

- (a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- (b) $I = \int_c (P dx + Q dy)$ is independent of the path of integration
- (c) $I = \oint_c (P dx + Q dy)$ is zero.
- Exact differential in three variables.
If $P dx + Q dy + R dz$ is an exact differential
- (a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$; $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$; $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$
- (b) $\int_c (P dx + Q dy + R dz)$ is independent of the path of integration.
- (c) $\oint_c (P dx + Q dy + R dz)$ is zero.
- Green's theorem
 $\oint_c (P dx + Q dy) = - \iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx dy$ and, for a simple closed curve,
- $\oint_c (x dy - y dx) = 2 \iint_R dx dy = 2A$
where A is the area of the enclosed figure.

Gradient of a scalar function
Del operator is given by $\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$

$$\nabla \phi = \text{grad } \phi = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z}$$

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

Div (Divergence of a vector function)

If $\mathbf{A} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$
then $\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k})$

$$\therefore \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

Note that

- (a) the grad operator ∇ acts on a scalar and gives a vector
- (b) the div operator $\nabla \cdot$ acts on a vector and gives a scalar.

Example

If $\mathbf{A} = x^2 y \mathbf{i} - xyz \mathbf{j} + yz^2 \mathbf{k}$ then

$$\text{Div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} (x^2 y) - \frac{\partial}{\partial y} (xyz) + \frac{\partial}{\partial z} (yz^2) = 2xy - xz + 2yz$$

Example

If $\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3)\mathbf{j} + 3y^2z^2\mathbf{k}$ determine $\nabla \cdot \mathbf{A}$ i.e. $\text{div } \mathbf{A}$.

$$\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = 4xy - 2(2xy + 3y^2z) + 6y^2z = 4xy - 4xy - 6y^2z + 6y^2z = 0$$

Such a vector $\mathbf{A}$ for which $\nabla \cdot \mathbf{A} = 0$ at all points, i.e. for all values of x, y, z , is called a solenoid vector. It is rather a special case.

Curl (curl of a vector function)

The curl operator denoted by $\nabla \times$, acts on a vector and gives another vector as a result.

If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $\text{curl } \mathbf{A} = \nabla \times \mathbf{A}$
i.e. $\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} \times (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$\therefore \nabla \times \mathbf{A} = \mathbf{i} \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + \mathbf{j} \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) + \mathbf{k} \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

Curl $\mathbf{A}$ is thus a vector function.

Example

If $\mathbf{A} = (y^4 - x^2z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} - x^2yz\mathbf{k}$, determine curl $\mathbf{A}$ at the point $(1, 3, -2)$.

$$\text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^4 - x^2z^2 & x^2 + y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla \times \mathbf{A} = \mathbf{i} \left\{ \frac{\partial}{\partial y} (-x^2yz) - \frac{\partial}{\partial z} (x^2 + y^2) \right\} - \mathbf{j} \left\{ \frac{\partial}{\partial x} (-x^2yz) - \frac{\partial}{\partial z} (y^4 - x^2z^2) \right\}$$

$$+ \mathbf{k} \left\{ \frac{\partial}{\partial x} (x^2 + y^2) - \frac{\partial}{\partial y} (y^4 - x^2z^2) \right\}$$

$$\nabla \times \mathbf{A} = \mathbf{i} \{-x^2z\} - \mathbf{j}\{-2xyz + 2x^2z\} + \mathbf{k}(2x - 4y^3). \therefore \text{At } (1, 3, -2),$$

$$\nabla \times \mathbf{A} = \mathbf{i}(2) - \mathbf{j}(12 - 4) + \mathbf{k}(2 - 108) = 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$$

Example

Determine curl $\mathbf{F}$ at the point $(2, 0, 3)$ given that $\mathbf{F} = ze^{2xy}\mathbf{i} + 2xyz\cos y\mathbf{j} + (x + 2y)\mathbf{k}$.

$$\text{In determine form, } \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x + 2y \end{vmatrix}$$

Now expand the determinant and substitute the values for x, y and z , finally obtaining curl

$$\nabla \times \mathbf{F} = \mathbf{i}\{2 - 2x \cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\{2z \cos y - 2xe^{2xy}\}$$

$$\therefore \text{At } (2, 0, 3) \quad \nabla \times \mathbf{F} = \mathbf{i}(2 - 4) - \mathbf{j}(1 - 1) + \mathbf{k}(6 - 12) = -2\mathbf{i} - 6\mathbf{k} = -2(\mathbf{i} + 3\mathbf{k})$$

Summary of grad, div and curl

- (a) Grad operator ∇ acts on a scalar field to give a vector field.
 (b) Div operator $\nabla \cdot$. Acts on a vector field to give a scalar field.
 (c) Curl operator $\nabla \times$ acts on a vector field to give a vector field.
 (d) With a scalar function $\phi(x, y, z)$

$$\text{Grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

- (e) With a vector function $\mathbf{A} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}$

$$(i) \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$(ii) \text{Curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

Multiple Operations

We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

EXAMPLE

If $\mathbf{A} = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$
 Then $\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot (x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$

$$= 2xy + z^3 + x^3 = \phi \text{ (say)}$$

Then $\text{grad}(\text{div } \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} = (2y + 3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$

i.e., $\text{grad div } \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) = (2y + 3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$

Example

If $\phi = xyz - 2y^2z + x^2z^2$ determine $\text{div grad } \phi$ at the point (2, 4, 1).

First find $\text{grad } \phi$ and then the div of the result.

$$\text{div grad } \phi = \nabla \cdot (\nabla \phi)$$

We have $\phi = xyz - 2y^2z + x^2z^2$

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} = (yz + 2xz^2)\mathbf{i} + (xz - 4yz)\mathbf{j} + (xy - 2y^2 + 2x^2z)\mathbf{k}$$

$$\therefore \text{div grad } \phi = \nabla \cdot (\nabla \phi) = 2z^2 - 4z + 2x^2$$

$$\therefore \text{At } (2, 4, 1), \text{div grad } \phi = \nabla \cdot (\nabla \phi) = 2 - 4 + 8 = 6$$

$$\text{grad } \phi = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

$$\text{Then } \text{div grad } \phi = \nabla \cdot (\nabla \phi) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \text{div grad } \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Example

If $F = x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine $\text{curl } F$ at the point (2, 1, 1). Determine an expression for $\text{curl } F$ in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

$$\text{Curl curl } F = \nabla \times (\nabla \times F) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$\text{For curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix} = (2yz - 2xyz)\mathbf{i} + x^2y\mathbf{j} + (yz^2 - x^2z)\mathbf{k}$$

$$\text{Then Curl Curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz - 2xyz & x^2y & yz^2 - x^2z \end{vmatrix} = z^2\mathbf{i} - (-2xz - 2y + 2xy)\mathbf{j} + (2xy - 2z + 2xz)\mathbf{k}$$

$$\therefore \text{At (2, 1, 1), curl curl } F = \nabla \times (\nabla \times F) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

Two interesting general results

(a) Curl $\text{grad } \phi$ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\therefore \text{curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i} \left\{ \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y} \right\} - \mathbf{j} \left\{ \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z} \right\} + \mathbf{k} \left\{ \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} \right\} = 0$$

$$\therefore \text{curl grad } \phi = \nabla \times (\nabla \phi) = 0$$

(b) Div $\text{curl } A$ where A is a vector.

$$A = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$\text{curl } A = \nabla \times A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \mathbf{i} \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) - \mathbf{j} \left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) + \mathbf{k} \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right)$$

$$\text{Then div curl } A = \nabla \cdot (\nabla \times A) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\nabla \times A \right)$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_z}{\partial x \partial y} + \frac{\partial^2 a_x}{\partial y \partial z} - \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial y \partial z} = 0$$

$$\therefore \text{div curl } A = \nabla \cdot (\nabla \times A) = 0$$

(c) Div $\text{grad } \phi$ where ϕ is a scalar.

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\text{Then div grad } \phi = \nabla \cdot (\nabla \phi) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \text{div grad } \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

This result is sometimes denoted by $\nabla^2 \phi$.

So these general results are

- (a) $\text{curl grad } \phi = \nabla \times (\nabla \phi) = 0$
- (b) $\text{div curl } \mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$
- (c) $\text{div grad } \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

Lecture No -35 Definite Integrals

Definite integral for $\sin^n x$ and $\cos^n x$, $0 \leq x \leq \pi/2$

$$\int_0^{\pi/2} \sin^2 x dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) dx = \frac{1}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{1}{2} \left[\frac{\pi}{2} - \frac{\sin \pi}{2} \right] = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \sin^2 x dx = \frac{1}{2} \frac{\pi}{2}$$

$$\int_0^{\pi/2} \cos^2 x dx = \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2x) dx = \frac{1}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sin \pi}{2} \right] = \frac{\pi}{4}$$

$$\int_0^{\pi/2} \cos^2 x dx = \frac{1}{2} \frac{\pi}{2}$$

$$\begin{aligned} \int_0^{\pi/2} \sin^3 x dx &= \int_0^{\pi/2} \sin^2 x \sin x dx = \int_0^{\pi/2} (1 - \cos^2 x) \sin x dx = \int_0^{\pi/2} \sin x dx + \int_0^{\pi/2} \cos^2 x (-\sin x) dx \\ &= -\cos x \Big|_0^{\pi/2} + \left[\frac{\cos^3 x}{3} \right]_0^{\pi/2} = -\cos \frac{\pi}{2} + \cos 0 + \frac{1}{3} \left[\cos^3 \frac{\pi}{2} - \cos^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/2} \cos^3 x dx &= \int_0^{\pi/2} \cos^2 x \cos x dx = \int_0^{\pi/2} (1 - \sin^2 x) \cos x dx = \int_0^{\pi/2} \cos x dx - \int_0^{\pi/2} \sin^2 x (\cos x) dx \\ &= \sin x \Big|_0^{\pi/2} - \left[\frac{\sin^3 x}{3} \right]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 - \frac{1}{3} \left[\sin^3 \frac{\pi}{2} - \sin^3 0 \right] = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \int_0^{\pi/2} \sin^4 x dx &= \int_0^{\pi/2} (\sin^2 x)^2 dx = \int_0^{\pi/2} \left[\frac{1 - \cos 2x}{2} \right]^2 dx = \frac{1}{4} \int_0^{\pi/2} (1 - 2 \cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4} \int_0^{\pi/2} \left(1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{4} \int_0^{\pi/2} \left(\frac{3}{2} - 2 \cos 2x + \frac{\cos 4x}{2} \right) dx \\ &= \frac{1}{4} \left[\frac{3}{2} x - \sin 2x + \frac{\sin 4x}{8} \right]_0^{\pi/2} = \frac{1}{4} \left[\frac{3\pi}{2} - \sin \pi + \frac{\sin 2\pi}{8} \right] \\ \int_0^{\pi/2} \sin^4 x dx &= \frac{1}{4} \left[\frac{3\pi}{2} \right] \quad \text{so} \quad \int_0^{\pi/2} \sin^4 x dx = \frac{3}{4} \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^4 x dx &= \int_0^{\frac{\pi}{2}} (\cos^2 x)^2 dx = \int_0^{\frac{\pi}{2}} \left[\frac{1 + \cos 2x}{2} \right]^2 dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 + 2 \cos 2x + \cos^2 2x) dx \\
&= \frac{1}{4} \int_0^{\frac{\pi}{2}} \left(1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) dx = \frac{1}{4} \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2 \cos 2x + \frac{\cos 4x}{2} \right) dx \\
&= \frac{1}{4} \left[\frac{3}{2} x + \sin 2x + \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{2}} = \frac{1}{4} \left[\frac{3\pi}{2} + \sin \pi + \frac{\sin 2\pi}{4} \right] \\
\int_0^{\frac{\pi}{2}} \cos^4 x dx &= \frac{1}{4} \left[\frac{3\pi}{2} \right] \quad \text{So} \quad \int_0^{\frac{\pi}{2}} \cos^4 x dx = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}
\end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \sin^5 x dx = \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^5 x dx = \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^6 x dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^7 x dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^8 x dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^9 x dx = \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \sin^{10} x dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \cos^{10} x dx = \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

Wallis Sine Formula

When n is even $\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot \dots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$

When n is odd $\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot \dots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$

$$\int_0^{\pi/2} \sin x dx = \frac{10.8.6.4.2}{11.9.7.5.3} \quad \text{and} \quad \int_0^{\pi/2} \cos x dx = \frac{10.8.6.4.2}{11.9.7.5.3}$$

$$\int_0^{2\pi} \sin x dx = \frac{11.9.7.5.3.1 \pi}{10.8.6.4.2 \cdot 2} \quad \text{and} \quad \int_0^{2\pi} \cos x dx = \frac{11.9.7.5.3.1 \pi}{10.8.6.4.2 \cdot 2}$$

Integration By Parts

$$\int UV dx = U \int V dx - \int \left[U \frac{dV}{dx} \right] dx$$

Example Evaluate $\int x \ln x dx$

$$\begin{aligned} \int x \ln x dx &= \ln x \int x dx - \int \left[\ln x \cdot \frac{d}{dx}(x) \right] dx \quad (\text{We are integrating by parts}) \\ &= \ln x \left(\frac{x^2}{2} \right) - \int \left(\frac{x^2}{2} \right) \left(\frac{1}{x} \right) dx = \left(\frac{x^2}{2} \right) \ln x - \int \left(\frac{x}{2} \right) dx = \left(\frac{x^2}{2} \right) \ln x - \frac{1}{2} \left(\frac{x^2}{2} \right) \end{aligned}$$

Example Evaluate $\int x \sin x dx$

$$\begin{aligned} \int x \sin x dx &= x \int \sin x dx - \int \left[x \frac{d}{dx}(\sin x) \right] dx \quad (\text{We are integrating by parts}) \\ &= x(-\cos x) - \int (-\cos x)(1) dx = -x(\cos x) + \int \cos x dx = -x(\cos x) + \sin x \end{aligned}$$

Line Integrals

Let a point p on the curve c joining A and B be denoted by the position vector $\mathbf{r}$ with respect to origin O. If q is a neighboring point on the curve with position vector

$\mathbf{r} + d\mathbf{r}$, then $\overline{PQ} = d\mathbf{r}$

The curve c can be divided up into many n such small arcs, approximating to $d\mathbf{r}_1, d\mathbf{r}_2, d\mathbf{r}_3, \dots, d\mathbf{r}_p, \dots$

so that $AB \approx \sum_{p=1}^n d\mathbf{r}_p$ where $d\mathbf{r}_p$ is a vector representing the element of the arc in both

magnitude and direction. If $d\mathbf{r} \rightarrow 0$, then the length of the curve $AB = \int_c d\mathbf{r}$.

Scalar Field

If a scalar field $V(\mathbf{r})$ exists for all points on the curve,

the $\sum_{p=1}^n V(\mathbf{r}) d\mathbf{r}_p$ with $d\mathbf{r} \rightarrow 0$, defines the line integral

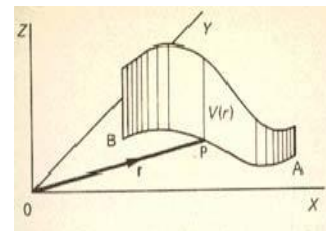
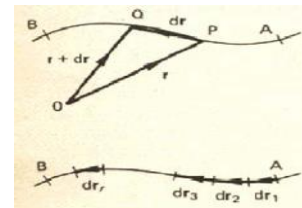
of V i.e line integral $= \int_c V(\mathbf{r}) d\mathbf{r}$.

We can illustrate this integral by erecting a continuous

Ordinate to $V(\mathbf{r})$ at each point of the curve $\int_c V(\mathbf{r}) d\mathbf{r}$ is then represented by the area of the

curved surface between the ends A and B the curve c. To evaluate a line integral, the integrand is expressed in terms of x, y, z with $d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$

In practice, x, y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. $x = x(u)$; $y = y(u)$; $z = z(u)$. From these, dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u.



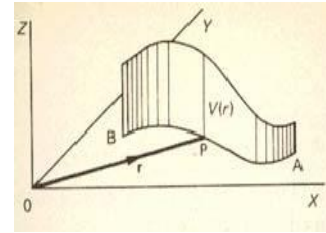
Lecture No -36 Scalar Field

Scalar Field

If a scalar field $V(r)$ exists for all points on the curve ,

the $\sum_{p=1}^n V(r) dr_p$ with $dr \rightarrow 0$, defines the line integral

of V i.e line integral $= \int_c V(r) dr$.



We can illustrate this integral by erecting a continuous

Ordinate to $V(r)$ at each point of the curve $\int_c V(r) dr$ is then represented by the area of the

curved surface between the ends A and B the curve c. To evaluate a line integral , the integrand is expressed in terms of x, y, z with $dr = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$

In practice , x, y and are often expressed in terms of parametric equation of a fourth variable (say u), i.e. $x = x(u)$; $y = y(u)$; $z = z(u)$. From these , dx, dy and dz can be written in terms of u and the integral evaluate in terms of this parameter u .

