Now, in order to find the mean, we will apply the limits 0 to ∞ to the expression we have obtained:

$$E(X) = \int_0^{\infty} x.\lambda e^{-\lambda x} dx = \left[-xe^{-\lambda x} \right]_0^{\infty} - \frac{1}{\lambda} \left[e^{-\lambda x} \right]_0^{\infty}$$

$$= -\left[xe^{-\lambda x} \right]_0^{\infty} - \frac{1}{\lambda} \left[\frac{1}{e^{\lambda x}} \right]_0^{\infty}$$

In order to apply the upper limit to the first expression, we need to utilize the L'Hospital Rule.

We have $xe^{-\lambda x} = \frac{x}{e^{\lambda x}}$.

Now, $\underset{x\to\infty}{\text{Lim}} \frac{x}{e^{\lambda x}} = \underset{x\to\infty}{\text{Lim}} \frac{1}{\lambda e^{\lambda x}} = \frac{1}{\lambda e^{\lambda(\infty)}} = \frac{1}{\lambda e^{(\infty)}} = \frac{1}{\lambda(\infty)} = 0.$

On the other hand, $\underset{x\to 0}{\text{Lim}} \frac{x}{e^{\lambda x}} = \frac{0}{e^{\lambda(0)}} = \frac{0}{e^{0}} = \frac{0}{1} = 0.$

Hence,

$$E(X) = -\left[\frac{x}{e^{\lambda x}}\right]_0^{\infty} - \frac{1}{\lambda}\left[\frac{1}{e^{\lambda x}}\right]_0^{\infty} = -(0-0) - \frac{1}{\lambda}\left[\frac{1}{e^{\lambda x}}\right]_0^{\infty}$$

$$= -0 - \frac{1}{\lambda}\left[\frac{1}{\infty} - \frac{1}{e^0}\right] = -\frac{1}{\lambda}\left[\frac{1}{\infty} - \frac{1}{e^0}\right] = -\frac{1}{\lambda}\left[0 - \frac{1}{1}\right]$$

$$= -\frac{1}{\lambda}[-1] = \frac{1}{\lambda}.$$

Thus, $E(X)$ *or* $\mu = \frac{1}{\lambda}$.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 155

Derivation of
the Variance of
the Exponential Distribution

A very interesting property:

The **mean and standard deviation** of the exponential distribution are equal.

We know that, for the exponential distribution given by

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \lambda > 0$$

the mean is

$$E(X) = \frac{1}{\lambda}$$

1. Derivation of the **Variance:**

According to the short-cut formula

$$\mathrm{Var}\left(X\right)=E\left(X^{2}\right)-\left[E\left(X\right)\right]^{2},$$

Now

$$E\left(X^2\right)=\int_0^\infty x^2.f\left(x\right)dx=\int_0^\infty x^2.\lambda e^{-\lambda x}dx$$

First of all we will apply the formula of integration by parts:

$$\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

Here, we let $u = x^2$ and $v = \lambda e^{-\lambda x} = -(-\lambda e^{-\lambda x})$

$$\text{so that } \int vdx = \int -(-\lambda e^{-\lambda x})dx = -\left(e^{-\lambda x}\right)$$

$$\text{and } \frac{du}{dx} = \frac{d}{dx}\left(x^2\right) = 2x.$$

Hence, we have

$$\int x^2.\lambda e^{-\lambda x}dx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

$$= x^2(-e^{-\lambda x}) - \int 2x.\left(-e^{-\lambda x}\right)dx = -x^2e^{-\lambda x} - 2\int\left(-xe^{-\lambda x}\right)dx$$

$$= -x^2e^{-\lambda x} - \frac{2}{\lambda}\int\left(\lambda\left(-xe^{-\lambda x}\right)\right)dx$$

$$= -x^2e^{-\lambda x} - \frac{2}{\lambda}\int x\left(-\lambda e^{-\lambda x}\right)dx$$

Hence, we have

$$\int x^2.\lambda e^{-\lambda x} dx = -x^2 e^{-\lambda x} + \frac{2}{\lambda}\int x\left(\lambda e^{-\lambda x}\right) dx$$

implying that

$$E\left(X^2\right) = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \frac{2}{\lambda} E\left(X\right)$$

or

$$E\left(X^2\right) = -\left[\frac{x^2}{e^{\lambda x}}\right]_0^{\infty} + \frac{2}{\lambda} E\left(X\right)$$

In order to apply the upper limit to $\left[\frac{x^2}{e^{\lambda x}}\right]$, we need to utilize the L'Hospital Rule.

We have $\frac{x^2}{e^{\lambda x}}$

Now, $\lim_{x\to\infty}\frac{x^2}{e^{\lambda x}} = \lim_{x\to\infty}\frac{2x}{\lambda e^{\lambda x}} = \lim_{x\to\infty}\frac{2}{\lambda^2 e^{\lambda x}} = \frac{2}{\lambda^2 e^{\lambda(\infty)}} = \frac{2}{\infty} = 0.$

On the other hand, $\lim_{x\to 0}\frac{x^2}{e^{\lambda x}} = \frac{(0)^2}{e^{\lambda(0)}} = \frac{0}{e^0} = \frac{0}{1} = 0.$

Hence, we have

$$E\left(X^2\right) = -\left[\frac{x^2}{e^{\lambda x}}\right]_0^{\infty} + \frac{2}{\lambda}E\left(X\right)$$

$$= -\left[0-0\right] + \frac{2}{\lambda}E\left(X\right)$$

$$= -0 + \frac{2}{\lambda}E\left(X\right) = \frac{2}{\lambda}E\left(X\right) = \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}$$

Hence,

$$\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{2-1}{\lambda^2} = \frac{1}{\lambda^2}$$

implying that

$$S.D.(X) = \sigma = \frac{1}{\lambda}.$$

Hence we have the interesting property that the **mean and standard deviation** of the exponential distribution are equal.

This, in turn, means that **the Coefficient of Variation of the Exponential distribution is always equal to 1.**

Explanation:

The coefficient of variation is defined as $CV = \frac{\sigma}{\mu}$.

Hence, for the exponential distribution given by

$f(x) = \lambda e^{-\lambda x}, 0 < x < \infty$

the coefficient of variation is

$$CV = \frac{\sigma}{\mu} = \frac{1/\lambda}{1/\lambda} = \frac{\lambda}{\lambda} = 1.$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 156

Application of
the Exponential Distribution
(explained through an example)

Example:

The duration of long-distance telephone calls is found to be exponentially distributed with a mean of 3 minutes. What is the probability that a call will last

(i) more than 3 minutes, (ii) more than 5 minutes?

Let X be the exponential r.v. with parameter θ.

Then the mean, i.e. $\frac{1}{\theta} = 3$, so that $\theta = \frac{1}{3}$.

and the pdf can be written as

$$f(x) = \int_3^{\infty} \left(\frac{1}{\theta} \right) e^{-x/\theta} dx$$

Now (i) the probability that a call will last more than 3 minutes, is given by

$$P(X > 3) = \int_3^{\infty} \left(\frac{1}{3}\right) e^{-x/3} dx = \left[-e^{-x/3}\right]_3^{\infty} = e^{-1} = 0.3679.$$

And (ii) the probability that a call will last more than 5 minutes, is given by

$$P(X>5)=\int_5^{\infty}\left(\frac{1}{3}\right)e^{-x/3}dx=\left[-e^{-x/3}\right]_5^{\infty}=e^{-1.7}=0.1827.$$

We say that, the probability of having call duration longer than 5 minutes its probability is almost half of the probability of getting a call duration greater than three minutes.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 157

MGF of the Exponential Distribution

We know that the probability density function (pdf) of an exponential distribution is

$$f\left(x;\lambda\right)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0,\ \ \lambda>0\\ 0 & x<0.\end{cases}$$

where λ is the only parameter of the distribution.

