vector field $\vec{A}$ is called the flux of $\vec{A}$ through S .

The other forms of surface integrals are

$$\iint_S \varphi d\vec{S} \text{ and } \iint_S \vec{A} \times d\vec{S}$$

Where φ is a scalar point function.

If S is a closed surface, then the surface integrals may be written as

$$\oint \vec{A} \cdot d\vec{S},$$

$$\oint \vec{A} \times d\vec{S},$$

Or

$$\oint \varphi d\vec{S}$$

Properties of Surface Integrals

Like the double integrals, surface integrals have the following properties:

- i. $\iint_S K \vec{A} \cdot d\vec{S} = K \iint_S \vec{A} \cdot d\vec{S}$ (K is any real constant)
- ii. $\iint_S (\vec{A} + \vec{B}) \cdot d\vec{S} = \iint_S \vec{A} \cdot d\vec{S} + \iint_S \vec{B} \cdot d\vec{S}$
- iii. $\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S}$

Where the surface S is subdivided into two smooth surfaces S_1 and S_2 having atmost a curve in common.

- iv. If the surface S is partitioned by the smooth curves into a finite number of non-overlapping smooth patches $S_1, S_2, \dots, S_n$ (i.e. S is piece-wise smooth), then the normal surface integral of $\vec{A}$ over S is the sum of the normal surface integrals of $\vec{A}$ over all the smooth patches, i.e.

$$\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S} + \dots + \iint_{S_n} \vec{A} \cdot d\vec{S}$$

Module No. 42

Evaluation of the Surface Integral

To evaluate the surface integral, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes.

Evaluation of surface integrals can be made by done by the following result

Theorem Statement

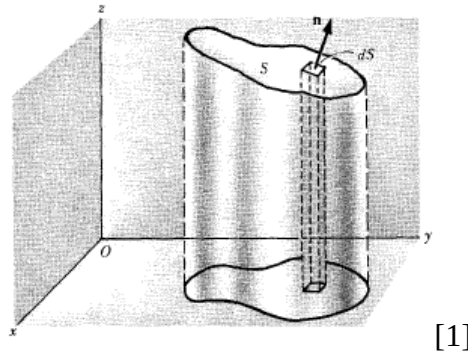
Let R be the projection of the surface S on the xy -plane, then prove that

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Proof

Let the surface S and its projection R on the xy -plane be as shown in figure.

Divide R into a rectangles of area ΔA_k , $k = 1, 2, \dots, n$ and erect a vertical column on each of these sub-regions to intersect S in an element of surface area ΔS_k .



Choose a point (x_k, y_k, z_k) on each surface element ΔS_k and draw the unit normal $\hat{n}_k$ to this element at this point. Let γ_k be the acute angle between this unit normal $\hat{n}_k$ and the positive z -axis.

If this surface element is sufficiently small, it can be regarded as a plane.

We know from geometry, that if two planes intersect at an acute angle, an area in one plane may be projected into the other by multiplying the cosine of the included angle as shown in figure.

Since the angle between two planes is the angle between their normal, therefore

$$\Delta S_k \cos \gamma_k \approx \Delta A_k$$

or

$$\Delta S_k \approx \sec \gamma_k \Delta A_k = \frac{\Delta x_k \Delta y_k}{|\hat{n} \cdot \hat{k}|}$$

Thus the sum (1) in the definition of unit normal surface integrals

$$\sum_{k=1}^n A_k \cdot \hat{n}_k \Delta S_k \approx \sum_{k=1}^n A_k \cdot \hat{n}_k \frac{\Delta x_k \Delta y_k}{|\hat{n} \cdot \hat{k}|}$$

and the limit of this sum can be written as

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx dy}{|\hat{n} \cdot \hat{k}|}$$

Similarly, we can prove that if R is the projection of the surface S on the yz-plane, then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dy dz}{|\hat{n} \cdot \hat{i}|}$$

And if R is the projection of S on the xz-plane, then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dz dx}{|\hat{n} \cdot \hat{j}|}$$

Module No. 43

Example to the previous topic: Surface Integrals

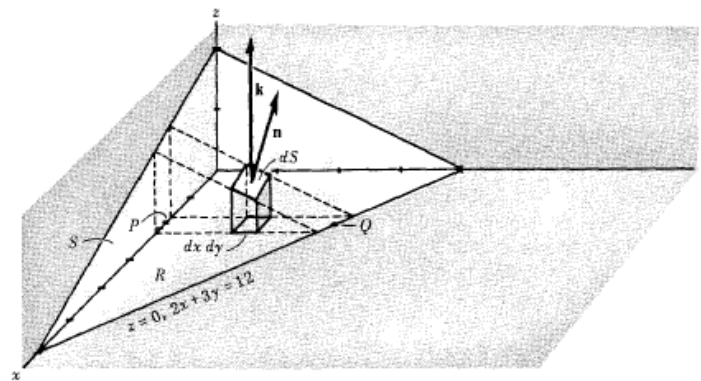
Problem Statement

Evaluate

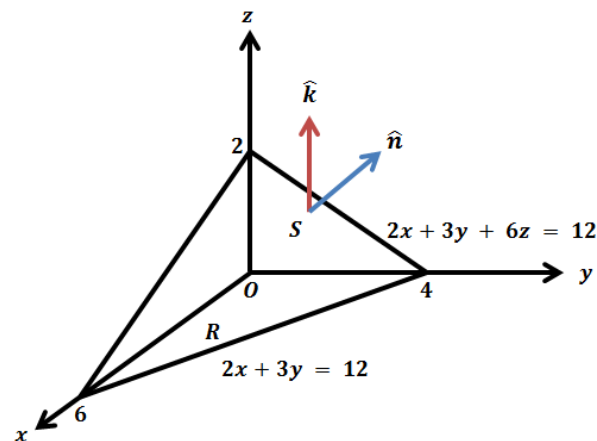
$\iint_S \vec{A} \cdot \hat{n} dS$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

Solution

The surface S and its projection R on the xy plane are shown in the figure below.



[1]



We know that

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Also we know that a normal vector to the surface $2x + 3y + 6z = 12$ is given by

$$\nabla(2x + 3y + 6z) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Then a unit normal $\hat{n}$ to any point of S is $\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

Thus $\hat{n} \cdot \hat{k} = \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \cdot \hat{k} = \frac{6}{7}$

and so

$$\frac{dxdy}{|\hat{n} \cdot \hat{k}|} = \frac{7}{6} dxdy$$

Also

$$\begin{aligned} \vec{A} \cdot \hat{n} &= (18z\hat{i} - 12\hat{j} + 3y\hat{k}) \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \\ &= \frac{36z - 36 + 18y}{7} \end{aligned}$$

Now from the equation of the surface S $2x + 3y + 6z = 12$,

$$z = \frac{12 - 2x - 3y}{6}$$

Therefore

$$\vec{A} \cdot \hat{n} = \frac{6(12 - 2x - 3y) - 36 + 18y}{7} = \frac{36 - 12x}{7}$$

Then

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|} \\ &= \iint_R \frac{36 - 12x}{7} \frac{7}{6} dxdy \\ &= \int_{x=0}^6 \int_{y=0}^{\frac{12-2x}{3}} (6 - 2x) dy dx \\ &= \int_0^6 (6 - 2x) \left(\frac{12 - 2x}{3}\right) dx \end{aligned}$$

$$\begin{aligned} &= \int_0^6 \left(24 - 12x + \frac{4}{3}x^2 \right) dx \\ &= \left| 24x - 6x^2 + \frac{4}{9}x^3 \right|_0^6 \\ &= 144 - 216 + 96 = 24 \end{aligned}$$

If we had chosen the positive unit normal $\hat{n}$ opposite to that in the figure above, we would have obtained the result — 24 .

Module No. 44

Further Example on Surface Integral

Problem Statement

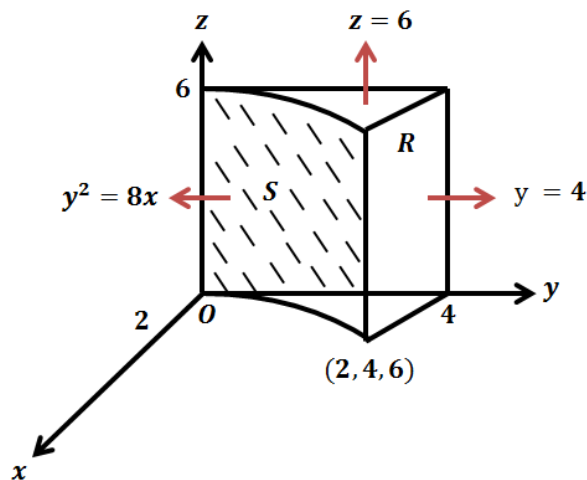
If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$,

Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

Solution

The surface S and its projection R on the yz -plane are shown in figure.



