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HANDOUT

MTH5101

Calculus I

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MTH5101

Calculus I

STUDY HANDOUT

A complete topic-wise study handout for MTH5101 Calculus I, prepared from the course video lectures (MTH5101 short-lecture series, Virtual University). Covers real numbers and functions, limits, continuity, differentiation and its applications, integration, and an introduction to sequences and series - with definitions, worked examples, key formulas and exam pointers.

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Lecture 1: Real Numbers, Intervals and Inequalities

Key terms: natural numbers, integers, rationals, irrationals, intervals, absolute value

Calculus begins with the real numbers: natural numbers (1, 2, 3, ...), integers (... , -2, -1, 0, 1, 2, ...), rational numbers (ratios p/q of integers) and irrational numbers (non-repeating decimals like π and $\sqrt{2}$). The reals form a complete ordered field - every bounded set has a least upper bound - which is why limits work.

Intervals describe sets of reals: open (a, b) , closed $[a, b]$, half-open, and infinite $(-\infty, a)$. Inequalities are solved by finding critical points and testing sign regions. The absolute value $|x|$ is the distance from x to 0; $|x| < a$ means $-a < x < a$.

- Properties of order: if $a < b$ then $a + c < b + c$; multiplying by a negative reverses the inequality.
- Triangle inequality: $|x + y| \leq |x| + |y|$ - very important for limit proofs.
- Example: solve $|2x - 3| < 5$: $-5 < 2x - 3 < 5$, so $-1 < x < 4$.

EXAM POINTER: Inequality and absolute-value questions appear in nearly every midterm. Practice sign-chart solving for rational inequalities like $(x-1)/(x+2) > 0$.

Lecture 2: Functions: Domain, Range and Graphs

Key terms: function, domain, range, vertical line test, even/odd functions

A function f from a set D (domain) to the reals assigns to each x in D exactly one value $f(x)$. The range is the set of all values $f(x)$. The vertical line test: a curve is a function's graph iff no vertical line meets it more than once. A function is even if $f(-x) = f(x)$ (symmetric about y -axis), odd if $f(-x) = -f(x)$ (symmetric about origin).

- Domain restrictions: denominators nonzero, even roots of nonnegatives, logarithms of positives.
- Piecewise functions are defined by different formulas on different intervals - check continuity at the joints.
- Example: $f(x) = \sqrt{x - 2}/(x - 5)$ has domain $[2, 5) \cup (5, \infty)$.

EXAM POINTER: Finding domains of combined and composite functions is a favourite short question. Always list every restriction.

Lecture 3: New Functions from Old: Transformations and Composition

Key terms: transformations, shifts, stretches, reflections, composition, inverse functions

From $y = f(x)$ we get: $y = f(x) + k$ (shift up k), $y = f(x - h)$ (shift right h), $y = cf(x)$ (vertical stretch if $c > 1$), $y = f(cx)$ (horizontal compression if $c > 1$), $y = -f(x)$ (reflect in x-axis), $y = f(-x)$ (reflect in y-axis). Composition $(f \circ g)(x) = f(g(x))$ applies g first, then f ; domain requires x in domain of g and $g(x)$ in domain of f .

A function has an inverse f^{-1} iff it is one-to-one (passes the horizontal line test). The graphs of f and f^{-1} are reflections in the line $y = x$. To find the inverse: write $y = f(x)$, solve for x , swap x and y .

- Order matters in composition: $f \circ g$ is generally different from $g \circ f$.
- Restricting the domain can make a function one-to-one (e.g. x^2 on $[0, \infty)$ has inverse $\sqrt{x}$).
- Example: $f(x) = 2x + 3$, $g(x) = x^2$: $(f \circ g)(x) = 2x^2 + 3$; $(g \circ f)(x) = (2x+3)^2 = 4x^2 + 12x + 9$.

Lecture 4: Polynomial, Rational and Algebraic Functions

Key terms: polynomials, degree, rational functions, asymptotes, algebraic functions

A polynomial $p(x) = a_n x^n + \dots + a_1 x + a_0$ has domain all reals; its end behaviour is governed by the leading term $a_n x^n$. Zeros are found by factoring; multiplicity affects whether the graph crosses (odd) or touches (even) the x-axis. Rational functions $f(x) = p(x)/q(x)$ have vertical asymptotes where $q(x) = 0$ (and $p(x) \neq 0$) and horizontal/slant asymptotes determined by comparing degrees.