So to find the MGF, we proceed as follows:

$$M_0(t) = E\left[e^{tX}\right] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{-(\lambda-t)x}dx$$

$$= \lambda\left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty = \lambda\left[\frac{-e^{-(\lambda-t)x}}{\lambda-t}\right]_0^\infty$$

Now, if $t < \lambda$ then $\lambda - t > 0$

and we have

$$M_0(t) = \lambda\left[\frac{-e^{-(\lambda-t)\infty}}{\lambda-t} - \frac{-e^{-(\lambda-t)0}}{\lambda-t}\right] = \lambda\left[\frac{-e^{-\infty}}{\lambda-t} - \frac{-e^{0}}{\lambda-t}\right]$$

$$= \lambda\left[\frac{0}{\lambda-t} - \frac{-1}{\lambda-t}\right] = \lambda\left[0 + \frac{1}{\lambda-t}\right] = \frac{\lambda}{\lambda-t}$$

Hence, the MGF of the Exponential Distribution is given by

$$M_0(t) = \frac{\lambda}{\lambda - t} \quad \text{for} \quad t < \lambda.$$

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Probability Distributions

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Topic No. 158

Gamma distribution
(PDF, CDF and shape of the distribution)

Formal definition:

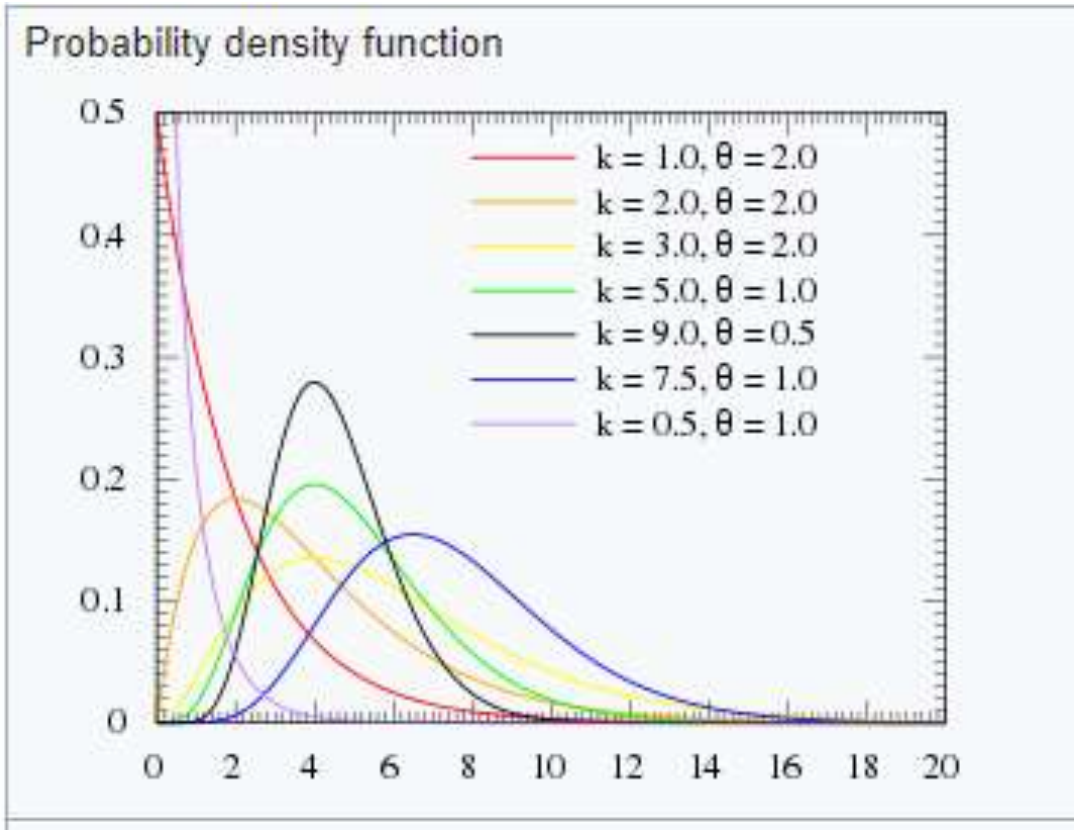
Gamma distribution

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta} & 0<x<\infty,\ \alpha>\ 0,\ \beta>0\\ 0 & \text{elsewhere}\end{cases}$$

is said to be the pdf of a gamma distribution with shape parameter α and scale parameter β and we write this as X has a $\Gamma(\alpha, \beta)$ distribution.

Shape of the distribution

- From the graph it is obvious that, for any fixed value of the scale parameter β, **as α increases**, the shape of the distribution tends to normality.



First we introduce the **gamma (Γ) function**:

In calculus, the integral

$$\int_0^\infty y^{\alpha-1}e^{-y}dy$$

exists for $\alpha > 0$ and that the value of the integral is a positive number.

The integral is called the **gamma function**, and we write

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y} dy$$

Properties of the Gamma Function

- If $\alpha = 1$, clearly (easy to verify through integration)

$$\Gamma(1) = \int_0^\infty e^{-y} dy = 1$$

- If $\alpha > 1$, an integration by parts show that

$$\Gamma(\alpha) = (\alpha - 1) \int_0^\infty y^{\alpha-2} e^{-y} dy = (\alpha - 1)\Gamma(\alpha - 1).$$

- Accordingly, if α is a positive **integer** greater than 1,

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)\cdots(3)(2)(1)\Gamma(1) = (\alpha - 1)!.$$

- Since $\Gamma(1) = 1$, this suggests we take **0! =1**.

- It is important to note that **the value of a gamma function is always positive**.

In the integral that defines $\Gamma(\alpha)$, let us introduce a new variable by writing $y = \frac{x}{\beta}$, where $\beta > 0$

Now

$$y = \frac{x}{\beta} \Rightarrow x = y\beta \Rightarrow \frac{dx}{dy} = \beta \Rightarrow \frac{dx}{\beta} = \text{dy}$$

Also, limits remain unchanged.

Hence, we have:

$$\Gamma(\alpha)=\int_0^{\infty}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-x/\beta}\left(\frac{1}{\beta}\right)dx,$$

Hence

$$1=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{1}{\beta^{\alpha-1+1}}x^{\alpha-1}e^{-x/\beta}dx=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{1}{\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}dx$$

$$=\int_0^{\infty}\frac{1}{\Gamma(\alpha)}\cdot\frac{1}{\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}dx$$

Now

$$\int_0^{\infty} \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx = 1$$

implies that

$$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \ 0 < x < \infty$$

is **eligible** to be called a **probability density function.**

CDF of the Gamma Distribution

The CDF of the distribution is given by

$$F(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\gamma\left(\alpha, \dfrac{x}{\beta}\right) & \alpha > 0, \beta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where this $\gamma\left(\alpha, \dfrac{x}{\beta}\right)$ is the incomplete gamma function.

Incomplete gamma function

In mathematics, the **upper** and **lower incomplete gamma functions** are types of special functions which arise as solutions to various mathematical problems such as certain integrals.

The upper incomplete gamma function is defined as:

$$\Gamma(\alpha, y) = \int_{y}^{\infty} t^{\alpha-1} e^{-t} dt,$$

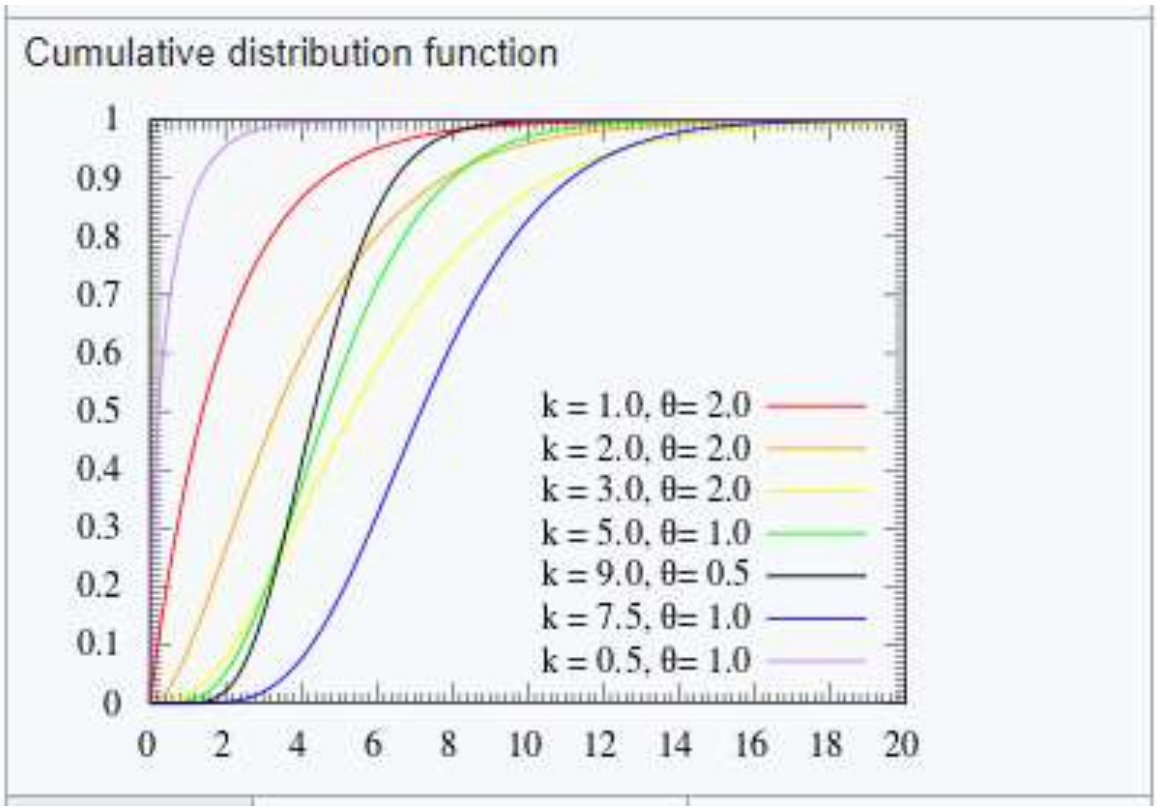
whereas the lower incomplete gamma function is defined as:

$$\gamma(\alpha, y) = \int_{0}^{y} t^{\alpha-1} e^{-t} dt.$$

It is interesting to note that $\gamma(\alpha, y) + \Gamma(\alpha, y) = \Gamma(\alpha)$.