[1]

A vector normal to S is

$$\nabla(8x - y^2) = 8\hat{i} - 2y\hat{j}$$

and therefore

$$\hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{8^2 + (2y)^2}} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{4(16 + y^2)}} = \frac{4\hat{i} - y\hat{j}}{\sqrt{16 + y^2}} = \frac{4\hat{i}}{\sqrt{16 + y^2}} + \frac{-y\hat{j}}{\sqrt{16 + y^2}}$$

Also,

$$\hat{n} \cdot \hat{i} = \frac{4}{\sqrt{16 + y^2}}$$

and

$$\begin{aligned}\vec{F} \cdot \hat{n} &= (2y\hat{i} - z\hat{j} + x^2\hat{k}) \cdot \left(\frac{4\hat{i}}{\sqrt{16+y^2}} + \frac{-y\hat{j}}{\sqrt{16+y^2}} \right) \\ &= \frac{8y + zy}{\sqrt{16+y^2}}\end{aligned}$$

Thus

$$\begin{aligned}\iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dzdy}{|\hat{n} \cdot \hat{i}|} \\ &= \int_0^4 \int_0^6 \frac{8y + zy}{\sqrt{16+y^2}} \frac{\sqrt{16+y^2}}{4} dzdy \\ &= \frac{1}{4} \int_0^4 \int_0^6 (8y + zy) dzdy = \frac{1}{4} \int_0^4 66y dy \\ &= \left| \frac{33}{4} y^2 \right|_0^4 = \frac{33}{4} 4^2 = 132 \\ \iint_S \vec{F} \cdot \hat{n} dS &= 132\end{aligned}$$

is the required result.

Module No. 45

Selected Example/Problem 1: Surface Integrals

Problem Statement

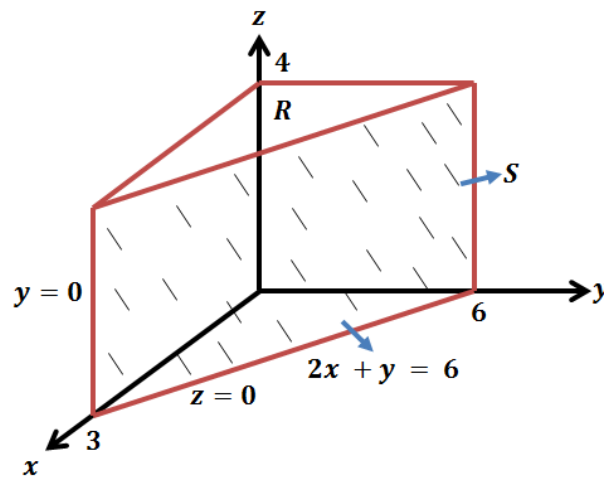
Evaluate

$$\iint_S \vec{A} \cdot \hat{n} dS$$

for $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$

Solution

The Surface S and its projection R in the yz -plane are shown in figure.



[1]

A vector normal to S is given by

$$\nabla(2x + y) = 2\hat{i} + \hat{j}$$

Therefore,

$$\hat{n} = \frac{2\hat{i} + \hat{j}}{\sqrt{2^2 + (1)^2}} = \frac{2\hat{i} + \hat{j}}{\sqrt{4 + 1}} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}$$

Also,

$$\hat{n} \cdot \hat{i} = \frac{2}{\sqrt{5}}$$

and

$$\begin{aligned}\vec{A} \cdot \hat{n} &= (y\hat{i} + 2x\hat{j} - z\hat{k}) \cdot \left(\frac{2\hat{i}}{\sqrt{5}} + \frac{\hat{j}}{\sqrt{5}} \right) \\ &= \frac{2y}{\sqrt{5}} + \frac{2x}{\sqrt{5}} = \frac{2}{\sqrt{5}}(x + y)\end{aligned}$$

and

$$\begin{aligned}\iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dzdy}{|\hat{n} \cdot \hat{i}|} \\ &= \int_{y=0}^6 \int_{z=0}^4 \frac{2}{\sqrt{5}}(x+y) \frac{\sqrt{5}}{2} dzdy \\ 2x + y &= 6 \Rightarrow x = \frac{6-y}{2} \\ &= \int_{y=0}^6 \int_{z=0}^4 y + \left(\frac{6-y}{2} \right) dzdy = \int_{y=0}^6 \int_{z=0}^4 \left(3 + \frac{y}{2} \right) dzdy \\ &= \int_{y=0}^6 \left(3 + \frac{y}{2} \right) |z|_0^4 dy \\ &= 4 \int_{y=0}^6 \left(3 + \frac{y}{2} \right) dy \\ &= \left| 3y + \frac{y^2}{4} \right|_0^6 \\ &= 4(18 + 9) = 108\end{aligned}$$

Module No. 46

Selected Example/Problem 2: Surface Integral

Problem Statement

Evaluate

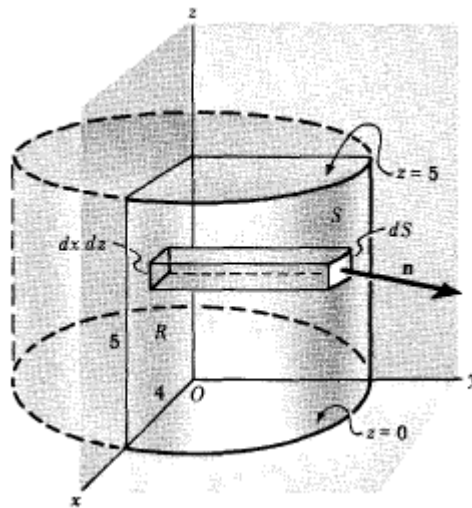
i. $\iint_S \vec{A} \cdot \hat{n} dS$

ii. $\iint_S \varphi \hat{n} dS$

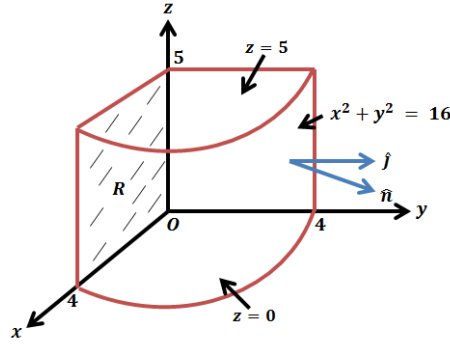
where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and $\varphi = \frac{3}{8}xyz$ S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution

- i. The surface S and its projection R on the xz -plane are shown in figure.



[1]



[1]

Then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx dz}{|\hat{n} \cdot \hat{j}|} \quad (1)$$

A normal vector to $x^2 + y^2 = 16$ is $\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$

Thus the unit normal $\hat{n}$ to S is

$$\begin{aligned} \hat{n} &= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(x^2 + y^2)}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(16)}} \\ &= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{64}} = \frac{2(x\hat{i} + y\hat{j})}{8} = \frac{(x\hat{i} + y\hat{j})}{4} \end{aligned}$$

Using $x^2 + y^2 = 16$ on S

Now,

$$\begin{aligned} \vec{A} \cdot \hat{n} &= (z\hat{i} + x\hat{j} - 3y^2z\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \\ &= \frac{1}{4}(zx + xy) \end{aligned}$$

And

$$\hat{n} \cdot \hat{j} = \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \cdot \hat{j} = \frac{y}{4}$$

Thus putting obtained values in equation (1)

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \frac{1}{4}(zx + xy) \frac{4}{y} dx dz \\ &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx + xy}{y} dx dz \\ &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx}{y} + x dx dz \end{aligned} \quad (2)$$

Utilizing the equation of the surface of the cylinder

$$x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2 \Rightarrow y = \sqrt{16 - x^2}$$

Substituting the value of y in equation (2)

$$\begin{aligned} &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx}{\sqrt{16-x^2}} + x \, dx \, dz \\ &= \int_0^5 \left| -z\sqrt{16-x^2} + \frac{x^2}{2} \right|_0^4 dz \\ &= \int_0^5 (4z + 8) dz = |2z^2 + 8z|_0^5 = 90 \end{aligned}$$

ii. $\iint_S \varphi \hat{n} dS$

$$\iint_S \varphi \cdot \hat{n} dS = \iint_R \varphi \hat{n} \frac{dx dz}{|\hat{n} \cdot \hat{j}|}$$

Substituting values, we get

$$\begin{aligned} \iint_S \varphi \cdot \hat{n} dS &= \iint_R \left(\frac{3}{8} xyz \right) \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \frac{4}{y} dx dz \\ &= \frac{3}{8} \int_{z=0}^5 \int_{x=0}^4 xz (x\hat{i} + y\hat{j}) dx dz \end{aligned}$$

Using the value of $y = \sqrt{16 - x^2}$ in above integral,

$$\begin{aligned} &= \frac{3}{8} \int_{z=0}^5 \int_{x=0}^4 \left(x^2 z \hat{i} + xz \sqrt{16 - x^2} \hat{j} \right) dx dz \\ &= \int_0^5 \left[z \hat{i} \left| \frac{x^3}{3} \right|_0^4 + z \hat{j} \left| \frac{-1}{3} (16 - x^2)^{3/2} \right|_0^4 \right] dz \\ &= \int_0^5 \left(\frac{64}{3} z \hat{i} + \frac{64}{3} z \hat{j} \right) dz = 8 \left| \frac{z^2}{2} \hat{i} + \frac{z^2}{2} \hat{j} \right|_0^5 \\ &= 8 \left(\frac{25}{2} \hat{i} + \frac{25}{2} \hat{j} \right) = 100\hat{i} + 100\hat{j} \end{aligned}$$

is required solution.