Example

If $V = xy^2z$, evaluate $\int_c V(r) dr$ along the curve c having parametric equations

$x = 3u$; $y = 2u^2$; $z = u^3$ between $A(0,0,0)$ and $B(3,2,1)$

$$V = xy^2z = (3u)(4u^4)(u^3) = 12u^8$$

$$dr = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k} \Rightarrow dr = 3du \mathbf{i} + 4u du \mathbf{j} + 3u^2 du \mathbf{k}$$

$$\text{for } x = 3u ; \therefore dx = 3du ; y = 2u^2 \therefore dy = 4u du ; z = u^3 \therefore dz = 3u^2 du$$

Limiting : $A(0,0,0)$ corresponds to $B(3,2,1)$ corresponds to u

$$A(0,0,0) \equiv u=0 ; B(3,2,1) \equiv u = 1$$

$$\int_c V(r) dr = \int_0^1 12u^8 (3 \mathbf{i} + 4u \mathbf{j} + 3u^2 \mathbf{k}) du = \left[36 \frac{u^9}{9} \mathbf{i} + 48 \frac{u^{10}}{10} \mathbf{j} + 36 \frac{u^{11}}{11} \mathbf{k} \right]_0^1 = 4\mathbf{i} + \frac{24}{5} \mathbf{j} + \frac{36}{11} \mathbf{k}$$

Example

If $V = xy + y^2z$ Evaluate $\int_c V(r) dr$ along the curve c defined by $x = t^2$; $y = 2t$; $z = t+5$

between $A(0,0,5)$ and $B(4,4,7)$. As before , expressing V and dr in term of the parameter t .

since $V = xy + y^2z$

$$= (t^2)(2t) + (4t^2)(t+5)$$

$$= 6t^3 + 20t^2$$

$$x = t^2 \quad dx = 2t \, dt$$

$$\left. \begin{array}{l} y = 2t \quad dy = 2 \, dt \\ z = t+5 \quad dz = dt \end{array} \right\}$$

$$\therefore dr = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$= 2t \, dt \mathbf{i} + 2 \, dt \mathbf{j} + dt \mathbf{k}$$

$$\therefore \int_c V dr = \int_0^1 (6t^3 + 20t^2) (2t \mathbf{i} + 2 \mathbf{j} + \mathbf{k}) dt$$

Limits: A (0, 0, 5) $\equiv t = 0$;

B (4, 4, 7) $\equiv t = 2$

$$\therefore \int_C \mathbf{V} d\mathbf{r} = \int_0^2 (6t^3 + 20t^2)(2t \mathbf{i} + 2 \mathbf{j} + \mathbf{k}) dt$$

$$\begin{aligned} \int_C \mathbf{V} d\mathbf{r} &= 2 \int_0^2 \{6t^4 + 20t^3\} \mathbf{i} + \{6t^3 + 20t^2\} \mathbf{j} \\ &\quad + \{3t^3 + 10t^2\} \mathbf{k} \} dt. \\ &= \frac{8}{15} (444\mathbf{i} + 290\mathbf{j} + 145\mathbf{k}) \end{aligned}$$

Vector Field

If a vector field $\mathbf{F}(\mathbf{r})$ exist for all points of the curve c , then for each element of arc we can form the scalar product $\mathbf{F} \cdot d\mathbf{r}$. Summing these products for all elements of arc, we have

$$\sum_{p=1}^n \mathbf{F} \cdot d\mathbf{r}_p$$

The line integral of $\mathbf{F}(\mathbf{r})$ from A to B along the stated curve = $\int_C \mathbf{F} \cdot d\mathbf{r}$.

In this case, since $\mathbf{F} \cdot d\mathbf{r}$ is a scalar product, then the line integral is a scalar.

To evaluate the line integral, $\mathbf{F}$ and $d\mathbf{r}$ are expressed in terms of x, y, z , and the curve in parametric form. We have

$$\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$$

$$\text{And } d\mathbf{r} = dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}$$

$$\text{Then } \mathbf{F} \cdot d\mathbf{r} = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \cdot (dx \mathbf{i} + dy \mathbf{j} + dz \mathbf{k}) = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

Now for an example to show it in operation.

Example

If $\mathbf{F}(\mathbf{r}) = x^2y \mathbf{i} + xz \mathbf{j} + 2yz \mathbf{k}$, Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ between A(0,0,0) and B(4,2,1) along the

curve c having parametric equations $x=4t$; $y=2t^2$; $z=t^3$

Expressing everything in terms of the parameter t , we have

$$dx = 4 dt; dy = 4t dt; dz = 3t^2 dt$$

$$x^2y = (16t^2)(2t^2) = 32t^4$$

$$x = 4t \quad \therefore dx = 4 dt$$

$$xz = (4t)(t^3) = 4t^4$$

$$y = 2t^2 \quad dy = 4t dt$$

$$2yz = (4t^2)(t^3) = 4t^5$$

$$z = t^3 \quad \therefore dz = 3t^2 dt$$

$$\mathbf{F} = 32t^4 \mathbf{i} + 4t^4 \mathbf{j} - 4t^5 \mathbf{k}$$

$$d\mathbf{r} = 4dt \mathbf{i} + 4t dt \mathbf{j} + 3t^2 dt \mathbf{k}$$

$$\text{Then } \int \mathbf{F} \cdot d\mathbf{r} = \int (32t^4 \mathbf{i} + 4t^4 \mathbf{j} - 4t^5 \mathbf{k}) \cdot$$

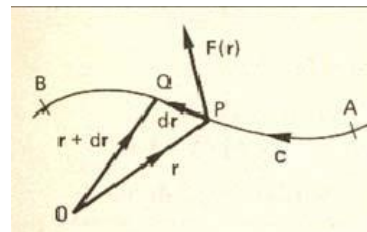
$$(4dt \mathbf{i} + 4t dt \mathbf{j} + 3t^2 dt \mathbf{k})$$

$$= \int (128t^4 + 16t^5 + 12t^7) dt$$

Limits: A(0,0,0) $\equiv t = 0$;

B (4, 2, 1) $\equiv t = 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (128t^4 + 16t^5 + 12t^7)dt = \frac{128}{5} t^5 + \frac{16}{6} t^6 + \frac{12}{8} t^8 = \frac{128}{5} + \frac{8}{3} + \frac{3}{2} = 29.76$$



Example

If $\mathbf{F}(\mathbf{r}) = x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ between A(0,0,0) and B(1,2,3)

B (1, 2, 3)

(a) along the straight line

c_1 from (0, 0, 0) to (1, 0, 0)

then c_2 from (1, 0, 0) to (1, 2, 0)

and c_3 from (1, 2, 0) to (1, 2, 3)

(b) along the straight line c_4 joining (0, 0, 0) to (1, 2, 3).

We first obtain an expression for $\mathbf{F} \cdot d\mathbf{r}$ which is

$$\mathbf{F} \cdot d\mathbf{r} = (x^2y\mathbf{i} + 2yz\mathbf{j} + 3z^2x\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$\mathbf{F} \cdot d\mathbf{r} = x^2y dx + 2yz dy + 3z^2x dz$$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int x^2y dx + \int 2yz dy + \int 3z^2x dz$$

Here the integration is made in three sections, along c_1 , c_2 and c_3 .

$$\begin{aligned} \text{(i) } c_1: y = 0, z = 0, dy = 0, dz = 0 \\ \therefore \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{(ii) } c_2: \text{The conditions along } c_2 \text{ are} \\ c_2: x = 1, z = 0, dx = 0, dz = 0 \\ \therefore \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{(iii) } c_3: x = 1, y = 2, dx = 0, dy = 0 \\ \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + \int_0^3 3z^2 dz = 27 \end{aligned}$$

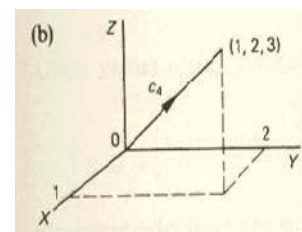
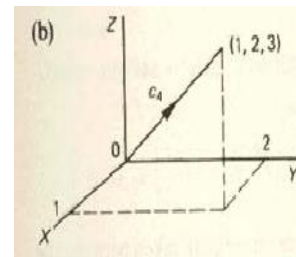
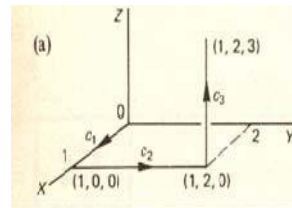
Summing the three partial results

$$\begin{aligned} \int_{(0,0,0)}^{(1,2,3)} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 27 = 27 \\ \therefore \int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} = 27 \end{aligned}$$

If t taken as the parameter, the parametric equation of c are $x = t; y = 2t; z = 3t$

$(0, 0, 0) \Rightarrow t = 0, (1, 2, 3) \Rightarrow t = 1$ and the limits of t are $t = 0$ and $t = 1$

$$\begin{aligned} \mathbf{F} &= 2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k} \\ d\mathbf{r} &= dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} = dt\mathbf{i} + 2dt\mathbf{j} + 3dt\mathbf{k} \\ \int_{c_4} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (2t^3\mathbf{i} + 12t^2\mathbf{j} + 27t^3\mathbf{k}) \cdot (dt\mathbf{i} + 2dt\mathbf{j} + 3dt\mathbf{k}) = \int_0^1 (2t^3 + 24t^2 + 81t^3) dt \\ &= \int_0^1 (83t^3 + 24t^2) dt = \left[83 \frac{t^4}{4} + 8t^3 \right]_0^1 = \frac{115}{4} = 28.75 \end{aligned}$$



So the value of the line integral depends on the path taken between the two end points A and B

$$(a) \quad \int \mathbf{F} \cdot d\mathbf{r} \text{ via } c_1, c_2 \text{ and } c_3 = 27$$

$$(b) \quad \int \mathbf{F} \cdot d\mathbf{r} \text{ via } c_4 = 28.75$$

Example

Evaluate $\int_V F dV$ where V is the region bounded by the planes $x = 0, y = 0, z = 0$ and

$2x + y + z = 2$, and $F = 2z \mathbf{i} + y \mathbf{k}$. To sketch the surface $2x + y + z = 2$, note that

when $z = 0, 2x + y = 2$ i.e. $y = 2 - 2x$

when $y = 0, 2x + z = 2$ i.e. $z = 2 - 2x$

when $x = 0, y + z = 2$ i.e. $z = 2 - y$

Inserting these in the planes

$x = 0, y = 0, z = 0$ will help.

The diagram is therefore.

So $2x + y + z = 2$ cuts the axes at

A(1,0,0); B(0, 2, 0); C(0, 0, 2).

Also $F = 2z\mathbf{i} + y\mathbf{k}$;

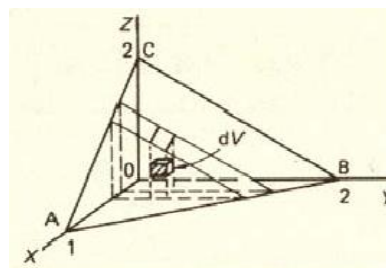
$$z = 2 - 2x - y = 2(1 - x) - y$$

$$\therefore \int_V F dV = \int_0^1 \int_0^{2(1-x)} \int_0^{2(1-x)-y} (2x\mathbf{i} + y\mathbf{k}) dz dy dx$$

$$= \int_0^1 \int_0^{2(1-x)} \left[z^2 \mathbf{i} + yz \mathbf{k} \right]_{z=0}^{z=2(1-x)-y} dy dx$$

$$= \int_0^1 \int_0^{2(1-x)} \{ [4(1-x)^2 - 4(1-x)y + y^2] \mathbf{i} + [2(1-x)y - y^2] \mathbf{k} \} dy dx$$

$$\int_V F dV = \frac{1}{3} (2 + \dots)$$



Lecture No -37 Examples

Example

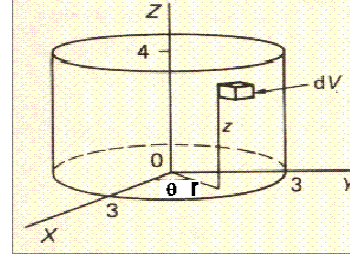
Evaluate $\int_V \mathbf{F} dV$ where $\mathbf{F} = 2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$ and V is the region bounded by the planes $z = 0$, $z = 4$ and the surface $x^2 + y^2 = 9$.

It will be convenient to use cylindrical polar coordinates (r, θ, z) so the relevant transformations are $x = r\cos\theta$; $y = r\sin\theta$ $z = z$; $dV = r dr d\theta dz$

$$\text{Then } \int_V \mathbf{F} dV = \iiint_V (2\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}) dx dy dz$$

Changing into cylindrical polar coordinates with appropriate change of limits this becomes

$$\begin{aligned} \int_V \mathbf{F} dV &= \int_{\theta=0}^{2\pi} \int_{r=0}^3 \int_{z=0}^4 (2\mathbf{i} + 2z\mathbf{j} + r\sin\theta\mathbf{k}) r dz dr d\theta = \int_{\theta=0}^{2\pi} \int_{r=0}^3 \left[2z\mathbf{i} + z^2\mathbf{j} + r\sin\theta z\mathbf{k} \right]_{z=0}^4 r dr d\theta \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^3 (8\mathbf{i} + 16\mathbf{j} + 4r\sin\theta\mathbf{k}) r dr d\theta = 4 \int_{\theta=0}^{2\pi} \int_{r=0}^3 (2r\mathbf{i} + 4r\mathbf{j} + r^2 \sin\theta\mathbf{k}) dr d\theta = 4 \int_{\theta=0}^{2\pi} \left[r^2\mathbf{i} + 2r^2\mathbf{j} + \frac{r^3}{3} \sin\theta\mathbf{k} \right]_{r=0}^3 d\theta \\ &= 4 \int_0^{2\pi} (9\mathbf{i} + 18\mathbf{j} + 9 \sin\theta\mathbf{k}) d\theta = 36 \int_0^{2\pi} (\mathbf{i} + 2\mathbf{j} + \sin\theta\mathbf{k}) d\theta = 36 \left[\theta\mathbf{i} + 2\theta\mathbf{j} - \cos\theta\mathbf{k} \right]_0^{2\pi} \\ &= 36 \{ (2\pi\mathbf{i} + 4\pi\mathbf{j} - \mathbf{k}) - (-\mathbf{k}) \} = 72\pi (\mathbf{i} + 2\mathbf{j}) \end{aligned}$$

**Scalar Fields**

A scalar field $F = xyz$ exists over the curved surface S defined by $x^2 + y^2 = 4$ between the planes $z = 0$ and $z = 3$ in the first octant.

Evaluate $\int_S \mathbf{F} d\mathbf{S}$ over this surface.

We have $F = xyz$ $S: x^2 + y^2 - 4 = 0$, $z = 0$ to $z = 3$

$$d\mathbf{S} = \hat{\mathbf{n}} dS \text{ where } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\text{Now } \nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$

$$|\nabla S| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{x\mathbf{i} + y\mathbf{j}}{2}$$

$$\therefore d\mathbf{S} = \hat{\mathbf{n}} dS = \frac{x\mathbf{i} + y\mathbf{j}}{2} dS$$

$$\therefore \int_S \mathbf{F} d\mathbf{S} = \int_S \mathbf{F} \hat{\mathbf{n}} dS = \frac{1}{2} \int_S xyz(x\mathbf{i} + y\mathbf{j}) dS = \frac{1}{2} \int_S (x^2 y z \mathbf{i} + x y^2 z \mathbf{j}) dS \quad (1)$$

We have to evaluate this integral over the prescribed surface.

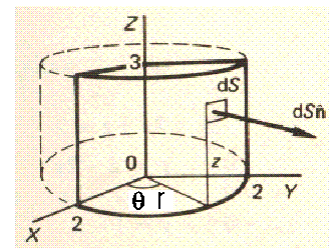
Changing to cylindrical coordinates with $r = 2$

$$x = 2 \cos \theta; \quad y = 2 \sin \theta; \quad z = z; \quad dS = 2 d\theta dz$$

$$\therefore x^2 y z = (4 \cos^2 \theta)(2 \sin \theta)(z) = 8 \cos^2 \theta \sin \theta z$$

$$x y^2 z = (2 \cos \theta)(4 \sin^2 \theta)(z) = 8 \cos \theta \sin^2 \theta z$$

$$\text{Then result } \int_S \mathbf{F} d\mathbf{S} = \frac{1}{2} \int_S (x^2 y z \mathbf{i} + x y^2 z \mathbf{j}) dS \text{ becomes.}$$



$$\begin{aligned}
 \int_S \mathbf{F} \cdot d\mathbf{S} &= \frac{8}{2} \int_0^{\pi/2} \int_0^3 (\cos^2 \theta \sin \theta \mathbf{i} + \cos \theta \sin^2 \theta \mathbf{j}) 2z dz d\theta = 4 \int_0^{\pi/2} \int_0^3 (\cos^2 \theta \sin \theta \mathbf{i} + \cos \theta \sin^2 \theta \mathbf{j}) 2z dz d\theta \\
 &= 4 \int_0^{\pi/2} (\cos^2 \theta \sin \theta \mathbf{i} + \cos \theta \sin^2 \theta \mathbf{j}) z^2 \Big|_0^3 d\theta = 4 \int_0^{\pi/2} (\cos^2 \theta \sin \theta \mathbf{i} + \cos \theta \sin^2 \theta \mathbf{j}) 9 d\theta \\
 &= 36 \left[-\frac{\cos^3 \theta}{3} \mathbf{i} + \frac{\sin^3 \theta}{3} \mathbf{j} \right]_0^{\pi/2} = 12 (\mathbf{i} + \mathbf{j})
 \end{aligned}$$

Vector Field

A vector field $\mathbf{F} = y\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ exists over a surface S defined by $x^2 + y^2 + z^2 = 9$ bounded by $x = 0, y = 0, z = 0$ in the first octant,

Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ over the surface indicated.

$$d\mathbf{S} = \hat{\mathbf{n}} dS; \quad \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

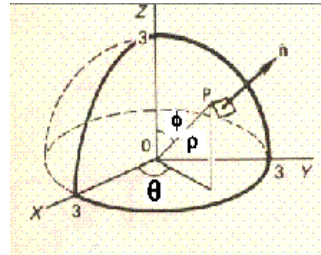
$$S: x^2 + y^2 + z^2 = 9$$

$$\nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\therefore |\nabla S| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{9} = 6$$

$$\therefore \hat{\mathbf{n}} = \frac{1}{6} (2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}) = \frac{1}{3} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int_S (y\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \cdot \frac{1}{3} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) dS = \frac{1}{3} \int_S (xy + 2y + z) dS$$