Shape of the CDF:

We can see that for smaller values of the shape parameter α, i.e. α =0.5 and 1.0, the shape of the CDF appears to be parabolic whereas, for larger values of α i.e. α =2.0, 3.0,5.0 etc , the CDF is S-shaped.



Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 159

Derivation of Mean and Variance of the Gamma distribution

Mean: $$\mu_X = E\left(X\right) = \alpha\beta$$

Variance: $$\sigma_X^2 = \alpha\beta^2$$

Derivation of the Mean

- If X has a gamma distribution with parameters α and β, then the mean of X is given by

$$\mu_X = E\left(X\right) = \alpha\beta$$

Proof:

$$\int_0^\infty x.\left(\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}\right)dx=\int_0^\infty\left(\frac{x^{\alpha}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}\right)dx=\frac{\alpha}{\alpha}\int_0^\infty\left(\frac{x^{\alpha}e^{-x/\beta}}{\beta^\alpha\Gamma(\alpha)}\right)dx$$

$$=\alpha\int_0^\infty\left(\frac{x^{\alpha}e^{-x/\beta}}{\beta^\alpha\alpha\,\Gamma(\alpha)}\right)dx=\alpha\int_0^\infty\left(\frac{x^{\alpha}e^{-x/\beta}}{\beta^\alpha\,\Gamma(\alpha+1)}\right)dx$$

$\because$ we know that, $\Gamma(\alpha+1)\ =\alpha\,\Gamma(\alpha).$

Hence,

$$E(X) = \alpha \int_0^{\infty} \left(\frac{x^{\alpha} e^{-x/\beta}}{\beta^{\alpha} \Gamma(\alpha+1)} \right) dx = \alpha\beta \int_0^{\infty} \left(\frac{x^{\alpha} e^{-x/\beta}}{\beta^{\alpha+1} \Gamma(\alpha+1)} \right) dx$$

$$= \alpha\beta \int_0^{\infty} f(x; \alpha+1, \beta) dx = \alpha\beta(1) = \alpha\beta$$

$\therefore$ The total area under the curve of a probability density function is always equal to 1.

Derivation of the Variance

If X has a gamma distribution with parameters α and β,

The variance of X is

$$\sigma_X^2 = E\left(X^2\right) - \left[E\left(X\right)\right]^2$$

$$E\left(X^2\right)=\int_0^{\infty} x^2.\left(\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}\right)dx=\alpha(\alpha+1)\beta^2$$

The variance of X is

$$\sigma_X^2 = E\left(X^2\right) - \left[E\left(X\right)\right]^2 = \alpha\left(\alpha+1\right)\beta^2 - \left(\alpha\beta\right)^2$$

$$= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 = \alpha\beta^2.$$

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Probability Distributions

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Dr. Saleha Naghmi Habibullah

Topic No. 160

Example of computation of probabilities for a gamma-distributed random variable

Example:

Let X have a gamma distribution with pdf

$$f(x) = \frac{1}{\beta^2} x e^{-x/\beta}, \quad 0 < x < \infty,$$

zero elsewhere. If $x = 2$ is the unique mode of the distribution, find the parameter β and $P(X < 9.46)$.

Solution:

The pdf of the gamma distribution is given by

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \alpha > 0, \beta > 0$$

Letting $\alpha = 2,$ we have

$$f(x) = \frac{1}{\Gamma(2)\beta^2} x^{2-1} e^{-x/\beta} = \frac{1}{1!\beta^2} x e^{-x/\beta}$$

$$= \frac{1}{\beta^2} x e^{-x/\beta}, \qquad 0 < x < \infty$$

Hence we can see that the given density function is the pdf of the gamma distribution having α =2.

- In general, it is easy to prove that, for $\alpha \geq 1$, the mode of the Gamma distribution is given by $(\alpha - 1)\beta.$.
- Here we have been given the information that the mode=2. Therefore,

$$Mode = (2-1)\beta = 2 \Rightarrow \beta = \frac{2}{2-1} = 2.$$

Hence,

$$f(x) = \frac{1}{\beta^2} x e^{-x/\beta} = \frac{1}{2^2} x e^{-x/2} = \frac{1}{4} x e^{-x/2}, \qquad 0 < x < \infty$$

Now to find $P(X < 9.46)$.

$$P\left(X < 9.46 \right) = \int_0^{9.46} \frac{1}{4} x e^{-x/2} dx = \frac{1}{4} \int_0^{9.46} x e^{-x/2} dx$$

$$P\left(X < 9.46\right) = \frac{1}{4}\left(-2e^{-x/2}x - 4e^{-x/2}\right)_0^{9.46}$$

$$= \frac{1}{4}\left[\left(-2e^{-9.46/2}(9.46) - 4e^{-9.46/2}\right) - \left(-2e^{-0/2}.0 - 4e^{-0/2}\right)\right]$$

$$= \frac{1}{4}\left[\left(0.16700 - 0.003531\right) - \left(-0 - 4e^{0}\right)\right]$$

$$= \frac{1}{4}\left[\left(-0.20230\right) - \left(-4\right)\right] = \frac{1}{4}\left[\left(-0.20230\right) + \left(4\right)\right] = \frac{3.79770}{4} = 0.9494.$$

In other words, the probability that $X < 9.46$ is 94.94% almost 95 %.

Virtual University of Pakistan
Probability Distributions

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Dr. Saleha Naghmi Habibullah

Topic No. 161

MGF of Gamma Distribution

The moment-generating function of the gamma distribution is

$$M(t) = (1 - \beta t)^{-\alpha} \quad for\ t < \frac{1}{\beta}$$

In order to derive the mgf, we begin as follows

$$M(t) = E\left(e^{tX}\right) = \int_0^{\infty} e^{tx} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} dx$$

PROOF:

- First let's combine the two exponential terms and move the gamma fraction out of the integral:

$$M(t) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{tx - x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-x(1-\beta t)/\beta} dx$$

Now we're ready to do a substitution in the integral.

$$\text{Let } u = \frac{x\left(1-\beta t\right)}{\beta} \Rightarrow x = \frac{\beta u}{1-\beta t}$$

$$\text{That means we have,} \quad du = \frac{1-\beta t}{\beta}\,dx \Rightarrow dx = \frac{\beta}{1-\beta t}\,du$$

Also, the limits remain unchanged.

Now substitute those in the integral :

$$M(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}x^{\alpha-1}e^{-x(1-\beta t)/\beta}dx$$

$$M(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}\left(\frac{\beta u}{1-\beta t}\right)^{\alpha-1}\left(e^{-u}\right)\frac{\beta}{1-\beta t}du$$

$$M(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^{\infty}(1-\beta t)^{-\alpha+1}(1-\beta t)^{-1}(\beta u)^{\alpha-1}\beta e^{-u}du$$

$$M(t)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_{0}^{\infty}(1-\beta t)^{-\alpha+1}(1-\beta t)^{-1}(\beta u)^{\alpha-1}\beta e^{-u}du$$

can be re-written as

$$M(t)=\frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)\beta^{\alpha}}\int_{0}^{\infty}(\beta u)^{\alpha-1}\beta e^{-u}du$$

$$M(t) = \frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} (\beta u)^{\alpha-1} \beta e^{-u} du$$

can be re-written as

$$M(t) = \frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} u^{\alpha-1} \beta^{\alpha} e^{-u} du = \frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)} \int_0^{\infty} u^{\alpha-1} e^{-u} du$$

The integral is now the gamma function: $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du.$

Making that substitution:

$$M(t) = \frac{(1-\beta t)^{-\alpha}}{\Gamma(\alpha)} \Gamma(\alpha)$$

Now, cancelling out the terms and we have the required **moment-generating function**:

$$M(t) = (1 - \beta t)^{-\alpha} \quad for\ t < \frac{1}{\beta}$$

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Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 162

Beta distribution
(PDF, CDF and shape of the distribution)

The **probability density function** of the β distribution is

$$f\left(x;\alpha,\beta\right)=\frac{1}{B\left(\alpha,\beta\right)}x^{\alpha-1}\left(1-x\right)^{\beta-1},\qquad 0\le x\le 1;\alpha,\beta>0$$

where,

$$B\left(\alpha,\beta\right)=\frac{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}{\Gamma\left(\alpha+\beta\right)}$$

and α and β are the **shape parameters** respectively.