Module No. 47

Selected Example/Problem 3: Surface Integral

Problem Statement

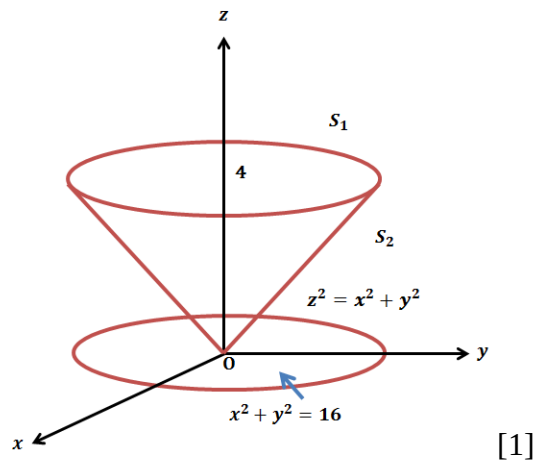
Evaluate $\iint_S \vec{A} \cdot \hat{n} dS$

over the entire surface of the region above the xy plane bounded by the cone S

$z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{A} = 4xz \hat{i} + xyz^2 \hat{j} + 3z \hat{k}$.

Solution

The surface S and its projection R on the xy-plane are shown in figure.



Then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{A} \cdot \hat{n} dS_2$$

For S_1 , $z = 4$, $\hat{n} = \hat{k}$ and $\hat{n} \cdot \hat{k} = 1$

Then

$$\begin{aligned} \iint_{S_1} \vec{A} \cdot \hat{n} dS &= \iint_{S_1} 12 dS_1 = 12 \iint_R dx dy \\ &= 12 \times \pi(4)^2 = 192\pi \end{aligned}$$

For S_2 , the normal vector is given by

$$\nabla(x^2 + y^2 - z^2) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

Therefore,

$$\begin{aligned}\hat{n} &= \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2(x\hat{i} + y\hat{j} - z\hat{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{x^2 + y^2 + z^2}} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2z^2}} \\ \hat{n} &= \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{z\sqrt{2}}\end{aligned}$$

Also,

$$\vec{A} \cdot \hat{n} = \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z) \text{ and } \hat{n} \cdot \hat{k} = \frac{-1}{\sqrt{2}}$$

Therefore

$$\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z) \cdot \sqrt{2} dy dx$$

Using $z^2 = x^2 + y^2$ in the above integral, we obtain

$$= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (4x^2 + xy^2\sqrt{x^2 + y^2} - 3\sqrt{x^2 + y^2}) dy dx$$

Now we will further solve it by using polar coordinates

Let $x = r \cos \theta$, $y = r \sin \theta$, therefore $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$

and

$$\begin{aligned}\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 &= \int_0^{2\pi} \int_0^4 (4r^2 \cos^2 \theta + r^4 \cos \theta \sin^2 \theta - 3r) r dr d\theta \\ &= \int_0^{2\pi} \int_0^4 (4r^3 \cos^2 \theta + r^5 \cos \theta \sin^2 \theta - 3r^2) dr d\theta\end{aligned}$$

integrating w.r.t r

$$\begin{aligned}&= \int_0^{2\pi} \left[r^4 \cos^2 \theta + \frac{r^6}{6} \cos \theta \sin^2 \theta - r^3 \right]_0^4 d\theta \\ &= \int_0^{2\pi} \left(256 \cos^2 \theta + \frac{4^6}{6} \cos \theta \sin^2 \theta - 4^3 \right) d\theta\end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \left(256 \cos^2 \theta + \frac{4^6}{6} \cos \theta \sin^2 \theta - 4^3 \right) d\theta \\
&= \int_0^{2\pi} \left(128(1 + \cos 2\theta) + \frac{2048}{3} \cos \theta \sin^2 \theta - 64 \right) d\theta
\end{aligned}$$

Now integrating w.r.t θ , we get

$$= \left| 128\theta + 64 \sin 2\theta + \frac{2048}{3} \sin^3 \theta - 64\theta \right|_0^{2\pi}$$

$$\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = 128\pi$$

Thus,

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{A} \cdot \hat{n} dS_2$$

$$\iint_S \vec{A} \cdot \hat{n} dS = 192\pi + 128\pi = 320\pi$$

is the required solution.

Module No. 48

Volume Integral

A volume integral refers to an integral over a 3-dimensional domain, that is, it is a special case of multiple integrals.

Definition

Let $\vec{A}$ be a given vector point function which is defined and continuous in a closed region R. Then

$$\iiint_R \vec{A} dV$$

If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then the above integral may be written as

$$\iiint_R \vec{A} dV = \hat{i} \iiint_R A_1 dV + \hat{j} \iiint_R A_2 dV + \hat{k} \iiint_R A_3 dV$$

If we have a scalar point function $\varphi(x, y, z)$ defined as continuous over the region R, then the volume integral becomes

$$\iiint_R \varphi dV$$

In rectangular coordinate system, $dV = dxdydz$ so the volume integral becomes

$$\iiint_R \varphi dV = \iiint_R \varphi(x, y, z) dxdydz$$

Which is ordinary triple integral of $\varphi(x, y, z)$ over the region R.

If $\varphi(x, y, z) = 1$, the volume V of the region R is given by

$$\iiint_R dxdydz = \iiint_R dV$$

Notations for other than Cartesian system

Volume integral can be expressed also in cylindrical and spherical coordinates as

- i. Volume integral in cylindrical coordinates:

$$\iiint_R g(\rho, \varphi, z) \rho d\rho d\varphi dz$$

ii. Volume integral in cylindrical coordinates:

$$\iiint_R g(r, \theta, \varphi) r^2 \sin \varphi dr d\theta d\varphi$$

Which are equivalent to ordinary triple integrals in cylindrical and spherical coordinates.

Module No. 49

Example on Volume Integral

Problem Statement

Let $\vec{F} = 2xzi - xj + y^2k$.

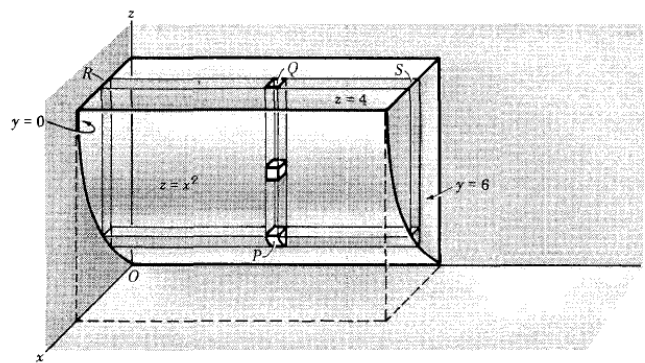
Evaluate

$$\iiint_R \vec{F} dV$$

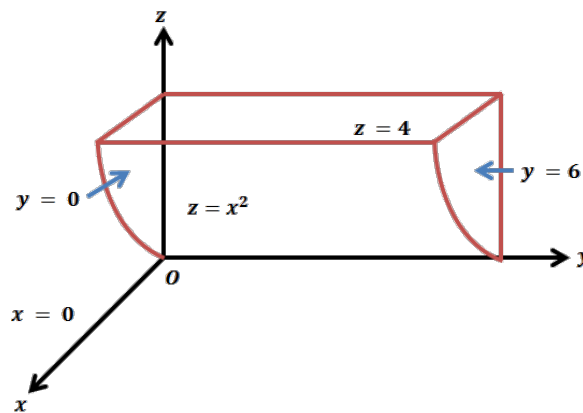
where V is the region bounded by the surfaces $x = 0, y = 0, y = 6, z = x^2, z = 4$.

Solution

We will evaluate the given problem through following scheme.



The region V is covered (i) by keeping x and y fixed and integrating from $z = x^2$ to $z = 4$ (base to top of column PQ),



- (ii) Then by keeping χ fixed and integrating from $y = 0$ to $y = 6$ (R to S in the slab),
 (iii) Finally integrating from $x = 0$ to $x = 2$ (where $z = x^2$ meets $z = 4$). Then the required integral is

$$\begin{aligned} \iiint_R \vec{F} dV &= \int_0^2 \int_0^6 \int_{x^2}^4 (2xz\hat{i} - x\hat{j} + y^2\hat{k}) dz dy dx \\ &= \hat{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx - \hat{j} \int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx + \hat{k} \int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx \end{aligned}$$

Solving integrals turn by turn

$$\begin{aligned} \int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx &= \int_0^2 \int_0^6 x(16 - x^4) dy dx = \int_0^2 x(16 - x^4) |y|_0^6 dx = 6 \int_0^2 x(16 - x^4) dx \\ &= 6 \int_0^2 (16x - x^5) dx = 6 \left| \frac{16x^2}{2} - \frac{x^6}{6} \right|_0^2 = 6 \left| \frac{64}{2} - \frac{64}{6} \right| = 128 \end{aligned}$$

$$\begin{aligned} \int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx &= \int_0^2 \int_0^6 x |z|_{x^2}^4 dy dx = \int_0^2 \int_0^6 x(4 - x^2) dy dx = \int_0^2 x(4 - x^2) |y|_0^6 dx \\ &= 6 \int_0^2 x(4 - x^2) dx = 6 \left| 2x^2 - \frac{x^4}{4} \right|_0^2 = 6 \left| 8 - \frac{16}{4} \right| = 24 \end{aligned}$$

$$\begin{aligned} \int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx &= \int_0^2 \int_0^6 y^2 |z|_{x^2}^4 dy dx = \int_0^2 \int_0^6 y^2(4 - x^2) dy dx = \int_0^2 \left| \frac{y^3}{3} \right|_0^6 (4 - x^2) dx \\ &= 72 \int_0^2 (4 - x^2) dx = 72 \left| 4x - \frac{x^3}{3} \right|_0^2 = 72 \left(8 - \frac{8}{3} \right) = 384 \end{aligned}$$

Hence

$$\iiint_R \vec{F} dV = \hat{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx - \hat{j} \int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx + \hat{k} \int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx$$

$$\iiint_R \vec{F} dV = 128\hat{i} - 24\hat{j} + 384\hat{k}$$

is the required solution.

