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**HANDOUT**

## **MTH403**

Calculus and Analytical Geometry - II

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## MTH403 - Calculus and Analytical Geometry - II

A structured study handout compiled from the course's lecture topics. Read each topic's objectives first, study the key concepts, then use the summary and exam tips for revision.

# MTH403

## Calculus and Analytical Geometry - II

Based on 45 lecture topics

Compiled for VU students as a structured study guide

*Concepts are summarized in the writer's own words;  
use alongside your lecture videos for best results.*

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# Course Introduction

Calculus II takes the ideas of limits, derivatives, and integrals into deeper and wider territory: advanced integration techniques, infinite sequences and series, parametric curves, polar coordinates, and an introduction to vectors and multivariable thinking.

Where Calculus I taught differentiation and basic integration, this course teaches power: computing integrals that resist elementary methods, determining whether infinite sums converge, describing curves that functions cannot, and extending calculus toward three dimensions.

The emphasis is on technique and understanding together: each method (substitution, parts, partial fractions) is motivated by why it works, series are understood through what convergence means, and geometry (polar, parametric, vectors) keeps the analysis visual and grounded.

**How to use this handout:** Each topic lists its learning objectives, explains the key concepts in simple words, defines important terms, gives an example where helpful, and closes with a summary plus exam tips. Topics marked **Very important** deserve extra revision.

## Unit 1: Integration Techniques

### Topic 1: Review: The Definite Integral

#### Learning Objectives

- Recall Riemann sums
- State the Fundamental Theorem
- Apply basic properties

#### Key Concepts

The definite integral as a limit of Riemann sums: area under a curve, accumulated quantity. The Fundamental Theorem connects differentiation and integration: if  $F' = f$ , then the integral of  $f$  from  $a$  to  $b$  equals  $F(b) - F(a)$  - differentiation reversed.

Properties: linearity, additivity over intervals, comparison, and symmetry tricks (even/odd functions over symmetric intervals halve or kill integrals). These simplify before technique begins.

Strategy mindset: integration is pattern recognition - identify the integral's form, choose the method that fits, and verify by differentiating. There is no universal algorithm; experience builds the repertoire.

#### Key Points

- Integral as Riemann limit; FTC links it to antiderivatives.
- Linearity, additivity, symmetry shortcuts.
- Integration is pattern recognition + verification.

#### Key Terms

**Riemann sum:** Approximating area with rectangles.

**Fundamental Theorem:** Linking differentiation and integration.

**Antiderivative:** A function whose derivative is given.

#### Example

- Integral of  $x^2$  from 0 to 1 =  $1/3$  via  $x^3/3$ .

**Summary:** The integral accumulates; the FTC computes.

### Exam Tips

- ★ Very important: FTC statement.
- ★ Remember even/odd shortcuts.

## Topic 2: Substitution Revisited

### Learning Objectives

- Apply u-substitution fluently
- Handle definite integrals
- Recognize substitution patterns

### Key Concepts

Substitution reverses the chain rule: spot an inner function  $g(x)$  whose derivative (up to constants) also appears, set  $u = g(x)$ , and transform. For definite integrals, change the limits with  $u$  - avoiding back-substitution.

Patterns: integrals of  $f(g(x))g'(x)$ ; rationalizing substitutions ( $u = \sqrt{x}$ ); trigonometric substitutions previewed. The art is choosing  $u$ : look for the 'inside' function, especially one whose derivative lurks nearby.

Common errors: forgetting  $du$ 's constant factors, not changing limits, and choosing substitutions that complicate. Practice builds the eye.

### Key Points

- $u$  = inner function;  $du$  absorbs the derivative.
- Change limits for definite integrals.
- Look for the inside function with its derivative nearby.

### Key Terms

**u-substitution:** Reversing the chain rule.

**Inner function:** The composed function's inside part.

**Back-substitution:** Returning from  $u$  to  $x$ .

### Example

- Integral of  $2x \cos(x^2)$ :  $u = x^2$ .

**Summary:** Substitution undoes composition.

### Exam Tips

- ★ Key point: changing limits.
- ★ Remember the pattern recognition.