- $\deg p < \deg q$: horizontal asymptote $y = 0$. $\deg p = \deg q$: $y =$ ratio of leading coefficients.
- $\deg p = \deg q + 1$: slant asymptote (divide).
- Holes occur at common factors cancelled from p and q .
- Example: $f(x) = (x^2 - 1)/(x - 1) = x + 1$ for $x \neq 1$: a hole at $(1, 2)$, not an asymptote.

EXAM POINTER: Sketching rational functions with all asymptotes and intercepts labelled is a standard 10-mark question.

Lecture 5: Exponential and Logarithmic Functions

Key terms: exponential growth, e , natural logarithm, logarithm laws

The exponential function $f(x) = a^x$ ($a > 0$, $a \neq 1$) grows fast for $a > 1$ and decays for $0 < a < 1$. The natural exponential e^x ($e = 2.718\dots$) is special because its derivative equals itself. Its inverse is the natural logarithm $\ln x$, defined for $x > 0$. Logarithm laws: $\ln(ab) = \ln a + \ln b$, $\ln(a/b) = \ln a - \ln b$, $\ln(a^r) = r \ln a$.

- Change of base: $\log_a x = \ln x / \ln a$.

- Exponential equations: take logs of both sides; logarithmic equations: exponentiate, then check domain.
- Example: solve $3^{(2x)} = 5$: $2x \ln 3 = \ln 5$, so $x = \ln 5 / (2 \ln 3)$.

EXAM POINTER: You must know the logarithm laws cold - they are used in logarithmic differentiation and integration later.

Lecture 6: Trigonometric Functions and Their Graphs

Key terms: radians, sine, cosine, tangent, amplitude, period, phase shift

Angles in calculus are measured in radians: π radians = 180 degrees. The sine and cosine functions have period 2π , range $[-1, 1]$; tangent has period π with vertical asymptotes at odd multiples of $\pi/2$. For $y = A \sin(Bx - C) + D$: amplitude $|A|$, period $2\pi/|B|$, phase shift C/B , vertical shift D .

- Key values: sin/cos of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ - memorise the unit circle.
- Pythagorean identities: $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$.
- Inverse trig functions: arcsin has domain $[-1, 1]$, range $[-\pi/2, \pi/2]$.

Lecture 7: Intuitive Idea of Limits

Key terms: limit, left-hand limit, right-hand limit, infinite limits

The limit $\lim_{x \rightarrow a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to (but not equal to) a . The function's value at a is irrelevant - the function need not even be defined at a . One-sided limits consider approach from the left ($x \rightarrow a^-$) or right ($x \rightarrow a^+$); the two-sided limit exists iff both one-sided limits exist and are equal.

- Numerical/graphical estimation: build a table of values approaching a from both sides.
- Infinite limits: $\lim_{x \rightarrow 0} 1/x^2 = \infty$ describes vertical asymptotes.
- Example: $f(x) = (x^2 - 4)/(x - 2)$ has limit 4 as $x \rightarrow 2$ even though $f(2)$ is undefined - factor and cancel.

EXAM POINTER: State the precise relationship between one-sided and two-sided limits - examiners ask this as a theory short question.

Lecture 8: Limit Laws and Computing Limits

Key terms: limit laws, direct substitution, factoring, rationalising, squeeze theorem

If $\lim f(x) = L$ and $\lim g(x) = M$, then limits of sums, differences, products, quotients ($M \neq 0$), powers and roots behave as expected. Strategy: try direct substitution first; if you get $0/0$,

factor and cancel, rationalise, or use known special limits. The squeeze theorem: if $g(x) \leq f(x) \leq h(x)$ near a and g, h have the same limit L at a , then f has limit L too.

- $\lim_{x \rightarrow 0} \sin x / x = 1$ and $\lim_{x \rightarrow 0} (1 - \cos x)/x = 0$ - the two fundamental trig limits.
- $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$ - the definition of e as a limit.
- Example: $\lim_{x \rightarrow 0} \sin(3x)/x = 3 * \lim_{x \rightarrow 0} \sin(3x)/(3x) = 3$.

EXAM POINTER: The 0/0 indeterminate form with trig functions almost always uses $\sin x / x = 1$. Learn to spot the substitution.

