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HANDOUT

MTH5205

Calculus III

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MTH5205

Calculus III

Study Handout

Complete lecture-wise notes for Virtual University students

About This Handout

This handout covers the complete MTH5205 Calculus III course: the calculus of several variables. It builds directly on Calculus I (single-variable differentiation and integration) and Calculus II (sequences, series, and techniques of integration). Each lesson states the key definitions and theorems, works through examples with full steps, and ends with exam pointers. For VU exams, practice is decisive — after reading each lesson, close the handout and reproduce the worked examples from memory.

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Lesson 1: Vectors in Three Dimensions

Learning Objectives

- Represent vectors in $\mathbb{R}^3$ and compute magnitude
- Add, subtract, and scale vectors geometrically and algebraically
- Find dot products and interpret the angle between vectors
- Compute cross products and apply them to area and volume

A vector in three dimensions is an ordered triple $\mathbf{v} = \langle a, b, c \rangle$ with magnitude $|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$ and direction. The standard basis vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$ span $\mathbb{R}^3$, so $\mathbf{v} = a \cdot \mathbf{i} + b \cdot \mathbf{j} + c \cdot \mathbf{k}$. Vector addition and scalar multiplication work componentwise and obey the same algebraic laws as in $\mathbb{R}^2$; geometrically, addition is the parallelogram/triangle rule and scalar multiplication stretches or reverses direction.

The dot product $\mathbf{u} \cdot \mathbf{v} = a_1a_2 + b_1b_2 + c_1c_2 = |\mathbf{u}||\mathbf{v}|\cos \theta$ gives the angle between vectors: $\cos \theta = (\mathbf{u} \cdot \mathbf{v})/(|\mathbf{u}||\mathbf{v}|)$. Orthogonality ($\mathbf{u} \cdot \mathbf{v} = 0$) is the vector form of perpendicularity, and the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$ is $(\mathbf{u} \cdot \mathbf{v})/|\mathbf{v}|$. The cross product $\mathbf{u} \times \mathbf{v}$ is the vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, with magnitude $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin \theta$ — the area of the parallelogram they span. Its components come from the determinant formula, and the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ gives the volume of the parallelepiped they span.

These tools are the language of everything that follows: gradients use dot products, surface normals use cross products, and work/flux integrals are built on them. Memorize: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ (cyclic), and reversing the order flips the sign (anti-commutativity: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$).

Key Terms & Definitions

- Magnitude — $|\mathbf{v}| = \sqrt{a^2+b^2+c^2}$; the length of the vector.
- Dot product — $\mathbf{u} \cdot \mathbf{v} = \sum a_i b_i = |\mathbf{u}||\mathbf{v}|\cos \theta$; measures alignment.
- Cross product — $\mathbf{u} \times \mathbf{v} \perp$ both $\mathbf{u}$ and $\mathbf{v}$; $|\mathbf{u} \times \mathbf{v}|$ = area of parallelogram.
- Scalar triple product — $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ = volume of the parallelepiped spanned by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

Example / Illustration

- $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle -1, 0, 2 \rangle$. $\mathbf{u} \cdot \mathbf{v} = -1 + 0 + 6 = 5$. $|\mathbf{u}| = \sqrt{14}$, $|\mathbf{v}| = \sqrt{5}$, so $\cos \theta = 5/\sqrt{70}$, $\theta \approx 53.3^\circ$. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{vmatrix} = \langle 4, -5, 2 \rangle$. Check: $\langle 4, -5, 2 \rangle \cdot \langle 1, 2, 3 \rangle = 4 - 10 + 6 = 0$ ✓.

Lesson Summary

- Vectors: magnitude, component arithmetic, basis $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
- Dot product → angles, projections, orthogonality.
- Cross product → normals, areas; triple product → volumes.

Exam Pointers — Very Important

- ★ Cross-product determinant computations and the angle formula appear constantly — drill until error-free.

Lesson 2: Lines and Planes in Space

Learning Objectives

- Write vector, parametric, and symmetric equations of a line
- Find equations of planes from a point and normal vector
- Compute distances from points to lines and planes
- Determine intersections of lines and planes

A line through point $P_0 = (x_0, y_0, z_0)$ in direction $v = \langle a, b, c \rangle$ has vector equation $r(t) = r_0 + t \cdot v$, parametric form $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$, and (when $a, b, c \neq 0$) symmetric form $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c$. Two lines may intersect, be parallel, or be skew (non-parallel, non-intersecting — possible only in 3D). The distance from a point to a line uses the cross product: $d = |PQ \times v|/|v|$.

A plane through P_0 with normal vector $n = \langle A, B, C \rangle$ has equation $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$, i.e., $Ax + By + Cz = D$. Given three non-collinear points, the normal is the cross product of two edge vectors. The distance from point (x_1, y_1, z_1) to plane $Ax+By+Cz=D$ is $|Ax_1+By_1+Cz_1-D|/\sqrt{A^2+B^2+C^2}$. The angle between planes is the angle between their normals.

Parallel planes share normals; perpendicular planes have orthogonal normals. A line meets a plane at one point unless the direction is orthogonal to the normal (line parallel to plane) — substitute the parametric line into the plane equation and solve for t .

Key Terms & Definitions

- Symmetric equations — $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c$; eliminate the parameter t .
- Skew lines — Non-parallel lines that never meet; exist only in 3D.
- Normal vector — $n = \langle A, B, C \rangle$ perpendicular to the plane $Ax+By+Cz=D$.
- Point-to-plane distance — $|Ax_1+By_1+Cz_1-D|/\sqrt{A^2+B^2+C^2}$.

Example / Illustration

- Plane through $(1,0,2), (0,1,1), (2,1,0)$: edge vectors $\langle -1,1,-1 \rangle, \langle 1,1,-2 \rangle$; normal = cross = $\langle -1,-3,-2 \rangle$. Equation: $-(x-1) - 3y - 2(z-2) = 0 \rightarrow x + 3y + 2z = 5$.

Lesson Summary

- Line: $r = r_0 + tv$; plane: $n \cdot (r - r_0) = 0$.
- Normal via cross product of two in-plane vectors.
- Distance formulas use cross product (line) and normal (plane).

Exam Pointers — Very Important

★ 'Find the plane through three points' is a standard 5-mark question — cross product, then point-normal form.

Lesson 3: Functions of Several Variables and Limits

Learning Objectives

- Describe domains and graphs of functions $f(x,y)$
- Compute limits of two-variable functions
- Use the two-path test to show a limit does not exist
- Define continuity for multivariable functions

A function $f(x,y)$ assigns a real number to each point (x,y) in its domain — typically a region of the plane. Its graph $z = f(x,y)$ is a surface in 3D: level curves $f(x,y) = c$ (contour lines, like on a topographic map) reveal the shape. The domain may be all of $\mathbb{R}^2$ or restricted (e.g., $f = \sqrt{x^2+y^2-1}$ needs $x^2+y^2 \geq 1$; $f = \ln(xy)$ needs $xy > 0$). Sketching level curves is the fastest way to visualize a surface.