## Topic 3: Integration by Parts

### Learning Objectives

- State the parts formula
- Choose  $u$  and  $dv$  (LIATE)

- Handle repeated parts

## Key Concepts

Integration by parts reverses the product rule:  $\int u \, dv = uv - \int v \, du$ . Choose  $u$  to simplify when differentiated,  $dv$  to remain integrable when integrated. LIATE guides: Logarithmic, Inverse trig, Algebraic, Trigonometric, Exponential - earlier letters become  $u$ .

Repeated parts: for  $e^x \sin x$ , apply twice and solve for the original integral algebraically. Tabular (DI) method speeds repeated cases.

Applications: integrals of  $\ln x$ ,  $\arctan x$ ,  $x^n e^x$ , and reduction formulas (e.g. for  $\sin^n x$ ). Parts is the workhorse for products of unlike functions.

## Key Points

- $uv - \int v \, du$ ; LIATE chooses  $u$ .
- Repeated parts may need algebraic solving.
- Products of unlike functions signal parts.

## Key Terms

**LIATE:** Priority order for choosing  $u$ .

**Reduction formula:** Recurrence lowering an integral's power.

**Tabular method:** Fast repeated integration by parts.

## Example

- *Integral of  $x \cdot e^x$ :  $u = x$ ,  $dv = e^x \, dx$ .*

**Summary:** Parts undoes the product rule.

## Exam Tips

- ★ Very important: LIATE.
- ★ Remember the twice-and-solve trick.

## Topic 4: Trigonometric Integrals

### Learning Objectives

- Integrate powers of  $\sin/\cos$
- Handle  $\sec/\tan$  powers
- Use power-reduction

### Key Concepts

Powers of sine and cosine: for odd powers, peel one factor and convert the rest with  $\sin^2 + \cos^2 = 1$ , then substitute  $u =$  the other function. For even powers, use half-angle identities (power reduction) to linearize.

Secant/tangent:  $\sec^2$ 's derivative is  $\sec^2 \tan$  - peel  $\sec^2$  for odd  $\sec$  powers; convert  $\tan^2 = \sec^2 - 1$ . The integral of  $\sec x$  (multiply by  $(\sec+\tan)/(\sec+\tan)$ ) is a classic trick.

Products  $\sin(mx)\cos(nx)$ : product-to-sum identities convert to easy integrals. Strategy: identities first, substitution second.

### Key Points

- Odd powers: peel and convert; even: half-angle reduce.
- sec/tan: exploit  $\sec^2$ 's derivative.
- Product-to-sum for mixed products.

### Key Terms

**Power reduction:** Lowering trig powers via identities.

**Half-angle identity:**  $\sin^2 = (1 - \cos 2x)/2$ .

**Product-to-sum:** Converting products to sums.

### Example

- *Integral of  $\sin^3 x$ : peel  $\sin x$ . convert  $\sin^2$ .*

**Summary:** Identities tame trigonometric integrals.

### Exam Tips

- ★ Key point: odd vs. even strategy.
- ★ Remember  $\sec x$  trick.

## Topic 5: Trigonometric Substitution

### Learning Objectives

- Apply trig substitution
- Match forms to substitutions
- Complete the method

### Key Concepts

For radicals of quadratics:  $\sqrt{a^2 - x^2} \rightarrow x = a \sin(\theta)$ ;  $\sqrt{a^2 + x^2} \rightarrow x = a \tan(\theta)$ ;  $\sqrt{x^2 - a^2} \rightarrow x = a \sec(\theta)$ . Each exploits a trig identity to eliminate the radical.

Procedure: substitute, simplify with identities, integrate in  $\theta$ , then back-substitute using a reference triangle (opposite/adjacent/hypotenuse from the substitution).

Completing squares extends the method to general quadratics. After substitution, the integral often needs parts or trig-integral techniques - methods combine.

### Key Points

- Three forms, three substitutions ( $\sin$ ,  $\tan$ ,  $\sec$ ).
- Reference triangles back-substitute.
- Complete the square for general quadratics.

### Key Terms

**Trigonometric substitution:** Removing radicals via trig identities.

**Reference triangle:** Right triangle encoding the substitution.

**Completing the square:** Rewriting quadratics as sums of squares.

### Example

- *$\sqrt{4 - x^2}$ :  $x = 2 \sin(\theta)$ .*

**Summary:** Trig substitution dissolves quadratic radicals.

### Exam Tips

- ★ Very important: match form to substitution.
- ★ Remember reference triangles.