Lecture 9: Limits at Infinity and End Behaviour

Key terms: limits at infinity, horizontal asymptotes, end behaviour

$\lim_{x \rightarrow \infty} f(x) = L$ means $f(x)$ approaches L as x grows without bound; the line $y = L$ is a horizontal asymptote. For rational functions, divide numerator and denominator by the highest power of x . For polynomials, the leading term dominates. Infinite limits at infinity (e.g. $x^3 \rightarrow \infty$) describe end behaviour.

- $\lim_{x \rightarrow \infty} 1/x^r = 0$ for $r > 0$ - the workhorse for rational limits at infinity.
- Exponential dominance: e^x grows faster than any power x^n ; $\ln x$ grows slower than any positive power.
- Example: $\lim_{x \rightarrow \infty} (3x^2 + 2x)/(5x^2 - 1) = 3/5$ (ratio of leading coefficients).

Lecture 10: Continuity

Key terms: continuity, removable/jump/infinite discontinuity, Intermediate Value Theorem

f is continuous at a if: (1) $f(a)$ is defined, (2) $\lim_{x \rightarrow a} f(x)$ exists, (3) the limit equals $f(a)$. Polynomials, rationals (on their domain), roots, trig, exponential and log functions are continuous where defined. Discontinuities: removable (hole - limit exists), jump (one-sided limits differ), infinite (vertical asymptote).

The Intermediate Value Theorem: if f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then $f(c) = N$ for some c in (a, b) . It guarantees roots: if $f(a)$ and $f(b)$ have opposite signs, f has a zero between a and b - the basis of bisection root-finding.

- Continuity on an interval plus differentiability are the hypotheses of the big theorems (Rolle, MVT).
- Example: $f(x) = (x^2 - 9)/(x - 3)$ for $x \neq 3$, $f(3) = 6$ is continuous everywhere (removable discontinuity filled).

EXAM POINTER: Be able to classify a discontinuity from a graph or formula and to apply IVT to prove a root exists in an interval.

Lecture 11: Tangent Lines and Rates of Change

Key terms: tangent line, secant line, average rate of change, instantaneous rate of change

The tangent line to $y = f(x)$ at $x = a$ is the limiting position of secant lines through $(a, f(a))$ and $(a+h, f(a+h))$ as $h \rightarrow 0$. Its slope - the instantaneous rate of change - is the limit of difference quotients $[f(a+h) - f(a)]/h$. Average velocity over $[t_1, t_2]$ is displacement/time; instantaneous velocity is the limit as the interval shrinks.

- Slope of tangent at a : $m = \lim_{h \rightarrow 0} [f(a+h) - f(a)]/h$.
- The tangent line equation: $y - f(a) = m(x - a)$; the normal line has slope $-1/m$.
- Example: $f(x) = x^2$ at $a = 1$: $m = \lim [(1+h)^2 - 1]/h = \lim (2h + h^2)/h = 2$; tangent: $y - 1 = 2(x - 1)$.

Lecture 12: The Derivative: Definition and Notation

Key terms: derivative, differentiability, Leibniz notation, dy/dx

The derivative $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)]/h$ gives the slope of the tangent (and instantaneous rate of change) at every x where the limit exists. Notation: $f'(x)$, dy/dx , $D_x y$, y' . If $f'(a) > 0$ the function is increasing at a ; if $f'(a) < 0$ it is decreasing there.

- Differentiability implies continuity, but not conversely: $|x|$ is continuous at 0 but not differentiable there (corner).
- Non-differentiable points: corners/cusps, vertical tangents, discontinuities.
- Higher derivatives: f'' is the derivative of f' - acceleration is the second derivative of position.

EXAM POINTER: Use the limit definition to differentiate $f(x) = 1/x$ or $\sqrt{x}$ - a guaranteed exam question testing the definition itself.

Lecture 13: Differentiation Rules: Power, Sum, Product, Quotient

Key terms: power rule, product rule, quotient rule

From the definition we prove the working rules: $d/dx [x^n] = n x^{(n-1)}$ (power rule); derivatives of sums/differences and constant multiples split termwise; the product rule $(fg)' = f'g + fg'$; the quotient rule $(f/g)' = (f'g - fg')/g^2$. Learn the quotient rule as 'low d-high minus high d-low over low squared'.

- Combine rules: differentiate term by term, then apply product/quotient where functions multiply or divide.
- Simplify before differentiating when possible: e.g. $(x^3 + 2x)/x = x^2 + 2$.
- Example: $d/dx [x^2 \sin x] = 2x \sin x + x^2 \cos x$.