Before integrating over the surface, we convert to spherical polar coordinates.

$$x = 3 \sin \phi \cos \theta; \quad y = 3 \sin \phi \sin \theta; \quad z = 3 \cos \phi; \quad dS = 9 \sin \phi d\phi d\theta$$

$$\text{Limits of } \phi \text{ and } \theta \text{ are } \phi = 0 \text{ to } \frac{\pi}{2}; \quad \theta = 0 \text{ to } \frac{\pi}{2}$$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \int_S (xy + 2y + z) dS$$

$$xy = 3 \sin \phi \cos \theta \cdot 3 \sin \phi \sin \theta = 9 \sin^2 \theta \cos \theta$$

$$2y = 2 \cdot 3 \sin \phi \sin \theta = 6 \sin \phi \sin \theta$$

$$z = 3 \cos \phi$$

$$dS = 9 \sin \phi d\phi d\theta$$

Putting these values we get

$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \int_0^{\pi/2} \int_0^{\pi/2} (9 \sin^2 \phi \sin \theta \cos \theta + 6 \sin \phi \sin \theta + 3 \cos \phi) 9 \sin \phi d\phi d\theta$$

$$= 9 \int_0^{\pi/2} \int_0^{\pi/2} (3 \sin^3 \phi \sin \theta \cos \theta + 2 \sin^2 \phi \sin \theta + \sin \phi \cos \phi) d\phi d\theta$$

As we know that

$$\int_0^{\pi/2} \sin^3 \phi d\phi = \frac{2}{3} \quad \text{by Wallis Formula}$$

$$\text{also } \int_0^{\pi/2} \sin^2 \phi d\phi = \frac{1}{2} \cdot \frac{\pi}{2}$$

So we get

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 9 \int_0^{\pi/2} \left(2 \sin \theta \cos \theta + \frac{\pi}{2} \sin \theta + \frac{1}{2} \right) d\theta = 9 \left[\sin^2 \theta - \frac{\pi}{2} \cos \theta + \frac{\theta}{2} \right]_0^{\pi/2} = 9 \left[\left(1 - 0 + \frac{\pi}{4} \right) - \left(0 - \frac{\pi}{2} + 0 \right) \right]$$

$$= 9 \left[1 + \frac{\pi}{4} + \frac{\pi}{2} \right] = 9 \left(1 + \frac{3\pi}{4} \right)$$

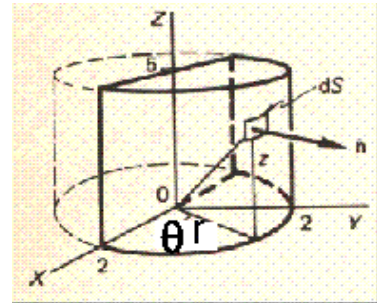
Example

Evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ and S is the surface $x^2 + y^2 = 4$ in the first two octants bounded by the planes $z = 0$, $z = 5$ and $y = 0$.

$$S : x^2 + y^2 - 4 = 0$$

$$\hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|}$$

$$\nabla S = \frac{\partial S}{\partial x} \mathbf{i} + \frac{\partial S}{\partial y} \mathbf{j} + \frac{\partial S}{\partial z} \mathbf{k} = 2x\mathbf{i} + 2y\mathbf{j}$$



$$\therefore |\nabla S| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{4} = 4$$

$$\therefore \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j}}{4} = \frac{1}{2} (x\mathbf{i} + y\mathbf{j})$$

$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \frac{1}{2} \int_S (2y^2) \, dS = \int_S y^2 \, dS$$

This is clearly a case for using cylindrical polar coordinates.

$$x = 2 \cos \theta; \quad y = 2 \sin \theta \quad z = z; \quad dS = 2d\theta \, dz$$

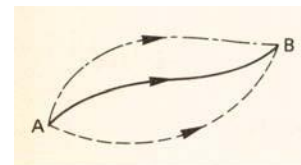
$$\therefore \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S y^2 \, dS = \iint_S 4\sin^2\theta \, 2d\theta \, dz = 8 \iint_S \sin^2\theta \, d\theta \, dz$$

Limits: $\theta = 0$ to $\theta = \pi$; $z = 0$ to $z = 5$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 4 \int_{z=0}^5 \int_{\theta=0}^{\pi} (1 - \cos 2\theta) \, d\theta \, dz = 4 \int_0^5 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} \, dz = 4 \int_0^5 \pi \, dz = 4\pi \left[z \right]_0^5 = 20\pi$$

Conservative Vector Fields

In general, the value of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field $\mathbf{F}$ is said to be conservative.



It follows that, for a closed path in a conservative

field, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

For, if the field is conservative,

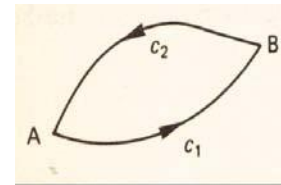
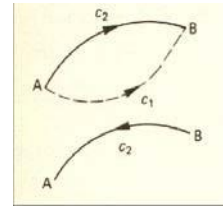
$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\text{But } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

Hence, for the closed path $AB_{C_1} + BA_{C_2}$, $\oint \mathbf{F} \cdot d\mathbf{r}$

$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

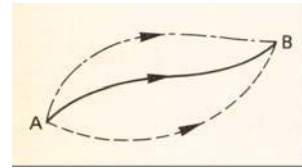
$$= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$$



Lecture No -38 Vector Field

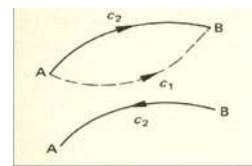
Conservative Vector Fields

In general, the value of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ between two stated points A and B depends on the particular path of integration followed. If, however, the line integral between A and B is independent of the path of integration between the two end points, then the vector field F is said to be conservative.



It follows that, for a closed path in a conservative field, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

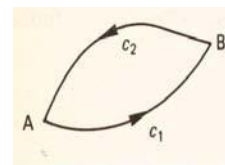
For, if the field is conservative, $\int_{C_1}^{(AB)} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2}^{(AB)} \mathbf{F} \cdot d\mathbf{r}$ But $\int_{C_2}^{(BA)} \mathbf{F} \cdot d\mathbf{r} = - \int_{C_1}^{(AB)} \mathbf{F} \cdot d\mathbf{r}$



Hence, for the closed path $AB_{C_1} + BA_{C_2}$, $\oint \mathbf{F} \cdot d\mathbf{r}$

$$= \int_{C_1}^{(AB)} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2}^{(BA)} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1}^{(AB)} \mathbf{F} \cdot d\mathbf{r} - \int_{C_1}^{(AB)} \mathbf{F} \cdot d\mathbf{r} = 0$$



Note that this result holds good only for a closed curve and when the vector field is a conservative field.

Now for some examples

Example

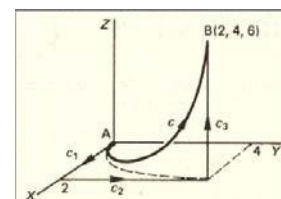
If $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$, evaluate the line integral $\int \mathbf{F} \cdot d\mathbf{r}$ between A(0,0,0) and B(2,4,6)

(a) along the curve c whose parametric equations are $x = u$, $y = u^2$, $z = 3u$

(b) along the three straight lines c_1 : (0,0,0) to (2, 0, 0); c_2 : (2, 0, 0) to (2, 4, 0); c_3 : (2, 4, 0) to (2, 4, 6).

Hence determine whether or not F is a conservative field.

First draw the diagram.



(a) $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$

$x = u$; $y = u^2$; $z = 3u$

$\therefore dx = du$; $dy = 2u du$; $dz = 3 du$.

$\mathbf{F} \cdot d\mathbf{r} = (2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$
 $= 2xyz dx + x^2z dy + x^2y dz$

Using the transformation shown above, we can now express $\mathbf{F} \cdot d\mathbf{r}$ in terms of u.

for $2xyz dx = (2u)(u^2)(3u) du = 6u^4 du$

$x^2z dy = (u^2)(3u)(2u) du = 6u^4 du$

$x^2y dz = (u^2)(u^2)3 du = 3u^4 du$

$\therefore \mathbf{F} \cdot d\mathbf{r} = 15u^4 du$

The limits of integration in u are $u = 0$ to $u = 2$

$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 15u^4 du = \left[3u^5 \right]_0^2 = 96$

(b) The diagram for (b) is as shown. We consider each straight

$$\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz \, dx + x^2z \, dy + x^2y \, dz)$$

c_1 : (0,0,0) to (2,0,0); $y = 0, z = 0, dy = 0,$

$$\therefore \int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0$$

In the same way, we evaluate the line integral along c_2 and c_3 .

$$\int_{c_1} \mathbf{F} \cdot d\mathbf{r} = 0;$$

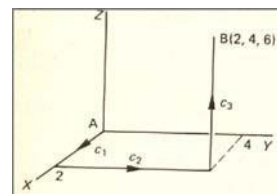
$$\int \mathbf{F} \cdot d\mathbf{r} = \int (2xyz \, dx + x^2z \, dy + x^2y \, dz)$$

c_2 : (2,0,0) to (2,4,0); $x = 2, z = 0, dx=0, \quad dz = 0$

$$\therefore \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 0 = 0 \quad \int_{c_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

c_3 : (2,4,0) to (2,4,6); $x = 2, y = 4, dx=0, \quad dy = 0$

$$\therefore \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + \int_0^6 16 \, dz = [16z]_0^6 = 96 \quad \therefore \int_{c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$



Collecting the three results together

$$\int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 96 \quad \therefore \int_{c_1+c_2+c_3} \mathbf{F} \cdot d\mathbf{r} = 96$$

In this particular example, the value of the line integral is independent of the two paths we have used joining the same two end points and indicates that $\mathbf{F}$ is a conservative field.

It follows that

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2z & x^2y \end{vmatrix} \\ &= (x^2 - x^2)\mathbf{i} - (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} = 0 \end{aligned}$$

$$\therefore \text{curl } \mathbf{F} = 0$$

So three tests can be applied to determine whether or not a vector field is conservative. They are

$$(a) \quad \oint \mathbf{F} \cdot d\mathbf{r} = 0$$

$$(b) \quad \text{curl } \mathbf{F} = 0$$

$$(c) \quad \mathbf{F} = \text{grad } V$$

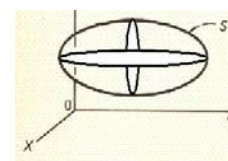
Any one of these conditions can be applied as is convenient.

Divergence Theorem (Gauss' theorem)

For a closed surface S , enclosing a region V in a vector field $\mathbf{F}$

$$\int_V \text{div } \mathbf{F} \, dV = \int_S \mathbf{F} \cdot d\mathbf{S}$$

In general, this means that the volume integral (triple integral) on the left-hand side can be expressed as a surface integral (double integral) on the right-hand side.



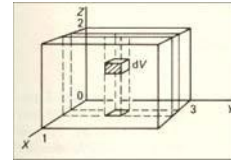
Example

Verify the divergence theorem for the vector field

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$ taken over the region bonded by the planes $z = 0, z = 2, x = 0, x = 1, y = 0, y = 3$.

$dV = dx dy dz$

we have to show that $\int_V \text{div } \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$



$$\begin{aligned} \text{(a) To find } \int_V \text{div } \mathbf{F} dV &= \int_V \left(\frac{\partial}{\partial x} (x^2) + \frac{\partial}{\partial y} (z) + \frac{\partial}{\partial z} (y) \right) dV \\ &= \int_V (2x + 0 + 0) dV = \int_V 2x dV \end{aligned}$$

$$\therefore \int_V \text{div } \mathbf{F} dV = \int_V 2x dV = \int_0^1 \int_0^3 \int_0^2 2x dz dy dx$$

$$\int_V \text{div } \mathbf{F} dV = \int_0^1 \int_0^3 \int_0^2 2x dz dy dx = \int_0^1 \int_0^3 [2xz]_0^2 dy dx = \int_0^1 [4xz]_0^3 dx = \int_0^1 12x dx = [6x^2]_0^1 = 6$$

(b) to find $\int_S \mathbf{F} \cdot d\mathbf{S}$ i.e. $\int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$

The enclosing surface S consists of six separate plane faces denoted as

$S_1, S_2, \dots, S_6$ as shown. We consider each face in turn.

$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$

(i) S_1 (base): $z = 0; \hat{\mathbf{n}} = -\mathbf{k}$ (outwards and downwards)

$$\therefore \mathbf{F} = x^2\mathbf{i} + y\mathbf{k} \quad dS_1 = dx dy$$

$$\therefore \int_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_1} (x^2\mathbf{i} + y\mathbf{k}) \cdot (-\mathbf{k}) dy dx = \int_0^1 \int_0^3 (-y) dy dx = \int_0^1 \left[-\frac{y^2}{2} \right]_0^3 dx = -\frac{9}{2}$$

(ii) S_2 (top): $z = 2; \hat{\mathbf{n}} = \mathbf{k} \quad dS_2 = dx dy$

$$\int_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_2} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \cdot (\mathbf{k}) dy dx = \int_0^1 \int_0^3 y dy dx = \frac{9}{2}$$

(iii) S_3 (right-hand end): $y = 3;$

$$\hat{\mathbf{n}} = \mathbf{j} \quad dS_3 = dx dz$$

$$\mathbf{F} = x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}$$

$$\therefore \int_{S_3} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_3} (x^2\mathbf{i} + z\mathbf{j} + 3\mathbf{k}) \cdot (\mathbf{j}) dz dx = \int_0^1 \int_0^2 z dz dx = \int_0^1 \left[\frac{z^2}{2} \right]_0^2 dx = \int_0^1 2 dx = 2$$

(iv) S_4 (left-hand end): $y = 0, \hat{\mathbf{n}} = -\mathbf{j}, \quad dS_4 = dx dz$

$$\therefore \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} dS = -2$$

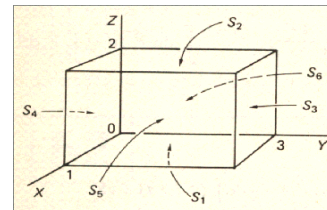
$$\text{for } \int_{S_4} \mathbf{F} \cdot \hat{\mathbf{n}} dS = \iint_{S_4} (x^2\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \cdot (-\mathbf{j}) dz dx = \int_0^1 \int_0^2 (-z) dz dx = \int_0^1 \left[-\frac{z^2}{2} \right]_0^2 dx$$

$$= \int_0^1 (-2) dx = -2$$

Now for the remaining two sides S_5 and S_6 . Evaluate these in the same manner, obtaining

$$\int_{S_5} \mathbf{F} \cdot \hat{\mathbf{n}} dS = 6; \quad \int_{S_6} \mathbf{F} \cdot \hat{\mathbf{n}} dS = 0$$

Check:



(v) S_5 (front): $x = 1$; $\hat{n} = \mathbf{i}$ $dS_5 = dy dz$

$$\therefore \int_{S_5} \mathbf{F} \cdot \hat{n} dS = \iint_{S_5} (\mathbf{i} + z\mathbf{j} + y\mathbf{k}) \cdot (\mathbf{i}) dy dz = \iint_{S_5} 1 dy dz = 6$$

(vi) S_6 (back): $x = 0$; $\hat{n} = -\mathbf{i}$ $dS_6 = dy dz$

$$\therefore \int_{S_6} \mathbf{F} \cdot \hat{n} dS = \iint_{S_6} (z\mathbf{j} + y\mathbf{k}) \cdot (-\mathbf{i}) dy dz = \iint_{S_6} 0 dy dz = 0$$

For the whole surface S we therefore have $\int_S \mathbf{F} \cdot d\mathbf{S} = -\frac{9}{2} + \frac{9}{2} - 2 + 6 + 0 = 6$

and from our previous work in section (a) $\int_V \text{div } \mathbf{F} dV = 6$

We have therefore verified as required that, in this example $\int_V \text{div } \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S}$

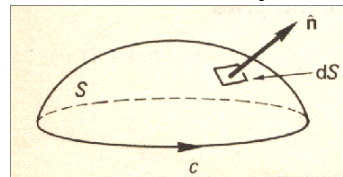
Stokes Theorem

If $\mathbf{F}$ is a vector field existing over an open surface S and around its boundary closed curve

$$C, \text{ then } \int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

This means that we can express a surface integral in terms of a line integral round the boundary curve.

The proof of this theorem is rather lengthy and is to be found in the Appendix. Let us demonstrate its application in the following examples.