## Topic 6: Partial Fractions

### Learning Objectives

- Decompose rational functions
- Handle repeated/irreducible factors
- Integrate the decomposition

### Key Concepts

Partial fractions split  $P(x)/Q(x)$  (proper:  $\deg P < \deg Q$ ; else divide first) into integrable pieces. Factor  $Q$ : linear factors give  $A/(x-a)$ ; repeated linears give chains  $A/(x-a) + B/(x-a)^2$ ; irreducible quadratics give  $(Ax+B)/(\text{quadratic})$ .

Find constants by equating coefficients or strategic  $x$ -values (cover-up for distinct linears). Each piece integrates elementarily: logs for linears, arctan for quadratics (after completing the square).

This is the method that 'always works' for rational functions - the theoretical guarantee that every rational function has an elementary antiderivative.

### Key Points

- Proper form first; factor denominator.
- Linear/repeated/quadratic factors get specific forms.
- Every rational function integrates elementarily.

### Key Terms

**Partial fractions:** Decomposing rational functions.

**Proper fraction:** Numerator degree below denominator's.

**Cover-up method:** Quick constants for distinct linears.

### Example

- $1/((x-1)(x+2)) = A/(x-1) + B/(x+2)$ .

**Summary:** Partial fractions guarantee rational integrals.

### Exam Tips

- ★ Very important: the three factor forms.
- ★ Remember polynomial division first.

## Topic 7: Improper Integrals

### Learning Objectives

- Define improper integrals
- Evaluate via limits
- Apply comparison tests

### Key Concepts

Improper integrals extend integration to infinite limits or unbounded integrands: replace the problem point with a limit and evaluate. Converges if the limit exists finitely; diverges otherwise.

Examples: integral of  $1/x^2$  from 1 to infinity converges ( $= 1$ ); integral of  $1/x$  diverges (logarithmically).  
The p-test:  $1/x^p$  converges at infinity iff  $p > 1$ ; near zero iff  $p < 1$ .

Comparison: if  $0 \leq f \leq g$  and  $g$  converges,  $f$  converges. Improper integrals model total quantities over infinite horizons (probabilities, work) - convergence is the question that matters.

### Key Points

- Infinite limits or unbounded integrands via limits.
- p-test:  $1/x^p$  converges at infinity iff  $p > 1$ .
- Comparison test transfers convergence.

### Key Terms

**Improper integral:** Integral over infinite/unbounded domains.

**Convergence:** The limit existing finitely.

**p-test:** Convergence criterion for  $1/x^p$ .

### Example

- *Integral  $1/x^2$  from 1 to  $\infty = 1$ .*

**Summary:** Limits extend integration to infinity.

### Exam Tips

- ★ Very important: the p-test both ways.
- ★ Remember limit definition.

## Unit 2: Sequences and Series

### Topic 8: Sequences

#### Learning Objectives

- Define sequences and limits
- Test convergence
- Recognize key sequences

#### Key Concepts

A sequence is an ordered list  $a_1, a_2, \dots$ ; convergence means terms approach a limit  $L$ . Techniques: divide by highest power for rationals, squeeze theorem, and recognizing forms ( $r^n$  converges iff  $|r| < 1$ ... careful:  $r=1$  gives 1,  $r=-1$  oscillates).

Monotone bounded sequences converge (completeness of reals) - useful when limits are hard to compute directly. Recursive sequences (Newton's method, continued fractions) need monotonicity analysis.

Sequences underlie series: a series converges iff its partial-sum sequence converges. Master sequences first; series follow.

#### Key Points

- Convergence: terms approaching a limit.
- $r^n$  converges iff  $-1 < r \leq 1$  (to 0 unless  $r=1$ ).
- Monotone + bounded implies convergence.

## Key Terms

**Sequence:** An ordered list of numbers.

**Limit:** The value terms approach.

**Squeeze theorem:** Bounding between converging sequences.

## Example

- $(n/(n+1)) \rightarrow 1$ ;  $(1/2)^n \rightarrow 0$ .

**Summary:** Sequences are series' foundation.

## Exam Tips

- ★ Key point:  $r^n$  cases.
- ★ Remember monotone-bounded theorem.

## Topic 9: Infinite Series: Basics

### Learning Objectives

- Define series convergence
- State the divergence test
- Sum geometric and telescoping series

### Key Concepts

A series sums infinitely: convergence means partial sums approach a finite limit. The divergence (nth-term) test: if terms don't approach 0, the series diverges - but terms approaching 0 guarantees nothing (harmonic series diverges!).