EXAM POINTER: Speed and accuracy with product and quotient rules decide your marks in every differentiation question - drill them.

Lecture 14: Derivatives of Trigonometric Functions

Key terms: derivatives of sin, cos, tan, sec, csc, cot

$\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ (from the limit definition using $\sin x/x \rightarrow 1$). The rest follow by the quotient rule: $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} \cot x = -\csc^2 x$, $\frac{d}{dx} \sec x = \sec x \tan x$, $\frac{d}{dx} \csc x = -\csc x \cot x$. Note the cofunctions (cos, cot, csc) take a minus sign.

- Mnemonic: derivatives of 'co-' functions get a negative sign.
- Combine with chain rule: $\frac{d}{dx} \sin(3x) = 3 \cos(3x)$.
- Example: $\frac{d}{dx} [x \tan x] = \tan x + x \sec^2 x$ (product rule).

Lecture 15: The Chain Rule

Key terms: chain rule, outside-inside, composite functions

The chain rule differentiates compositions: $\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$ - derivative of the outside evaluated at the inside, times the derivative of the inside. In Leibniz form, $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$. Apply repeatedly for triple compositions; it is the single most-used differentiation rule.

- General power rule: $\frac{d}{dx} [u^n] = n u^{(n-1)} * u'$.
- Example: $\frac{d}{dx} (3x^2 + 1)^5 = 5(3x^2+1)^4 * 6x = 30x(3x^2+1)^4$.
- Example: $\frac{d}{dx} \sin^2(x) = 2 \sin x \cos x = \sin(2x)$.

EXAM POINTER: Most chain-rule errors come from forgetting to multiply by the derivative of the inside - always do the 'times u-prime' step explicitly.

Lecture 16: Implicit Differentiation

Key terms: implicit differentiation, related rates preparation

When y is defined implicitly by an equation like $x^2 + y^2 = 25$, differentiate both sides with respect to x , treating y as a function of x : every y -term gets a chain-rule factor $\frac{dy}{dx}$ (e.g. $\frac{d}{dx}[y^2] = 2y \frac{dy}{dx}$). Then solve the resulting linear equation for $\frac{dy}{dx}$.

- $\frac{d}{dx} [xy] = y + x \frac{dy}{dx}$ (product rule plus chain rule).
- Second derivatives: differentiate $\frac{dy}{dx}$ implicitly again.
- Example: $x^2 + y^2 = 25 \rightarrow 2x + 2y y' = 0 \rightarrow y' = -x/y$. At $(3, 4)$: $y' = -3/4$.

Lecture 17: Derivatives of Logarithmic and Exponential Functions

Key terms: $d/dx \ln x = 1/x$, $d/dx e^x = e^x$, logarithmic differentiation

$d/dx \ln x = 1/x$ and $d/dx e^x = e^x$; with the chain rule, $d/dx \ln(u) = u'/u$ and $d/dx e^u = e^u \cdot u'$. Logarithmic differentiation handles products, quotients and powers elegantly: take $\ln$ of both sides, differentiate, solve for y' . It is essential for functions like x^x : $\ln y = x \ln x$ gives $y' = x^x (\ln x + 1)$.

- $d/dx \log_a x = 1/(x \ln a)$; $d/dx a^x = a^x \ln a$.
- Use log differentiation for $y = (\text{complicated product/quotient})^{\text{power}}$.
- Example: $y = x^3 (2x+1)^2/(x-1)$: $\ln y = 3 \ln x + 2 \ln(2x+1) - \ln(x-1)$, then $y'/y = 3/x + 4/(2x+1) - 1/(x-1)$.

EXAM POINTER: Logarithmic differentiation of x^x type functions is a classic 'show that' question - memorise the steps.

Lecture 18: Related Rates

Key terms: related rates, implicit differentiation with respect to time

In related-rates problems, several quantities change with time and are linked by a geometric equation. Strategy: (1) draw and label, (2) write the relating equation, (3) differentiate implicitly with respect to t , (4) substitute the known instantaneous values (never before differentiating!), (5) solve for the unknown rate.