Example

A hemisphere S is defined by $x^2 + y^2 + z^2 = 4$ ($z \geq 0$). A vector field $\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$ exists over the surface and around its boundary C . Verify Stoke's theorem that

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

$$S : x^2 + y^2 + z^2 - 4 = 0$$

$$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k} \quad C \text{ is the circle } x^2 + y^2 = 4.$$

$$\begin{aligned} \text{(a)} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \int_C (2ydx - xdy + xzdz) \end{aligned}$$

Converting to polar coordinates. $x = 2 \cos \theta$; $y = 2 \sin \theta$; $z = 0$

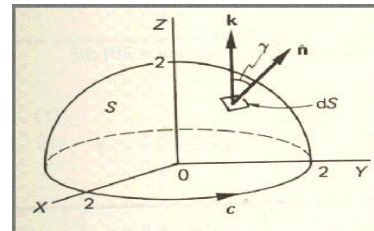
$dx = -2 \sin \theta d\theta$; $dy = 2 \cos \theta d\theta$; Limits $\theta = 0$ to 2π

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (4 \sin \theta [-2 \sin \theta d\theta] - 2 \cos \theta [2 \cos \theta d\theta] + 0) = -4 \int_0^{2\pi} (2 \sin^2 \theta + \cos^2 \theta) d\theta \\ &= -4 \int_0^{2\pi} (1 + \sin^2 \theta) d\theta = -2 \int_0^{2\pi} (3 - \cos 2\theta) d\theta = -2 \left[3\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} = -12\pi \quad (1) \end{aligned}$$

(b) Now we determine $\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_S \text{curl } \mathbf{F} \cdot \hat{n} dS$$

$$\mathbf{F} = 2y\mathbf{i} - x\mathbf{j} + xz\mathbf{k}$$



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \mathbf{i}(0-0) - \mathbf{j}(z-0) + \mathbf{k}(-1-2) = -z\mathbf{j} - 3\mathbf{k}$$

$$\text{Now } \hat{\mathbf{n}} = \frac{\nabla S}{|\nabla S|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\text{Then } \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} dS = \int_S (-z\mathbf{j} - 3\mathbf{k}) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \right) dS = \frac{1}{2} \int_S (-yz - 3z) dS$$

$$x = 2 \sin \phi \cos \theta; y = 2 \sin \phi \sin \theta; z = 2 \cos \phi, dS = 4 \sin \phi d\phi d\theta$$

$$\begin{aligned} \therefore \int_S \text{curl } \mathbf{F} \cdot \hat{\mathbf{n}} dS &= \frac{1}{2} \iint_S (-2 \sin \phi \sin \theta \cdot 2 \cos \phi - 6 \cos \phi) 4 \sin \phi d\phi d\theta \\ &= - \int_0^{2\pi} \int_0^{\pi/2} (2 \sin^2 \phi \cos \phi \sin \theta + 3 \sin \phi \cos \phi) d\phi d\theta = -4 \int_0^{2\pi} \left[\frac{2 \sin^3 \phi \sin \theta}{3} + \frac{3 \sin^2 \phi}{2} \right]_0^{\pi/2} d\theta \\ &= -4 \int_0^{2\pi} \left(\frac{2}{3} \sin \theta + \frac{3}{2} \right) d\theta = -12\pi \quad (2) \end{aligned}$$

So we have from our two results (1) and (2)

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Lecture No -39 Periodic Functions

Periodic functions

A function $f(x)$ is said to be periodic if its function values repeat at regular intervals of the independent variable. The regular interval between repetitions is the period of the oscillations. $f(x + p) = f(x)$

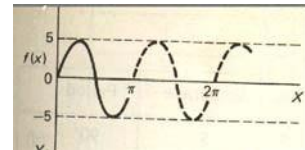
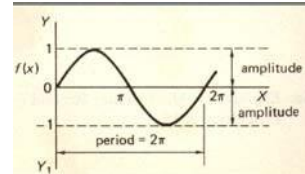
Graphs of $y = A \sin nx$

(a) $y = \sin x$ The obvious example of a periodic function is $y = \sin x$, which goes through its complete range of values while x increases from 0° to 360° .

The period is therefore 360° or 2π radians and the amplitude, the maximum displacement from the position of rest.

$$y = 5 \sin 2x$$

The amplitude is 5. The period is 180° and there are thus 2 complete cycles in 360° .



Example

Functions	Amplitude	Period
$y = 3 \sin 5x$	3	72°
$y = 2 \cos 3x$	2	120°
$y = \sin \frac{x}{2}$	1	720°
$y = 4 \sin 2x$	4	180°

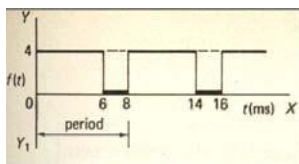
$$y = A \sin nx$$

Thinking along the same lines, the function $y = A \sin nx$

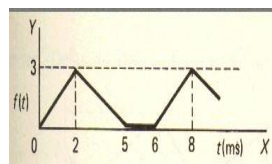
has amplitude = A ; period = $\frac{360^\circ}{n} = \frac{2\pi}{n}$; n cycles in 360° .

Graphs of $y = A \cos nx$ have the same characteristics

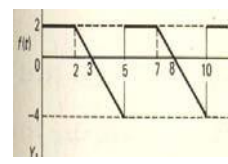
Example



period = 8 ms



period = 6 ms



period = 5 ms

Analytical description of a periodic function

A periodic function can be defined analytically in many cases.

Example

(a) Between $x = 0$ and $x = 4$, $y = 3$,

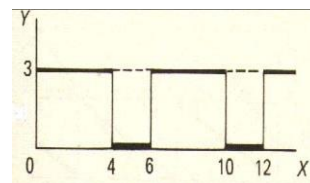
$$\text{i.e. } f(x) = 3 \quad 0 < x < 4$$

(b) Between $x = 4$ and $x = 6$, $y = 0$,

$$\text{i.e. } f(x) = 0 \quad 4 < x < 6.$$

So we could define the function by

$$f(x) = 3 \quad 0 < x < 4$$



$$f(x) = 0 \quad 4 < x < 6$$

$$f(x) = f(x + 6)$$

the last line indicating that the function is periodic with period 6 units

$$f(x) = 2 - x \quad 0 < x < 3$$

$$f(x) = -1 \quad 3 < x < 5$$

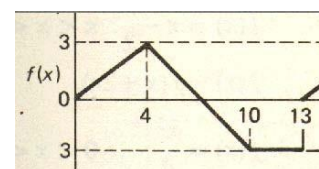
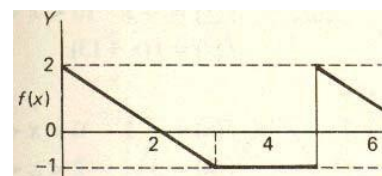
$$f(x) = f(x+5)$$

$$f(x) = \frac{3x}{4} \quad 0 < x < 4$$

$$f(x) = 7 - x \quad 4 < x < 10$$

$$f(x) = -3 \quad 10 < x < 13$$

$$f(x) = f(x + 13)$$



Example

Sketch the graphs of the following inserting relevant values.

$$1. \quad f(x) = 4 \quad 0 < x < 5$$

$$f(x) = 0 \quad 5 < x < 8$$

$$f(x) = f(x + 8)$$

$$f(x) = 3x - x^2 \quad 0 < x < 3$$

$$f(x) = f(x + 3)$$

$$f(x) = 2 \sin x \quad 0 < x < \pi$$

$$f(x) = 0 \quad \pi < x < 2\pi$$

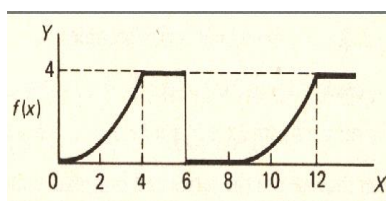
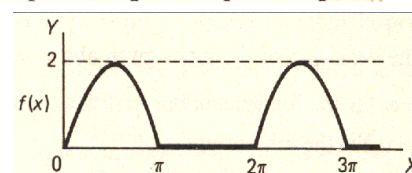
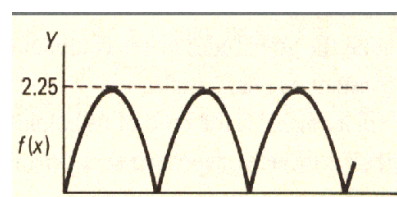
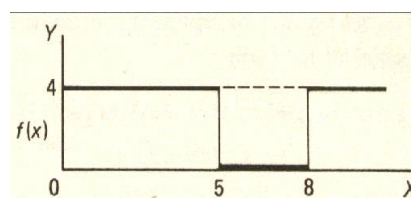
$$f(x) = f(x + 2\pi)$$

$$f(x) = \frac{x^2}{4} \quad 0 < x < 4$$

$$f(x) = 4 \quad 4 < x < 6$$

$$f(x) = 0 \quad 6 < x < 8$$

$$f(x) = f(x + 8)$$



Useful integrals

The following integrals appear frequently in our work on Fourier series, so it will help if we obtain the result in readiness. In each case, m and n are integers other than zero.

$$(a) \quad \int_{-\pi}^{\pi} \sin nx \, dx = \left[\frac{-\cos nx}{n} \right]_{-\pi}^{\pi} = \frac{1}{n} \{-\cos n\pi + \cos n\pi\} = 0$$

$$(b) \quad \int_{-\pi}^{\pi} \cos nx \, dx = \left[\frac{\sin nx}{n} \right]_{-\pi}^{\pi} = \frac{1}{n} \{\sin n\pi + \sin n\pi\} = 0$$

$$(c) \quad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos 2nx) \, dx = \frac{1}{2} \left[x - \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \pi \quad (n \neq 0)$$

$$(d) \int_{-\pi}^{\pi} \cos^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx = \frac{1}{2} \left[x + \frac{\sin 2nx}{2n} \right]_{-\pi}^{\pi} = \pi \quad (n \neq 0)$$

$$(e) \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\sin(n+m)x + \sin(n-m)x\} \, dx = \frac{1}{2} \{0 + 0\} = 0$$

from result (a) with $n \neq m$

$$(f) \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n+m)x + \cos(n-m)x\} \, dx = \frac{1}{2} \{0 + 0\} = 0$$

from result (b) with $n \neq m$

Note:

If $n = m$, then

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx \text{ becomes } \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad \text{from (d) above.}$$

$$(g) \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = -\frac{1}{2} \int_{-\pi}^{\pi} (-2) \sin nx \sin mx \, dx = -\frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n+m)x - \cos(n-m)x\} \, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \{0 - 0\} \, dx = 0 \quad \text{from result (b) with } n \neq m$$

Note:

If $n = m$, then

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx \text{ becomes } \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \quad \text{from (c) above}$$

Summary of integrals

$$(a) \int_{-\pi}^{\pi} \sin nx \, dx = 0, \quad (b) \int_{-\pi}^{\pi} \cos nx \, dx = 0, \quad (c) \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi \quad (n \neq 0)$$

$$(d) \int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad (n \neq 0) \quad (e) \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0$$

$$(f) \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m) \quad (g) \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \pi \quad (n = m) \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \quad (n = m)$$

Note

We have evaluated the integrals between $-\pi$ and π , but, provided integration is carried out over a complete periodic interval of 2π , the results are the same. Thus, the limits could just as well be $-\pi$ to π , 0 to 2π , $-\pi/2$ to $3\pi/2$, etc. We can therefore choose the limits to suit the particular problems.

Fourier series**Periodic functions of period 2π**

The basis of a Fourier series is to represent a periodic function by a trigonometrical series of the form.

$$f(x) = A_0 + c_1 \sin(x + \alpha_1) + c_2 \sin(2x + \alpha_2) + c_3 \sin(3x + \alpha_3) + \dots + c_n \sin(nx + \alpha_n) + \dots$$

where A_0 is a constant term

$c_1, c_2, c_3, \dots, c_n$ denote the amplitudes of the compound sine terms $\alpha_1, \alpha_2, \alpha_3, \dots$ are constant auxiliary angles.

Each sine term, $c_n \sin(nx + \alpha_n)$ can be expanded thus:

$$c_n \sin(nx + \alpha_n) = c_n \{ \sin nx \cos \alpha_n + \cos nx \sin \alpha_n \} = (c_n \sin \alpha_n) \cos nx + (c_n \cos \alpha_n) \sin nx \\ = a_n \cos nx + b_n \sin nx$$

the whole series becomes.

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

Lecture No- 40 Fourier Series

As we know that $A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\};$

which can be written as in the expanded form

$$A_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots + (a_n \cos nx + b_n \sin nx) + \dots$$

$$A_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots$$

$$f(x) = A_0 + a_1 \cos x + a_2 \cos 2x + \dots + a_n \cos nx + \dots + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots$$

Fourier coefficients

We have defined Fourier series in the form

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}; \quad n \text{ a positive integer}$$

(a) To find A_0 , we integrate $f(x)$ with respect to x from $-\pi$ to π .

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} A_0 dx + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nx dx + \int_{-\pi}^{\pi} b_n \sin nx dx \right\} = [A_0 x]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \{0 + 0\} = 2A_0 \pi$$

$$2A_0 \pi = \int_{-\pi}^{\pi} f(x) dx$$

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2} a_0; \quad \text{Where } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

(b) To find a_n we multiply $f(x)$ by $\cos mx$ and integrate from $-\pi$ to π .

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \int_{-\pi}^{\pi} A_0 \cos mx dx + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nx \cos mx dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx dx \right\}$$

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = A_0 \{0\} + \sum_{n=1}^{\infty} \{a_n (0) + b_n (0)\} = 0 \quad \text{for } n \neq m$$

$$= 0 + a_n \pi + 0 = a_n \pi \quad \text{for } n = m$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

(c) To find b_n we multiply $f(x)$ by $\sin mx$ and integrate from $-\pi$ to π .

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = \int_{-\pi}^{\pi} A_0 \sin mx dx + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nx \sin mx dx + \int_{-\pi}^{\pi} b_n \sin nx \sin mx dx \right\}$$

$$\int_{-\pi}^{\pi} f(x) \sin mx dx = A_0 \{0\} + \sum_{n=1}^{\infty} \{a_n (0) + b_n (0)\} = 0 \quad \text{for } n \neq m$$

$$= 0 + 0 + b_n \pi = b_n \pi \quad \text{for } n = m$$

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Result For Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\};$$

$$(a) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 2 \times \text{mean value of } f(x) \text{ over a period}$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 2 \times \text{mean value of } f(x) \cos nx \text{ over a period.}$$

$$(c) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 2 \times \text{mean value of } f(x) \sin nx \text{ over a period.}$$

In each case, $n = 1, 2, 3, \dots$

Example

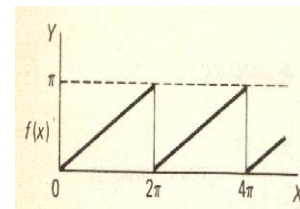
Determine the Fourier series to represent the periodic function shown.

It is more convenient here to take the limits as 0 to 2π .

The function can be defined as

$$f(x) = \frac{x}{2} \quad 0 < x < 2\pi$$

$$f(x) = f(x + 2\pi) \quad \text{period} = 2\pi.$$



Now to find the coefficients

$$(a) \quad a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{x}{2}\right) dx = \frac{1}{4\pi} [x^2]_0^{2\pi} = \pi$$

$$a_0 = \pi$$

$$(b) \quad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{x}{2}\right) \cos nx dx$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} x \cos nx dx = \frac{1}{2\pi} \left\{ \left[\frac{x \sin nx}{n} \right]_0^{2\pi} - \int_0^{2\pi} \sin nx dx \right\} = \frac{1}{2\pi} \{ (0 - 0) - \left[-\frac{\cos nx}{n} \right]_0^{2\pi} \}$$

$$\therefore a_n = 0$$

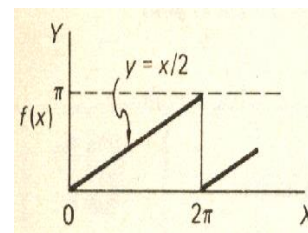
$$(a) \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx \quad \text{So we now have}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2} \sin nx dx = \frac{1}{2\pi} \left\{ \left[-\frac{x \cos nx}{n} \right]_0^{2\pi} + \int_0^{2\pi} \cos nx dx \right\} = -\frac{1}{2\pi n} [2\pi - 0] = -\frac{1}{n}$$

$$\mathbf{a_0 = \pi; \quad a_n = 0; \quad b_n = -\frac{1}{n}}$$

Now the general expression for a Fourier series is

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \quad \text{Therefore in this case}$$



$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \{b_n \sin nx\} = \frac{\pi}{2} + \left\{ -\frac{1}{1} \sin x - \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x - \dots \right\} \quad \text{since } a_n = 0$$

$$f(x) = \frac{\pi}{2} - \left\{ \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \dots \right\}$$

Dirichlet Conditions

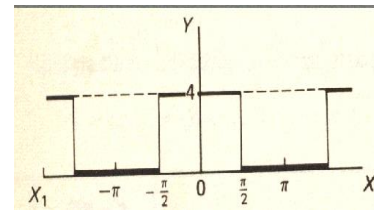
If the Fourier series is to represent a function $f(x)$, then putting $x = x_1$ will give an infinite series in x_1 and the value of this should converge to the value of $f(x_1)$ as more and more terms of the series are evaluated. For this to happen, the following conditions must be fulfilled.

- (a) The function $f(x)$ must be defined and single-valued.
- (b) $f(x)$ must be continuous or have a finite number of finite discontinuities within a periodic interval.
- (c) $f(x)$ and $f'(x)$ must be piecewise continuous in the periodic interval.

If these Dirichlet conditions are satisfied, the Fourier series converges to $f(x_1)$, if $x = x_1$ is a point of continuity

Example

Find the Fourier series for the function shown.



Consider one cycle between $x=0$ and $x=\pi$.