Geometric series: sum of  $ar^n = a/(1-r)$  for  $|r| < 1$  - the workhorse (finance, probabilities, fractals). Telescoping series collapse via cancellation (partial fractions reveal them).

The harmonic series  $(1/n)$  diverges - slowly (logarithmically). This cautionary example motivates the convergence tests ahead.

### Key Points

- Convergence via partial sums; nth-term test for divergence.
- Geometric:  $a/(1-r)$  for  $|r| < 1$ .
- Harmonic diverges despite terms  $\rightarrow 0$ .

## Key Terms

**Series:** Sum of a sequence's terms.

**Partial sum:** Sum of the first  $n$  terms.

**Divergence test:** Terms not  $\rightarrow 0$  implies divergence.

## Example

- $1 + 1/2 + 1/4 + \dots = 2$ .

**Summary:** Infinite sums need convergence proof.

## Exam Tips

- ★ Very important: geometric sum formula.
- ★ Remember harmonic caution.

## Topic 10: Positive-Term Tests

### Learning Objectives

- Apply integral, comparison, ratio, root tests
- Choose tests strategically
- Handle p-series

### Key Concepts

Integral test:  $\sum f(n)$  converges iff integral of  $f$  converges ( $f$  positive, decreasing) - links series to improper integrals. P-series  $(1/n^p)$ : converges iff  $p > 1$ .

Comparison tests: direct (bound by known series) and limit comparison (asymptotic equivalence - compare growth rates). Ratio test: if  $|a_{n+1}/a_n| \rightarrow L$ , converges if  $L < 1$ , diverges if  $L > 1$  (inconclusive at 1) - ideal for factorials and exponentials. Root test: similar with  $n$ th roots - ideal for  $n$ th powers.

Strategy: recognize the dominant growth (factorials beat exponentials beat polynomials beat logs), then choose the test that exploits it.

### Key Points

- Integral test; p-series converges iff  $p > 1$ .
- Comparison and limit comparison.
- Ratio (factorials), root (powers); inconclusive at 1.

### Key Terms

**p-series:** Sum of  $1/n^p$ .

**Ratio test:** Using successive-term ratios.

**Limit comparison:** Comparing asymptotic growth.

### Example

- $\sum n!/n^n$ : ratio test.

**Summary:** Tests classify series by growth rates.

### Exam Tips

- ★ Very important: p-series rule.
- ★ Remember ratio-test use cases.

## Topic 11: Alternating Series and Absolute Convergence

### Learning Objectives

- State the alternating series test
- Distinguish absolute/conditional
- Estimate remainders

### Key Concepts

Alternating series (signs alternate): converge if terms decrease to 0 (Leibniz test) - e.g. the alternating harmonic series converges (to  $\ln 2$ ). The remainder is bounded by the first omitted term - easy error estimates.

Absolute vs. conditional: absolutely convergent ( $\sum |a_n|$  converges) series behave perfectly - rearrangement-safe, product-safe. Conditionally convergent series (convergent but not absolutely) are delicate: Riemann showed rearrangement can sum to anything!

Moral: prove absolute convergence when possible; handle conditional cases with the alternating test and care.

### Key Points

- Alternating test: decreasing terms  $\rightarrow 0$ .
- Absolute convergence is robust; conditional is delicate.
- Remainder bounded by first omitted term.

### Key Terms

**Absolute convergence:** Sum of absolute values converges.

**Conditional convergence:** Convergent but not absolutely.

**Leibniz test:** The alternating series test.

### Example

- Alternating harmonic converges conditionally to  $\ln 2$ .

**Summary:** Signs matter: absolute safety vs. conditional delicacy.

### Exam Tips

- ★ Very important: absolute vs. conditional.
- ★ Remember remainder estimate.

## Topic 12: Power Series

### Learning Objectives

- Define power series
- Find radius/interval of convergence
- Differentiate and integrate termwise

### Key Concepts

Power series  $\sum c_n (x-a)^n$ : functions as infinite polynomials. The ratio/root test gives a radius  $R$ : converges absolutely inside  $(a-R, a+R)$ , diverges outside; endpoints checked individually - the interval of convergence.

Inside the interval, power series differentiate and integrate termwise - making them calculus-friendly representations. Operations (addition, multiplication, composition) work within radii.