- Common relations: $A = \pi r^2$, $V = (4/3) \pi r^3$, Pythagoras $x^2 + y^2 = z^2$, similar triangles.
- Units: the answer's units are (units of quantity) per (unit of time).
- Example: spherical balloon $V = (4/3)\pi r^3$, $dV/dt = 100 \text{ cm}^3/\text{s}$; find dr/dt when $r = 5$: $dV/dt = 4 \pi r^2 dr/dt \rightarrow dr/dt = 100/(100 \pi) = 1/\pi \text{ cm/s}$.

EXAM POINTER: The number-one error is substituting the instant's values before differentiating - differentiate first, substitute after.

Lecture 19: Linear Approximations and Differentials

Key terms: linearisation, differentials, error estimation

Near $x = a$, $f(x) \approx f(a) + f'(a)(x - a)$ - the tangent line approximates the curve. Differentials: $dy = f'(x) dx$ estimates the change in y for a small change dx . This gives quick estimates ($\sqrt{4.02} \approx 2 + (1/4)(0.02) = 2.005$) and error propagation formulas used in science and engineering.

- Approximation improves as x gets closer to a ; error depends on f'' (curvature).
- Relative error in y approx $|f'(x)/f(x)|$ times absolute error in x .
- Example: estimate $(1.03)^5$ via $f(x) = x^5$ at $a = 1$: $1 + 5(0.03) = 1.15$ (true 1.159).

Lecture 20: Maximum and Minimum Values

Key terms: absolute extrema, local extrema, critical numbers, Extreme Value Theorem

An absolute maximum is the largest value of f on its domain; a local maximum is the largest nearby. Critical numbers are interior points where $f'(x) = 0$ or f' is undefined - local extrema can only occur there (or at endpoints). The Extreme Value Theorem: a continuous function on a closed interval $[a, b]$ attains an absolute max and min - find them by evaluating f at critical numbers and endpoints.

- Fermat's theorem: if f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.
- Procedure for $[a, b]$: find critical numbers in (a, b) , evaluate f there and at a, b ; largest/smallest wins.
- Example: $f(x) = x^3 - 3x^2$ on $[-1, 3]$: $f' = 3x(x-2)$, critical 0, 2; values $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, $f(3) = 0$: abs max 0, abs min -4.

EXAM POINTER: Closed-interval absolute extrema problems are mechanical - never forget to evaluate the endpoints.

Lecture 21: The Mean Value Theorem and Its Consequences

Key terms: Rolle's theorem, Mean Value Theorem, increasing/decreasing test

Rolle's theorem: if f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then $f'(c) = 0$ for some c in (a, b) . The Mean Value Theorem generalises: some c in (a, b) has $f'(c) = [f(b) - f(a)]/(b - a)$ - the tangent is parallel to the secant. Consequences: if $f' > 0$ on an interval, f is increasing there; if $f' = 0$ everywhere, f is constant.

- MVT justifies speed-limit logic: average speed 120 km/h implies instantaneous speed 120 km/h at some moment.
- First-derivative test for local extrema follows from the increasing/decreasing test.
- Hypotheses matter: continuity on the closed interval, differentiability on the open interval.

EXAM POINTER: Be ready to verify the MVT hypotheses for a given function and find the guaranteed point c explicitly.

Lecture 22: First and Second Derivative Tests; Curve Sketching

Key terms: first derivative test, concavity, inflection points, second derivative test

First derivative test: f' changes + to - at c gives a local max; - to + gives a local min. Concavity: $f'' > 0$ means concave up (cup), $f'' < 0$ concave down (cap). Inflection points are where concavity changes ($f'' = 0$ or undefined, with a sign change). Second derivative test: $f'(c) = 0$ and $f''(c) > 0$ implies a local min; $f''(c) < 0$ a local max; $f''(c) = 0$ inconclusive.

- Sketching checklist: domain, intercepts, asymptotes, intervals of increase/decrease, local extrema, concavity, inflection points.
- Example: $f(x) = x^3 - 3x$: $f' = 3(x^2 - 1)$, local max at $x = -1$, local min at $x = 1$; $f'' = 6x$, inflection at 0.

EXAM POINTER: Full curve-sketching questions combine everything - organise your answer under the checklist headings for full marks.

Lecture 23: Optimization Problems

Key terms: optimisation, objective function, constraint

Strategy: (1) understand and draw, (2) write the quantity to maximise/minimise as a function of one variable using the constraint to eliminate others, (3) find the domain from physical sense, (4) use the closed-interval or first/second derivative test to identify the optimum, (5) answer the question asked with units.