The limit $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ requires f to approach L along every path to (a,b) — far stricter than single-variable limits. The two-path test: find two paths giving different limits, and the limit does not exist. Classic example: $f(x,y) = xy/(x^2+y^2)$ gives 0 along the axes but $1/2$ along $y = x$, so the limit at $(0,0)$ DNE. Along $y = mx$ the limit may depend on m — another way to kill a limit. Proving a limit exists usually needs the squeeze theorem or conversion to polar coordinates (show the expression $\rightarrow L$ as $r \rightarrow 0$ regardless of θ).

Continuity means the limit equals the function value; polynomials, rationals (where defined), roots, trig, exponentials and their compositions are continuous on their domains — so most limits are found by direct substitution. Only suspicious points (typically $(0,0)$ in quotients) need path analysis.

Key Terms & Definitions

- Level curve — The curve $f(x,y) = c$; contour map of the surface.
- Two-path test — Different limits along two paths $\Rightarrow$ the limit does not exist.
- Polar coordinates — $x = r \cos \theta$, $y = r \sin \theta$; limit exists if expression $\rightarrow L$ as $r \rightarrow 0$ for all θ .
- Continuity — $\lim_{(x,y) \rightarrow (a,b)} f = f(a,b)$; elementary functions are continuous on their domains.

Example / Illustration

- $f(x,y) = (x^2-y^2)/(x^2+y^2)$ at $(0,0)$: along $y=0 \rightarrow 1$; along $x=0 \rightarrow -1$. Different $\Rightarrow$ limit DNE.

Lesson Summary

- Graph via level curves; watch domain restrictions.
- Two-path test proves non-existence; polar/squeeze proves existence.
- Continuity $\Rightarrow$ direct substitution works almost everywhere.

Exam Pointers — Very Important

- ★ Two-path limit questions at (0,0) are guaranteed exam material — always test $y = x$ and $y = mx$.

Lesson 4: Partial Derivatives

Learning Objectives

- Compute partial derivatives from the definition and by rules
- Interpret partials as slopes and rates of change
- Find second-order partials and verify Clairaut's theorem
- Use partials in applied rate problems

The partial derivative $f_x = \partial f / \partial x$ treats y as a constant and differentiates with respect to x — it is the slope of the surface in the x -direction, i.e., the rate of change of f when only x varies. Formally $f_x(a,b) = \lim_{h \rightarrow 0} \{ [f(a+h,b) - f(a,b)] / h \}$. Computationally, just apply the ordinary differentiation rules while holding other variables fixed: for $f(x,y) = x^3y^2 + \sin(xy)$, $f_x = 3x^2y^2 + y \cdot \cos(xy)$ and $f_y = 2x^3y + x \cdot \cos(xy)$.

Second-order partials: f_{xx} , f_{yy} , and the mixed partials $f_{xy} = \partial / \partial y (\partial f / \partial x)$, f_{yx} . Clairaut's theorem: if the mixed partials are continuous, $f_{xy} = f_{yx}$ — the order of differentiation doesn't matter. This is both a computational shortcut and a frequent exam check ('verify Clairaut's theorem for f' '). Higher-order and multi-variable ($f(x,y,z)$) partials work identically.

Notation fluency matters: f_x , $\partial f / \partial x$, $D_x f$ all mean the same; subscripts stack for higher orders ($f_{xxy} = \partial^3 f / \partial y \partial x^2$ — differentiate right to left). In applications, partials give marginal effects: in economics $\partial P / \partial x$ is marginal productivity; in physics, partials of a potential give force components.

Key Terms & Definitions

- Partial derivative — $\partial f / \partial x$: differentiate w.r.t. x holding others constant.
- Clairaut's theorem — $f_{xy} = f_{yx}$ when mixed partials are continuous.
- Mixed partials — f_{xy} vs f_{yx} ; order of subscripts is right-to-left for ∂ notation.
- Marginal effect — Applied meaning: rate of change holding other inputs fixed.

Example / Illustration

- $f(x,y) = e^{xy} + x^2y$. $f_x = y \cdot e^{xy} + 2xy$; $f_y = x \cdot e^{xy} + x^2$. $f_{xy} = e^{xy} + xy \cdot e^{xy} + 2x = f_{yx}$ ✓.

Lesson Summary

- Hold other variables constant; use ordinary rules.
- Clairaut: mixed partials equal (continuous case).
- Partial = directional slopes and marginal rates.

Exam Pointers — Very Important

- ★ Compute-all-partial + Clairaut verification is a standard question — watch the chain rule inside mixed partials.

Lesson 5: Chain Rule and Implicit Differentiation

Learning Objectives

- Apply the multivariable chain rule (tree diagrams)
- Differentiate compositions with several intermediate variables
- Perform implicit differentiation via partials
- Solve related-rates problems in several variables

When $z = f(x,y)$ with $x = g(t)$, $y = h(t)$, the chain rule is $dz/dt = (\partial z/\partial x)(dx/dt) + (\partial z/\partial y)(dy/dt)$ — follow every path from z to t in the tree diagram, multiply along each path, and add. With two independent variables, $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$: $\partial z/\partial s = (\partial z/\partial x)(\partial x/\partial s) + (\partial z/\partial y)(\partial y/\partial s)$, and similarly for $\partial z/\partial t$. Draw the tree first; it prevents the most common error (missing a path).

Implicit differentiation becomes one formula: if $F(x,y) = 0$ defines y implicitly, then $dy/dx = -F_x/F_y$. For $F(x,y,z) = 0$ defining z as a function of x,y : $\partial z/\partial x = -F_x/F_z$, $\partial z/\partial y = -F_y/F_z$. This is faster and less error-prone than differentiating term-by-term — a genuine exam time-saver. Example: $x^3 + y^3 - 3xy = 0$ gives $dy/dx = -(3x^2 - 3y)/(3y^2 - 3x) = (y - x^2)/(y^2 - x)$.

Related rates: a quantity depending on several changing variables differentiates via the chain rule — e.g., the volume $V = \pi r^2 h$ of a cylinder with dr/dt , dh/dt known gives $dV/dt = 2\pi r h \cdot dr/dt + \pi r^2 \cdot dh/dt$.

Key Terms & Definitions

- Tree diagram — Maps dependencies; each root-to-leaf path contributes a product term.
- Chain rule — $dz/dt = \sum (\partial z/\partial x_i)(dx_i/dt)$ over all intermediate variables.
- Implicit formula — $dy/dx = -F_x/F_y$; $\partial z/\partial x = -F_x/F_z$ for $F(x,y,z)=0$.
- Related rates — d/dt of a multivariable quantity via the chain rule.