Power series are the bridge to Taylor series: every sufficiently smooth function may equal its Taylor expansion - analysis's most powerful representation.

### Key Points

- Radius  $R$  via ratio/root; check endpoints separately.
- Termwise differentiation/integration inside interval.
- Functions as infinite polynomials.

### Key Terms

**Power series:** Sum of  $c_n(x-a)^n$ .

**Radius of convergence:** Distance of guaranteed convergence.

**Interval of convergence:** Full set of convergence points.

### Example

- $\sum x^n/n!$  converges for all  $x$  ( $R = \infty$ ).

**Summary:** Power series: calculus with infinite polynomials.

### Exam Tips

- ★ Very important: endpoint checking.
- ★ Remember termwise operations.

## Topic 13: Taylor and Maclaurin Series

### Learning Objectives

- Construct Taylor series
- Memorize key Maclaurin series
- Estimate with Taylor's inequality

### Key Concepts

Taylor series:  $f(x) = \sum f^{(n)}(a)(x-a)^n/n!$  - matching all derivatives at  $a$ . Maclaurin is  $a = 0$ . Key series:  $e^x$ ,  $\sin x$ ,  $\cos x$ ,  $1/(1-x)$ ,  $\ln(1+x)$ ,  $(1+x)^k$  (binomial) - memorize these; derive others by substitution, differentiation, integration.

Taylor's inequality bounds the remainder:  $|R_n| \leq M|x-a|^{n+1}/(n+1)!$  - quantifying approximation accuracy, essential for numerical work.

Applications: limits (via series expansion), approximations ( $\sin x \approx x - x^3/6$ ), proving  $e$ 's irrationality, and defining functions ( $e^x$  for complex  $x$ ). Series are analysis's microscope.

### Key Points

- Taylor: match derivatives; Maclaurin at 0.
- Memorize  $e^x$ ,  $\sin$ ,  $\cos$ ,  $1/(1-x)$ ,  $\ln(1+x)$ .
- Taylor's inequality bounds approximation error.

### Key Terms

**Taylor series:** Expansion matching derivatives at a point.

**Maclaurin series:** Taylor series at 0.

**Remainder:** The truncation error.

### Example

- $e^x = 1 + x + x^2/2! + \dots$

**Summary:** Taylor series: functions revealed as infinite polynomials.

### Exam Tips

- ★ Very important: memorize the key series.
- ★ Remember remainder bound.

# Unit 3: Parametric, Polar, and Vectors

## Topic 14: Parametric Curves

### Learning Objectives

- Parametrize curves
- Compute parametric derivatives
- Find arc lengths

### Key Concepts

Parametric equations  $(x(t), y(t))$  describe curves functions cannot: circles, cycloids, projectile paths - with  $t$  as time, tracing direction matters. Eliminating  $t$  recovers Cartesian form when possible.

Derivatives:  $dy/dx = (dy/dt)/(dx/dt)$  - tangents via parameter; second derivatives need the quotient rule in  $t$ . Arc length: integral of  $\sqrt{(dx/dt)^2 + (dy/dt)^2} dt$  - distance traveled.

Applications: motion (velocity/acceleration vectors from position), Bezier curves in design, and plane curves in engineering. Parameters add the dimension of 'when'.

### Key Points

- $(x(t), y(t))$ : curves with direction and time.
- $dy/dx = y'(t)/x'(t)$ ; arc length via speed integral.
- Motion and design applications.

### Key Terms

**Parameter:** The variable tracing the curve.

**Cycloid:** Curve traced by a rolling wheel.

**Arc length:** Curve's length via integral.

### Example

- Circle:  $x = \cos t$ ,  $y = \sin t$ .

**Summary:** Parameters describe motion, not just shape.

### Exam Tips

- ★ Key point: the  $dy/dx$  formula.
- ★ Remember arc length integrand.

## Topic 15: Polar Coordinates

### Learning Objectives

- Plot in polar coordinates
- Convert between systems
- Graph polar curves

### Key Concepts

Polar coordinates  $(r, \theta)$ : position by distance and angle - natural for circular/spiral symmetry. Conversion:  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ;  $r^2 = x^2 + y^2$ ,  $\tan(\theta) = y/x$  (mind quadrants!).