The function can be defined by

$$f(x) = 0 \quad -\pi < x < -\frac{\pi}{2}$$

$$f(x) = 4 \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = 0 \quad \frac{\pi}{2} < x < \pi$$

$$f(x) = f(x + 2\pi)$$

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

The expression for a_0 is $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$ This gives

$$a_0 = \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} 0 dx + \int_{-\pi/2}^{\pi/2} 4 dx + \int_{\pi/2}^{\pi} 0 dx \right\} = \frac{1}{\pi} [4x]_{-\pi/2}^{\pi/2} = 4$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} (0) \cos nx dx + \int_{-\pi/2}^{\pi/2} 4 \cos nx dx + \int_{\pi/2}^{\pi} (0) \cos nx dx \right\} = \frac{1}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} 4 \cos nx dx \right\}$$

$$= \frac{4}{\pi n} \sin nx \Big|_{-\pi/2}^{\pi/2} = \frac{8}{\pi n} \sin \frac{n\pi}{2}$$

$$a_n = \frac{8}{\pi n} \sin \frac{n\pi}{2}$$

Then considering different integer values of n , we have

If n is even $a_n = 0$

If $n = 1, 3, 5, \dots$ $a_n = \frac{8}{n\pi}$

If $n = 3, 7, 11, \dots$ $a_n = -\frac{8}{n\pi}$

(c) To find b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left\{ \int_{-\pi/2}^{\pi/2} (0) \sin nx \, dx + \int_{\pi/2}^{\pi} 4 \sin nx \, dx + \int_{\pi}^{2\pi} (0) \sin nx \, dx \right\}$$

$$b_n = \frac{4}{\pi} \int_{\pi/2}^{\pi} \sin nx \, dx = \frac{4}{\pi} \left[-\frac{\cos nx}{n} \right]_{\pi/2}^{\pi} = -\frac{4}{n\pi} \left\{ \cos \frac{n\pi}{2} - \cos \left(\frac{-n\pi}{2} \right) \right\} = 0$$

$$b_n = 0$$

So with $a_0 = 4$: a_n as stated above; $b_n = 0$;

The Fourier series is $\frac{1}{\pi} \{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \}$

In this particular example, there are, in fact, no sine terms.

Effect Of Harmonics

It is interesting to see just how accurately the Fourier series represents the function with which it is associated. The complete representation requires an infinite number of terms, but we can, at least, see the effect of including the first few terms of the series.

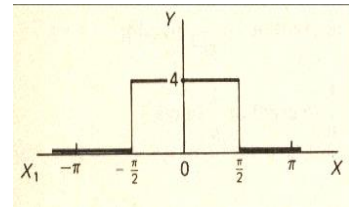
Let us consider the waveform shown. We established earlier that the function

$$f(x) = 0 \quad -\pi < x < -\frac{\pi}{2}$$

$$f(x) = 4 \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f(x) = 0 \quad \frac{\pi}{2} < x < \pi$$

$$f(x) = f(x + 2\pi)$$

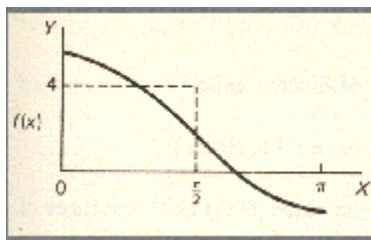


gives the Fourier series

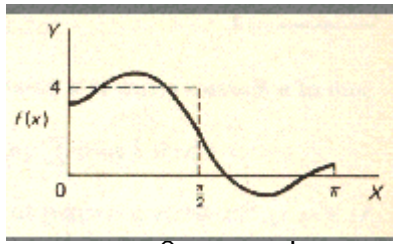
$$f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \right\}$$

If we start with just one cosine term, we can then see the effect of including subsequent harmonics. Let us restrict our attention to just the right-hand half of the symmetrical waveform. Detailed plotting of points gives the following development.

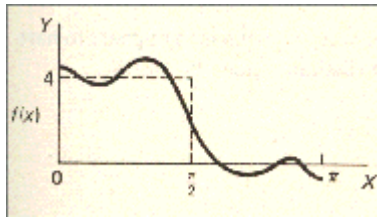
$$(1) f(x) = 2 + \frac{8}{\pi} \cos x$$



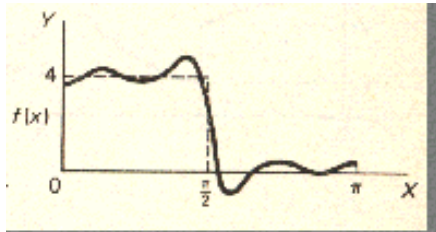
$$(2) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x \right\}$$



$$(3) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x \right\}$$



$$(4) f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x \right\}$$



As the number of terms is increased, the graph gradually approaches the shape of the original square waveform. The ripples increase in number and decrease in amplitude, but a perfectly square waveform is unattainable in practice. For practical purpose, the first few terms normally suffice to give an accuracy of acceptable level.

Lecture No -41 Examples

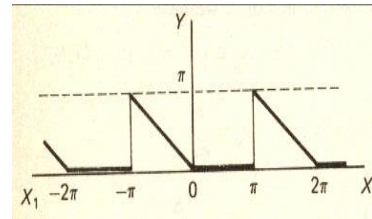
Example

Find the Fourier series for the function defined by

$$f(x) = -x \quad -\pi < x < 0$$

$$f(x) = 0 \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

The general expressions for a_0 , a_n , b_n are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) dx + \frac{1}{\pi} \int_0^{\pi} 0 dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) dx = \frac{1}{\pi} \left[-\frac{x^2}{2} \right]_{-\pi}^0 = \frac{\pi}{2}$$

(b) To find a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} 0 dx = -\frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx$$

$$= -\frac{1}{\pi} \left\{ \left[x \frac{\sin nx}{n} \right]_{-\pi}^0 - \frac{1}{n} \int_{-\pi}^0 \sin nx dx \right\} = -\frac{1}{\pi} \left\{ (0 - 0) - \frac{1}{n} \left[-\frac{\cos nx}{n} \right]_{-\pi}^0 \right\}$$

$$= -\frac{1}{\pi} \left\{ \frac{1}{n} \left[\frac{\cos nx}{n} \right]_{-\pi}^0 \right\} = -\frac{1}{\pi n^2} \left\{ \left[\frac{\cos nx}{n} \right]_{-\pi}^0 \right\} = -\frac{1}{\pi n^2} [\cos 0 - \cos n\pi]$$

$$= -\frac{1}{\pi n^2} \{1 - \cos n\pi\}$$

But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$$a_n = -\frac{2}{\pi n^2} \text{ (n odd) and } a_n = 0 \text{ (n even)}$$

(c) Now to find b_n

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \sin nx dx = -\frac{1}{\pi} \int_{-\pi}^0 x \sin nx dx$$

$$= -\frac{1}{\pi} \left\{ \left[x \left(-\frac{\cos nx}{n} \right) \right]_{-\pi}^0 + \frac{1}{n} \int_{-\pi}^0 \cos nx dx \right\} = -\frac{1}{\pi} \left\{ \frac{\pi \cos n\pi}{n} + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_{-\pi}^0 \right\} = -\frac{\cos n\pi}{n}$$

$$b_n = -\frac{1}{n} \text{ (n even) and } b_n = \frac{1}{n} \text{ (n odd)}$$

So we have

$$a_0 = \frac{\pi}{2}; \quad a_n = 0 \text{ (n even) and } a_n = -\frac{2}{\pi n^2} \text{ (n odd)}$$

$$b_n = -\frac{1}{n} \text{ (n even) and } b_n = \frac{1}{n} \text{ (n odd)}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi} \left(\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right) + \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots \right)$$

It is just a case of substituting $n = 1, 2, 3$, etc.

In this particular example, we have a constant term and both sine and cosine terms.

Odd And Even Functions

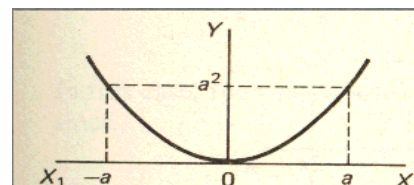
(a) Even functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$ i.e. the function value for a particular negative value of x is the same as that for the corresponding positive value of x . The graph of an even function is therefore symmetrical about the y -axis.

$y = f(x) = x^2$ is an even function
since

$$f(-2) = 4 = f(2)$$

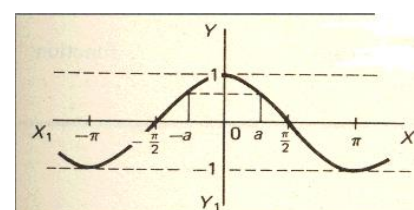
$$f(-3) = 9 = f(3) \quad \text{etc.}$$



$y = f(x) = \cos x$ is an even function

since $\cos(-x) = \cos x$

$$f(-a) = \cos a = f(a)$$



(b) Odd functions

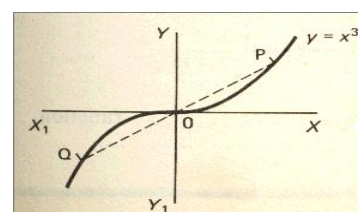
A function $f(x)$ is said to be odd if $f(-x) = -f(x)$

i.e. the function value for a particular negative value of x is numerically equal to that for the corresponding positive value of x but opposite in sign. The graph of an odd function is thus symmetrical about the origin.

$y = f(x) = x^3$ is an odd function since

$$f(-2) = -8 = -f(2)$$

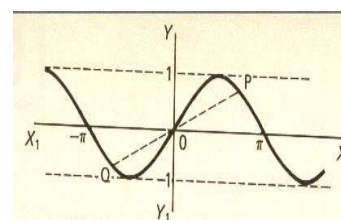
$$f(-5) = -125 = -f(5) \quad \text{etc.}$$



$y = f(x) = \sin x$ is an odd function

Since $\sin(-x) = -\sin x$

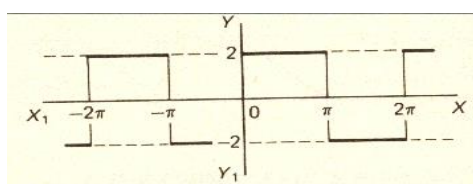
$$f(-a) = -f(a).$$



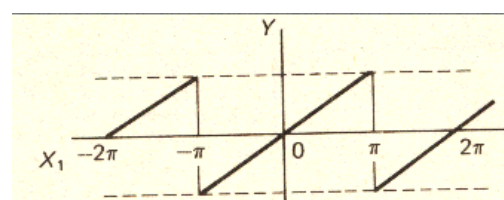
So, for an even function $f(-x) = f(x)$

symmetrical about the y -axis

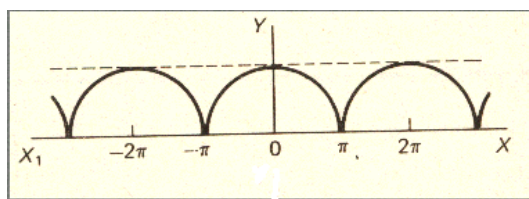
for an odd function $f(-x) = -f(x)$ symmetrical about the origin.



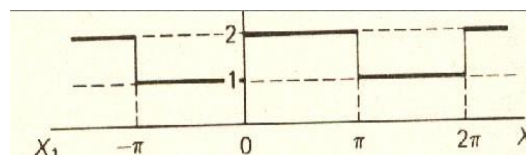
odd.



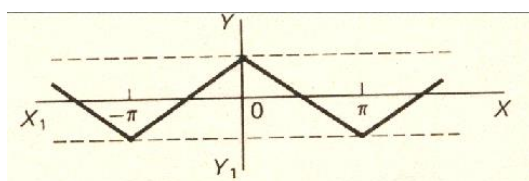
Odd



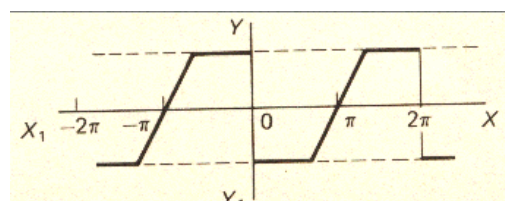
Even



neither



Even



Odd

Products Of Odd And Even Functions

The rules closely resemble the elementary rules of signs.

(even) × (even) = (even) like $(+) \times (+) = (+)$; (odd) × (odd) = (even) $(-) \times (-) = (+)$;

(odd) × (even) = (odd) $(-) \times (+) = (-)$

The results can easily be proved.

(a) Two even functions

Let $F(x) = f(x) g(x)$ where $f(x)$ and $g(x)$ are even functions.

Then $F(-x) = f(-x) g(-x) = f(x) g(x)$ since $f(x)$ and $g(x)$ are even.

$\therefore F(-x) = F(x)$

$F(x)$ is even

(b) Two odd functions

Let $F(x) = f(x) g(x)$ where $f(x)$ and $g(x)$ are odd functions.

Then $F(-x) = f(-x) g(-x) = \{-f(x)\} \{-g(x)\}$

since $f(x)$ and $g(x)$ are odd.

$= f(x) g(x) = F(x)$

$\therefore F(-x) = F(x)$

$F(x)$ is even

Finally

(c) One odd and one even function

Let $F(x) = f(x) g(x)$ where $f(x)$ is odd and $g(x)$ even.

Then $F(-x) = f(-x) g(-x) = -f(x) g(x) = -F(x)$

$\therefore F(-x) = -F(x)$

$F(x)$ is odd

So if $f(x)$ and $g(x)$ are both even, then $f(x) g(x)$ is even and if $f(x)$ and $g(x)$ are both odd, then $f(x) g(x)$ is even but if either $f(x)$ or $g(x)$ is even and the other odd. Then $f(x) g(x)$ is odd.

Example

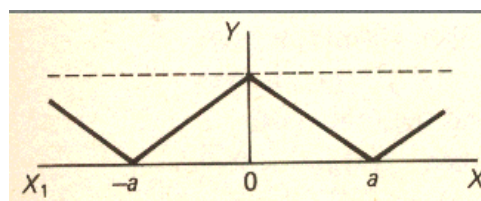
State whether each of the following products is odd, even, or neither.

1. $x^2 \sin 2x$ odd (E) (O) = (O)
2. $x^3 \cos x$ odd (O) (E) = (O)
3. $\cos 2x \cos 3x$ even (E) (E) = (E)
4. $x \sin nx$ even (O) (O) = (E)
5. $3 \sin x \cos 4x$ odd (O) (E) = (O)
6. $(2x + 3) \sin 4x$ neither (N) (O) = (N)
7. $\sin^2 x \cos 3x$ even (E) (E) = (E)
8. $x^3 e^x$ neither (O) (N) = (N)
9. $(x^4 + 4) \sin 2x$ odd (E) (O) = (O)

Two useful facts emerge from odd and even functions.

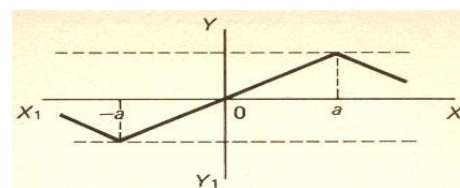
(a) For an even function

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$



(b) For an odd function

$$\int_{-a}^a f(x) dx = 0$$

**THEOREM 1**

If $f(x)$ is defined over the interval $-\pi < x < \pi$ and $f(x)$ is even, then the Fourier series for $f(x)$ contains cosine terms only. Included in this is a_0 which may be regarded as $a_n \cos nx$ with $n = 0$.

Proof:

$$(a) \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx \quad \therefore a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx.$$

But $f(x) \cos nx$ is the product of two even functions and therefore itself even.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx. \quad \therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$(c) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Since $f(x) \sin nx$ is the product of an even function and an odd function, it is itself odd.

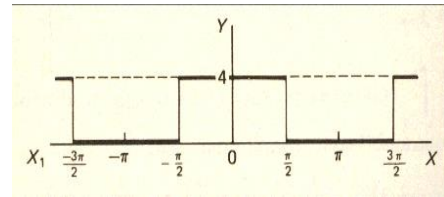
$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = 0.$$

$$\therefore b_n = 0$$

Therefore, there are no sine terms in the Fourier series for $f(x)$.

Example

The waveform shown is symmetrical about the y-axis. The function is therefore even and there will be no sine terms in the series.



$$\therefore f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$(a) a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi/2} 4 dx$$

$$= \frac{2}{\pi} [4x]_0^{\pi/2} = 4$$

$$(b) a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi/2} 4 \cos nx dx = \frac{8}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi/2}$$

$$= \frac{8}{\pi n} \sin \frac{n\pi}{2}$$

But $\sin \frac{n\pi}{2} = 0$ for n even

$$= 1 \text{ for } n = 1, 5, 9, \dots$$

$$= -1 \text{ for } n = 3, 7, 11, \dots$$

$$a_n = 0 \quad (n \text{ even});$$

$$a_n = \frac{8}{\pi n} \quad (n = 1, 5, 9, \dots);$$

$$a_n = -\frac{8}{\pi n} \quad (n = 3, 7, 11, \dots)$$

(c) $b_n = 0$, since $f(x)$ is an even function. Therefore, the required series is

$$f(x) = 2 + \frac{8}{\pi} \left\{ \cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \frac{1}{7} \cos 7x + \dots \right\}$$

Theorem 2:

If $f(x)$ is an odd function defined over the interval $-\pi < x < \pi$, then the Fourier series for $f(x)$ contains sine terms only.

Proof:

Since $f(x)$ is an odd function

$$\int_{-\pi}^0 f(x) dx = - \int_0^{\pi} f(x) dx.$$

$$(a) a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx.$$

But $f(x)$ is odd

$$\therefore a_0 = 0$$

$$(b) a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

Remembering that $f(x)$ is odd and $\cos nx$ is even, the product $f(x) \cos nx$ is odd.