Example / Illustration

- $z = x^2y$, $x = \sin t$, $y = e^t$: $dz/dt = 2xy \cdot \cos t + x^2 \cdot e^t = 2t \cdot (\sin t)(e^t) \cos t \dots$
(substitute): $= 2e^t \sin t \cos t + e^t \sin^2 t$.

Lesson Summary

- Tree diagram $\rightarrow$ multiply along paths, add across paths.
- Implicit: $dy/dx = -F_x/F_y$ (memorize this).
- Related rates are chain-rule applications.

Exam Pointers — Very Important

- ★ The $-F_x/F_y$ formula is high-yield; examiners expect you to know it rather than derive each time.

Lesson 6: Directional Derivatives and the Gradient

Learning Objectives

- Compute directional derivatives via the gradient
- Find the gradient vector and interpret it
- Determine maximum rate of change and its direction
- Find tangent planes using the gradient

The directional derivative $D_u f(a,b)$ is the rate of change of f in the direction of unit vector u : $D_u f = \nabla f \cdot u$, where the gradient $\nabla f = \langle f_x, f_y \rangle$ (in 3D, $\langle f_x, f_y, f_z \rangle$). The gradient packages all first-order information: the maximum rate of increase of f at a point is $|\nabla f|$, attained in the direction of ∇f itself; the maximum decrease is $-|\nabla f|$ in the direction $-\nabla f$; and $D_u f = 0$ in directions orthogonal to ∇f (moving along a level curve).

This has immediate applied meaning: ∇T points toward fastest temperature increase (heat flows along $-\nabla T$); in machine learning, gradient descent follows $-\nabla f$ to minimize a loss. The gradient is always perpendicular to level curves/surfaces — the geometric fact behind tangent planes: the tangent plane to $z = f(x,y)$ at (a,b) is $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, equivalently $F(x,y,z) = 0$ with normal ∇F for implicit surfaces.

Computation: normalize the direction vector first (u must be a unit vector!), then dot with ∇f . Forgetting to normalize is the classic error.

Key Terms & Definitions

- Gradient — $\nabla f = \langle f_x, f_y, f_z \rangle$; points in the direction of steepest increase.
- Directional derivative — $D_u f = \nabla f \cdot u$ (u unit); rate of change in direction u .
- Steepest ascent — $\max D_u f = |\nabla f|$ in direction $\nabla f/|\nabla f|$.
- Tangent plane — $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$; normal $= \nabla F$.

Example / Illustration

- $f(x,y) = x^2 + 3xy - y^2$ at $(1,2)$, direction $v = \langle 3,4 \rangle$. $\nabla f = \langle 2x+3y, 3x-2y \rangle = \langle 8,-1 \rangle$. $u = \langle 3/5, 4/5 \rangle$. $D_u f = 8(3/5) - 1(4/5) = 4$. Max rate = $|\nabla f| = \sqrt{65}$ in direction $\langle 8,-1 \rangle/\sqrt{65}$.

Lesson Summary

- $D_u f = \nabla f \cdot u$; always normalize u .
- ∇f : steepest increase, $\perp$ level curves.
- Tangent plane from gradient of $F(x,y,z) = f(x,y) - z$.

Exam Pointers — Very Important

- ★ Direction MUST be a unit vector — examiners plant non-unit directions as traps.

Lesson 7: Tangent Planes and Linear Approximation

Learning Objectives

- Write equations of tangent planes to surfaces
- Use differentials for linear approximation
- Estimate errors with differentials
- Define differentiability for $f(x,y)$

Zooming in on a smooth surface, it looks like its tangent plane — the basis of linear approximation: $f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ near (a,b) . The total differential $dz = f_x dx + f_y dy$... ($dz = f_x dx + f_y dy$) estimates the actual change Δz when x,y change by small dx, dy . Example: $f(x,y) = \sqrt{x^2+y^2}$ at $(3,4)$: $f = 5$, $f_x = 3/5$, $f_y = 4/5$; $f(3.02, 3.98) \approx 5 + 0.6(0.02) + 0.8(-0.02) = 4.996$.

Differentials also estimate measurement error: if r and h of a cylinder are measured with errors dr, dh , the volume error is $\approx dV = 2\pi r h \cdot dr + \pi r^2 \cdot dh$, and relative error $dV/V \approx 2dr/r + dh/h$ — a practical formula worth memorizing. Differentiability (stronger than mere existence of partials) requires the linear approximation error to vanish faster than the distance; continuous partials guarantee differentiability — which is why 'fx, fy continuous $\Rightarrow$ differentiable $\Rightarrow$ continuous' is the key theorem chain.

The tangent plane to a level surface $F(x,y,z) = k$ at P_0 has normal $\nabla F(P_0)$: equation $\nabla F(P_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$. The normal line is $r(t) = r_0 + t\nabla F(P_0)$.

Key Terms & Definitions

- Linear approximation — $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.
- Total differential — $dz = f_x dx + f_y dy$; estimates Δz and measurement error.
- Differentiability — Linear approximation valid; implied by continuous partials.
- Normal line — Parallel to $\nabla F(P_0)$; perpendicular to the tangent plane.

Example / Illustration

- $V = \pi r^2 h$, $r=10 \pm 0.1$, $h=20 \pm 0.2$: $dV/V \approx 2(0.1/10) + 0.2/20 = 0.03 \rightarrow \sim 3\%$ relative error.

Lesson Summary

- Tangent plane = graph of the linearization.
- dz estimates small changes and propagated error.
- Continuous partials $\Rightarrow$ differentiable $\Rightarrow$ continuous.

Exam Pointers — Very Important

- ★ Error-propagation via differentials is a favourite applied question — learn dV/V style shortcuts.

Lesson 8: Maxima and Minima of Functions of Two Variables

Learning Objectives

- Find critical points by solving $f_x = f_y = 0$
- Apply the second-derivative (D) test
- Classify saddles, local max, local min
- Solve applied optimization problems

Local extrema of $f(x,y)$ occur at critical points where $f_x = f_y = 0$ (or where partials don't exist). The second-derivative test uses $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ at the critical point: $D > 0$ and $f_{xx} > 0 \rightarrow$ local min; $D > 0$ and $f_{xx} < 0 \rightarrow$ local max; $D < 0 \rightarrow$ saddle point; $D = 0 \rightarrow$ inconclusive (need other methods). Geometrically, $D > 0$ means the surface curves the same way in all directions (bowl up or down); $D < 0$ means it curves up in one direction and down in another — the saddle.

Absolute extrema on a closed bounded region: evaluate f at interior critical points AND on the boundary (parametrize each boundary piece, reducing to single-variable calculus), then take the largest/smallest values. Skipping the boundary is the classic error — the absolute max often sits on the edge. Applied problems (maximize volume, minimize cost/surface area) follow: define the function, find critical points, verify with D or boundary analysis.