About the Beta function B

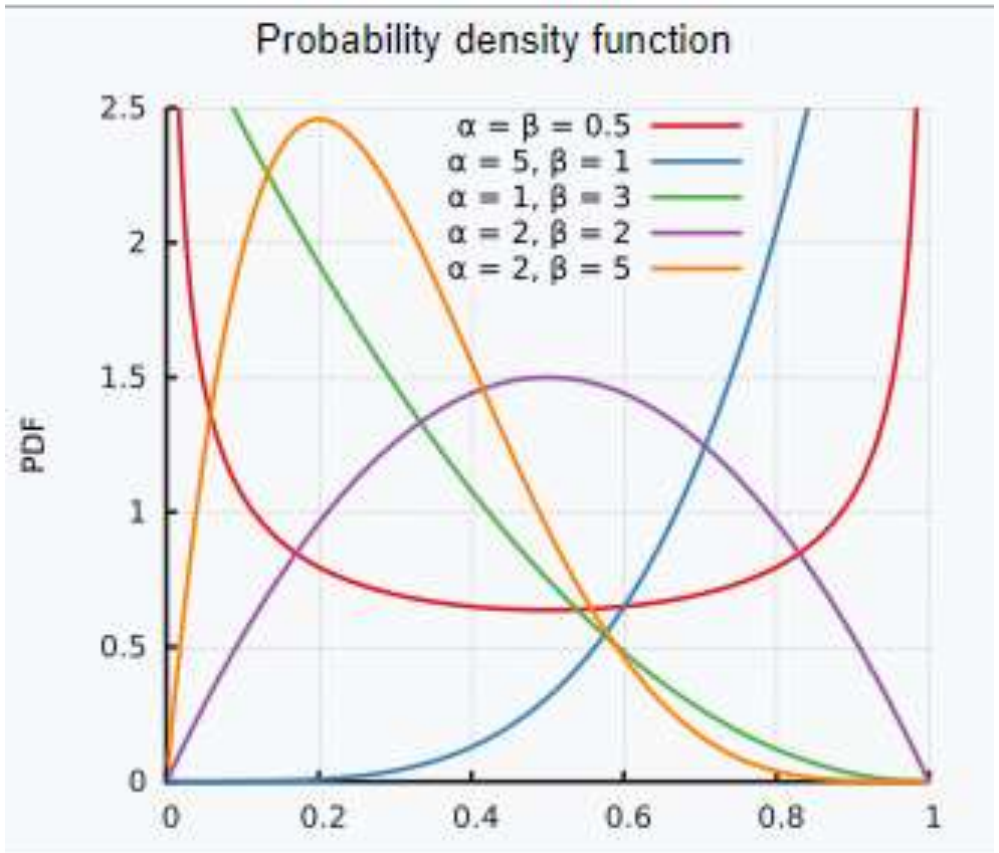
- The Beta function B in the denominator plays the role of a “**normalizing constant**” which assures that the total area under the density curve equals 1.

- The Beta function is equal to a **ratio** of Gamma functions:

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

- Keeping in mind that for integers, $\Gamma(k) = (k-1)!$, one can do some checking and get an idea of what the shape might be.

Shape of the distribution:



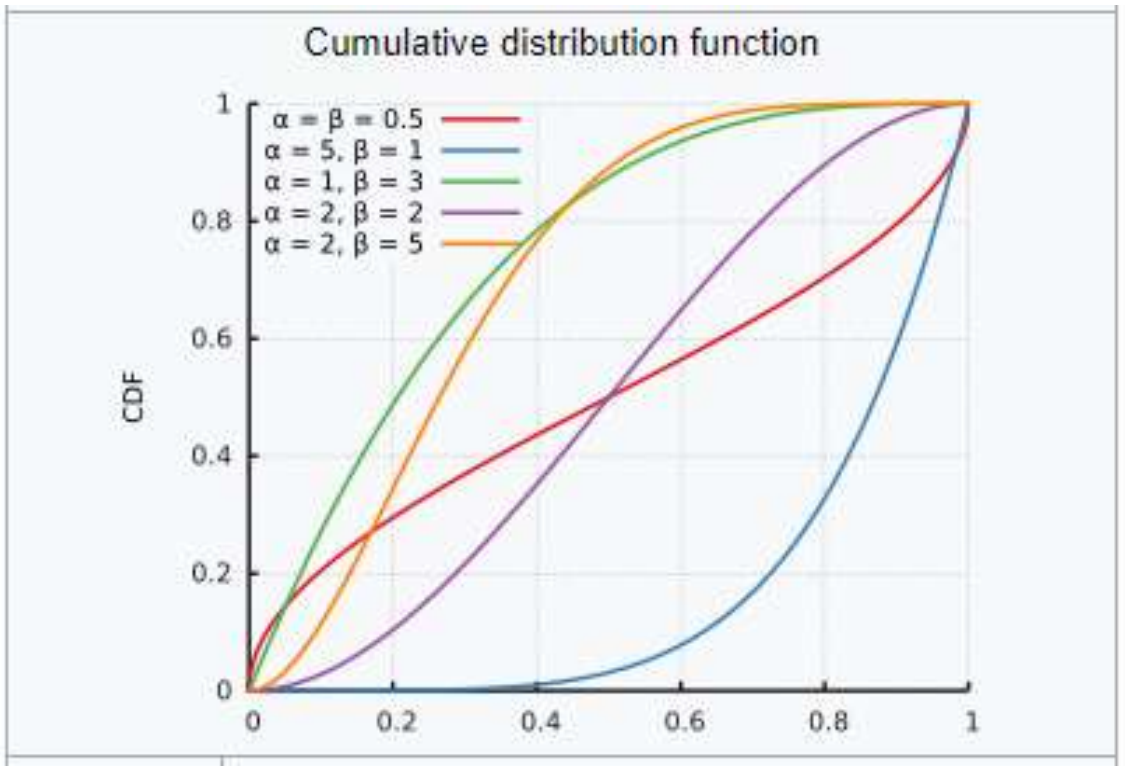
Cumulative distribution Function

- The formula for the cumulative distribution function of the beta distribution is also called the **incomplete β function ratio** (commonly denoted by I_x) and is defined as

$$F(x) = I_x(\alpha, \beta) = \frac{\int_0^x t^{\alpha-1}(1-t)^{\beta-1}}{B(\alpha, \beta)}, \qquad 0 \le x \le 1; \alpha, \beta > 0$$

where B is the beta function defined above.

Shape of the CDF:



Some Properties:

- **Mean** $=\mu$

$$= E(x) = \alpha/\alpha+\beta$$

- **Variance(*x*)**

$$= \alpha\beta / (\alpha+\beta)^2 (\alpha+\beta+1)$$

Mode:

- If $\alpha > 1$and $\beta > 1$, the peak of the density is in the interior of [0,1] and mode of the Beta distribution is

$$\textbf{mode} = \alpha - 1/\alpha + \beta - 2$$

If α or $\beta < 1$, the mode may be at an edge.

Relation to Uniform Distribution

Proposition 1:

A Beta distribution with parameters α and β is a uniform distribution on the interval .

Proof: When α=1 and β=1 , we have,

$$\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$$

$$=\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}x^{1-1}(1-x)^{1-1}=\frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}x^{0}(1-x)^{0}$$

$$=\frac{(2-1)!}{0!0!}1.(1)=1. \qquad \because \Gamma(n)=(n-1)!$$

- Therefore, the probability density function of a Beta distribution with parameters α and β can be written as

$$f_X\left(x;\alpha=1,\beta=1\right)=1, \qquad if\,, x\in\left[0,1\right]$$
$$=0, \qquad f\,, x\notin\left[0,1\right]$$

which is the probability density function of a **uniform distribution** of X on the **interval** [0,1].