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{odd function}) dx$$

$$\therefore a_n = 0$$

$$(b) \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

and since $f(x)$ and $\sin nx$ are each odd, the product $f(x) \sin nx$ is even.

$$\text{Then } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{even function}) \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

So,

If $f(x)$ is odd function then $a_0 = 0$; $a_n = 0$;

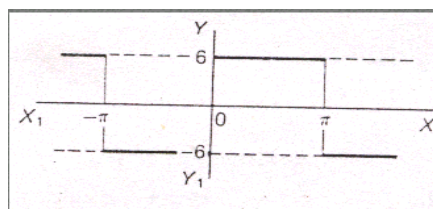
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \text{ i.e. the Fourier series contains sine terms only.}$$

Example

$$f(x) = -6 \quad -\pi < x < 0$$

$$f(x) = 6 \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$



We can see that this is an odd function;

$$a_0 = 0 \text{ and } a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$f(x) \sin nx$ is a product of two odd functions and is therefore even.

$$\begin{aligned} \therefore b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} 6 \sin nx \, dx = \frac{12}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi} = \frac{12}{\pi} \left[\frac{\cos nx}{n} \right]_0^{\pi} = \frac{12}{\pi} [\cos 0 - \cos n\pi] \\ &= \frac{12}{\pi n} (1 - \cos n\pi). \end{aligned}$$

$$b_n = 0 \quad (n \text{ even}) \quad b_n = \frac{24}{\pi n} \quad (n \text{ odd})$$

$$\text{So the series is } f(x) = \frac{24}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

Of course,

if $f(x)$ is neither an odd nor an even function, then we must obtain expressions for a_0 , a_n and b_n in full.

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

At $x = x_1$, the series converges to the value $f(x_1)$ as the number of terms including increases to infinity.

A particular point of interest occurs at a point of finite discontinuity or 'jump' of the function $y = f(x)$. At $x = x_1$, the function appears to have two distinct values, y_1 and y_2 .

If we approach $x = x_1$ from below that value, the limiting value of $f(x)$ is y_1 .

If we approach $x = x_1$ from above that value, the limiting value of $f(x)$ is y_2 .

To distinguish between these two values we write $y_1 = f(x_1 - 0)$ denoting immediately before $x = x_1$
 $y_2 = f(x_1 + 0)$ denoting immediately after $x = x_1$.

In fact, if we substitute $x = x_1$ in the Fourier series for $f(x)$, it can be shown that the series converges to the value

$$\frac{1}{2} \{f(x_1 - 0) + f(x_1 + 0)\} = \frac{1}{2} (y_1 + y_2), \text{ the average of } y_1 \text{ and } y_2.$$

Example

Consider the function

$$f(x) = 0 \quad -\pi < x < 0$$

$$f(x) = a \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

$$\begin{aligned} (a) \quad a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_0^{\pi} a dx \\ &= \frac{1}{\pi} [ax]_0^{\pi} = a \end{aligned}$$

$$\therefore a_0 = a$$

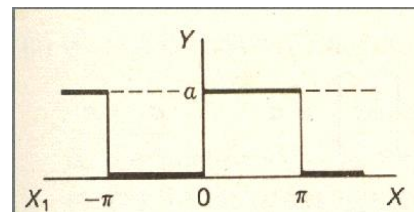
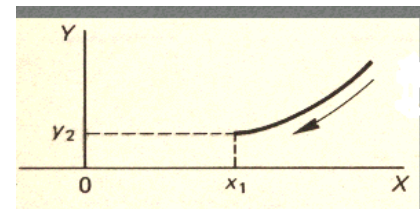
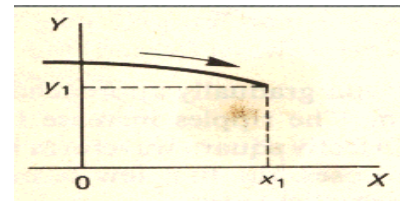
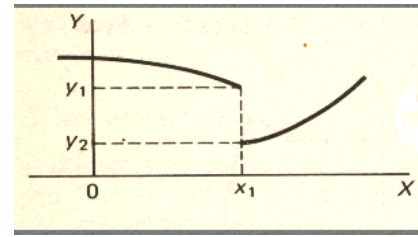
$$(b) \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} a \cos nx dx = \frac{a}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} = 0$$

$$\therefore a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} a \sin nx dx = \frac{a}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} = \frac{a}{n\pi} (1 - \cos n\pi)$$

But $\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$$b_n = 0 \quad (n \text{ even}) \quad \text{and} \quad b_n = \frac{2a}{n\pi} \quad (n \text{ odd})$$

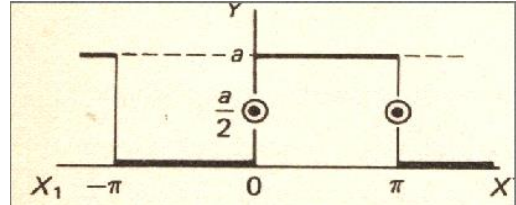


$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{a}{2} + \frac{2a}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\}$$

A finite discontinuity, or 'jump', occurs at $x = 0$. If we substitute $x = 0$ in the series obtained, all the sine terms vanish and we get $f(x) = a/2$, which is, in fact, the average of the two function values at $x = 0$.

Note also that at $x = \pi$, another finite discontinuity occurs and substituting $x = \pi$ in the series gives the same result. Because of this ambiguity, the function is said to be 'undefined' at $x = 0$, $x = \pi$, etc.



Half-Range Series

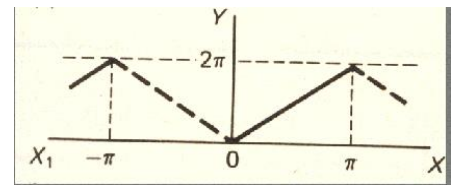
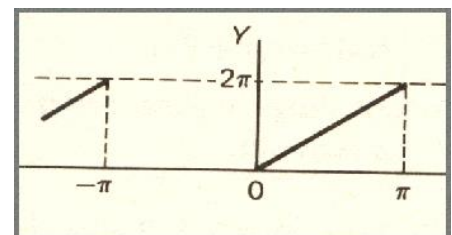
Sometime a function of period 2π is defined over the range 0 to π , instead of the normal $-\pi$ to π , or 0 to 2π . We then have a choice of how to proceed.

For example, if we are told between

$x = 0$ and $x = \pi$, $f(x) = 2x$,

then, since the period is 2π , we have no evidence of how the function behave between $x = -\pi$ and $x = 0$.

If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have only cosine terms (including possibly a_0).



On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have only sine terms.

Example

A function $f(x)$ is defined by

$$f(x) = 2x \quad 0 < x < \pi$$

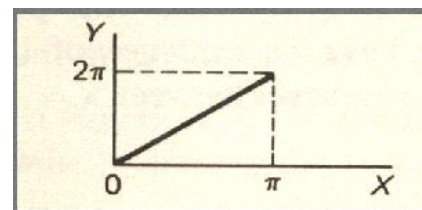
$$f(x) = f(x + 2\pi)$$

Obtain a half-range cosine series to represents the function.

To obtain a cosine series, i.e. a series with no sine terms, we need an even function.

Therefore, we assume the waveform between

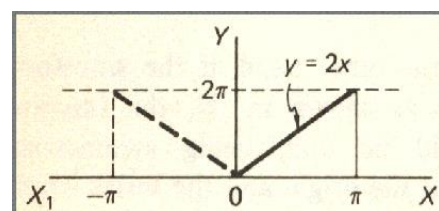
$x = -\pi$ and $x = 0$ to be as shown, making the total graph symmetrical about the y-axis.



Now we can find expressions for the Fourier coefficients as usual.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} 2x dx = \frac{2}{\pi} \left[x^2 \right]_0^{\pi} = 2\pi$$

$$\therefore a_0 = 2\pi$$



$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
 a_n &= \frac{2}{\pi} \int_0^{\pi} 2x \cos nx \, dx = \frac{4}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{4}{\pi} \left\{ \left[\frac{x \sin nx}{n} \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx \, dx \right\} \\
 &= \frac{4}{\pi} \left\{ (0-0) - \frac{1}{n} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \right\} = \frac{4}{\pi n^2} (\cos n\pi - 1) \\
 \cos n\pi &= 1 \quad (n \text{ even}) \text{ and } \cos n\pi = -1 \quad (n \text{ odd}) \\
 \therefore a_n &= 0 \quad (n \text{ even}) \text{ and } a_n = -\frac{8}{\pi n^2} \quad (n \text{ odd})
 \end{aligned}$$

All that now remains is b_n which is zero, since $f(x)$ is an even function, i.e. $b_n = 0$

So $a_0 = 2\pi$, $a_n = 0$ (n even) and $a_n = -\frac{8}{\pi n^2}$ (n odd),

$$\begin{aligned}
 b_n &= 0. \\
 f(x) &= \pi - \frac{8}{\pi} \left\{ \cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right\}
 \end{aligned}$$

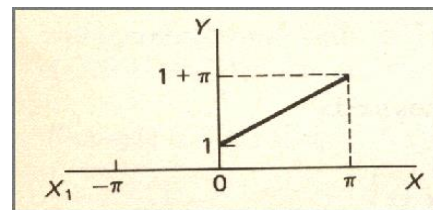
Example

Determine a half-range sine series to represent the function $f(x)$ defined by

$$f(x) = 1 + x \quad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

We choose the waveform between $x = -\pi$ and $x = 0$ so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.



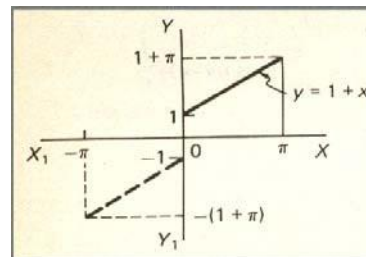
$$\therefore a_0 = 0 \quad \text{and} \quad a_n = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^{\pi} (1+x) \sin nx \, dx = \frac{2}{\pi} \left\{ \left[\frac{\sin nx}{n} \right]_0^{\pi} + \int_0^{\pi} \cos nx \, dx \right\} \\
 &= \frac{2}{\pi} \left\{ -\frac{1+\pi}{n} \cos n\pi + \frac{1}{n} \left[\frac{\sin nx}{n} \right]_0^{\pi} \right\} = \frac{2}{\pi} \left\{ \frac{1}{n} - \frac{1+\pi}{n} \cos n\pi \right\} = \frac{2}{\pi n} \{ 1 - (1+\pi) \cos n\pi \} \\
 \cos n\pi &= 1 \quad (n \text{ even}) \text{ and } \cos n\pi = -1 \quad (n \text{ odd}) \\
 \therefore b_n &= -\frac{2}{n} \quad (n \text{ even}) \\
 \therefore b_n &= \frac{4+2\pi}{\pi n} \quad (n \text{ odd})
 \end{aligned}$$

Substituting in the general expression $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ we have



$$f(x) = \frac{4 + 2\pi}{\pi} \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\} - 2 \left\{ \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x + \dots \right\}$$

and the required series obtained

$$f(x) = \left(\frac{4}{\pi} + 2 \right) \left\{ \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right\} - 2 \left\{ \frac{1}{2} \sin 2x + \frac{1}{4} \sin 4x + \frac{1}{6} \sin 6x + \dots \right\}$$

So knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.

Lecture No.-43 Functions With Periods Other Than 2π

So far, we have considered functions $f(x)$ with period 2π . In practice, we often encounter functions defined over periodic intervals other than 2π , e.g. from 0 to T , $-\frac{T}{2}$ to $\frac{T}{2}$ etc.

Functions With Period T

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$, i.e. has a period T , we can convert this to an interval of 2π by changing the units of the independent variable.

In many practical cases involving physical oscillations, the independent variable is time (t) and the periodic interval is normally denoted by T , i.e.

$$f(t) = f(t + T)$$

Each cycle is therefore completed in T seconds and the frequency f hertz (oscillations per second) of the

periodic function is therefore given by $f = \frac{1}{T}$. If the

angular velocity, ω radians per seconds, is defined by $\omega = 2\pi f$, then

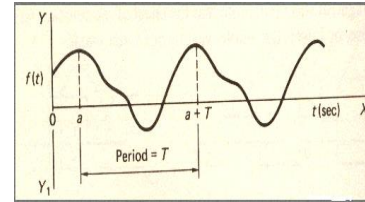
$$\omega = \frac{2\pi}{T} \text{ and } T = \frac{2\pi}{\omega}$$

The angle, x radians, at any time t is therefore $x = \omega t$ and the Fourier series to represent the function can be expressed as

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

which can also be written in the form

$$f(t) = \frac{1}{2} A + \sum_{n=1}^{\infty} B_n \sin(n\omega t + \phi_n) \quad n = 1, 2, 3, \dots$$



Fourier Coefficients

With the new variable, the Fourier coefficients become:

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T} \int_0^T f(t) dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega t dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \cos n\omega t dt$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega t dt = \frac{\omega}{\pi} \int_0^{2\pi/\omega} f(t) \sin n\omega t dt$$

We can see that there is very little difference between these expressions and those that have gone before. The limits can, of course, be 0 to T , $-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc.

as is convenient, so long as they cover a complete period.

Example

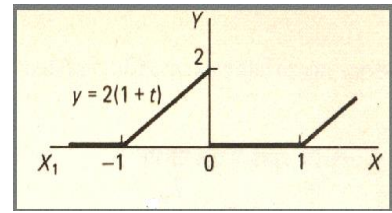
Determine the Fourier series for a periodic function defined by

$$f(t) = 2(1+t) \quad -1 < t < 0$$

$$f(t) = 0 \quad 0 < t < 1$$

$$f(t) = f(t+2)$$

The first step is to sketch the wave which is.



$$f(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \{a_n \cos n\omega t + b_n \sin n\omega t\}$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{2}{2} \int_{-1}^0 2(1+t) dt + \int_0^1 (0) dt = \left[2t + t^2 \right]_{-1}^0 = 1$$

$$\begin{aligned} \therefore a_0 &= 1 \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega t dt = \frac{2}{2} \int_{-1}^0 2(1+t) \cos n\omega t dt \\ a_n &= 2 \left\{ \left[(1+t) \frac{\sin n\omega t}{n\omega} \right]_{-1}^0 - \frac{1}{n\omega} \int_{-1}^0 \sin n\omega t dt \right\} = 2 \left\{ (0-0) - \frac{1}{n\omega} \left[-\frac{\cos n\omega t}{n\omega} \right]_{-1}^0 \right\} \\ &= \frac{2}{n^2 \omega^2} (1 - \cos n\omega) \end{aligned}$$

$$\text{Now } T = \frac{2\pi}{\omega}$$

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \therefore a_n = \frac{2}{n^2 \omega^2} (1 - \cos n\pi)$$

$$\begin{aligned} \therefore a_n &= 0 \quad (n \text{ even}) \\ &= \frac{4}{n^2 \omega^2} \quad (n \text{ odd}) \end{aligned}$$

Now for b_n

$$\begin{aligned} b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega t dt \\ b_n &= \frac{2}{2} \int_{-1}^0 2(1+t) \sin n\omega t dt = 2 \left\{ \left[(1+t) \frac{-\cos n\omega t}{n\omega} \right]_{-1}^0 + \frac{1}{n\omega} \int_{-1}^0 \cos n\omega t dt \right\} \\ &= 2 \left\{ -\frac{1}{n\omega} + \frac{1}{n\omega} \left[\frac{\sin n\omega t}{n\omega} \right]_{-1}^0 \right\} = 2 \left\{ -\frac{1}{n\omega} + \frac{1}{n^2 \omega^2} (\sin n\omega) \right\} \end{aligned}$$

As before $\omega = \pi$

$$\therefore b_n = -\frac{2}{n\omega}$$

So the first few terms of the series give

$$f(t) = \frac{1}{2} + \frac{4}{\omega^2} \left\{ \cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \dots \right\} - \frac{2}{\omega} \left\{ \sin \omega t + \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{4} \sin 4\omega t + \dots \right\}$$

Half-Range Series

The theory behind the half-range sine and cosine series still applies with the new variable.

(a) Even function

Half-range cosine series

$$y = f(t) \quad 0 < t < \frac{T}{2}$$

$$f(t) = f(t + T)$$

symmetrical about the y-axis.

With an even function, we know that $b_n = 0$

$$\therefore f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$\text{where } a_0 = \frac{4}{T} \int_0^{T/2} f(t) dt \quad \text{and} \quad a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega t dt$$

(b) Odd function

Half-range sine series

$$y = f(t) \quad 0 < t < \frac{T}{2}$$

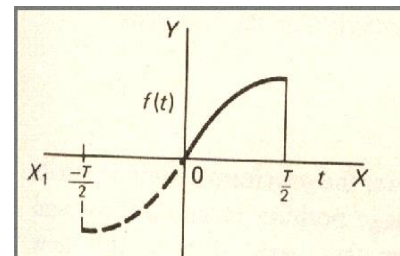
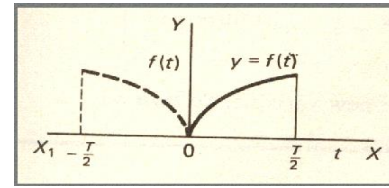
$$f(t) = f(t + T)$$

symmetrical about the origin.

$$\therefore a_0 = 0 \text{ and } a_n = 0$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t;$$

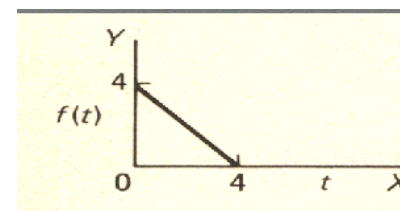
$$b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t dt$$

**Example**

A function $f(t)$ is defined by

$$f(t) = 4 - t, \quad 0 < t < 4.$$

We have to form a half-range cosine series to represent the function in this interval.