Example: $f(x,y) = x^3 + y^3 - 3xy$. $f_x = 3x^2 - 3y = 0$, $f_y = 3y^2 - 3x = 0 \rightarrow$ critical points $(0,0)$, $(1,1)$. $D = 36xy - 9$: at $(1,1)$, $D = 27 > 0$, $f_{xx} = 6 > 0 \rightarrow$ local min $f = -1$; at $(0,0)$, $D = -9 < 0 \rightarrow$ saddle.

Key Terms & Definitions

- Critical point — $f_x = f_y = 0$ (or partials undefined); candidate for extrema.
- D test — $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$; sign pattern classifies the point.
- Saddle point — $D < 0$; min in one direction, max in another.
- Closed region method — Check interior critical points AND the boundary.

Lesson Summary

- Solve $f_x = f_y = 0$; classify with D.
- Absolute extrema need the boundary too.
- Set up applied problems as unconstrained optimization first.

Exam Pointers — Very Important

- ★ D-test classification with full computation is standard; also practice boundary parametrization for absolute extrema.

Lesson 9: Lagrange Multipliers (Constrained Optimization)

Learning Objectives

- Set up the Lagrange equations $\nabla f = \lambda \nabla g$
- Solve constrained optimization problems
- Handle two constraints with two multipliers
- Interpret λ economically/geometrically

To optimize $f(x,y)$ subject to $g(x,y) = 0$, introduce a multiplier λ and solve $\nabla f = \lambda \nabla g$ together with $g = 0$. The geometry: at the optimum, the level curves of f and g are tangent — their gradients are parallel. This converts a constrained problem into a system of equations. Example: maximize $f(x,y) = xy$ subject to $x^2 + y^2 = 1$. $\nabla f = \langle y, x \rangle = \lambda \langle 2x, 2y \rangle$ gives $y = 2\lambda x$, $x = 2\lambda y \rightarrow x = \pm y$; with the constraint, optima at $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$; $\max f = 1/2$.

With two constraints $g = 0$, $h = 0$ in 3D: $\nabla f = \lambda \nabla g + \mu \nabla h$. The method extends to any number of variables. Always verify the constraint qualification ($\nabla g \neq 0$ at the solution) and check boundary/end behavior when the feasible set isn't closed — Lagrange finds interior critical points of the constraint manifold, not endpoints at infinity.

Applications: utility maximization subject to a budget (λ = marginal utility of money), minimizing cost subject to output quota, maximizing volume with fixed surface area. The multiplier λ itself has meaning: $\partial(\text{optimal value})/\partial(\text{constraint constant})$ — the shadow price of the constraint.

Key Terms & Definitions

- Lagrange multiplier — λ in $\nabla f = \lambda \nabla g$; enforces tangency of level sets.
- Constraint qualification — Need $\nabla g \neq 0$ at candidate points.
- Shadow price — λ = rate of change of the optimum with the constraint level.
- Two constraints — $\nabla f = \lambda \nabla g + \mu \nabla h$ in three variables.

Example / Illustration

- Minimize $f = x^2 + y^2 + z^2$ subject to $x + y + z = 1$: $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \langle 1, 1, 1 \rangle \rightarrow x = y = z = \lambda/2$; constraint $\rightarrow 3\lambda/2 = 1 \rightarrow x = y = z = 1/3$; $\min f = 1/3$.

Lesson Summary

- $\nabla f = \lambda \nabla g$ plus the constraint; solve the system.
- Optimum where level curves are tangent.
- λ measures the constraint's marginal value.

Exam Pointers — Very Important

- ★ Budget/utility and box-volume Lagrange problems are exam staples — set up $\nabla f = \lambda \nabla g$ cleanly.

Lesson 10: Double Integrals over Rectangles

Learning Objectives

- Define the double integral as a limit of Riemann sums
- Evaluate iterated integrals over rectangles
- Apply Fubini's theorem
- Compute volumes under surfaces

The double integral $\iint_R f(x,y) \, dA$ over a rectangle R is the limit of Riemann sums $\sum f(x_i^*, y_j^*) \Delta A$ — the volume under $z = f(x,y)$ above R (for $f \geq 0$). Fubini's theorem: if f is continuous on the rectangle, the double integral equals either iterated integral: $\int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$. Compute inside-out: integrate with respect to the inner variable treating the outer as constant.

Example: $\iint_R (x^2 + y) \, dA$, $R = [0,2] \times [1,3]$. Inner: $\int_1^3 (x^2 + y) \, dy = [x^2 y + y^2/2]_1^3 = 3x^2 + 9/2 - (x^2 + 1/2) = 2x^2 + 4$. Outer: $\int_0^2 (2x^2 + 4) \, dx = [2x^3/3 + 4x]_0^2 = 16/3 + 8 = 40/3$. Properties mirror single integrals: linearity, additivity over regions, and $\iint_R 1 \, dA = \text{area}(R)$.

When $f(x,y) = g(x)h(y)$ separates, the double integral factors: $(\int g)(\int h)$ — a useful shortcut. And $\iint_R f \, dA$ with $f \leq 0$ gives negative (signed) volume, exactly as in one variable.

Key Terms & Definitions

- Double integral — Limit of Riemann sums; signed volume under the surface.
- Iterated integral — Compute inside-out, holding the outer variable fixed.
- Fubini's theorem — Continuous $f \Rightarrow$ both iteration orders agree.
- Separable integrand — $f = g(x)h(y) \Rightarrow$ integral factors into a product.

Example / Illustration

- $\iint_{[0,1] \times [0,2]} xy \, dA = (\int_0^1 x \, dx)(\int_0^2 y \, dy) = (1/2)(2) = 1.$

Lesson Summary

- $\iint_R f \, dA =$ iterated integrals (Fubini).
- Inside-out computation; linearity and additivity hold.
- Separable integrands factor.

Exam Pointers — Very Important

- ★ Iterated-integral computations are mechanical marks — be careful with inner/outer variable bookkeeping.

Lesson 11: Double Integrals over General Regions

Learning Objectives

- Describe Type I and Type II regions
- Set up limits for non-rectangular regions
- Change the order of integration
- Find areas and volumes via double integrals

For general regions, classify as Type I (y between functions of x): $D = \{(x,y): a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, giving $\iint_D f \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f \, dy \, dx$; or Type II (x between functions of y): $D = \{(x,y): h_1(y) \leq x \leq h_2(y), c \leq y \leq d\}$, giving $\int_c^d \int_{h_1(y)}^{h_2(y)} f \, dx \, dy$. The skill is reading the region: sketch it, decide which description is simpler, and set the limits so the inner integral's bounds may depend on the outer variable but never vice versa.

Changing the order of integration: sketch the region from the given limits, then re-describe it the other way. Example: $\int_0^1 \int_{x^2}^1 e^{y^2} \, dy \, dx$ — the inner integral is non-elementary, but reversing gives $\int_0^1 \int_0^y e^{y^2} \, dx \, dy = \int_0^1 y \cdot e^{y^2} \, dy = (e-1)/2$. This trick is an exam favourite: if the inner integral looks impossible, reverse the order.