Module No. 50

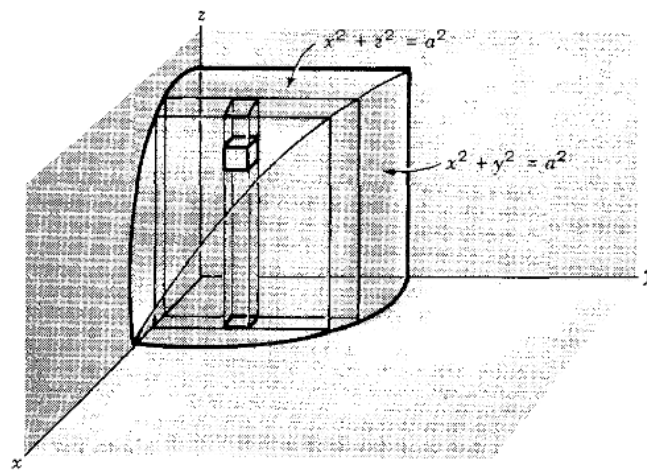
Selected Example/Problem 1: Volume Integral

Problem Statement

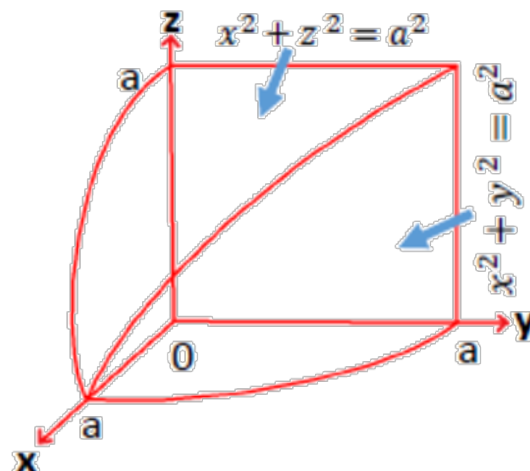
Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution

Let M be the required common region then



[1]



[1]

Required volume = 8 times volume of region shown in above figure.

$$\begin{aligned}
 \iiint_V M dV &= 8 \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dz dy dx \\
 &= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} z dy dx = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dy dx = \int_{x=0}^a (a^2-x^2) dx \\
 &= \frac{16a^3}{3}
 \end{aligned}$$



Lecture Handouts

on

**VECTORS AND CLASSICAL MECHANICS
(MTH-622)**

**Virtual University of Pakistan
Pakistan**

About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

Contents

Module No. 79.....	1
Introduction to Classical Mechanics	1
Module No. 80.....	2
Introductions to the Basics of Newton's Law	2
Module No. 81.....	4
Introductions to Rectangular Components of Velocity & Acceleration	4
Module No. 82.....	6
Introduction to Tangential and Normal Components of Velocity & Acceleration	6
Module No. 83.....	8
Example of Tangential and Normal Acceleration.....	8
Module No. 84.....	10
Curvature and Radius of Curvature	10
Module No. 85.....	12
Introduction to Radial and Transverse Components of Velocity & Acceleration	12
Module No. 86.....	14
Example of Radial and Transverse Components of Velocity and Acceleration.....	14
Module No. 87.....	15
Reference System or Inertial Frame	15
Module No. 88.....	16
Example of inertial Frame	16
Module No. 89.....	18
Newton's First and Second Laws.....	18
Module No. 90.....	20
The Third Law and Law of Conservation of Momentum	20
Module No. 91.....	23
Validity of Newton's Law	23
Module No. 92.....	24
Example of Newton's Law 1, 2.....	24
Module No. 93.....	27
Example of Newton's Law: 3	27
Module No. 94.....	29
Introduction to Energy: Kinetic Energy	29

Module No. 95.....	30
Introduction to Work – Theorem	30
Module No. 96.....	33
Example of Work Done	33
Module No. 97.....	34
Conservative Force Field.....	34
Module No. 98.....	36
Example of Conservative Field	36
Module No. 99.....	37
Related Topic of Conservative Force Fields	37
Module No. 100.....	39
Non-Conservative Force Field	39

Module No. 79

Introduction to Classical Mechanics

Mechanics is the science of motion. It studies the states of rest and motion and the laws governing rest, equilibrium and motion.

A broad division of mechanics is made through the terms classical mechanics and quantum mechanics.

Classical Mechanics

The science of classical mechanics deals with the motion of objects through absolute space and time in the Newtonian sense. Classical mechanics deals with the macroscopic object i.e. those bodies which we encounter in everyday life. Such bodies may be those bodies in the form of billiard balls, parts of machinery and astronomical bodies.

Classical mechanics itself has two broad divisions in the form of non-relativistic and relativistic mechanics. Non-relativistic mechanics, based on the laws of Newton is concerned with bodies moving at speeds and velocity negligibly small as compared to the speed of light $c = 3 \times 10^5$ per kilometer per second. Relativistic mechanics is dispensable for particles and bodies moving with speed comparable to the speed of light.

Quantum mechanics

The quantum mechanics is the mechanics of microscopic object. Its basic concepts, principals and laws are of entirely different from those of classical mechanics.

Division of Classical Mechanics

Three major divisions of classical mechanics are the following:

Mechanics of particles and rigid bodies: It is based on newton's law. Basic concepts and terms are space, time and mass; particle and body; velocity, momentum and acceleration; force and energy.

Mechanics of fluid: it is also based on newton's law and their extensions and deal with the behavior of the fluid (liquid and gases) in motion. Its two well-known branches are hydrodynamics (for fluid) and Aerodynamics (for gases).

Mechanics of elastic solids: it deals with the behavior of solids when they undergo deformation under forces.

Module No. 80

Introductions to the Basics of Newton's Law

Newton's three laws of motion are formulated in terms of crucial underlying concepts: the notions of space, time, mass and force.

In this section we will review all of the factors of formulation of Newton's law.

Space and time:

We shall assume that space and time are described strictly in the Newtonian sense.

- **Space.** This is closely related to the concepts of point, position, direction and displacement sense. Three-dimensional space is Euclidian, and positions of points in that space are specified by a set of three numbers (x,y,z) relative to the origin (0,0,0) of a rectangular Cartesian coordinate system. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.
- **Time.** This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc. the basic unit of time in SI, began as an arbitrary fraction ($1/86,400$) of a mean solar day ($24 \times 60 \times 60 = 86,400$).
- **Mass** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called its mass. The mass of an object is characterized by its inertia. It's the resistance to be accelerated. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time. Length, mass and time are often called dimensions from which other physical quantities are constructed.

- **Force** The informal notion of force as a push or pull. It is a vector quantity. As the unit of force we naturally adopt the newton (abbreviated N) defined as the magnitude of any single force that accelerates a standard kilogram mass with an acceleration of 1 m/s^2 . If we apply a given force F (and no other forces) to any object at rest, the direction of F is defined as the direction of the resulting acceleration, that is, the direction in which the body moves off.

Now that we know, at least in principle, what we mean by positions, times, masses, and forces, we can proceed to discuss the cornerstone of our subject — Newton's three laws of motion.

Module No. 81

Introductions to Rectangular Components of Velocity & Acceleration

Rectangular Components

The process of splitting a vector into various parts or components is called “Resolution of vector” and these parts are called components of vector. If we split a vector in a rectangular plane OXY, such components are called rectangular components of a vector.

Component Along x-axis is called horizontal component of vector.

Component Along y-axis is called vertical component of vector.

Position vector

It is often convenient to describe the motion of a particle in terms of its x, y or rectangular components, relative to a fixed frame of reference. In a given reference system, the position of a particle can be specified by a single vector, namely, the displacement of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates (see Figure), the position vector is simply

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

The components of the position vector of a moving particle are functions of the time, namely,

$$x = x(t), y = y(t)$$

RECTANGULAR COMPONENTS: VELOCITY

In above Equation we gave the formal definition of the derivative of any vector with respect to some parameter. In particular, if the vector is the position vector $\mathbf{r}$ of a moving particle and the parameter is the time t , the derivative of $\mathbf{r}$ with respect to t is called the velocity, which we shall denote by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

where the dots indicate differentiation with respect to t .

where

$$v_x = \dot{x}, v_y = \dot{y}, v_z = \dot{z}$$

The magnitude of the velocity is called the speed. In rectangular components the speed is just

The magnitude of the velocity vector is

$$v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$$

The direction of $\mathbf{v}$ is tangent to the path of motion.