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Probability Distributions

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Topic No. 163

Derivation of the Mean and Variance of the Beta distribution

- As we know that the mean of a distribution of a continuous random variable is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x.f(x)dx$$

$$\mu = E\left(X\right) = \int_0^1 x.\frac{1}{B\left(\alpha,\beta\right)}.x^{\alpha-1}\left(1-x\right)^{\beta-1}\,dx$$

$$\mu = E(X) = \int_0^1 x.\frac{1}{B(\alpha,\beta)}.x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$= \int_0^1 \frac{1}{B(\alpha,\beta)}.x^{\alpha-1+1}(1-x)^{\beta-1}\,dx = \int_0^1 \frac{1}{B(\alpha,\beta)}.x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

Now we know that the function is given by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

Hence,

$$\mu=\frac{1}{B(\alpha,\beta)}\int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx=\frac{1}{B(\alpha,\beta)}.B(\alpha+1,\beta)$$

But we know that,

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Hence

$$\mu=\frac{B(\alpha+1,\beta)}{B(\alpha,\beta)}=\frac{\dfrac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)}}{\left(\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\right)}=\frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}.\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$=\frac{\alpha}{(\alpha+\beta)\Gamma(\alpha+\beta)}.\Gamma(\alpha+\beta)=\frac{\alpha}{(\alpha+\beta)}.$$

- The variance of this distribution is $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ respectively.

They are computed as below:

$$Var\left(X\right)=\sigma^2=E\left(X^2\right)-\left[E\left(X\right)\right]^2,\ \text{where}$$

$$E\left(X^2\right)=\int_0^1 x^2 f\left(x\right)dx$$

$$=\int_0^1 x^2\frac{1}{B\left(\alpha,\beta\right)}.x^{\alpha-1}\left(1-x\right)^{\beta-1}dx$$

$$=\int_0^1\frac{1}{B\left(\alpha,\beta\right)}.x^{(\alpha+2)-1}\left(1-x\right)^{\beta-1}dx=\frac{1}{B\left(\alpha,\beta\right)}\int_0^1 x^{(\alpha+2)-1}\left(1-x\right)^{\beta-1}dx$$

$$=\frac{B\left(\alpha+2,n\right)}{B\left(\alpha,\beta\right)}=\frac{\alpha\left(\alpha+1\right)}{\left(\alpha+\beta\right)\left(\alpha+\beta+1\right)}$$

Therefore,

$$\sigma^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{\alpha+\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 164

Example of Application of the Beta Distribution

Important Note

It is important to note that actually there are two types of Beta distribution:

- Beta distribution of the first kind
- Beta distribution of the first kind is defined on the interval from 0 to 1

Important Note

It is important to note that actually there are two types of Beta distribution:

- Beta distribution of the second kind.

- Beta distribution of the second kind is defined on the interval from 0 to infinity.

I will explain this to you with the help of an example for the beta distribution of the first kind.

It is both interesting and important to note that the Beta distribution of the first kind (also called “Beta distribution”) is applicable in those situations where the **random variable** X goes from 0 to 1.

This may happen in a number of situations but the one that might come to mind first of all is the case when X represents the **proportions** of successes in a large sample.

Question:

Suppose that the proportion of defective DVDs in a large shipment follow a Beta distribution with $\alpha = 2$ and $\beta = 5$.

What is the probability that the shipment has 20% to 30% defective DVDs ?

The **largeness** of the sample will facilitate the assumption that our random variable X is **continuous**.

Question:

Suppose that the proportion of defective DVDs in a large shipment follow a Beta distribution with $\alpha = 2$ and $\beta = 5$.

What is the probability that the shipment has 20% to 30% defective DVDs ?

Solution:

Let X represent the proportion of defective DVDs in this large shipment.

(Here **defectiveness** denotes **‘Success’.)**

Then, obviously, X will go from **0 to 1.**

The probability density function of the Beta distribution is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, 0 \le x \le 1, \alpha > 0, \beta > 0$$

In this example, $\alpha = 2$ and $\beta = 5$.

Hence, our PDF is

$$f(x) = \frac{x^{2-1}(1-x)^{5-1}}{B(2,5)}, 0 \leq x \leq 1$$

We need to compute

$$P(20\% \le x \le 30\%) = P(0.20 \le x \le 0.30)$$

which is given by

$$P(0.2 \le x \le 0.3) = \int_{0.2}^{0.3} \frac{x^{2-1}(1-x)^{5-1}}{B(2,5)} dx$$

Solving the beta function

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

So,

$$B(2,5)=\frac{\Gamma(2)\Gamma(5)}{\Gamma(2+5)}=\frac{\Gamma(2)\Gamma(5)}{\Gamma(7)}=\frac{(2-1)!(5-1)!}{(7-1)!}$$

$$=\frac{(1)!(4)!}{(6)!}=0.03333$$

We need to compute

$$P(20\% \le x \le 30\%) = P(0.20 \le x \le 0.30)$$

which is given by

$$P(0.2 \le x \le 0.3) = \int_{0.2}^{0.3} \frac{x^{2-1}(1-x)^{5-1}}{B(2,5)} dx = \int_{0.2}^{0.3} \frac{x^{2-1}(1-x)^{5-1}}{0.033333} dx = 0.2352$$

Interpretation:

- Hence we can say that the probability that the shipment has 20% to 30% defective DVDs is 23.52% i.e. a little less than 25%.

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Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 165

Normal Distribution
(PDF, CDF and Shape of the distribution)

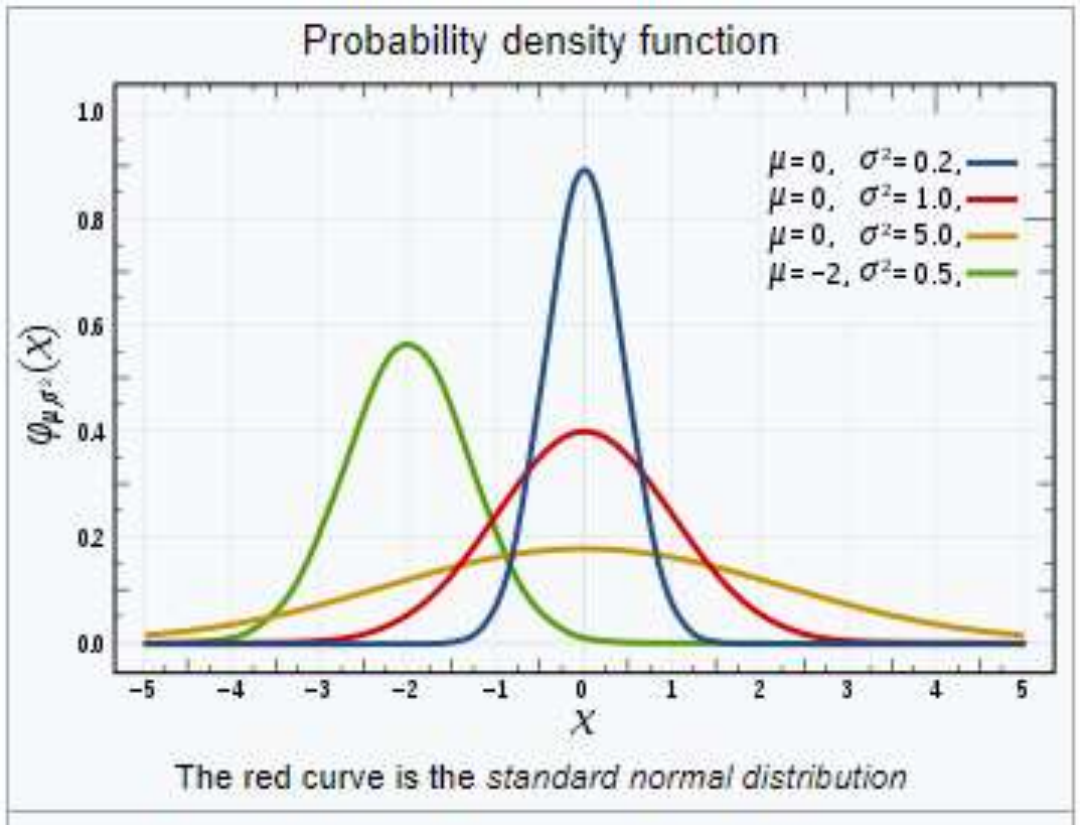
Definition:

A random variable X is said to have a normal distribution if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty.$$