Curves: circles ( $r = a \cos(\theta)$ ), cardioids ( $r = a(1 - \cos \theta)$ ), roses ( $r = a \cos(n \theta)$ :  $n$  petals, or  $2n$  if  $n$  even), spirals ( $r = \theta$ ), lemniscates. Symmetry tests (about  $x$ -axis,  $y$ -axis, origin) simplify graphing.

Derivatives and tangents in polar need the parametric view ( $x(\theta)$ ,  $y(\theta)$ ). Areas:  $(1/2) \int r^2 d(\theta)$  - sectors, not rectangles.

### Key Points

- $(r, \theta)$ : distance and angle; conversion formulas.
- Cardioids, roses, spirals, lemniscates.
- Area =  $(1/2) \int r^2 d(\theta)$ .

### Key Terms

**Polar coordinates:** Position by radius and angle.

**Cardioid:** Heart-shaped  $r = a(1 - \cos \theta)$ .

**Rose:** Petal curve  $r = a \cos(n \theta)$ .

### Example

- $r = 2$ : circle radius 2 centered at origin.

**Summary:** Polar coordinates suit circular symmetry.

### Exam Tips

- ★ Very important: area formula.
- ★ Remember rose petal counts.

## Topic 16: Vectors in the Plane and Space

### Learning Objectives

- Perform vector operations
- Explain dot and cross products
- Describe lines and planes

### Key Concepts

Vectors: magnitude and direction; operations: addition (tip-to-tail), scalar multiplication, components. In 3D: with  $i, j, k$  basis.

Dot product:  $a \cdot b = |a||b|\cos(\theta)$  - measures alignment; zero means perpendicular; projections decompose vectors. Cross product:  $a \times b$  (3D only) - perpendicular vector with  $|a \times b| = \text{area of parallelogram}$ ; direction by right-hand rule.

Lines:  $r = r_0 + t v$  (point + direction). Planes:  $n \cdot (r - r_0) = 0$  (point + normal). Distances: point-to-plane via projection. Vectors geometrize 3D.

### Key Points

- Dot: alignment/projection; zero = perpendicular.
- Cross: perpendicular vector; magnitude = parallelogram area.
- Lines: point +  $t$ \*direction; planes: point + normal.

### Key Terms

**Dot product:** Scalar measure of alignment.

**Cross product:** Perpendicular vector product.

**Normal vector:** Perpendicular to a plane.

### Example

- *Work = force . displacement.*

**Summary:** Vectors: algebra for geometry.

### Exam Tips

- ★ Very important: dot vs. cross meanings.
- ★ Remember plane equation.

## Topic 17: Vector Calculus Preview

### Learning Objectives

- Differentiate vector functions
- Explain velocity/acceleration
- Preview multivariable ideas

### Key Concepts

Vector functions  $\mathbf{r}(t)$  : derivatives give velocity  $\mathbf{v} = \mathbf{r}'$ , speed  $= |\mathbf{v}|$ , acceleration  $\mathbf{a} = \mathbf{v}'$ . Unit tangent  $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ ; curvature measures bending - the geometry of motion.

Projectile motion:  $\mathbf{r}(t)$  from constant gravitational acceleration - parametric equations with physics. Kepler's laws emerge from central-force motion (optional enrichment).

Preview: functions of several variables  $f(x, y)$  - partial derivatives, gradients pointing uphill, and double integrals - Calculus III's territory, glimpsed here.

### Key Points

- $\mathbf{r}'(t)$  = velocity; speed  $= |\text{velocity}|$ .
- Unit tangent and curvature describe motion geometry.
- Gradient and partial derivatives preview what's next.

### Key Terms

**Velocity:** Derivative of position.

**Curvature:** Measure of bending.

**Partial derivative:** Derivative holding others fixed.

### Example

- *Projectile:  $\mathbf{r}(t) =$  .*

**Summary:** Vector calculus: motion mathematized.

### Exam Tips

- ★ Key point: velocity/acceleration relations.
- ★ Remember gradient direction.

## Topic 18: Applications of Integration

## Learning Objectives

- Compute areas and volumes
- Explain work and average value
- Model with integrals

## Key Concepts

Areas between curves: integral of (top - bottom). Volumes: disks/washers (revolve around axis) and shells (cylindrical) - choose by which gives simpler integrals; known cross-sections generalize.