RECTANGULAR COMPONENTS: ACCELERATION

Acceleration represents the rate of change in the velocity of a particle.

If a particle's velocity changes from $\mathbf{v}$ to $\mathbf{v}'$ over a time increment Δt , the average acceleration during that increment is: avg $\mathbf{a} = \Delta \mathbf{v} / \Delta t = (\mathbf{v} - \mathbf{v}') / \Delta t$ the instantaneous accelerations the time derivative of velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = i\mathbf{a}_x + j\mathbf{a}_y + k\mathbf{a}_z$$

In rectangular components,

$$\mathbf{a} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

$$\mathbf{a} = i\ddot{x} + j\ddot{y} + k\ddot{z}$$

$$a_x = \ddot{x}, a_y = \ddot{y}, a_z = \ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

The direction of $\mathbf{a}$ is usually not tangent to the path of the particle.

Module No. 82

Introduction to Tangential and Normal Components of Velocity & Acceleration

Introduction To Tangential And Normal Components Of Vector

In mathematics, given a vector at a point on a curve, that vector can be decomposed uniquely as a sum of two vectors, one tangent to the curve, called the tangential component of the vector, and another one perpendicular to the curve, called the normal component of the vector.

Introduction to tangential and normal components of velocity

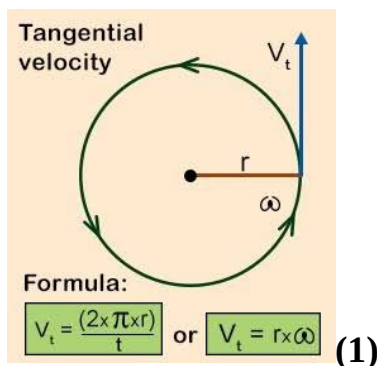
The tangential velocity is the velocity measured at any point tangent to a circular path. Thus tangential velocity, V_t is related to the angular velocity of the wheel, ω , and the radius of the wheel, r .

$$V_t = \omega r$$

V_t = tangential velocity

ω = angular velocity

r = radius of wheel



Total velocity of a particle at a point on a curve can be write as

$$v = v_t + v_N$$

We can determine normal component of velocity by

$$v_N = v - v_t$$

Tangential and normal components of Acceleration

The rate of change of tangential velocity of an object traveling in a circular orbit or path is known as Tangential acceleration. It is directed towards tangent to the path of a body.

Total acceleration of a particle at a point on a curve can be write as

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_N$$

where

$$a_t = \frac{dv_t}{dt}, a_N = \frac{v^2}{r}$$

where v is total velocity of the object and r is radius of circular path.

Module No. 83

Example of Tangential and Normal Acceleration

Example

Consider the space curve C with the position vector $\mathbf{r}$ given to be at time t

$$\mathbf{r} = 5 \cos 3t \mathbf{i} + 3 \sin 3t \mathbf{j} + 10t \mathbf{k}$$

(a) Find a unit tangent vector $\mathbf{T}$ to the curve.

(b) Verify that $\mathbf{v} = v \mathbf{T}$.

Solution:

(a) Tangent vector to the space curve C is

$$\frac{d\mathbf{r}}{dt} = -15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}$$

The magnitude of this vector is

$$\begin{aligned} v = \left| \frac{d\mathbf{r}}{dt} \right| &= \frac{ds}{dt} = \sqrt{(-15 \sin 3t)^2 + (9 \cos 3t)^2 + (10)^2} \\ &= \sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100} \end{aligned}$$

The unit tangent vector to C is

$$\mathbf{T} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \frac{-15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}}$$

(b) From the first part, it follows that This follows directly from (a) since

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}$$

$$\begin{aligned}
 \boldsymbol{v} &= -15 \sin 3t \boldsymbol{i} + 9 \cos 3t \boldsymbol{j} + 10k \frac{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}} \\
 \boldsymbol{v} &= \sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100} \frac{(-15 \sin 3t \boldsymbol{i} + 9 \cos 3t \boldsymbol{j} + 10k)}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}} \\
 &= v \boldsymbol{T}
 \end{aligned}$$

Module No. 84

Curvature and Radius of Curvature

Curvature is the amount by which a geometric object such as a surface differs from being a flat plane, or a curve from being straight as in the case of a line.

Statement:

If $\mathbf{T}$ is a unit tangent vector to a space curve C , then show that $\frac{d\mathbf{T}}{ds}$ is normal to $\mathbf{T}$.

Proof:

Since $\mathbf{T}$ is a unit vector, we have $\mathbf{T} \cdot \mathbf{T} = 1$. Then differentiating with respect to s , we obtain

$$\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} + \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 0$$

$$2 \frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 0$$

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 0$$

Which states that $\frac{d\mathbf{T}}{ds}$ is normal, i.e. perpendicular to $\mathbf{T}$.

If $\mathbf{N}$ is a unit vector in the direction of $\frac{d\mathbf{T}}{ds}$, we have

$$\frac{d\mathbf{T}}{ds} = k\mathbf{N}$$

and we call $\mathbf{N}$ the unit principal normal to C . The scalar $k = \left| \frac{d\mathbf{T}}{ds} \right|$ is called the curvature,

while $R = 1/k$ is called the radius of curvature.

Example:

Find the

- Curvature,
- Radius of curvature
- Unit principal normal $\mathbf{N}$ to any point of the space curve of given unit tangent vector is

$$T = -\frac{3}{5} \sin 2t \mathbf{i} + \frac{3}{5} \cos 2t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

moving with 10 ms^{-1} .

Solution:

Since given tangential vector is

$$T = -\frac{3}{5} \sin 2t \mathbf{i} + \frac{3}{5} \cos 2t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

Then,

$$\begin{aligned} \frac{dT}{ds} &= \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{-(6/5) \cos 2t \mathbf{i} - (6/5) \sin 2t \mathbf{j}}{10} \\ &= -\frac{3}{25} \cos 2t \mathbf{i} - \frac{3}{25} \sin 2t \mathbf{j} \end{aligned}$$

Thus the curvature is $k = \left| \frac{dT}{ds} \right| = \sqrt{\left(-\frac{3}{25} \cos 2t\right)^2 + \left(-\frac{3}{25} \sin 2t\right)^2} = \frac{3}{25}$

Radius of curvature $= \frac{1}{k} = \frac{25}{3}$

We have unit principal normal $N = \frac{1}{k} \cdot \frac{dT}{ds} = \frac{25}{3} \left(-\frac{3}{25} \cos 2t \mathbf{i} - \frac{3}{25} \sin 2t \mathbf{j}\right)$
 $= -\cos 2t \mathbf{i} - \sin 2t \mathbf{j}$

Module No. 85

Introduction to Radial and Transverse Components of Velocity & Acceleration

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector:

$$\mathbf{r} = r\mathbf{e}_r$$

As the particle moves, both r and $\mathbf{e}_r$ vary; thus, they are both functions of the time.

Hence, if we differentiate with respect to t , we have

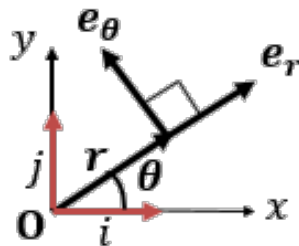
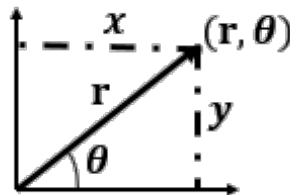
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

From fig $\mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ and $\mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$

So, $\dot{\mathbf{e}}_r = -\sin \theta \dot{\theta} \mathbf{i} + \cos \theta \dot{\theta} \mathbf{j}$ or $\dot{\mathbf{e}}_r = (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})\dot{\theta}$

This implies $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$



$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $\frac{d\mathbf{e}_r}{dt}$ and $\frac{d\mathbf{e}_\theta}{dt}$ are given by Equations

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{k} \times \mathbf{e}_r = \dot{\theta}\mathbf{e}_\theta \text{ and } \dot{\mathbf{e}}_\theta = \dot{\theta}\mathbf{k} \times \mathbf{e}_\theta = -\dot{\theta}\mathbf{e}_r$$

and yield the following equation for the acceleration vector in plane polar coordinates:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Module No. 86

Example of Radial and Transverse Components of Velocity and Acceleration

Statement:

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratic ally with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

The equation of acceleration in plane polar coordinates is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \quad (1)$$

As we have $r = bt^2$ and $\theta = \omega t$, therefore $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$.

By putting these values in equation (1), we have

$$\begin{aligned} \mathbf{a} &= (2b - bt^2\omega^2)\mathbf{e}_r + (bt^2(0) + 2(2bt)\omega)\mathbf{e}_\theta \\ \mathbf{a} &= b(2 - t^2\omega^2)\mathbf{e}_r + 4bt\omega\mathbf{e}_\theta \end{aligned}$$

Which is required acceleration of bug.