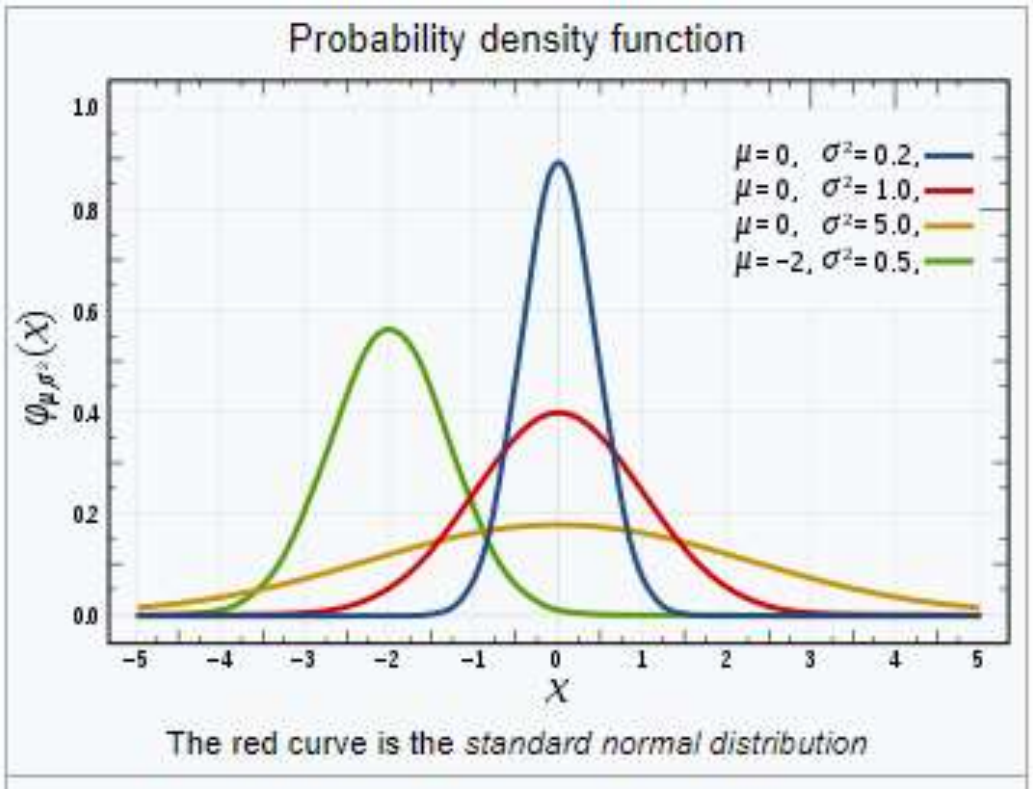
- μ is the mean or expectation of the distribution (and also its median and mode),
- $\sigma > 0$ is the standard deviation, and
- σ^2 is the variance.

Shape of the distribution:



The graph points to the following **two basic properties** of the normal distribution:

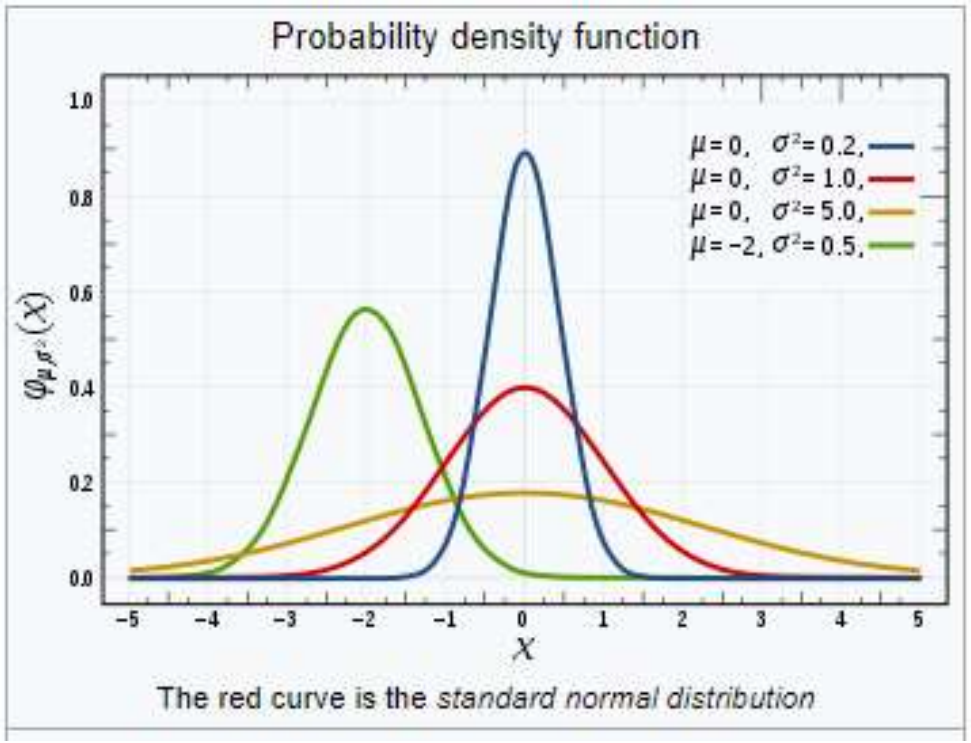
1. The normal curve is **asymptotic** to the x-axis,



2. The maximum ordinate
(i.e. the **modal ordinate**)
of the normal distribution
at $x=\mu$ is equal to

$$1/(\sigma\sqrt{2\pi})$$

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for} \quad -\infty<x<\infty.$$



Cumulative Distribution Function

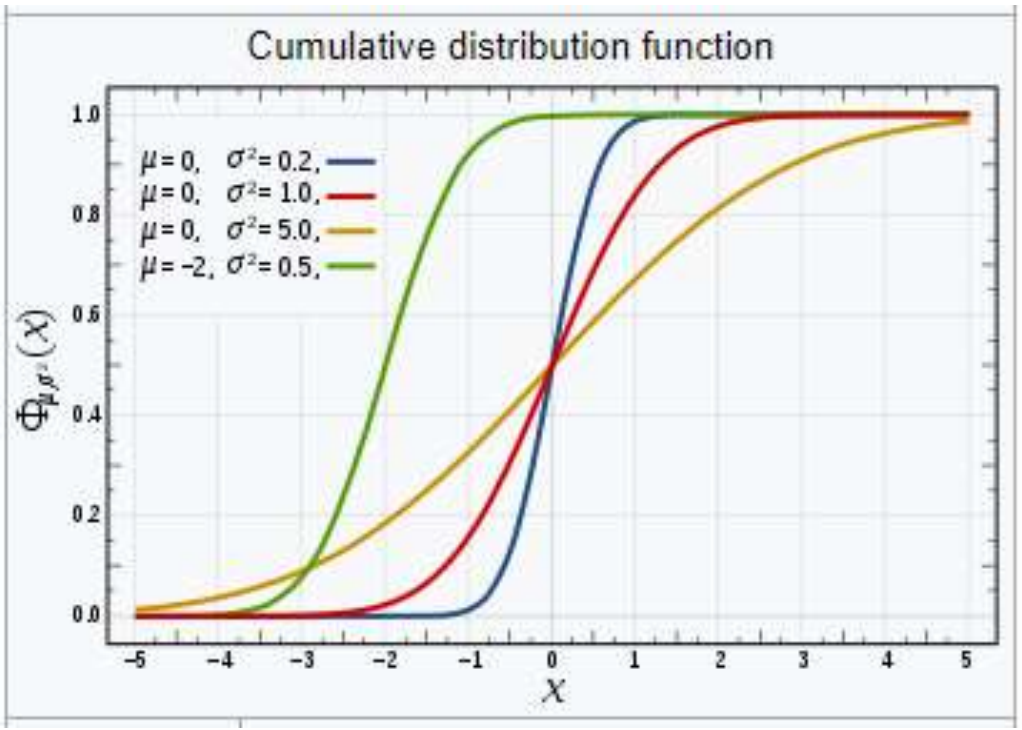
The formula for the **cumulative distribution function** of the normal distribution is

where $$F(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right] \quad \text{for } -\infty < x < \infty.$$

erf(z) is the “error function”

defined by $$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt.$$

The shape of the **cumulative distribution function** of the normal distribution is **S-shaped**:



Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 166

Standard Normal Distribution

- The standard normal distribution is a **special** case of the normal distribution .

- It is the distribution that occurs when a normal random variable has a **mean** of zero and a **standard deviation** of one.

In other words, if the random variable Z follows a normal distribution with parameters (0, 1), i.e. $Z \sim N(0, 1)$ then we say Z is a standard normal random variable.

The density of standard normal distribution is therefore,

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{z^2}{2}\right)}, \quad -\infty < z < \infty$$

Derivation of the Mean of the Standard Normal Distribution

The mean of the standard normal random variable Z is given by

$$E(Z) = \int_{-\infty}^{\infty} z f_Z(z)\, dz$$

We know that, for any **odd function** g i.e. a function for which

$$g(-x) = -g(x) \ \forall x \in \mathbb{R}$$

the integral $\int_{-\infty}^{\infty} g(x)\,dx$ is equal to zero.

Here, it is easy to see that the function

$$zf_Z(z) = z\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{z^2}{2}\right)} = \frac{z}{\sqrt{2\pi}}e^{\left(-\frac{z^2}{2}\right)}$$

is an **odd function**.