Applications: area = $\iint_D 1 \, dA$; volume under $z = f(x,y)$ over D ; average value = $(1/\text{area}(D)) \iint_D f \, dA$; mass of a lamina with density $\rho(x,y)$: $m = \iint_D \rho \, dA$.

Key Terms & Definitions

- Type I region — y-simple: y between two functions of x.
- Type II region — x-simple: x between two functions of y.
- Reversing order — Sketch the region, re-describe; rescues non-elementary inner integrals.
- Area/volume — $\iint_D 1 \, dA = \text{area}$; $\iint_D f \, dA = \text{volume under } f$.

Example / Illustration

- D bounded by $y = x^2$ and $y = x$: Type I: $0 \leq x \leq 1$, $x^2 \leq y \leq x$. Area = $\int_0^1 (x - x^2) \, dx = 1/6$.

Lesson Summary

- Sketch first; choose Type I or II.
- Inner limits may use the outer variable only.
- Reverse the order when the inner integral is impossible.

Exam Pointers — Very Important

- ★ Order-reversal problems are high-value — practice sketching from limits quickly.

Lesson 12: Double Integrals in Polar Coordinates

Learning Objectives

- Convert between rectangular and polar coordinates
- Explain the $r \, dr \, d\theta$ area element
- Set up polar integrals over circular regions
- Evaluate integrals using polar symmetry

When the region or integrand involves $x^2 + y^2$, switch to polar: $x = r \cos \theta$, $y = r \sin \theta$, $x^2 + y^2 = r^2$, and crucially $dA = r \, dr \, d\theta$ — the extra r is the Jacobian, accounting for how area stretches. Forgetting the r is the single most common error. The region description becomes r -simple: e.g., the disk $x^2 + y^2 \leq a^2$ is $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$; the cardioid $r = 1 + \cos \theta$ is $0 \leq r \leq 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$.

The classic application: $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} \, dA$. In polar: $\int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta = 2\pi \cdot [-e^{-r^2}/2]_0^\infty = \pi$ — which famously proves $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$. Whenever you see $x^2 + y^2$ in an integrand or circles in the region, reach for polar coordinates.

Limits in polar need care: a circle $x^2 + y^2 = 2x$ becomes $r = 2 \cos \theta$; the region inside it and above $y = 0$ is $0 \leq r \leq 2 \cos \theta$, $0 \leq \theta \leq \pi/2$. Always sketch or convert the boundary equations.

Key Terms & Definitions

- Polar conversion — $x = r \cos \theta$, $y = r \sin \theta$; $dA = r \, dr \, d\theta$.
- Jacobian r — The area element stretches by r — never omit it.
- r -simple region — r between functions of θ ; θ between constants.
- Gaussian integral — $\iint e^{-r^2} dA = \pi \Rightarrow \int e^{-x^2} dx = \sqrt{\pi}$.

Example / Illustration

- $\iint_D (x^2 + y^2) dA$, $D: x^2 + y^2 \leq 4$: $= \int_0^{2\pi} \int_0^2 r^2 \cdot r \, dr \, d\theta = 2\pi \cdot [r^4/4]_0^2 = 8\pi$.

Lesson Summary

- $x^2 + y^2$ or circular regions $\Rightarrow$ polar.
- $dA = r \, dr \, d\theta$ (the r is mandatory).
- Convert boundary curves to $r = f(\theta)$.

Exam Pointers — Very Important

- ★ Polar setup + the Jacobian r is tested every session — write $dA = r \, dr \, d\theta$ explicitly.

Lesson 13: Triple Integrals

Learning Objectives

- Define the triple integral over a box and general solids
- Set up iterated integrals in x , y , z
- Compute volumes of solids
- Find mass, center of mass with density

The triple integral $\iiint_E f(x,y,z) \, dV$ extends the same Riemann-sum idea to three dimensions: for $f = 1$ it gives volume; for $f = \text{density } \delta(x,y,z)$ it gives mass. Over a box, Fubini gives six equivalent iterated orders. Over a general solid, describe E as z -simple: $E = \{(x,y,z): (x,y) \in D, u_1(x,y) \leq z \leq u_2(x,y)\}$, then $\iiint_E f \, dV = \iint_D \int_{u_1}^{u_2} f \, dz \, dA$ — reduce to a double integral over the shadow D in the xy -plane.

Example: E under $z = 4 - x^2 - y^2$ above the xy -plane. Shadow $D: x^2 + y^2 \leq 4$. Volume $= \iint_D (4 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta = 2\pi [2r^2 - r^4/4]_0^2 = 8\pi$. Choosing the projection plane (xy , yz , or xz) to match the solid's description is the key setup skill.

Applications: mass $m = \iiint_E \delta \, dV$; center of mass $(\bar{x}, \bar{y}, \bar{z}) = (1/m)(\iiint x\delta \, dV, \iiint y\delta \, dV, \iiint z\delta \, dV)$; moments of inertia e.g. $I_z = \iiint (x^2 + y^2)\delta \, dV$. With constant density these reduce to volume and geometric centroid.

Key Terms & Definitions

- Triple integral — $\iiint_E f \, dV$; volume ($f=1$), mass ($f=\text{density}$).
- z-simple solid — z between surfaces over a shadow region D .
- Shadow region — Projection of E onto a coordinate plane.
- Center of mass — $(1/m) \iiint (x,y,z) \delta \, dV$.

Example / Illustration

- $E: 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq xy$. $\iiint_E z \, dV = \int_0^1 \int_0^1 \int_0^{xy} z \, dz \, dy \, dx = \int_0^1 \int_0^1 x^2 y^2 / 2 \, dy \, dx = 1/18$.

Lesson Summary

- Describe E simply (often z -simple over a shadow D).
- Iterate: inner dz , then double integral over D .
- Volume/mass/center of mass are the standard applications.

Exam Pointers — Very Important

- ★ Setting up the solid description and limits is where marks are won or lost — sketch always.

Lesson 14: Cylindrical and Spherical Coordinates

Learning Objectives

- Convert among rectangular, cylindrical, spherical
- Use $dV = r \, dz \, dr \, d\theta$ and $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
- Choose the right system for a solid
- Evaluate triple integrals in both systems

Cylindrical coordinates (r, θ, z) : $x = r \cos \theta$, $y = r \sin \theta$, $z = z$ — polar in the base, height unchanged. Volume element $dV = r \, dz \, dr \, d\theta$. Use for solids with cylindrical symmetry: cylinders, paraboloids $z = x^2 + y^2$ ($z = r^2$), cones. Spherical coordinates (ρ, θ, ϕ) : $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, where $\rho \geq 0$ is distance from origin, θ the azimuth (xy-plane angle), ϕ the angle from the positive z -axis ($0 \leq \phi \leq \pi$). Volume element $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$ — memorize both Jacobians.