Module No. 87

Reference System or Inertial Frame

Framework

A framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system. a system of geometric axes in relation to which measurements of size, position, or motion can be made.

Inertial frames

A frame of reference in which a body remains at rest or moves with constant linear velocity unless acted upon by forces: any frame of reference that moves with constant velocity relative to an inertial system is itself an inertial system. These are also called inertial frame.

Mechanics is based on Newton's laws of motion. These laws are usually stated without explicitly discussing the role of frames of references. It is to be emphasized that Newton's laws are not valid w.r.t every frame of references e.g. they are not true in a rotating frame or coordinate system. They hold only in restricted class of frames, so called the inertial frames (Newtonian frames). Such frames are assumed to be un-accelerated i.e. neither moving in straight line with variable velocity nor rotating. In other words, it is supposed to be absolutely at rest (or moving with uniform velocity w.r.t absolutely-at-rest frames). We can easily see that if S is an inertial frame, then any other frame S' which is in uniform motion relative to S is also an inertial frame. To all observers in inertial frames, the force acting on a particle will be the same i.e. the law of motion will be invariant under transformation connecting inertial frames. In fact all laws of mechanics are same in all frames of reference. This statement is called Newtonian principal of relativity.

Module No. 88

Example of inertial Frame

In order to illustrate the concept of inertia, we will consider the example of earth.

Statement:

Calculate the centripetal acceleration relative to the acceleration due to gravity g , of

- a) A point on the surface of the Earth's equator (the radius of the Earth is $R_e = 6.4 \times 10^6 \text{ km}$)
- b) The Earth in its orbit about the Sun (the radius of the Earth's orbit is
 - a. $a_e = 150 \times 10^6 \text{ km}$)
- c) The Sun in its rotation about the center of the galaxy (the radius of the Sun's orbit about the center of the galaxy is $R_G = 2.8 \times 10^4 \text{ LY}$. its orbital speed is $v_G = 220 \text{ km/s}$)

Solution:

The centripetal acceleration of a point rotating in a circle of radius R is given by

$$a_c = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \frac{4\pi R^2}{T^2}$$

where T is period of one complete rotation ($T = 3.16 \times 10^7$). Thus, relative to g (gravitational acceleration), we have

$$\frac{a_c}{g} = \frac{4\pi^2 R^2}{gT^2} \quad (1)$$

- a) By putting value from (a) in (1), we get

$$\begin{aligned} \frac{a_c}{g} &= \frac{4(3.14)^2(6.4 \times 10^6)^2}{(9.8)(3.16 \times 10^7)^2} \\ \frac{a_c}{g} &= 3.4 \times 10^{-3} \end{aligned}$$

- b) By putting values from (b) in (1), here

$$R = a_e = 150 \times 10^6 \text{ km}$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \times 10^6)^2}{(9.8)(3.16 \times 10^7)^2}$$

$$\frac{a_c}{g} = 6 \times 10^4$$

c) By putting values from (c),

$$R = R_G = 2.8 \times 10^4 \text{ LY}$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \times 10^6)^2}{(9.8)(2.8 \times 10^4)^2}$$

$$\frac{a_c}{g} = 1.5 \times 10^{-12}$$

Module No. 89

Newton's First and Second Laws

Sir Issac Newton derives the Newton's laws of motion which are the three physical laws which together forms the foundation of classical mechanics.

Newton's 1st law of Motion

Newton's 1st law of motion is also called the law of inertia. This law measures the force of an object qualitatively.

We can state it as

“In the absence of an external force, an object continues its state of rest or motion with constant velocity.”

The symbolic form of first law of motion is

$$\sum F = 0 \Rightarrow ma = 0$$

as the mass of the object is non-zero, therefore the acceleration of the concerned object must be zero

$$a = 0 \Rightarrow \frac{dv}{dt} = 0 \Rightarrow v = \text{constant}$$

As the consequence of first law of motion

- An object that is at rest will stay at rest unless a force acts upon it.
- An object continues its constant motion until an external force abrupt its state of motion.

The first law of motion says that if the net force acting on a body is zero then the velocity of object is said to be constant. The velocity of an object is vector quantity so if we say that the velocity of an object is constant we conclude that the speed and direction of the object is fixed.

Examples

Some of the examples of first law of motion are

- i. A ball kicked in a ground.
- ii. A car moving with constant velocity
- iii. A book lying in a book shelf.

Newton's 2nd law of Motion

Newton's 2nd law of motion describes the relationship among the force, mass and acceleration of the given object.

We can state the 2nd law of motion as

For any particle of mass m , the net force F on the particle of mass m times the particle's acceleration.

The symbolic form of first law of motion is

$$F = ma$$

The second law can be rephrased in terms of the particle's momentum, defined as

$$p = mv$$

by differentiating we get

$$\dot{p} = m\dot{v} = ma$$

hence we can state it as

$$F = \dot{p}$$

Consequently we can say the net force applied on a body is equals to the rate of change of momentum of the particle.

Examples

- i. When we apply same force to move a truck and a bicycle, the bicycle will have more acceleration than the truck, because the mass of bicycle is less than the truck.
- ii. An empty shopping cart is much easier to move than a full one, because the empty one has less mass.

Module No. 90

The Third Law and Law of Conservation of Momentum

Newton's first two laws concern the response of a single object to applied forces. The third law addresses a quite different issue: Every force on an object inevitably involves a second object the object that exerts the force. The nail is hit by the hammer; the cart is pulled by the horse, and so on. Newton realized that if an object 1 exerts a force on another object 2, then object 2 always exerts a force (the "reaction" force) back on object 1.

Newton's third law can be stated very compactly:

To every action there is an equal and opposite reaction.

The third law states that all forces between two objects exist in equal magnitude and opposite direction: if one object 1 exerts a force F_{12} on a second object 2, then Simultaneously exerts a force F_{21} on 1, and the two forces are equal in magnitude and opposite in direction: $F_{12} = -F_{21}$. This law is sometimes referred to as the *action-reaction law*, with F_{12} called the "action" and F_{21} the "reaction".

In other situations the magnitude and directions of the forces are determined jointly by both bodies and it isn't necessary to identify one force as the "action" and the other as the "reaction". The action and the reaction are simultaneous, and it does not matter which is called the *action* and which is called *reaction*; both forces are part of a single interaction, and neither force exists without the other.

Examples

A variety of action-reaction force pairs are evident in nature.

- i. A fish's thrust through the water.
- ii. A bird's fly in the air.
- iii. A rocket's launch.
- iv. The car moving on a road.
- v. The nail hit by hammer.

Law of Conservation of Momentum

Definition of Linear Momentum

If a body of mass m is moving with velocity, then the momentum of that body is equals to the product of mass and velocity of the specific body.

Mathematically, we can write is as

$$\vec{p} = m\vec{v}$$

it is a vector quantity.

It S.I unit is $kg \cdot s^{-1}$.

Momentum of system of particles

The total momentum of a system of particles is the vector sum of the momenta of the individual particles:

$$\vec{p}_{system} = \vec{p}_1 + \vec{p}_2 + \cdots \dots + \vec{p}_n = \sum_{i=1}^n \vec{p}_i$$

where the system consists of n particles. we can also express it as

$$\vec{p}_{system} = \sum_{i=1}^n \vec{p}_i = \sum_{i=1}^n m_i \vec{v}_i$$

where m_i is the mass of i th particle and $\vec{v}_i$ is corresponding velocity.

Statement “Law of conservation of momentum”

Law of conservation of momentum can be stated as:

If the sum of the external forces on a system is zero, the total momentum of the system does not change.

Mathematically, we can write is as

if

$$F_{ext} = 0 \Rightarrow \vec{p} = 0 \Rightarrow \sum_i m_i v_i = Mv = \text{constant}$$

Momentum is always conserved (even if forces are non-conservative).

As we stated earlier the second law of motion,

$$F_{ext} = ma$$

We can deduct the law of conservation of momentum from the last expression as well.

$$F_{ext} = 0 \Rightarrow ma = 0 \Rightarrow m \frac{dv}{dt} = 0$$

as mass is non-zero constant, therefore

$$\frac{dv}{dt} = 0 \text{ or } v = \text{constant} \Rightarrow mv = \text{constant.}$$

Explanation

In simple terms, momentum is considered to be a quantity of motion. This quantity is measurable because if an object is moving and has mass, then it has momentum. Something that has a large mass has a large momentum or something that is moving very fast has a large momentum. The momentum of individual component may change but the total momentum of system remains conserved.

Example

- i. A 3000 kg vehicle moving at 30 m/sec has a momentum of 90,000 kgm/sec as a result of product of the mass and the velocity.
- ii. Two hockey players of equal mass are traveling towards each other, one is moving at 9 m/sec and the other at 5 m/sec. The one moving with the faster velocity has a greater momentum and will knock the other one backwards.
- iii. A bullet fired from a gun, although small in mass, has a large momentum because of an extremely large velocity.

Module No. 91

Validity of Newton's Law

- It is known that the Newton's laws are not valid universally after the quantum mechanics and theory of relativity.
- Initially Newton's laws were considered as fundamental.
- But now quantum laws are fundamental and Newton's laws can be observed as derived from the quantum world.
- So, we can say that

Quantum Mechanics

Classical Mechanics

- In the Classical phenomena, Newton's first two laws are always valid.