Physics: work = integral of force (variable forces: springs, pumping water); center of mass and centroids; hydrostatic pressure; average value of a function  $(1/(b-a)) \int_a^b f(x) dx$ .

Modeling: integrals accumulate - from areas to probabilities (continuous distributions) to economics (consumer surplus). The integral is applied mathematics' accumulator.

## Key Points

- Volumes: disks, washers, shells - choose wisely.
- Work, center of mass, average value.
- Integrals accumulate: area, work, probability.

## Key Terms

**Method of shells:** Volume via cylindrical shells.

**Centroid:** Geometric center of mass.

**Average value:** Mean of a function over an interval.

## Example

- *Sphere volume via disks:*  $(4/3) \pi r^3$ .

**Summary:** Integration computes the physical world's totals.

## Exam Tips

- ★ Key point: disks vs. shells choice.
- ★ Remember work integral.

## Topic 19: Differential Equations Preview

### Learning Objectives

- Recognize basic ODEs
- Solve separable equations
- Model with ODEs

### Key Concepts

Separable equations:  $dy/dx = g(x)h(y)$  - separate and integrate:  $\int dy/h(y) = \int g(x) dx$ . Exponential growth/decay ( $dy/dt = ky$ ), Newton's cooling, logistic growth (with carrying capacity) - the classic models.

First-order linear:  $dy/dx + P(x)y = Q(x)$  - integrating factor  $\mu = e^{\int P dx}$  makes the left an exact derivative. Mixing problems and circuits apply it.

Direction fields visualize solutions without solving - qualitative ODE theory previewed. Differential equations are calculus's purpose: describing change.

## Key Points

- Separable: split and integrate both sides.
- Linear first-order: integrating factor method.
- Growth, cooling, logistic, mixing models.

## Key Terms

**Separable equation:** Splittable into x- and y-parts.

**Integrating factor:** Multiplier enabling direct integration.

**Direction field:** Slope visualization of ODE solutions.

## Example

- $dv/dx = 2x$  with  $v(0)=1$ :  $v = x^2 + 1$ .

**Summary:** Differential equations: calculus describing change.

## Exam Tips

- ★ Key point: separable procedure.
- ★ Remember integrating factor formula.

## Topic 20: Problem-Solving Strategies and Review

### Learning Objectives

- Choose integration methods
- Plan series problems
- Review the course

### Key Concepts

Integration strategy: simplify first (algebra, trig identities), then classify - substitution (composition), parts (products), trig techniques, trig substitution (quadratic radicals), partial fractions (rationals). When stuck: try a different method, differentiate to check, or consider numerical/special functions.

Series strategy: check nth term; identify type (geometric, p-series, alternating); choose test by growth form; for power/Taylor: radius first, endpoints second.

The course's arc: from antidifferentiation techniques through infinite summation to new geometries - each unit generalizing the last. Mastery means method selection, not just execution.

## Key Points

- Simplify, classify, choose method, verify.
- Series: nth-term, type, test, endpoints.
- Method selection is the real skill.

## Key Terms

**Strategy:** Planned method selection.

**Verification:** Checking by differentiation.

**Classification:** Identifying problem type.

## Example

- Seeing  $\sqrt{x^2+9}$  and choosing  $x = 3 \tan(\theta)$ .

**Summary:** Calculus II: a toolkit wielded with judgment.

### Exam Tips

- ★ Key point: the integration decision tree.
- ★ Remember verify-by-differentiation.

## Glossary of Key Terms

**Antiderivative:** A function whose derivative is the given function.

**u-substitution:** Integration by reversing the chain rule.

**Integration by parts:** Reversing the product rule:  $uv - \int v \, du$ .

**Partial fractions:** Decomposing rational functions for integration.

**Improper integral:** Integral over infinite or unbounded domains.

**p-series:** Sum of  $1/n^p$ ; converges iff  $p > 1$ .

**Ratio test:** Convergence via successive-term ratios.

**Absolute convergence:** Convergence of absolute values.

**Power series:** Infinite polynomial sum  $\sum c_n(x-a)^n$ .

**Taylor series:** Function expansion matching derivatives.

**Polar coordinates:** Position by radius  $r$  and angle  $\theta$ .

**Dot product:** Scalar product measuring alignment.

**Cross product:** Vector product perpendicular to both factors.

**Separable equation:** ODE splittable into single-variable parts.

**Parametric equations:** Curves described via a parameter  $t$ .