Use spherical for balls, spheres, cones through the origin ($\phi = \text{constant}$ is a cone!), and integrands with $x^2 + y^2 + z^2 = \rho^2$. Example: volume of the ball $\rho \leq a$: $\int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot 2 \cdot a^3 / 3 = 4\pi a^3 / 3$ ✓. A solid inside the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$: ϕ ranges 0 to $\pi/4$ (cone is $\phi = \pi/4$), ρ 0 to 2.

Choosing: cylindrical symmetry about z -axis $\rightarrow$ cylindrical; spherical symmetry or cones/spheres $\rightarrow$ spherical. Convert boundary surfaces first (sphere $\rightarrow \rho = a$; cone $z = \sqrt{x^2 + y^2} \rightarrow \phi = \pi/4$; paraboloid $\rightarrow z = r^2$), then read off limits.

Key Terms & Definitions

- Cylindrical — (r, θ, z) ; $dV = r \, dz \, dr \, d\theta$.
- Spherical — (ρ, θ, ϕ) ; $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$; $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.
- ϕ convention — Angle from +z-axis, $0 \leq \phi \leq \pi$; $\phi = \text{const}$ is a cone.
- System choice — Cylinders/paraboloids $\rightarrow$ cylindrical; balls/cones $\rightarrow$ spherical.

Example / Illustration

- $\iiint_E z \, dV$, $E: x^2 + y^2 + z^2 \leq 1, z \geq 0$ (upper half-ball). Spherical: $\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot (1/2) \cdot (1/4) = \pi/4$.

Lesson Summary

- Two Jacobians: r (cylindrical), $\rho^2 \sin \phi$ (spherical).
- Convert surfaces, then read limits.
- Match the system to the symmetry.

Exam Pointers — Very Important

- ★ Spherical setup with correct ϕ limits for cones is a classic hard question — draw the ϕ angle.

Lesson 15: Vector Fields, Line Integrals of Scalar Functions

Learning Objectives

- Define vector fields and sketch simple ones
- Parametrize curves for line integrals
- Evaluate $\int_C f \, ds$ (scalar line integral)
- Compute arc length via line integrals

A vector field $F(x, y) = \langle P(x, y), Q(x, y) \rangle$ assigns a vector to each point — velocity fields, force fields, gradients. Sketch by sampling: $F(x, y) = \langle -y, x \rangle$ rotates counterclockwise (draw arrows tangent to circles). Conservative fields are gradients $F = \nabla f$ of some potential f ; they will matter greatly soon. A curve C is parametrized as $r(t)$, $a \leq t \leq b$; then $ds = |r'(t)|dt$ is the arc-length element.

The scalar line integral $\int_C f(x, y) \, ds = \int_a^b f(r(t)) |r'(t)| \, dt$ integrates a scalar function along the curve — e.g., mass of a wire with density f , or average temperature along a path. It is independent of parametrization direction. Computation: substitute the parametrization, multiply by the speed $|r'(t)|$, integrate in t . Example: $\int_C (x^2 + y^2) \, ds$, C the unit circle $r(t) = \langle \cos t, \sin t \rangle$: integrand = 1, $|r'| = 1$, integral = $\int_0^{2\pi} 1 \, dt = 2\pi$.

Arc length itself is $\int_C 1 \, ds$. Line integrals also give surface area of a 'curtain' $z = f(x, y)$ above C : $\int_C f \, ds$.

Key Terms & Definitions

- Vector field — $F(x,y) = \langle P,Q \rangle$; a vector at every point.
- Parametrization — $r(t)$, $a \leq t \leq b$; traces the curve.
- Scalar line integral — $\int_C f \, ds = \int_a^b f(r(t))|r'(t)|dt$; mass of a wire.
- ds element — $ds = |r'(t)|dt$; the speed factor.

Example / Illustration

- Wire $y = x^2$, $0 \leq x \leq 1$, density $\delta = x$. $r(t) = \langle t, t^2 \rangle$, $|r'| = \sqrt{1+4t^2}$. Mass = $\int_0^1 t\sqrt{1+4t^2} \, dt = [(1+4t^2)^{3/2}/12]_0^1 = (5\sqrt{5}-1)/12$.

Lesson Summary

- Parametrize, substitute, multiply by $|r'(t)|$, integrate.
- Scalar line integrals: mass, averages, curtain area.
- Independent of orientation.

Exam Pointers — Very Important

- ★ Distinguish $\int_C f \, ds$ (scalar, needs $|r'|$) from $\int_C F \cdot dr$ (vector, no $|r'|$) — mixing them up is the top error.

Lesson 16: Line Integrals of Vector Fields (Work)

Learning Objectives

- Evaluate $\int_C F \cdot dr$ (work integral)
- Relate work to circulation
- Use the $F \cdot T$ ds and $F \cdot n$ ds forms
- Decide when work is path-independent

The vector line integral $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) \, dt$ measures work done by force F along C (also circulation when C is closed). Unlike the scalar version, there is no $|r'(t)|$ — instead dot F with the velocity $r'(t)$. Reversing orientation flips the sign: $\int_{-C} F \cdot dr = -\int_C F \cdot dr$. Two alternative forms: $\int_C F \cdot T \, ds$ (tangential component) and $\int_C F \cdot n \, ds$ (flux across C , normal component) in the plane.

Example: $F(x,y) = \langle y, -x \rangle$ (clockwise rotation field), C the unit circle counterclockwise, $r(t) = \langle \cos t, \sin t \rangle$: $F(r) = \langle \sin t, -\cos t \rangle$, $r' = \langle -\sin t, \cos t \rangle$, $\text{dot} = -\sin^2 t - \cos^2 t = -1$; work = $\int_0^{2\pi} -1 \, dt = -2\pi$ (negative: field opposes the motion). For a closed curve, $\oint_C F \cdot dr$ is the circulation — net tendency of F to circulate around C .

Path independence: if $F = \nabla f$ (conservative), then $\int_C F \cdot dr = f(B) - f(A)$ depends only on endpoints (Fundamental Theorem for Line Integrals) — preview of the next lesson's test for conservativeness.

Key Terms & Definitions

- Work integral — $\int_C \mathbf{F} \cdot d\mathbf{r} = \int F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$; no $|\mathbf{r}'|$ factor.
- Orientation — Reversing C negates the integral.
- Circulation — $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around a closed curve.
- Flux across C — $\int_C \mathbf{F} \cdot \mathbf{n} \, ds$; normal component in the plane.

Example / Illustration

- $\mathbf{F} = \langle x, y \rangle$ (radial), C : line from $(0,0)$ to $(1,1)$, $\mathbf{r}(t) = \langle t, t \rangle$: $\mathbf{F} \cdot \mathbf{r}' = \langle t, t \rangle \cdot \langle 1, 1 \rangle = 2t$; $\int_0^1 2t \, dt = 1$. (Also $= f(1,1) - f(0,0)$ with $f = (x^2 + y^2)/2$ ✓.)