Examples

- If the speed is extremely increased, even than first law remains valid.
- Newton's second law $F = ma$ and its another form in terms of the particle's momentum $p = mv$ are no longer equivalent in relativity.

Module No. 92

Example of Newton's Law 1, 2

NEWTON'S 1st LAW

Example:

Because of the force field, a particle of mass 4 moves along a space curve C whose position vector is given as a function of time t by

$$r = (3t^3 + t^2)\hat{i} + (t^4 + 8t)\hat{j} - 5t\hat{k}$$

Find

- i. The velocity,
- ii. The acceleration
- iii. The momentum,
- iv. The force field at any time t .

Solution:

- i. Velocity = $v = \frac{dr}{dt}$

$$= \frac{d}{dt}((3t^3 + t^2)\hat{i} + (t^4 + 8t)\hat{j} - 5t\hat{k})$$

$$= (9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k}$$
- ii. Acceleration = $a = \frac{dv}{dt} = \frac{d}{dt}((9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k})$

$$= (18t + 2)\hat{i} + 12t^2\hat{j}$$
- iii. Momentum = $p = mv = 4((9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k})$

$$= (36t^2 + 8t)\hat{i} + (16t^3 + 32)\hat{j} - 20\hat{k}$$
- iv. Force = $F = \frac{dp}{dt} = \frac{d}{dt}((36t^2 + 8t)\hat{i} + (16t^3 + 32)\hat{j} - 20\hat{k})$

$$= (72t + 8)\hat{i} + 48t^2\hat{j}$$

Newton's 2nd Law

Example:

A particle of mass 4 units moves in a force field given by

$$F = 4t^2\hat{i} + (16t - 4)\hat{j} - 12t\hat{k}$$

Assume that initially, i.e at $t = 0$ the particle is located at $r_0 = i + 2j - 3k$ and has velocity $v_0 = i + 4j - 9k$, find

- a) The velocity
- b) The position at any time t .

Solution:

- i. Since we know that

$$F = ma$$

As we have $m = 4$, therefore

$$F = 4a = 4 \frac{dv}{dt} = 4t^2\hat{i} + (16t - 4)\hat{j} - 12t\hat{k}$$

$$\frac{dv}{dt} = t^2\hat{i} + (4t - 1)\hat{j} - 3t\hat{k}$$

Integrating the above equation gives the velocity

$$v = \frac{1}{3}t^3\hat{i} + (2t^2 - t)\hat{j} + \frac{3}{2}t^2\hat{k} + C$$

where C is the constant of integration.

It is given that at $t = 0$, $v_0 = i + 4j - 9k$ at $t = 0$, therefore

$$i + 4j - 9k = C$$

So,

$$v = \frac{1}{3}t^3\hat{i} + (2t^2 - t)\hat{j} + \frac{3}{2}t^2\hat{k} + i + 4j - 9k$$

$$v = \left(\frac{1}{3}t^3 + 1\right)\hat{i} + (2t^2 - t + 4)\hat{j} + \left(\frac{3}{2}t^2 - 9\right)\hat{k}$$

- ii. Since we know that

$$v = \frac{d\vec{r}}{dt} = \left(\frac{1}{3}t^3 + 1\right)\hat{i} + (2t^2 - t + 4)\hat{j} + \left(\frac{3}{2}t^2 - 9\right)\hat{k}$$

There integrating w.r.to t gives the position

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + C'$$

As $r_0 = i + 2j - 3k$ at $t = 0$, we get $C' = i + 2j - 3k$

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + i + 2j - 3k$$

$$\vec{r} = \left(\frac{1}{12}t^4 + t + 1\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t + 2\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t - 3\right)\hat{k}$$

Module No. 93

Example of Newton's Law: 3

Example 1:

Determine the constant force needed to bring a 4000 lb mass moving at a speed of 90 ft/sec to rest in 3 sec?

Solution:

We know that the motion can be in any direction but for simplicity, we assume that the mass is moving in a straight axis. We have the information that initially the mass is moving, and finally it has to be at rest, therefore we can write as

$$V_1 = 90 \text{ ft./sec,}$$

$$V_2 = 0 \text{ ft./sec,}$$

where

$$t = 4 \text{ sec}$$

Then using newton's law $F = ma$

$$\text{Where } a = \frac{V_2 - V_1}{t} \quad (\text{by using first equation of motion})$$

$$\text{Now, } F = m \left(\frac{V_2 - V_1}{t} \right) = (4000) \left(\frac{0 - 90}{3} \right) = -120000 \text{ ft lb/sec}^2$$

This is important to note that there is a negative sign in the expression for F . This negative sign is because of the fact that the force is opposite to the direction of motion.

Example

A person of mass 85 kg is standing in a lift which is accelerating downwards at 0.45 ms^{-2} .

Draw a diagram to show the forces acting on the person and calculate the force the person exerts on the floor of the lift. (© mathcentre 2009)

Solution

The weight of the person is $85\,g$, where g is the acceleration due to gravity.
where

$$W = 85\,g = 85 \times 9.81\,N$$

$$a = 0.45\,\text{ms}^{-2}$$

The resultant force is $W - R$ and using Newton's second law gives:

$$F = ma$$

$$W - R = F$$

$$W - R = 85 \times 0.45$$

$$R = W - 85 \times 0.45$$

$$R = 85 \times 9.81 - 85 \times 0.45$$

$$R = 795.6\,N = 800\,N$$

Therefore, using Newton's third law the force the person exerts on the floor of the lift is equal to the force of the floor acting on the person, i.e. R , which equals $800\,N$.

Module No. 94

Introduction to Energy: Kinetic Energy

Energy:

Energy is defined as the ability to do the work by the object. It is a measurable characteristic of a system which may be in the form of kinetic energy or potential.

There exist many forms of energy. The energy neither can be created nor be destroyed but can be converted from one form to another.

In mechanics, energy is the characteristic that transferred from one particle to another. The SI unit of energy is the joule; 1 joule can be defined as the energy transferred to an object by the work done of moving it a distance of 1 meter against a force of 1 newton.

The forms of energy include kinetic energy, potential energy, elastic energy, chemical energy, thermal energy and many others.

Kinetic energy is the energy stored in a body due to its motion. It can be transferred from one objects to another and transformed into other kinds of energy.

In classical mechanics, the kinetic energy is equal to 1/2 the product of the mass and the square of the speed. In formula form:

$$K.E = T = \frac{1}{2}mv^2$$

The measuring unit of kinetic energy is the joule.

It is denoted by T .

The kinetic energy increases with the square of the velocity. If a car is moving with double velocity then we can say that it has four times as much kinetic energy. As a consequence of this quadrupling, it takes four times the work to double the velocity.

if p denotes momentum of the object and m is the mass then we can symbolize the kinetic energy in the form of momentum as

$$T = \frac{p^2}{m}$$

Module No. 95

Introduction to Work – Theorem

When some external force is applied on an object, work is done by this force in the direction of force. Also when some work is done by the applied force, energy transferred from one place to another.

The work done can be defined as a product of force and the displacement in the direction of applied force. The amount of work done can be expressed as the following equation:

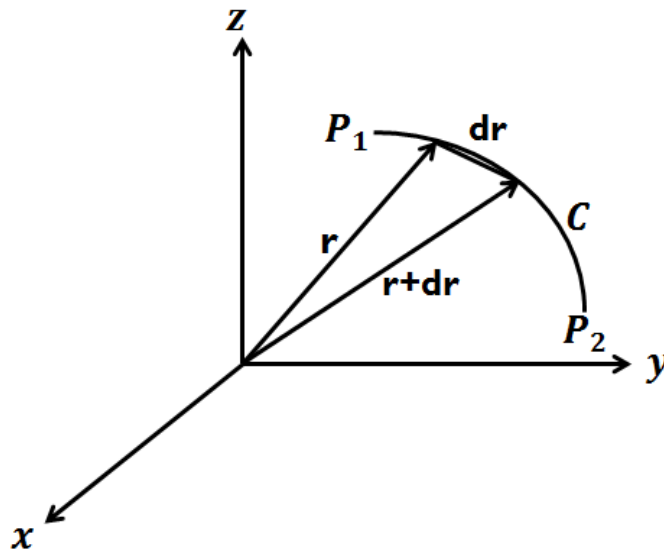
$$\text{Work} = \text{Applied Force} \times \text{Distance}$$

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton through a displacement of one meter.

If a force $\vec{F}$ acting on a particle gives it a displacement $d\vec{r}$, then the work done by the force on the particle is defined as

$$dW = \vec{F} \cdot d\vec{r}$$

Since only the component of $\vec{F}$ in the direction of $d\vec{r}$ is effective in producing the motion.



The total work done by a force field (vector field) $\vec{F}$ in moving the particle from point P_1 to point P_2 along the curve C of Fig. is given by the line integral

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$$

Where r_1 and r_2 are the position vectors of P_1 and P_2 respectively.