First we form an even function, i.e. symmetrical about the y-axis.

$$a_0 = \frac{4}{T} \int_0^{T/2} f_1(t) dt = \frac{4}{T} \int_0^4 (4 - t) dt = \frac{1}{2} \int_0^4 (4 - t) dt$$

$$= \frac{1}{2} \left[4t - \frac{t^2}{2} \right]_0^4 = \frac{1}{2} \left[4(4) - \frac{(4)^2}{2} \right] = \frac{1}{2} [16 - 8] = \frac{1}{2} (8) = 4$$

$$a_n = \frac{4}{T} \int_0^{T/2} f_1(t) \cos n\omega t dt = \frac{4}{8} \int_0^4 (4 - t) \cos n\omega t dt$$

Simple integration by parts gives

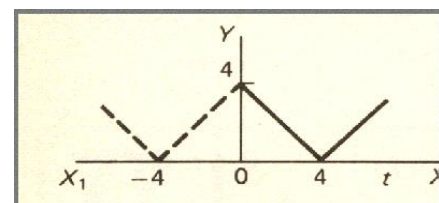
$$a_n = \frac{1}{2} \left\{ -\frac{2 \sin 4n\omega}{n\omega} - \frac{1}{n^2 \omega^2} (\cos 4n\omega - 1) \right\}$$

$$\text{But } \omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\therefore a_n = \frac{1}{2} \left\{ -\frac{2 \sin n\pi}{n\omega} - \frac{1}{n^2 \omega^2} (\cos n\pi - 1) \right\}$$

$$n = 1, 2, 3, \dots$$

$$\sin n\pi = 0;$$



$$\cos n\pi = 1 \quad (n \text{ even}); \quad \cos n\pi = -1 \quad (n \text{ odd})$$

$$\therefore a_n = 0 \quad (n \text{ even})$$

$$a_n = \frac{1}{n^2 \omega^2} \quad (n \text{ odd})$$

$$\therefore f(t) = 2 + \frac{1}{\omega^2} \left\{ \cos \omega t + \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \dots \right\}$$

$$\text{where } \omega = \frac{\pi}{4}.$$

Example

A function $f(t)$ is defined by

$$f(t) = 3 + t \quad 0 < t < 2$$

$$f(t) = f(t + 4)$$

Obtain the half-range sine series for the function in this range.

Sine series required. Therefore, we form an function, symmetrical about the origin.

$$a_0 = 0; \quad a_n = 0; \quad T = 4$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$\therefore b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega t \, dt = \int_0^2 (3+t) \sin n\omega t \, dt$$

$$\int_0^2 (3+t) \sin n\omega t \, dt = \left[\frac{(3+t) \cos n\omega t}{-n\omega} \right]_0^2 - \int_0^2 \frac{\cos n\omega t}{-n\omega} \, dt$$

$$= \frac{(3+2) \cos n\omega 2}{-n\omega} - \frac{3}{-n\omega} + \frac{1}{n\omega} \left[\frac{\sin n\omega t}{n\omega} \right]_0^2 = \frac{3}{n\omega} - \frac{5}{n\omega} \cos 2n\omega + \frac{1}{n^2 \omega^2} \left[\frac{\sin 2n\omega}{n\omega} - 0 \right]$$

$$\text{But } T = \frac{2\pi}{\omega} \quad \therefore \omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$= \frac{3}{n\omega} - \frac{5}{n\omega} \cos n\pi + \frac{1}{n^2 \omega^2} \left[\frac{\sin n\pi}{n\omega} \right]$$

$$b_n = \frac{1}{n\omega} (3 - 5 \cos 2n\omega) + \frac{1}{n^2 \omega^2} (\sin 2n\omega)$$

$$\therefore b_n = \frac{1}{n\omega} (3 - 5 \cos n\pi) + \frac{1}{n^2 \omega^2} (\sin n\pi) = -\frac{2}{n\omega} \quad (n \text{ even})$$

$$= \frac{8}{n\omega} \quad (n \text{ odd})$$

$$\therefore f(t) = \frac{2}{\omega} \left\{ 4 \sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{4}{3} \sin 3\omega t - \frac{1}{4} \sin 4\omega t + \dots \right\}$$

Half-Wave Rectifier

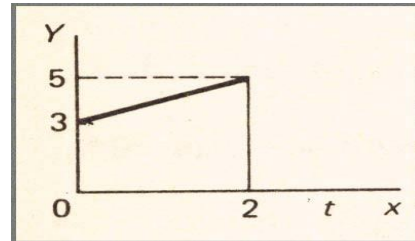
A sinusoidal voltage $E \sin \omega t$, where t is time, is passed through a half-wave rectifier that clips the negative portion of the wave

Find the Fourier series of the resulting periodic functions.

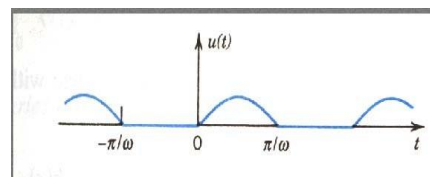
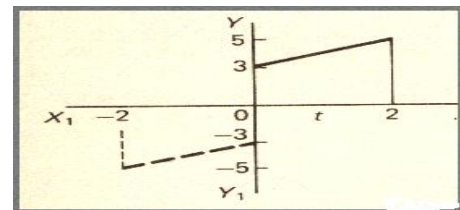
$$u(t) = 0 \quad \text{if} \quad -T/2 < t < 0$$

$$= E \sin \omega t \quad 0 < t < T/2$$

$$\text{here } T = \frac{2\pi}{\omega}$$



odd



$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} u(t) dt = \frac{2}{T} \int_{-T/2}^0 0 dt + \frac{2}{T} \int_0^{T/2} E \sin \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t dt = \frac{\omega}{\pi} \int_0^{\pi/\omega} E \sin \omega t dt$$

$$= \frac{\omega}{\pi} E \left[\frac{-\cos \omega t}{\omega} \right]_0^{\pi/\omega} = \frac{2E}{\pi}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \cos n \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t \cos n \omega t dt = \frac{\omega E}{2\pi} \int_0^{\pi/\omega} 2 \sin \omega t \cos n \omega t dt$$

$$= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\sin (1+n) \omega t + \sin (1-n) \omega t] dt$$

If $n = 1$ then integral on the right is zero

and if $n = 2, 3, \dots$ then we obtain.

$$a_n = \frac{\omega E}{2\pi} \left[\frac{\cos (1+n) \omega t}{(1+n)\omega} - \frac{\cos (1-n) \omega t}{(1-n)\omega} \right]_0^{\pi/\omega} = \frac{\omega E}{2\pi} \left[\frac{-\cos (1+n) \pi + 1}{(1+n)\omega} + \frac{-\cos (1-n) \pi + 1}{(1-n)\omega} \right]$$

$$= \frac{\omega E}{2\pi\omega} \left[\frac{-\cos (1+n) \pi + 1}{(1+n)} + \frac{-\cos (1-n) \pi + 1}{(1-n)} \right]$$

if n is odd then $a_n = 0$

$$\text{if } n \text{ is even then } a_n = \frac{\omega E}{2\pi} \left(\frac{2}{(1+n)} + \frac{2}{(1-n)} \right) = \frac{E}{2\pi} \left[\frac{2 - 2n + 2 + 2n}{(1+n)(1-n)} \right] = \frac{2E}{(1-n^2)\pi}$$

For b_n we have

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} u(t) \sin n \omega t dt = \frac{2}{T} \int_0^{T/2} E \sin \omega t \sin n \omega t dt = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} 2 \sin \omega t \sin n \omega t dt$$

$$= -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos (1+n) \omega t - \cos (1-n) \omega t] dt$$

If $n = 1$

$$b_n = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos 2\omega t - 1] dt = -\frac{\omega E}{2\pi} \left[\frac{\sin 2\omega t}{2\omega} - t \right]_0^{\pi/\omega} = -\frac{\omega E}{2\pi} (-\pi/\omega) = E/2$$

if $n \neq 1$

$$b_n = -\frac{\omega E}{2\pi} \int_0^{\pi/\omega} [\cos (1+n) \omega t - \cos (1-n) \omega t] dt = -\frac{\omega E}{2\pi} \left[\frac{\sin (1+n) \omega t}{(1+n)\omega} - \frac{\sin (1-n) \omega t}{(1-n)\omega} \right]_0^{\pi/\omega}$$

$$= -\frac{\omega E}{2\pi} \left[\frac{\sin (1+n) \pi}{(1+n)\omega} - \frac{\sin (1-n) \pi}{(1-n)\omega} \right] = 0$$

for $n = 2, 3, 4, \dots$

$$u(t) = \frac{1}{2} a_0 + \sum_{n=2}^{\infty} a_n \cos n \omega t$$

$$u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left(\frac{1}{1.3} \cos 2\omega t + \frac{1}{3.5} \cos 4\omega t + \dots \right)$$

Lecture No -44 Laplace Transforms

$$L\{e^{3t}\} = \frac{1}{s-3}; L\{t^3\} = \frac{3!}{s^4}; L\{2\sin 3t + \cos 3t\} = \frac{s+6}{s^2+9}$$

And $L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}; L^{-1}\left\{\frac{3!}{s^4}\right\} = t^3; L^{-1}\left\{\frac{s+6}{s^2+9}\right\} = 2\sin 3t + \cos 3t$

We show that $L\{t^3\} = \frac{3!}{s^4}$ For this consider the integral

$$\begin{aligned} L(t^3) &= \int_0^{\infty} t^3 e^{-st} dt = \left[t^3 \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} 3t^2 e^{-st} \frac{1}{-s} dt = \frac{1}{s} \left[t^3 e^{-st} \right]_0^{\infty} + \frac{3}{s} \int_0^{\infty} t^2 e^{-st} dt \\ &= 0 + \frac{3}{s} \left[t^2 \frac{e^{-st}}{-s} \right]_0^{\infty} + \frac{2}{s} \int_0^{\infty} t e^{-st} dt = \frac{3}{s} \frac{2}{s} \int_0^{\infty} t e^{-st} dt = \frac{3 \cdot 2}{s^2} \left\{ t \frac{e^{-st}}{-s} \right]_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt \} \\ &= \frac{3 \cdot 2}{s^3} \left[t e^{-st} \right]_0^{\infty} + \frac{3 \cdot 2}{s^3} \int_0^{\infty} e^{-st} (-s) dt = \frac{-3 \cdot 2 \cdot 1}{s^3} \left[e^{-st} \right]_0^{\infty} = \frac{-3 \cdot 2 \cdot 1}{s^3} (0 - 1) = \frac{3 \cdot 2 \cdot 1}{s^3} = \frac{3!}{s^3} \end{aligned}$$

So that $L\{t^3\} = \frac{3!}{s^4}$

Laplace Transform

The Laplace Transform of a function $F(t)$ is denoted by $L\{F(t)\}$ and is defined as the integral of $F(t) e^{-st}$ between the limits $t=0$ and $t = \infty$

$$L\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt.$$

In all cases, the constant parameter s is assumed to be positive and large enough to ensure that the product $F(t) e^{-st}$ converges to zero as $t \rightarrow \infty$, whatever the function $F(t)$.

In determining the transform of any function, you will appreciate that the limits are substituted for t , so that the result will be a function of s .

Laplace Transform of $F(t) = a$ (constant).

That is

$$\begin{aligned} L(a) &= \int_0^{\infty} a e^{-st} dt = a \int_0^{\infty} e^{-st} dt \\ &= a \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{-a}{s} \{0 - 1\} = \frac{a}{s} \end{aligned}$$

Example

Find the laplace transform of the form e^{at} that $F(t) = e^{at}$ where a is a constant.

$$\begin{aligned} L(e^{at}) &= \int_0^{\infty} e^{at} e^{-st} dt = \int_0^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \\ &= - \frac{1}{s-a} \left[e^{-(s-a)t} \right]_0^{\infty}, \quad s > a \\ &= - \frac{1}{s-a} [0 - 1] = \frac{1}{s-a} \end{aligned}$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

So we already have two standard transforms

$$\begin{aligned}
 L\{a\} &= \frac{a}{s}; & L\{e^{at}\} &= \frac{1}{s-a}; \\
 L\{4\} &= \frac{4}{s}; & L\{e^{4t}\} &= \frac{1}{s-4}; \\
 L\{-5\} &= \frac{-5}{s}; & L\{e^{-2t}\} &= \frac{1}{s+2};
 \end{aligned}$$

Laplace transform is always a function of s.

Complex Numbers Power of i

Every time a factor i^4 occurs, it can be replaced by the factor 1, so that the power of i is reduced to one of the four results above.

$$\begin{aligned}
 i^9 &= (i^4)^2 i = (1)^2 i = 1 \cdot i = i \\
 i^{20} &= (i^4)^5 = (1)^5 = 1 \\
 i^{30} &= (i^4)^7 i^2 = (1)^7 (-1) = 1(-1) = -1 \\
 i^{15} &= (i^4)^3 i^3 = 1(-i) = -i
 \end{aligned}$$

Complex Numbers

$z = 3 + 5i$, is called a complex number where 3 is real part and 5 is imaginary part of the complex number.

In general $z = a + bi$, is called a complex number where a is real part and b is imaginary part of the complex number. So,

$$\text{Complex Number} = (\text{Real Part}) + i(\text{Imaginary Part})$$

Conjugate complex numbers

For a complex number $a + bi$, the complex number $a - bi$ is called the conjugate of $a + bi$. Conjugate complex numbers are identical except the signs in the middle for the brackets.

- $(4 + 5i)$ and $(4 - 5i)$ are conjugate complex numbers
- $(6 + 2i)$ and $(2 + 6i)$ are not conjugate complex numbers
- $(5 - 3i)$ and $(-5 + 3i)$ are not conjugate complex numbers

Remember

The product of complex number by its conjugate is always entirely real.

$$(3 + 4i)(3 - 4i) = 9 + 16 = 25$$

$$(a + bi)(a - bi) = a^2 + b^2$$

Euler Formula

As we know that the series expansion of e^x , $\cos x$ and $\sin x$ are given as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Replace x by (it), we get

$$e^{(it)} = 1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \dots$$

$$e^{(it)} = 1 + (it) - \frac{(t)^2}{2!} - \frac{i(t)^3}{3!} + \frac{(t)^4}{4!} + \frac{i(t)^5}{5!} - \frac{(t)^6}{6!} - \frac{i(t)^7}{7!} + \dots$$

$$e^{(it)} = \left[1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \dots \right] + i \left[t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \dots \right]$$

where

$$\cos t = 1 - \frac{(t)^2}{2!} + \frac{(t)^4}{4!} - \frac{(t)^6}{6!} + \dots$$

$$\sin t = t - \frac{(t)^3}{3!} + \frac{(t)^5}{5!} - \frac{(t)^7}{7!} + \dots$$

$$e^{it} = \cos t + i \sin t$$

$$R(e^{it}) = \cos t \quad \text{and} \quad I(e^{it}) = \sin t$$

The Laplace transform of $F(t) = \sin at$

$$L(\sin at) = L(I(e^{iat})) = I \int_0^{\infty} e^{iat} e^{-st} dt = I \int_0^{\infty} e^{-(s-ia)t} dt = I \left\{ \left[\frac{e^{-(s-ia)t}}{-(s-ia)} \right]_0^{\infty} \right\}$$

$$= I \left\{ \frac{1}{-(s-ia)} (0-1) \right\} = I \left\{ \frac{1}{(s-ia)} \right\}$$

$$L\{\sin at\} = I \left\{ \frac{s+ia}{s^2+a^2} \right\} = I \left[\frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2} \right]$$

$$\therefore L\{\sin at\} = \frac{a}{s^2+a^2}$$

We can use the same method to determine $L\{\cos at\}$.

Since $\cos at$ is the real part of e^{iat} , written as $R(e^{iat})$

$$L\{\cos at\} = R\left\{ \frac{s+ia}{s^2+a^2} \right\} = R\left[\frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2} \right]$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

The Transform of $F(t) = t^n$ where n is a positive integer.