Hence, the mean $$E(Z) = \int_{-\infty}^{\infty} zf_Z(z)dz = 0.$$

Next: **Derivation of the Variance of the Standard Normal Distribution**

Using the short-cut formula of the variance, we have

$$\mathrm{Var}(Z) = E(Z^2) - \left[E(Z)\right]^2$$

Now we know that,

for the standard normal distribution, $E(Z) = 0.$

So

$$\mathrm{Var}(Z) = E\left(Z^2\right) - [0]^2 = E\left(Z^2\right) = \int_{-\infty}^{\infty} z^2 \frac{e^{\left(-\frac{z^2}{2}\right)}}{\sqrt{2\pi}}\, dz$$

Now, we know that, for any **even function** g i.e. a function for which $g(-x) = g(x) \ \forall x \in \mathbb{R}$

the integral $\int_{-\infty}^{\infty} g(x)\,dx$ is equal to $2\int_{0}^{\infty} g(x)\,dx$.

Now $z^2 \frac{e^{\left(-\frac{z^2}{2}\right)}}{\sqrt{2\pi}}$ is an even function of z.

Hence

$$\text{Var}(Z) = E\left(Z^2\right) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{\left(-\frac{z^2}{2}\right)} dz.$$

Let $w = \frac{z^2}{2}$, then $2w = z^2 \Rightarrow \sqrt{2w} = z \Rightarrow (2w)^{1/2} = z$

$\Rightarrow (2)^{1/2}(w)^{1/2} = z \Rightarrow \sqrt{2}(w)^{1/2} = z.$

Therefore

$$\frac{dz}{dw} = \sqrt{2}\,\frac{1}{2}(w)^{(1/2)-1} \Rightarrow \frac{dz}{dw} = \frac{1}{\sqrt{2}}(w)^{-1/2}$$

$$\Rightarrow \frac{dz}{dw} = \frac{1}{\sqrt{2w}} \Rightarrow dz = \frac{1}{\sqrt{2w}}\,dw.$$

And

$$\mathrm{Var}(Z)=\frac{2}{\sqrt{2\pi}}\int_0^{\infty} z^2 e^{\left(-\frac{z^2}{2}\right)}dz=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}(2w)e^{(-w)}\frac{1}{\sqrt{2w}}dw$$

$$=\frac{2(2)}{\sqrt{2}\sqrt{2\pi}}\int_0^{\infty}(w)e^{(-w)}\frac{1}{\sqrt{w}}dw=\frac{2(2)}{\sqrt{2}\sqrt{2}\sqrt{\pi}}\int_0^{\infty}(w)e^{(-w)}\frac{1}{\sqrt{w}}dw$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{\pi}}\int_0^{\infty}\sqrt{w}e^{(-w)}dw=\frac{2}{\sqrt{\pi}}\int_0^{\infty}w^{1/2}e^{(-w)}dw=\frac{2}{\sqrt{\pi}}\int_0^{\infty}w^{(3/2)-1}e^{(-w)}dw$$

$$=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)=\frac{2}{\sqrt{\pi}}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{2}{\sqrt{\pi}}.\frac{1}{2}.\sqrt{\pi}\quad\left[\sin ce\,\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right]$$

$$=1.$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 167

The Process of Standardization,
Area Table of the Standard Normal Distribution
and
Computation of probabilities pertaining to a normal random variable
(explained through an example)

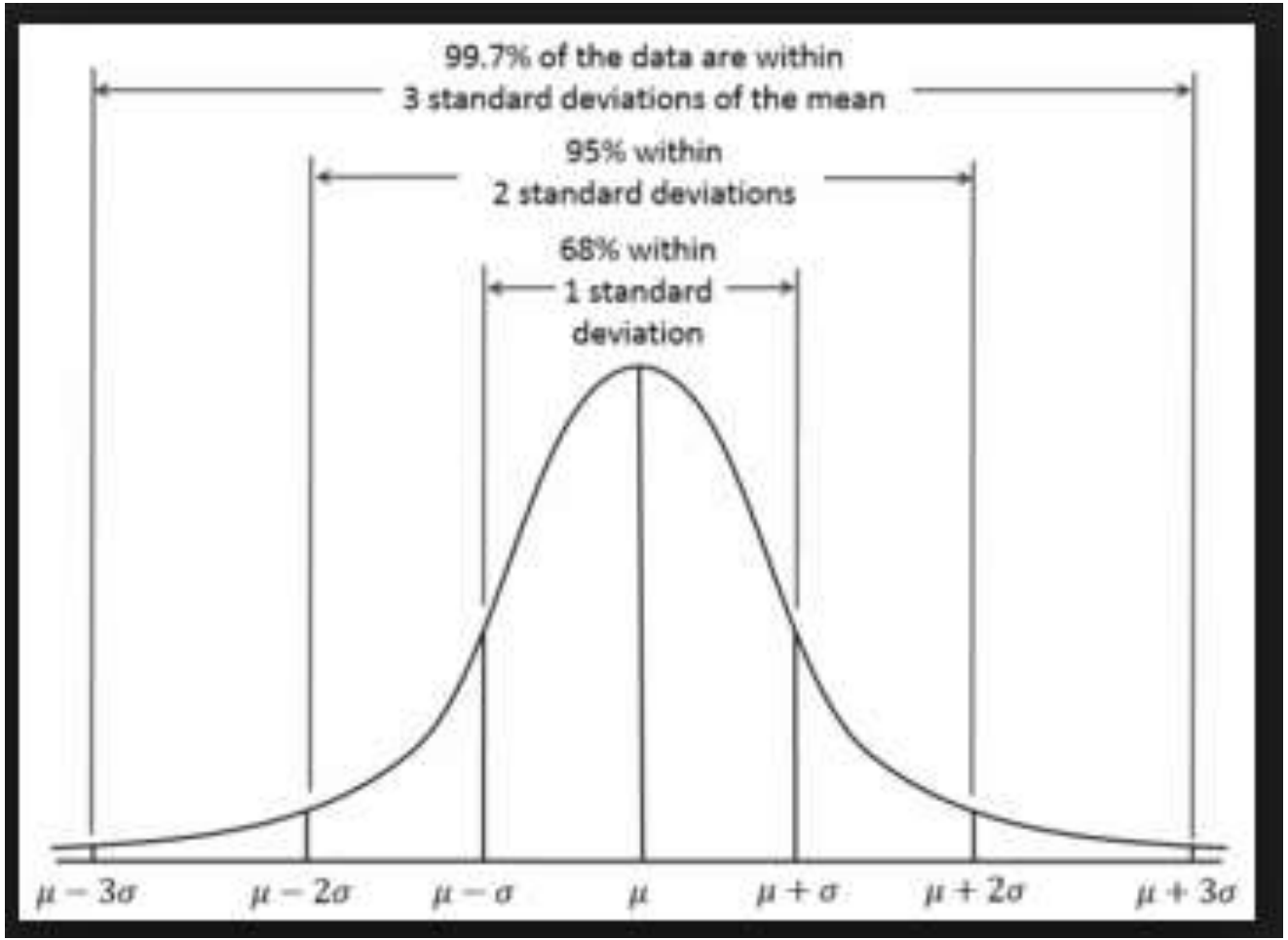
Normal Distribution

A random variable X is said to have a normal distribution if its pdf is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

where μ is the mean of the distribution and $\sigma > 0$ is the standard deviation.

We say that X has the $N(\mu, \sigma^2)$ distribution.


99.7% of the data are within
3 standard deviations of the mean
95% within
2 standard deviations
68% within
1 standard
deviation
$\mu - 3\sigma$
$\mu - 2\sigma$
$\mu - \sigma$
μ
$\mu + \sigma$
$\mu + 2\sigma$
$\mu + 3\sigma$

1. For this purpose, we let $Z = \dfrac{X - \mu}{\sigma}$.

Then, it is easy to prove that $Z \sim N(0,1)$.

2. Area Table of the Standard Normal Distribution

Table I

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 1: Standard Normal Cumulative Probabilities

The Process of Standardization

Let X be $N(\mu,\sigma)$ and suppose that we are interested in computing the probability $P\left(a < X < b\right)$.

In order to be able to do so, we will need to convert our $N(\mu,\sigma^2)$ random variable X into the standard normal random variable Z~$N(0,1)$.

3. The Computation of Probabilities

Example:

Let X be $N(2, 25)$ and suppose that we are interested in computing the probability

$P\left(0 < X < 10\right) = F(10) - F(0).$

Applying the standardization formula $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-2}{5}$,

we obtain

$$P\left(0 < X < 10\right) = F(10) - F(0)$$

$$= \Phi\left(\frac{10-2}{5}\right) - \Phi\left(\frac{0-2}{5}\right)$$

$$= \Phi\left(\frac{8}{5}\right) - \Phi\left(\frac{-2}{5}\right) = \Phi\left(1.60\right) - \Phi\left(-0.40\right).$$

The **standard normal table**, gives us values of the distribution function of a standard normal variable,

i.e. it gives the values of $\Phi(z)$ for various values of z.