- Classic types: fence/enclosure area, box volume, closest point, minimum cost/surface area, maximum profit.
- If the domain is an open interval, use the first or second derivative test rather than endpoint comparison.
- Example: rectangle of perimeter 100 with max area: $A = x(50 - x)$, $A' = 50 - 2x = 0 \rightarrow x = 25$, a 25x25 square, $A = 625$.

EXAM POINTER: Always verify it is a maximum (not minimum) with a derivative test, and state the final answer in the problem's words.

Lecture 24: Antiderivatives and the Indefinite Integral

Key terms: antiderivative, indefinite integral, constant of integration

F is an antiderivative of f if $F' = f$. Antiderivatives differ by a constant, so the indefinite integral is $\int f(x) dx = F(x) + C$. Basic formulas: $\int x^n dx = x^{(n+1)}/(n+1) + C$ ($n \neq -1$); $\int 1/x dx = \ln|x| + C$; $\int e^x dx = e^x + C$; $\int \sin x dx = -\cos x + C$; $\int \cos x dx = \sin x + C$; $\int \sec^2 x dx = \tan x + C$.

- The $+ C$ is not optional - an indefinite integral without it is marked wrong.

- Initial-value problems: use $f(x_0) = y_0$ to solve for C .
- Example: $f'(x) = 3x^2 + 2$, $f(1) = 5$: $f(x) = x^3 + 2x + C$, $1 + 2 + C = 5 \rightarrow C = 2$.

Lecture 25: The Definite Integral and Area

Key terms: definite integral, Riemann sums, area under curve

The definite integral $\int_a^b f(x) dx$ is the limit of Riemann sums: partition $[a, b]$ into n subintervals, $\sum f(x_i^*) \Delta x$, let $n \rightarrow \infty$. For $f \geq 0$ it equals the area under the curve. Properties: linearity, additivity over adjacent intervals, reversal flips the sign, and $\int_a^a f(x) dx = 0$. The integrand's units times dx 's units give the integral's units.

- Left, right and midpoint sums approximate; more subintervals give better accuracy.
- If f is odd and limits symmetric: $\int_{-a}^a f(x) dx = 0$. If f is even: twice $\int_0^a f(x) dx$.
- Example: $\int_0^2 x^2 dx$ via geometry-free evaluation comes in the next lecture.

Lecture 26: The Fundamental Theorem of Calculus

Key terms: FTC Part 1, FTC Part 2, differentiation of integrals

FTC Part 1: if $g(x) = \int_a^x f(t) dt$ with f continuous, then $g'(x) = f(x)$ - differentiation undoes integration (use the chain rule when the upper limit is a function of x). FTC Part 2: $\int_a^b f(x) dx = F(b) - F(a)$ where F is any antiderivative - this is how definite integrals are actually computed.

- $\frac{d}{dx} \int_1^{x^2} \sin t dt = \sin(x^2) * 2x$ (chain rule!).
- Example: $\int_0^2 x^2 dx = [x^3/3]_0^2 = 8/3$.
- Net signed area: parts below the x -axis count negatively.

EXAM POINTER: FTC Part 1 with a non- x upper limit is an exam favourite - remember to multiply by the derivative of the limit.

Lecture 27: Integration by Substitution

Key terms: u-substitution, change of variables

Substitution reverses the chain rule: for $\int f(g(x)) g'(x) dx$, set $u = g(x)$, $du = g'(x) dx$, giving $\int f(u) du$. For definite integrals, change the limits to u -values (or substitute back before evaluating). Choose u as the 'inside' function whose derivative also appears.

- If the derivative of your u -choice is missing, try another u or another method.
- Definite example: $\int_0^1 2x(x^2+1)^3 dx$; $u = x^2+1$, $du = 2x dx$; limits $1 \rightarrow 2$: $\int_1^2 u^3 du = 15/4$.
- Trigonometric integrals like $\int \sin^3 x \cos x dx$ fall to $u = \sin x$.

Lecture 28: Areas Between Curves

Key terms: area between curves, top minus bottom, intersection points

The area between $y = f(x)$ (top) and $y = g(x)$ (bottom) from a to b is $\int_a^b [f(x) - g(x)] dx$. Find intersection points to get the limits; if the curves cross, split the integral at each crossing so the integrand stays nonnegative. When functions of y are easier, integrate with respect to y : $\int_c^d [\text{right} - \text{left}] dy$.