Lesson Summary

- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int F(\mathbf{r}) \cdot \mathbf{r}' \, dt$; orientation matters (sign flips).
- Work, circulation, flux interpretations.
- Conservative fields give path independence (next lesson).

Exam Pointers — Very Important

- ★ Work-integral computations with careful $\mathbf{r}(t) \cdot \mathbf{r}'(t)$ algebra are core exam numericals.

Lesson 17: Conservative Fields and Potential Functions

Learning Objectives

- Test whether $\mathbf{F}$ is conservative ($P_y = Q_x$)
- Find potential functions f with $\mathbf{F} = \nabla f$
- Apply the Fundamental Theorem for Line Integrals
- Understand simply-connected domains

A field $\mathbf{F} = \langle P, Q \rangle$ is conservative if $\mathbf{F} = \nabla f$ for some potential f — equivalently, work between two points is path-independent. Test (on an open simply-connected domain): $\mathbf{F}$ is conservative $\iff P_y = Q_x$ (the cross-partial match). This is both necessary and sufficient where the domain has no holes. In 3D: $\mathbf{F} = \langle P, Q, R \rangle$ conservative $\iff \text{curl } \mathbf{F} = \mathbf{0}$ (on simply-connected domains).

Finding f : integrate P with respect to x (treat y constant), then differentiate the result with respect to y and match to Q to determine the 'constant' function of y . Example: $\mathbf{F} = \langle 2xy, x^2 + 3y^2 \rangle$: $P_y = 2x = Q_x$ ✓. $f = \int 2xy \, dx = x^2y + g(y)$; $f_y = x^2 + g'(y) = x^2 + 3y^2 \rightarrow g(y) = y^3$; $f(x,y) = x^2y + y^3$. Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ from A to $B = f(B) - f(A)$ — no parametrization needed!

The hole caveat: $\mathbf{F} = \langle -y/(x^2+y^2), x/(x^2+y^2) \rangle$ has $P_y = Q_x$ but is NOT conservative on $\mathbb{R}^2$ minus the origin (circulation around the unit circle is $2\pi \neq 0$) — the domain isn't simply connected. Examiners love this counterexample.

Key Terms & Definitions

- Conservative field — $F = \nabla f$; path-independent work.
- Test — $P_y = Q_x$ (2D) or $\text{curl } F = 0$ (3D), on simply-connected domains.
- Potential function — f with $\nabla f = F$; found by partial integration.
- FTLI — $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$.

Example / Illustration

- $F = (e^x \cos y, -e^x \sin y)$: $P_y = -e^x \sin y = Q_x \checkmark \Rightarrow$ conservative. $f = e^x \cos y + C$. Work from $(0,0)$ to $(1,\pi)$: $f(1,\pi) - f(0,0) = -e - 1$.

Lesson Summary

- Test $P_y = Q_x$ (mind the domain holes).
- Integrate $P \, dx$, match Q to get $g(y)$.
- FTLI replaces parametrization for conservative fields.

Exam Pointers — Very Important

- ★ 'Show conservative, find f , evaluate the integral via FTLI' is a complete exam question — master the workflow.

Lesson 18: Green's Theorem

Learning Objectives

- State Green's theorem (circulation and flux forms)
- Convert closed line integrals to double integrals
- Compute areas via line integrals
- Choose when Green simplifies a computation

Green's theorem: for positively oriented (counterclockwise) closed C bounding region D , $\oint_C P \, dx + Q \, dy = \iint_D (Q_x - P_y) \, dA$. It converts a line integral around the boundary into a double integral over the interior — often dramatically easier. Example: $\oint_C (x^2 - y) \, dx + (y^2 + x) \, dy$ around the unit circle: $Q_x - P_y = 1 - (-1) = 2$; integral = $\iint_D 2 \, dA = 2\pi$. Direct parametrization would be far messier.

The flux form: $\oint_C F \cdot \mathbf{n} \, ds = \iint_D (P_x + Q_y) \, dA = \iint_D \text{div } F \, dA$ — net outward flow equals the integral of divergence. Area shortcut: $\text{area}(D) = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C (x \, dy - y \, dx)$ — take $P = -y/2$, $Q = x/2$, then $Q_x - P_y = 1$. Requirements: C piecewise smooth, simple, closed, positively oriented; P, Q have continuous partials on D .

When to use: the line integral looks hard but $Q_x - P_y$ simplifies beautifully, or you're asked for area/flux directly. Orientation check: clockwise C introduces a minus sign.

Key Terms & Definitions

- Green's theorem — $\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$ (CCW positive).
- Flux form — $\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div } \mathbf{F} dA$.
- Area via Green — $\text{Area} = \frac{1}{2} \oint (x dy - y dx)$.
- Positive orientation — Counterclockwise; region on the left as you traverse.

Example / Illustration

- $\oint_C y^2 dx + x^2 dy$, C : square $[0,1] \times [0,1]$ CCW. $Q_x - P_y = 2x - 2y$. $\iint = \int_0^1 \int_0^1 (2x - 2y) dy dx = 0$. (Symmetric cancellation — Green reveals it instantly.)

Lesson Summary

- Boundary line integral $\leftrightarrow$ interior double integral.
- Flux form uses divergence.
- Area and circulation shortcuts fall out directly.

Exam Pointers — Very Important

- ★ Green's theorem conversions are extremely high-yield — always check $Q_x - P_y$ for simplification first.

Lesson 19: Surface Integrals and Flux

Learning Objectives

- Parametrize surfaces (especially graphs and spheres)
- Compute the surface element dS
- Evaluate $\iint_S f dS$ (scalar surface integral)
- Evaluate flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$

A surface S parametrized by $\mathbf{r}(u,v)$ has surface element $dS = |\mathbf{r}_u \times \mathbf{r}_v| du dv$. For a graph $z = g(x,y)$ over D : $dS = \sqrt{(g_x)^2 + (g_y)^2 + 1} dA$ — the standard workhorse. The scalar surface integral $\iint_S f dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| du dv$ gives mass of a shell (f = density) or surface area ($f = 1$). Orientation matters only for vector integrals.

Flux $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$ measures net flow of $\mathbf{F}$ through S . For a graph with upward orientation: $\mathbf{F} \cdot \mathbf{n} dS = (-P \cdot g_x - Q \cdot g_y + R) dA$ — no square roots needed! Example: $\mathbf{F} = \langle x, y, z \rangle$ through $z = 4 - x^2 - y^2$ (upper paraboloid), upward: $g_x = -2x$, $g_y = -2y$; integrand $= -x(-2x) - y(-2y) + z = 2x^2 + 2y^2 + (4 - x^2 - y^2) = x^2 + y^2 + 4$; over disk $r \leq 2$: $\int_0^{2\pi} \int_0^2 (r^2 + 4)r dr d\theta = 2\pi [r^4/4 + 2r^2]_0^2 = 24\pi$.