By the definition $L(t^n) = \int_0^{\infty} t^n e^{-st} dt$ integrating by parts

$$L(t^n) = \int_0^{\infty} t^n e^{-st} dt = \left[t^n \frac{e^{-st}}{-s} \right]_0^{\infty} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt = -\frac{1}{s} \left[t^n e^{-st} \right]_0^{\infty} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt$$

$$\therefore \left[t^n \frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - 0 = 0 \quad \therefore L\{t^n\} = \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} dt \quad \text{----- (1)}$$

you will notice that $\int_0^{\infty} t^{n-1} e^{-st} dt$ is identical to $\int_0^{\infty} t^n e^{-st} dt$ except that n is replaced by $(n-1)$

If $I_n = \int_0^{\infty} t^n e^{-st} dt$, then $I_{n-1} = \int_0^{\infty} t^{n-1} e^{-st} dt$ and the result (1) becomes

$$I_n = \frac{n}{s} I_{n-1}$$

This is reduction formula and, if we now replace n by $(n-1)$ we get

$$I_{n-1} = \frac{n-1}{s} I_{n-2}$$

If we replace n by $(n-1)$ again in this last result, we have

$$I_{n-2} = \frac{n-2}{s} I_{n-3}$$

$$\text{So } I_n = \int_0^{\infty} t^n e^{-st} dt = \frac{n}{s} I_{n-1} = \frac{n}{s} \cdot \frac{n-1}{s} I_{n-2} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} I_{n-3} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} I_{n-4}$$

So finally, we have

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} I_0$$

But

$$I_0 = L\{t^0\} = L\{1\} = \frac{1}{s}$$

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \cdots \frac{2}{s} \cdot \frac{1}{s} I_0 = \frac{n!}{s^{n+1}}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{t\} = \frac{1}{s^2}; L\{t^2\} = \frac{2!}{s^3}; L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

Laplace Transform of $F(t) = \sinh at$ and $F(t) = \cosh at$

Starting from the exponential definitions of $\sinh at$ and $\cosh at$

$$\text{i.e. } \sinh at = \frac{1}{2}(e^{at} - e^{-at}) \text{ and } \cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

We proceed as follow

$$a) \quad F(t) = \sinh at$$

$$\begin{aligned} \sinh at &= \int_0^{\infty} \sinh at e^{-st} dt = \frac{1}{2} \int_0^{\infty} (e^{at} - e^{-at}) e^{-st} dt = \frac{1}{2} \int_0^{\infty} (e^{at} e^{-st} - e^{-at} e^{-st}) dt = \frac{1}{2} \int_0^{\infty} (e^{-(s-a)t} - e^{-(s+a)t}) dt \\ &= \frac{1}{2} \left\{ \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} - \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \right\} = \frac{1}{2} \left\{ \frac{1}{-(s-a)} (0-1) - \frac{1}{-(s+a)} (0-1) \right\} = \frac{1}{2} \left\{ \frac{1}{(s-a)} - \frac{1}{(s+a)} \right\} \\ &= \frac{1}{2} \left\{ \frac{s+a-s+a}{(s-a)(s+a)} \right\} = \frac{1}{2} \left\{ \frac{2a}{s^2 - a^2} \right\} = \frac{a}{s^2 - a^2} \end{aligned}$$

$$b) \quad F(t) = \cosh at$$

$$\begin{aligned}
L(\cosh at) &= L\left(\frac{1}{2}(e^{at} + e^{-at})\right) = \frac{1}{2} \int_0^{\infty} (e^{at} + e^{-at})e^{-st} dt = \frac{1}{2} \left[\int_0^{\infty} e^{-(s-a)t} dt + \int_0^{\infty} e^{-(s+a)t} dt \right] \\
&= \frac{1}{2} \left\{ \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} + \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} \right\} = \frac{1}{2} \left\{ \frac{1}{-(s-a)} (0-1) + \frac{1}{-(s+a)} (0-1) \right\} = \frac{1}{2} \left\{ \frac{1}{(s-a)} + \frac{1}{(s+a)} \right\} \\
&= \frac{1}{2} \left\{ \frac{s+a+s-a}{(s-a)(s+a)} \right\} = \frac{1}{2} \left\{ \frac{2s}{s^2-a^2} \right\} = \left\{ \frac{s}{s^2-a^2} \right\} \\
L(\cosh at) &= \left\{ \frac{s}{s^2-a^2} \right\}
\end{aligned}$$

Several Standard Results

$$L\{a\} = \frac{a}{s}; \quad L\{e^{at}\} = \frac{1}{s-a}; \quad L\{t^n\} = \frac{n!}{s^{n+1}}$$

$$L\{\sin at\} = \frac{a}{s^2 + a^2}; \quad L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$L\{\sinh at\} = \frac{a}{s^2 - a^2}; \quad L\{\cosh at\} = \frac{s}{s^2 - a^2}$$

We can, of course, combine these transforms by adding or subtracting as necessary, but they must not be multiplied together to form the transform of a product.

Example

$$a) L\{2 \sin 3t + \cos 3t\} = 2L(\sin 3t) + L(\cos 3t) = 2 \cdot \frac{3}{s^2 + 9} + \frac{s}{s^2 + 9} = \frac{s+6}{s^2 + 9}$$

$$\begin{aligned}
b) L\{4e^{2t} + 3 \cosh 4t\} &= 4L(e^{2t}) + L(3 \cosh 4t) = 4 \cdot \frac{1}{s-2} + 3 \cdot \frac{s}{s^2-16} = \frac{4}{s-2} + \frac{3s}{s^2-16} \\
&= \frac{7s^2 - 6s - 64}{(s-2)(s^2-16)}
\end{aligned}$$

Lecture No -45 Theorems

Theorem 1

The First Shift Theorem

The first Shift theorem states that if $L\{F(t)\} = f(s)$ then $L\{e^{(-at)} F(t)\} = f(s+a)$

The transform $L\{e^{(-at)} F(t)\}$ is thus the same as $L\{F(t)\}$ with s everywhere in the result replaced by $(s+a)$

Example

$$L\{\sin 2t\} = \frac{2}{s^2 + 4} \quad \text{then} \quad L\{e^{-3t} \sin 2t\} = \frac{2}{(s+3)^2 + 4} = \frac{2}{s^2 + 6s + 13}$$

Example

$$L\{t^2\} = \frac{2}{s^3}; \quad L\{t^2 e^{4t}\} \text{ is the same with } s \text{ replaced by } (s-4)$$

$$\text{So } L\{t^2 e^{4t}\} = \frac{2}{(s-4)^3}$$

Theorem 2

Multiplying by t

If $L\{F(t)\} = f(s)$ then $L\{t(F(t))\} = -\frac{d}{ds}\{f(s)\}$

Example

$$L\{\sin 2t\} = \frac{2}{s^2 + 4}$$

And $L\{t \sin 2t\} = -\frac{d}{ds} \left[\frac{2}{s^2 + 4} \right] = \frac{4s}{(s^2 + 4)^2}$

Example

$$L\{t \cos 3t\} = -\frac{d}{ds} \left[\frac{s}{s^2 - 9} \right] = -\frac{(s^2 - 9) - s(2s)}{(s^2 - 9)^2} = -\frac{s^2 - 9 - 2s^2}{(s^2 - 9)^2} = \frac{s^2 + 9}{(s^2 - 9)^2}$$

We could, if necessary, take this a stage further and find

$$L\{t \cos 3t\} = -\frac{d}{ds} \left[\frac{s^2 + 9}{(s^2 - 9)^2} \right] = \frac{2s(s^2 + 27)}{(s^2 - 9)^3}$$

Theorem Obviously extends the range of function that we can deal with. So, in general

If $L\{F(t)\} = f(s)$ then $L\{t^n (F(t))\} = (-1)^n \frac{d^n}{ds^n} \{f(s)\}$

Theorem

Dividing by t

If $L\{F(t)\} = f(s)$ then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s) ds$

Example Determine $L\left\{\frac{\sin at}{t}\right\}$

$$\text{As } L\{\sin at\} = \frac{a}{s^2 + a^2}$$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\left(\frac{s}{a}\right) \right]_s^\infty = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \tan^{-1}\left(\frac{a}{s}\right)$$

Example Determine $L\left\{\frac{1-\cos 2t}{t}\right\}$

$$\text{As } L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$$

Then by Theorem 3,

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \left[\ln s - \frac{1}{2} \ln(s^2+4) \right]_s^\infty = \left[\frac{1}{2} \ln s - \frac{1}{2} \ln(s^2+4) \right]_s^\infty$$

$$= \left[\frac{1}{2} \ln s^2 - \frac{1}{2} \ln(s^2+4) \right]_s^\infty = \frac{1}{2} \left[\ln s - \ln(s^2+4) \right]_s^\infty = \frac{1}{2} \left[\ln \frac{s^2}{(s^2+4)} \right]_s^\infty$$

$$\text{When } s \rightarrow \infty \text{ then } \ln \frac{s^2}{(s^2+4)} \rightarrow \ln 1 = 0$$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_0^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = -\frac{1}{2} \left[\ln \frac{s^2}{(s^2+4)} \right] = \ln \left[\frac{s^2}{(s^2+4)} \right]^{\frac{1}{2}} = \ln \sqrt{\frac{s^2+4}{s^2}}$$

Standard Forms

F(t)	$L\{F(t)\} = f(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{s^{n+1}}$

(n a positive integer)

Theorem 1 The First Shift Theorem

If $L\{F(t)\} = f(s)$ then $L\{e^{(-at)} F(t)\} = f(s+a)$

Theorem 2 Multiplying by t

If $L\{F(t)\} = f(s)$ then $L\{tF(t)\} = -\frac{d}{ds}\{f(s)\}$

Theorem 3 Dividing by t

If $L\{F(t)\} = f(s)$ then $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s) ds$

Provided $\lim_{t \rightarrow 0} \left\{ \frac{F(t)}{t} \right\}$ exists.

Inverse Transforms

Here we have the reverse process i.e. given a Laplace transform, we have to find the function of t to which it belongs. For example, we know that $\frac{a}{s^2 + a^2}$ is the Laplace Transform of **sin at**, so we can now write $L^{-1}\left[\frac{a}{s^2 + a^2}\right] = \sin at$, the symbol L^{-1}

indicating the inverse transform and not a reciprocal.

$$(a) \quad L^{-1}\left[\frac{1}{s+2}\right] = e^{-2t}$$

$$(b) \quad L^{-1}\left\{\frac{s-2}{s^2+25}\right\} = \cos 5t$$

$$(c) \quad L^{-1}\left\{\frac{4}{s}\right\} = 4$$

$$(d) \quad L^{-1}\left\{\frac{12}{s^2-9}\right\} = 4 \sinh 3t$$

But what about $L^{-1}\left\{\frac{3s}{s^2-s-6}\right\}$, it happens that we can write $\frac{3s}{s^2-s-6}$ as the sum of

two simpler functions $\frac{1}{s+2} + \frac{1}{s-3}$ which, of course, makes all the difference, since we can now proceed. $L^{-1}\left\{\frac{3s}{s^2-s-6}\right\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{1}{s-3}\right\} = e^{-2t} + 2e^{3t}$

Rules of Partial Fractions

1. The numerator must be of lower degree than denominator. If it is not, then we first divide out.
2. Factorise the denominator into its prime factors. These determine the shapes of the partial fraction.
3. A linear factor $(s+a)$ gives a partial fraction $\frac{A}{s+a}$ is a constant to be determined.
4. A repeated factor $(s+a)^2$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2}$
5. Similarly $(s+a)^3$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$
6. A quadratic Factor $(s^2 + ps + q)$ gives $\frac{Ps + Q}{s^2 + ps + q}$
7. Repeated quadratic Factor $(s^2 + ps + q)^2$ gives $\frac{Ps + Q}{s^2 + ps + q} + \frac{Rs + T}{(s^2 + ps + q)^2}$

So $\frac{s-19}{(s-2)(s+5)}$ has partial fraction of the form $\frac{A}{(s-2)} + \frac{B}{(s+5)}$ and $\frac{s-19}{(s+3)(s-2)^2}$ has partial fraction $\frac{A}{(s+3)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$

Example

To determine $L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\}$

a) First we check that the numerator is of lower degree than the denominator. In fact this is so.

b) Factorise the denominator

$$\frac{5s+1}{s^2-s-12} = \frac{5s+1}{(s-4)(s+3)} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

We therefore have an identity

$$\frac{5s+1}{s^2-s-12} = \frac{A}{(s-4)} + \frac{B}{(s+3)}$$

which is true for any value of s we care to substitute

If we multiply through by the denominator (s^2-s-12) we have

$$5s+1 \equiv A(s+3) + B(s-4)$$

We now substitute convenient values for s

i) Let $(s-4)=0$ that is $s=4$ therefore $21 = A(7) + B(0) \Rightarrow A=3$

ii) Let $(s+3)=0$ that is $s=-3$ therefore $B=2$

$$\text{So } \frac{5s+1}{s^2-s-12} = \frac{3}{(s-4)} + \frac{2}{(s+3)}$$

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = 3e^{4t} + 2e^{-3t}$$

Example Determined $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$

$$L\{F(t)\} = \frac{9s-8}{s^2-2s}$$

a) Numerator of first degree ; denominator of second degree.

$$b) \frac{9s-8}{s^2-2s} = \frac{A}{s} + \frac{B}{s-2}$$

c) Multiply by $s(s-2)$

$$\therefore 9s-8 \equiv A(s-2) + B(s)$$

d) Put $s=0$

$$\therefore -8 \equiv A(-2) + B(0) \therefore A=4$$

e) Put $s=2$, i.e. $s=2$

$$\therefore 10 = A(0) + B(0) \therefore B=5$$

$$\therefore F(t) = L^{-1}\left\{\frac{4}{s} + \frac{5}{s-2}\right\} = 4 + 5e^{2t}$$

Table of inverse transforms

Standard transforms

$f(s)$	$F(t)$
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}
$\frac{n!}{s^{n+1}}$	t^n (n a positive integer)
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ (n a positive integer)
$\frac{a}{s^2 + a^2}$	$\sin at$
$\frac{s}{s^2 + a^2}$	$\cos at$
$\frac{a}{s^2 - a^2}$	$\sinh at$
$\frac{s}{s^2 - a^2}$	$\cosh at$

Transforms Of Derivatives

Let $F'(t)$ denote the first derivative of $F(t)$ with respect to t , $F''(t)$ denote the second derivative of $F(t)$ with respect to t , etc.

Then $L\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t) dt$ by definition,

Integrating By Parts

$$L\{F'(t)\} = \left[e^{-st} F(t) \right]_0^{\infty} - \int_0^{\infty} F(t) (-se^{-st}) dt$$

when $t \rightarrow \infty$, $e^{-st} F(t) \rightarrow 0$

$$L\{F'(t)\} = -F(0) + s \int_0^{\infty} e^{-st} F(t) dt$$

$$L\{F'(t)\} = -F(0) + sL\{F(t)\}$$

$$L\{F''(t)\} = -F'(0) + sL\{F'(t)\} = -F'(0) + s[-F(0) + sL\{F(t)\}]$$

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$$

$$L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - sF'(0) - F''(0)$$

$$L\{F^{(iv)}(t)\} = s^4 L\{F(t)\} - s^3 F(0) - s^2 F'(0) - sF''(0) - F'''(0)$$

Differential Equation And Its Solution

$$\frac{dx}{dt} - 2x = 4 \text{----- (1)}$$

Its Solution is $x = -2 + 3e^{2t}$, To verify it we find $\frac{dx}{dt}$

$$\frac{dx}{dt} = 6e^{2t} \text{ then}$$

$$\frac{dx}{dt} - 2x = 6e^{2t} - 2(-2 + 3e^{2t})$$

$$= 6e^{2t} + 4 - 6e^{2t} = 4$$

So equation (1) is satisfied. Hence $x = -2 + 3e^{2t}$ is solution of $\frac{dx}{dt} - 2x = 4$

Example Solve the differential equation $\frac{dx}{dt} - 2x = 4$ given that at $t = 0$, $x = 1$

Taking Laplace transform as

$$L\left[\frac{d}{dt}(x(t)) - 2L(x(t))\right] = L(4) \Rightarrow L\left[\frac{d}{dt}(x(t))\right] - 2L(x(t)) = L(4) \Rightarrow sL(x(t)) - x(0) - 2L(x(t)) = \frac{4}{s}$$

$$(s-2)L(x(t)) - x(0) = \frac{4}{s} \Rightarrow (s-2)L(x(t)) - 1 = \frac{4}{s} \Rightarrow (s-2)L(x(t)) = \frac{4}{s} + 1 \Rightarrow (s-2)L(x(t)) = \frac{4+s}{s}$$

$$\Rightarrow L(x(t)) = \frac{4+s}{s(s-2)} \Rightarrow x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] \text{----- (1)}$$

First we do the partial fraction of $\frac{4+s}{s(s-2)}$

$$\frac{4+s}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$\Rightarrow 4+s = A(s-2) + B(s) \text{----- (2)}$$

Put $s = 0$ in equation (2) ; $4 = -2A$; $A = -2$

Put $s = 2$ in equation (2) ; $6 = B(2)$; $B = 3$

$$\text{So } \frac{4+s}{s(s-2)} = \frac{-2}{s} + \frac{3}{s-2}$$

Equation# (1) becomes

$$x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right] = L^{-1}\left[\frac{-2}{s}\right] + L^{-1}\left[\frac{3}{s-2}\right] = -2 + 3e^{2t}$$

Solution of differential equation by laplace transforms

To solve a differential equation by Laplace transforms, we go through Laplace transforms, we go through four distinct stages.

- Re- write the equation in term of Laplace transforms.
- Insert the given initial conditions.
- Rearrange the equation algebraically to give the transform of the solution.

(d) Determine the inverse transform to obtain the particular solution.

Solve the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t} \text{ given that at } t = 0, x = 5 \text{ and } \frac{dx}{dt} = 7$$

$$x''(t) - 3x'(t) + 2x(t) = 2e^{3t}$$

$$\text{Given } x(0) = 5, x'(0) = 7$$

$$L(x''(t)) - 3L(x'(t)) + 2L(x(t)) = 2L(e^{3t})$$

$$s^2L\{x(t)\} - sx(0) - x'(0) - 3\{sL\{x(t)\} - x(0)\} + 2L\{x(t)\} = \frac{2}{s-3}$$

We rewrite the equation in term of its transforms.

$$L\left[\frac{d^2x}{dt^2}\right] - 3L\left[\frac{dx}{dt}\right] + 2L[x] = 2L[e^{3t}]$$

$$[s^2L(x(t)) - sx(0) - x'(0)] - 3[sL(x(t)) - x(0)] + 2L(e^{3t})$$

$$\text{At } t = 0, x = 5, \frac{dx}{dt} = 7$$

$$\text{So } x(0) = 5, x'(0) = 7$$

$$s^2L(x(t)) - s(5) - 7 - 3\{sL(x(t)) + 3(5)\} + 2L(x(t)) = \frac{2}{s-3}$$

$$s^2L(x(t)) - 3sL(x(t)) + 2L(x(t)) = \frac{2}{s-3} - 8 + 5s$$

$$(s^2 - 3s + 2)L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{s-3}$$

$$L(x(t)) = \frac{2 - 8s + 24 + 5s^2 - 15s}{(s-1)(s-2)(s-3)} = \frac{5s^2 - 23s + 24}{(s-1)(s-2)(s-3)}$$

Making Partial fraction of R.H.S, We have

$$L(x(t)) = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

After solving these we get $A = 3, B = 2$ and $C = 0$

$$\text{So } L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)} + \frac{0}{(s-3)}$$

$$L(x(t)) = \frac{3}{(s-1)} + \frac{2}{(s-2)}$$

$$x(t) = L^{-1}\left\{\frac{3}{(s-1)}\right\} + L^{-1}\left\{\frac{2}{(s-2)}\right\} = 3e^t + 2e^{2t}$$