- Sketch first - the picture tells you which function is on top and where they cross.
- Example: $y = x^2$ and $y = x$ cross at 0 and 1; area = $\int_0^1 (x - x^2) dx = 1/6$.
- Total area enclosed always uses absolute value of each piece.

Lecture 29: Volumes by Slicing: Disks and Washers

Key terms: disk method, washer method, volumes of revolution

Revolve a region about an axis to get a solid of revolution. Disk method: $V = \int \pi [R(x)]^2 dx$ where R is the radius from curve to axis. Washer method (region with a hole): $V = \int \pi ([R_{\text{outer}}]^2 - [R_{\text{inner}}]^2) dx$. Revolve around the y -axis using dy , or use cylindrical shells (next lecture).

- Radius = (curve) - (axis of rotation); be careful with axes like $y = -1$ or $x = 2$.
- Example: $y = \sqrt{x}$, $0 \leq x \leq 4$, about x -axis: $V = \int_0^4 \pi x dx = 8\pi$.
- Washer example: region between $y = x$ and $y = x^2$ about x -axis: $V = \pi \int_0^1 (x^2 - x^4) dx = 2\pi/15$.

Lecture 30: Volumes by Cylindrical Shells

Key terms: shell method, radius times height

The shell method is ideal when revolving around a vertical axis using dx (or horizontal axis using dy): $V = \int 2\pi (\text{radius})(\text{height}) dx$. Radius is the distance from the slice to the axis; height is the vertical extent of the region. Choose disks/washers when the slice is perpendicular to the axis, shells when it is parallel.

- Example: region under $y = x^2$, $0 \leq x \leq 2$, about y -axis: $V = \int_0^2 2\pi x \cdot x^2 dx = 8\pi$.
- For rotation about $x = -1$, radius = $x + 1$.
- Guideline: if integrating the 'natural' variable gives messy inverse functions, try shells.

Lecture 31: Exponential Growth, Logarithms and L'Hopital's Rule

Key terms: L'Hopital's rule, indeterminate forms 0/0 and inf/inf

L'Hopital's rule: if $\lim f(x)/g(x)$ is $0/0$ or ∞/∞ , then it equals $\lim f'(x)/g'(x)$ provided the latter exists. Differentiate top and bottom separately (not the quotient rule!). Other indeterminate forms - $0 \cdot \infty$, $\infty - \infty$, 0^0 , 1^∞ , ∞^0 - are first converted to $0/0$ or ∞/∞ , often via logarithms.

- Check the form is indeterminate BEFORE applying - the commonest mistake.
- Example: $\lim_{x \rightarrow 0} (e^x - 1)/x = \lim e^x/1 = 1$.
- Example: $\lim_{x \rightarrow \infty} \ln x / x = \lim (1/x)/1 = 0$ - logs grow slower than powers.

EXAM POINTER: Rewrite 1^∞ forms as $e^{\{\dots\}}$ using $\ln$: $\lim (1 + 1/x)^x = e$ is the model solution.

Lecture 32: Sequences

Key terms: sequence, convergence, divergence, monotonic, bounded

A sequence $\{a_n\}$ is an ordered list of numbers. It converges to L if $a_n \rightarrow L$ as $n \rightarrow \infty$; otherwise it diverges (to ∞ , $-\infty$, or by oscillation). A convergent sequence must be bounded; a bounded monotone sequence always converges. Squeeze theorem for sequences and L'Hopital-style limit evaluation (treat n as continuous x) are the main tools.

- $a_n = (1 + 1/n)^n \rightarrow e$; $a_n = n^{1/n} \rightarrow 1$; $a_n = (-1)^n$ diverges by oscillation.
- Recursive sequences: find fixed points of the recursion for the limit, then prove convergence.
- Example: $a_n = (2n+1)/(3n-2) \rightarrow 2/3$ (divide by n).

Lecture 33: Infinite Series: Convergence Tests I

Key terms: infinite series, partial sums, geometric series, p-series, divergence test

The series $\sum a_n$ converges if its partial sums S_N approach a finite limit. Geometric series $\sum ar^n$ converges iff $|r| < 1$, to $a/(1-r)$. The divergence (nth-term) test: if a_n does not tend to 0, the series diverges - but $a_n \rightarrow 0$ does NOT guarantee convergence (harmonic series diverges). The p-series $\sum 1/n^p$ converges iff $p > 1$.