Theorem Statement

A particle of constant mass m moves in space under the influence of a force field F . Assuming that at times t_1 and t_2 the velocity is $\vec{v}_1$ and $\vec{v}_2$ respectively, prove that the work done is the change in kinetic energy, i.e.,

$$\text{Work done} = \int_{t_1}^{t_2} \vec{F} \cdot d\vec{r} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Proof

$$\begin{aligned} \text{L. H. S} = \text{Work done} &= \int_{t_1}^{t_2} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt \end{aligned}$$

$$\therefore \frac{d\vec{r}}{dt} = \vec{v}$$

$$= \int_{t_1}^{t_2} m\vec{a} \cdot \vec{v} dt$$

$$\therefore F = ma$$

$$\begin{aligned} &= m \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_{t_1}^{t_2} \vec{v} \cdot d\vec{v} \\ &= \frac{1}{2}m \int_{t_1}^{t_2} d(\vec{v} \cdot \vec{v}) = \frac{1}{2}m|\vec{v}^2|_{t_1}^{t_2} \end{aligned}$$

$$\text{Work done} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Hence the required equation.

Module No. 96

Example of Work Done

Example 1:

Find the work done in moving an object along a vector $r = 8\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $F = 2\hat{i} - \hat{j} - \hat{k}$.

Solution:

As we know the formula for the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{r} \\ &= (2\hat{i} - \hat{j} - \hat{k}) \cdot (8\hat{i} + 2\hat{j} - 5\hat{k}) \\ &= 16 - 2 + 5 = 19 \text{ joule} \end{aligned}$$

Work done = 19 joule

Example 2:

Find the work done by the force of a body of mass $m = 10^4 \text{ kg}$ with initial velocity $v_1 = 1.5 \times 10^3 \text{ ms}^{-1}$ to the final velocity $v_2 = 3.0 \times 10^3 \text{ ms}^{-1}$ at any time t.

Solution:

As we studied the result that the work done is equals to the difference of kinetic energy of the initial and final points

$$\begin{aligned} \text{Work done} &= \frac{1}{2} m \vec{v}_2^2 - \frac{1}{2} m \vec{v}_1^2 \\ &= \frac{1}{2} m (\vec{v}_2^2 - \vec{v}_1^2) \\ &= \frac{1}{2} (10^4) [(3.0 \times 10^3)^2 - (1.5 \times 10^3)^2] \\ &= 3.38 \times 10^{10} \text{ kgms}^{-1} \\ &= 3.38 \times 10^{10} \text{ N} \end{aligned}$$

Hence the required work done is $3.38 \times 10^{10} \text{ N}$.

Module No. 97

Conservative Force Field

In order to understand the concept of conservative force field, we must understand the phenomena of path independence.

Path Independence

Independence of path is defined as that the work done by an object remains same between initial point and the final destination of the particle, it does not matter which path is taken by the object. The example of path independence is work done by the gravitation force. When an object freely falls from same height, the work done remains same no matter what path chose, when the object hits the ground due to the fixed gravitational force and the perpendicular distance between the object and the ground. The symbolic statement of path independence is

If a particle is moving along a curve C from P_1 to P_2 is, then the total work done can be expressed as

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} -(\nabla V) \cdot d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V(P_1) - V(P_2)$$

In such case we can observe that the work done by the object depend only on the end points of the curve P_1 and P_2 .

Now we can define the conservative force field in terms of path independence of a particle moving along a curve that:

“A force field is said to conservative if the total work done by the particle moving along a curve is independent of the path taken by the particle and depend upon the end points of the curve only.”

Necessary and sufficient conditions for a Conservative Force Field

Conservative force fields conserve the following properties:

- i. A force field F is conservative if and only if there exists a continuously differentiable scalar field V such that $\vec{F} = -\nabla V$ or, equivalently, if and only if

$$\text{curl } \vec{F} = \nabla \times \vec{F} = 0$$

identically

- ii. A continuously differentiable force field F is conservative if and only if for any closed non-intersecting curve C (simple closed curve)

$$W = \oint_C \vec{F} \cdot d\vec{r} = 0$$

i.e. the total work done in moving a particle around any closed path is zero.

Examples of Conservative Forces

- Gravitational force is an example of a conservative force.
- Elastic spring force is example of conservative force.
- The work done of a particle moving along a closed path is zero and the force which causes such motion is conservative.

Module No. 98

Example of Conservative Field

Problem:

Prove that the force field

$$\vec{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$$

is conservative.

Solution:

We know that the force field is conservative iff $\text{curl } \vec{F}$ is zero;

i.e. $\nabla \times \vec{F} = 0$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix} \\ &= \hat{i} \left[\frac{\partial}{\partial y} (6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z} (3 + 2xy - x^2z^3) \right] - \hat{j} \left[\frac{\partial}{\partial x} (6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z} (y^2 - 2xyz^3) \right] \\ &\quad + \hat{k} \left[\frac{\partial}{\partial x} (3 + 2xy - x^2z^3) - \frac{\partial}{\partial y} (y^2 - 2xyz^3) \right] \\ &= (-3x^2z^2 + 3x^2z^2)\hat{i} - (-6xyz^2 + -6xyz^2)\hat{j} + (2y - 2xz^2 - 2y + 2xz^2)\hat{k} \\ &= 0\end{aligned}$$

As we obtain $\nabla \times \vec{F} = 0$

We conclude that $\vec{F}$ is conservative.

Module No. 99

Related Topic of Conservative Force Fields

Conservation of Energy Theorem in Case of Conservative Force Field

The concept of energy is a useful alternative to the concept of force. We will derive the law of conservation of energy using the concept of conservative force field.

If P_1 and P_2 are any two points on the trajectory of a particle, the total work done by the total force acting on the particle is given by

$$\begin{aligned}
 W &= \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \frac{d}{dt} mv \cdot \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} m \frac{dv}{dt} \cdot v dt \\
 &= \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} (v^2) dt = \frac{1}{2} m |v^2|_{t_1}^{t_2} = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2 = T_2 - T_1
 \end{aligned} \tag{1}$$

where $v(t_1) = v_1$ & $v(t_2) = v_2$

Now we suppose that the applied force F on the particle is conservative. Then we can express it as $F = -\nabla V$, and the integral for W can be written as

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} -(\nabla V) \cdot d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V_1 - V_2 \tag{2}$$

where V_1 and V_2 corresponds to P.E. at initial and final positions of the particle. By comparing expression (1) and (2), we have

$$T_2 - T_1 = V_1 - V_2 \quad \text{or} \quad T_1 + V_1 = T_2 + V_2 \tag{3}$$

Expression (3) shows that the quantity $T_1 + V_1$ is a constant of motion. It is called total energy of the particle. Denoting the total energy by E , we have

$$E = T + V = \frac{1}{2} mv^2 + V$$

Conservative Systems and Orbits of Particles

A single particle moving in a conservative field of forces may perform an important type of motion. Suppose the total energy E of the system is a constant of motion i.e.

$$\frac{1}{2}m\dot{r}^2 + V(r) = E$$

where E is some constant denoting total energy of the system.

Suppose the particle's motion is such that it returns to the same position, represented by the position vector r_0 , at a later time. Then it must have the same K.E. and therefore the same speed. It follows that in a conservative system it is possible for closed trajectories to occur. This fact is very relevant in the study of Earth's motion about the Sun.

Module No. 100

Non-Conservative Force Field

Forces that cannot be expressed in the term of a potential energy function are called non-conservative forces. We can also state that forces that do not store energy are called non-conservative or dissipative forces. If there is no scalar function V such that $F = -\Delta V$ [or, equivalently, if $\nabla \times F = 0$], then F is called a non-conservative force field.

Friction is a non-conservative force, and there are others. It is always opposed to the direction of motion and is not a single valued function of position alone.

Similarly the impulse (time dependent force) is also non-conservative and cannot be derived from a scalar point function.

An example of non-conservative force, we have $F = kv$, where v is the velocity of the particle, then,

$$\begin{aligned}\oint \vec{F} \cdot d\vec{r} &= \int \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = k \int_{t_1}^{t_2} v^2 dt > 0\end{aligned}$$

which shows the integral is not equals to zero. Hence the force is non-conservative.

Work-Energy relation and Non-conservative Forces

We have already shown that for any general force F

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = T_2 - T_1$$

When the force F can be broken into conservative and non-conservative parts

$$\vec{F} = \vec{F}^{(c)} + \vec{F}^{(nc)}$$

we have

$$\int_{P_1}^{P_2} \vec{F}^{(c)} \cdot d\vec{r} + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 - T_1 \quad (1)$$

But $\vec{F}^{(c)} = -\nabla V$ and equation (1) can be written as

$$V_1 - V_2 + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 - T_1$$

which can also be written as

$$T_1 + V_1 + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 + V_2 \quad (2)$$

The work done in overcoming friction is always negative, because $\vec{F}^{(nc)}$ is opposite to the displacement relation (2) proves that the influence of friction is dissipative and therefore decrease the total mechanical energy of the system.

Alternatively (2) can be expressed as

$$(V_2 - V_1) + (T_2 - T_1) = \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r}$$

or

$$\Delta(V + T) = \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r}$$

It is interesting to remember that the process in which work is converted into internal energy (due to friction) are irreversible.