For closed surfaces, orientation is outward by convention. Parametrized surfaces (spheres: $\mathbf{r}(\phi, \theta)$) use $|\mathbf{r}_\phi \times \mathbf{r}_\theta|$ directly — for the sphere of radius a , $dS = a^2 \sin \phi d\phi d\theta$.

Key Terms & Definitions

- Surface element — $dS = |r_u \times r_v| du dv$; for graphs $\sqrt{(g_x^2 + g_y^2 + 1)} dA$.
- Scalar surface integral — $\iint_S f dS$; mass/area of a shell.
- Flux — $\iint_S F \cdot dS$; upward-graph shortcut $(-Pg_x - Qg_y + R)dA$.
- Orientation — Upward/outward by convention; flips sign if reversed.

Example / Illustration

- Area of $z = \sqrt{(x^2 + y^2)}$ (cone), $0 \leq z \leq 1$: $g_x = x/r$, $g_y = y/r$; $\sqrt{(1+1)} = \sqrt{2}$; area = $\iint_{\{r \leq 1\}} \sqrt{2} dA = \sqrt{2} \cdot \pi$.

Lesson Summary

- Graph shortcut $dS = \sqrt{(1 + g_x^2 + g_y^2)} dA$.
- Flux upward: $(-Pg_x - Qg_y + R)dA$.
- Closed surfaces: outward normal.

Exam Pointers — Very Important

- ★ Flux through a paraboloid/hemisphere via the graph shortcut is a standard long question.

Lesson 20: The Divergence Theorem and Stokes' Theorem

Learning Objectives

- State the divergence theorem and apply it
- State Stokes' theorem and apply it
- Choose between direct flux and the divergence theorem
- Relate Green, Stokes, and divergence as one pattern

Divergence theorem: for closed S bounding solid E , $\iint_S F \cdot dS$ (outward) = $\iiint_E \text{div } F dV$, where $\text{div } F = P_x + Q_y + R_z$. It turns a hard flux integral into a volume integral — spectacular when $\text{div } F$ simplifies. Example: $F = \langle x^3, y^3, z^3 \rangle$ over the unit sphere: $\text{div } F = 3(x^2 + y^2 + z^2) = 3\rho^2$; flux = $\int_0^{2\pi} \int_0^\pi \int_0^1 3\rho^2 \cdot \rho^2 \sin \phi d\rho d\phi d\theta = 3 \cdot 2\pi \cdot 2 \cdot (1/5) = 12\pi/5$. Direct surface integration would be brutal.

Stokes' theorem: for oriented surface S with boundary C , $\oint_C F \cdot dr = \iint_S (\text{curl } F) \cdot dS$ — circulation around the edge equals flux of curl through any spanning surface. Choose the simplest spanning surface! Example: circulation of F around the circle $x^2 + y^2 = 1$, $z = 0$: cap it with the flat disk; if $\text{curl } F = \langle 0, 0, 2 \rangle$, circulation = $\iint_{\text{disk}} 2 dA = 2\pi$. Requirements: S piecewise smooth, C compatibly oriented (right-hand rule).

The grand pattern: Green's theorem is Stokes in the plane; the divergence theorem is its 'volume' cousin. All say: integral over boundary = integral of derivative over interior. When an exam asks for flux through a closed surface, compute $\text{div } F$ first — if it's simple,

use the divergence theorem without hesitation.

Key Terms & Definitions

- Divergence theorem — $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$ (closed S , outward).
- Stokes' theorem — $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ (right-hand orientation).
- $\operatorname{div} \mathbf{F} = P_x + Q_y + R_z$; net outflow per unit volume.
- $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$; circulation per unit area; pick the easiest spanning surface.

Example / Illustration

- $\mathbf{F} = \langle x, y, z \rangle$, S unit cube: $\operatorname{div} \mathbf{F} = 3$; flux = $\iiint 3 \, dV = 3 \cdot 1 = 3$. (Direct: 6 faces — much longer.)

Lesson Summary

- Closed-surface flux $\Rightarrow$ try divergence theorem first.
- Boundary circulation $\Rightarrow$ Stokes with the simplest cap.
- Green $\subset$ Stokes; all are boundary-interior dualities.

Exam Pointers — Very Important

- ★ 'Use the divergence theorem' / 'Use Stokes' theorem' questions test setup (div/curl, limits, orientation) more than computation.

Lesson 21: Revision: Multivariable Calculus Drill

Learning Objectives

- Run a timed mixed-topic drill
- Identify your weak spots across chapters
- Recall every major theorem statement verbatim
- Apply the right tool to each problem type

Drill (one per lesson): (1) angle between $\langle 1,2,3 \rangle$, $\langle -1,0,2 \rangle$; (2) plane through three points; (3) two-path limit at origin; (4) all first and mixed partials + Clairaut; (5) $dy/dx = -F_x/F_y$; (6) directional derivative with normalization; (7) linearization and error estimate; (8) D-test classification; (9) Lagrange with one constraint; (10) iterated integral over rectangle; (11) Type I/II setup + order reversal; (12) polar integral with $r \, dr \, d\theta$; (13) triple integral volume; (14) spherical setup for a ball/cone; (15) scalar line integral with $|r'|$; (16) work integral $\mathbf{F} \cdot \mathbf{r}'$; (17) conservative test + potential; (18) Green's theorem conversion; (19) flux via graph shortcut; (20) divergence theorem vs direct flux.

Theorem spine (recite cold): Fubini; Clairaut ($f_{xy} = f_{yx}$); chain rule; $D_u f = \nabla f \cdot \mathbf{u}$; D-test; Lagrange $\nabla f = \lambda \nabla g$; Green $\oint Pdx + Qdy = \iint (Q_x - P_y)$; FTLI; divergence $\iint \mathbf{F} \cdot d\mathbf{S} = \iiint \operatorname{div} \mathbf{F}$; Stokes $\oint \mathbf{F} \cdot d\mathbf{r} = \iint \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. Common traps: forgetting to normalize direction; omitting r in polar / $\rho^2 \sin \phi$ in spherical; scalar vs vector line integrals; orientation signs; skipping boundaries in absolute extrema.

Final advice: in every problem, first name the tool ('this is a divergence-theorem problem'), then set up carefully, then compute. Setup marks are the majority of marks in MTH5205 — a correct setup with an arithmetic slip still passes; a wrong setup with perfect arithmetic does not.

Key Terms & Definitions

- Theorem spine — The ten boxed results above — memorize verbatim.
- Setup first — Name the tool, set up, then compute.
- Trap list — Normalize u ; Jacobians; orientation; boundaries; ds vs dr .
- Timed drill — 20 problems, ~6 minutes each — exam pace.

Lesson Summary

- Twenty-problem drill covering every lesson.
- Recite the theorem spine from memory.
- Setup earns the marks; arithmetic is secondary.

Exam Pointers — Very Important

- ★ If you can state all ten theorems and run this drill cleanly, you are ready for the MTH5205 final.