- Harmonic series $\sum 1/n$ diverges (integral test); $\sum 1/n^2 = \pi^2/6$ converges.
- Integral test: $\sum f(n)$ and $\int_1^\infty f(x) dx$ converge or diverge together (f positive decreasing).
- Example: $\sum_{n \geq 1} 3(1/2)^n = 3 \cdot (1/2)/(1-1/2) = 3$.

EXAM POINTER: Always run the divergence test first - it is the cheapest way to prove divergence.

Lecture 34: Infinite Series: Convergence Tests II

Key terms: comparison tests, ratio test, root test, alternating series

Direct/limit comparison: compare with a known benchmark (geometric or p-series). Ratio test: $L = \lim |a_{n+1}/a_n|$; $L < 1$ converges absolutely, $L > 1$ diverges, $L = 1$ inconclusive - best for factorials and exponentials. Root test: $L = \lim |a_n|^{1/n}$ - best for nth powers. Alternating series test (Leibniz): for $\sum (-1)^n b_n$ with b_n decreasing to 0, the series converges (conditionally).

- Ratio test: $\sum n!/n^n$ and $\sum n!/10^n$ fall instantly.
- Absolute vs conditional convergence: absolute convergence implies convergence; alternating harmonic converges conditionally.
- Alternating series remainder: $|\text{error}| \leq \text{first omitted term } b_{N+1}$.

EXAM POINTER: Choose the test from the series' shape: factorials $\rightarrow$ ratio; nth powers $\rightarrow$ root; alternating signs $\rightarrow$ Leibniz; rational-like $\rightarrow$ comparison.

Lecture 35: Taylor and Maclaurin Series

Key terms: Taylor series, Maclaurin series, radius of convergence

The Taylor series of f at a is $\sum f^{(n)}(a)/n! (x-a)^n$; at $a = 0$ it is the Maclaurin series. Memorise: $e^x = \sum x^n/n!$, $\sin x = \sum (-1)^n x^{2n+1}/(2n+1)!$, $\cos x = \sum (-1)^n x^{2n}/(2n)!$, $1/(1-x) = \sum x^n (|x| < 1)$, $\ln(1+x) = \sum (-1)^{n+1} x^n/n$. The radius of convergence comes from the ratio test; check endpoints separately for the interval of convergence.

- Substitution trick: $e^{-x^2} = \sum (-1)^n x^{2n}/n!$ - no fresh differentiation needed.
- Taylor polynomials approximate functions; the remainder estimates the error.
- Example: approximate $\sin(0.1)$ by $x - x^3/6$: $0.1 - 0.000167 = 0.099833$.

EXAM POINTER: Deriving a new series by substituting into a memorised one is faster and safer than differentiating from scratch.

Lecture 36: Revision: Midterm Practice Set

Attempt these under timed conditions, then verify each step against the lectures above.

1. Find the domain of $f(x) = \ln((x+1)/(x-2))$. (5 marks)
2. Evaluate $\lim_{x \rightarrow 0} (\tan x - x)/x^3$ using known limits or l'Hopital's rule. (5 marks)
3. Use the limit definition to find $f'(x)$ for $f(x) = \sqrt{x}$. (5 marks)
4. Differentiate $y = (x^2 + 1)^3 \sin(2x)$. (5 marks)

5. A ladder 5 m long slides with its foot moving at 0.4 m/s. How fast is the top descending when the foot is 3 m from the wall? (10 marks)
6. Find and classify all local extrema of $f(x) = x^4 - 4x^3$. (10 marks)

Lecture 37: Revision: Final-Term Practice Set

Comprehensive practice covering integration and series - the heart of the final term.

1. Evaluate $\int (3x^2 + 2x - 1/x) dx$ and $\int_1^4 (\sqrt{x} + 1/x^2) dx$. (10 marks)
2. Find the area enclosed by $y = x^2$ and $y = 2x$. (10 marks)
3. Use the washer method to find the volume when the region bounded by $y = x^2$ and $y = 4$ is revolved about the y-axis. (10 marks)
4. Evaluate $\lim_{x \rightarrow 0^+} x^x$. (5 marks)
5. Test $\sum_{n=1}^{\infty} n^{2/2^n}$ for convergence, naming the test used. (5 marks)
6. Find the Maclaurin series for $f(x) = x e^x$ up to the x^4 term. (10 marks)

End of Handout - Good luck with your exams!