Area Table of the Standard Normal Distribution

Table I

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
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0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
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0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
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0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
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1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 1: Standard Normal Cumulative Probabilities

Area Table of the
Standard Normal Distribution

z	.00	.01	
0	0.5	0.504	(
0.1	0.5398	0.5438	0
0.2	0.5793	0.5832	0
0.3	0.6179	0.6217	0
0.4	0.6554	0.6591	0
0.5	0.6915	0.695	0
0.6	0.7257	0.7291	0
0.7	0.758	0.7611	0
0.8	0.7881	0.791	0
0.9	0.8159	0.8186	0
1	0.8413	0.8438	0
1.1	0.8643	0.8665	0
1.2	0.8849	0.8869	0
1.3	0.9032	0.9049	0
1.4	0.9192	0.9207	0
1.5	0.9332	0.9345	0
1.6	0.9452	0.9463	0
1.7	0.9554	0.9564	0
1.8	0.9641	0.9649	0
1.9	0.9713	0.9719	0

$$P(0 < X < 1.0) = \Phi(1.00) - \Phi(-0.40).$$

So

$$P\left(0 < X < 10\right) = \Phi\left(\frac{10-2}{5}\right) - \Phi\left(\frac{0-2}{5}\right)$$

$$= \Phi\left(1.60\right) - \Phi\left(-0.40\right) = 0.945 - (1 - 0.655) = 0.600 = 60\%.$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 168

Computation of Some Specific Percentiles of a Normal Distribution (explained through an example)

Computing Percentiles

The **standard** normal distribution can also be used for computing **percentiles**.

For example, the median is the 50th percentile (P_{50}), the first quartile is the 25th percentile (P_{25}), and the third quartile is the 75th percentile (P_{75}).

- In some instances it may be of interest to compute other percentiles, for example the 5th or 95th.

Let us Recall the Process of Standardization

Let X be $N(\mu,\sigma)$ and suppose that we are interested in computing the probability $P\left(a < X < b\right)$.

In order to be able to do so, we will need to convert our $N(\mu,\sigma^2)$ random variable X into the standard normal random variable Z~$N(0,1)$.

For this purpose, we let $Z = \dfrac{X - \mu}{\sigma}$.

Then, it is easy to prove that $Z \sim N(0,1)$.

The **formula** below is used to compute percentiles of a normal distribution is obtained by "inverting" the standardization formula given above.

$$Z = \frac{X - \mu}{\sigma} \Rightarrow X - \mu = \sigma Z \Rightarrow X = \mu + \sigma Z$$

where, μ is the mean and σ is the standard deviation of the variable X,

and Z is the value from the standard normal distribution for the desired percentile.

Example:

The BMI for men aged 60 is normally distributed with mean equal to 29 with a standard deviation of 6 whereas the BMI for women aged 60 is normally distributed with mean 28 and standard deviation 7.

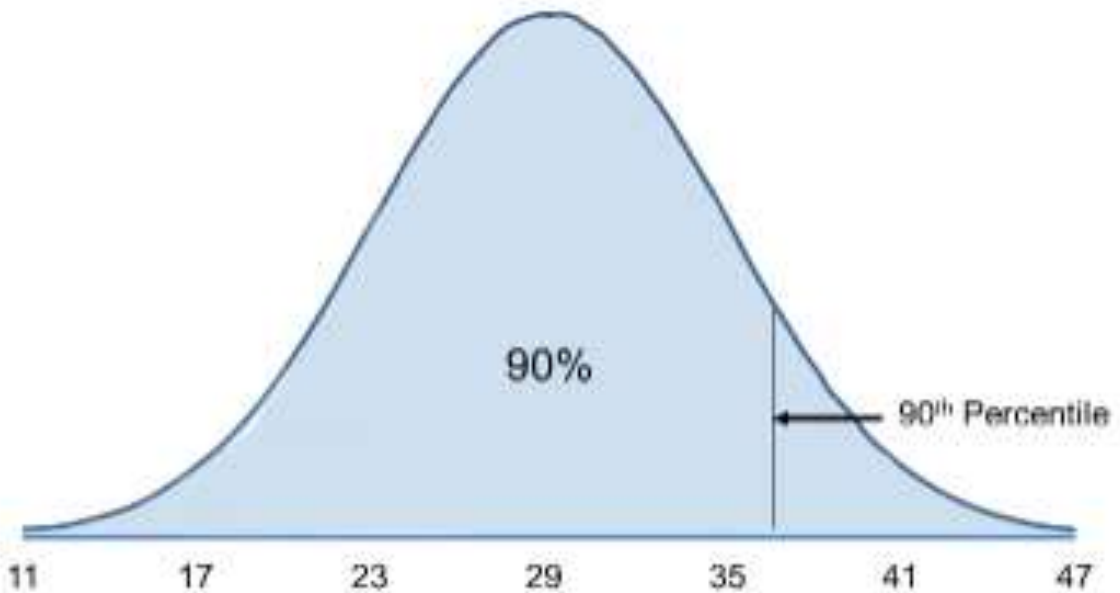
What is the 90th percentile of BMI for men and what does it mean ?

First of all, let us see what is <u>meant</u> by BMI:

- **Body Mass Index** (**BMI**) is a person's weight in kilograms divided by the square of height in meters.

- A high BMI can be an indicator of **high body <u>fatness.</u>**

- The 90th percentile is the BMI that holds 90% of the BMIs below it and 10% above it, as illustrated in the figure below.



- To compute the 90th percentile, we use the formula $X = \mu + \sigma Z$, and we will use the **standard normal distribution area table**, except that we will work in the **opposite** direction.

- Previously we started with a particular "*X*-value" and used the table to find the probability.

- However, in this case we want to start with a 90% probability and find the value of "*X*" that represents it.

- So we begin by going into the **interior** of the standard normal distribution area table to find the area under the curve closest to 0.90 i.e. 0.9000, and from this we can determine the corresponding *Z*-score.

- When we go to the table, we find that the value 0.9000 is not there exactly, however, the values 0.8997 and 0.9015 are there and correspond to Z values of 1.28 and 1.29 respectively.

- Since 0.8997 is much closer to 0.9000 than 0.9015, therefore we choose the z-score corresponding to 0.8997.

- The z-score is 1.28 (i.e., 89.97% of the area under the standard normal curve is below 1.28.

- Now that we have found z=1.28, we can use the equation

  $x = \mu + \sigma z = 29+6(1.28)$

  because we already know that the mean and standard deviation are 29 and 6, respectively.

So, using z =1.28 the 90th percentile of BMI for men comes out to be:

$$x = 29 + 6(1.28) = 29 + 7.68 = 36.68$$

Interpretation:

Ninety percent of the BMIs in men aged 60 are 36.68 or less. Ten percent of the BMIs in men aged 60 are 36.68 or higher.

REMARK:

The exact Z value holding 90% of the values below it is 1.282 which can be determined from a table of standard normal probabilities with **more** precision.

Using Z =1.282 the 90th percentile of BMI for men is:

$$X = 29 + 6(1.282) = 29 + 7.69 = 36.69$$

--- very close.

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by

Dr. Saleha Naghmi Habibullah

Topic No. 169

The function *f*(x) defining the normal distribution is a proper PDF

The function $f(x)$ defining the normal distribution is a proper pdf

i.e.

$$f(x) \geq 0$$

and

the total area under the normal curve is unity.

Recall:

A random variable X is said to have a normal distribution if its pdf is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{for } -\infty < x < \infty.$$

where μ is the mean or expectation of the distribution and $\sigma > 0$ is the standard deviation.

(i) **Clearly** $f(x)$ is always non-negative.

(ii) The total probability (i.e. the total area under the curve) is

$$\text{Area} = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Let, $z = \dfrac{x-\mu}{\sigma} \Rightarrow \sigma z = x - \mu \Rightarrow x = \mu + \sigma z$

so that $\dfrac{dx}{dz} = 0 + \sigma = \sigma \Rightarrow dx = \sigma dz.$

Limits remain unchanged.

Substituting these values, we get

$$\text{Area} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \sigma dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{-z^2/2} dz + \int_{0}^{\infty} e^{-z^2/2} dz \right]$$

The function $\int_{-\infty}^{0} e^{-z^2/2} dz$ being an even function of z,

can be by letting $w = -z$, written as $\int_{-\infty}^{0} e^{-z^2/2} dz = \int_{0}^{\infty} e^{-w^2/2} dw.$

Then,

$$\text{Area} = 2.\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$

Let $v = \frac{1}{2} z^2$ so that $\frac{dv}{dz} = \frac{1}{2} 2z = z \Rightarrow dv = z dz$

implying that

$$dz = \frac{dv}{z} = \frac{dv}{\sqrt{2v}}$$

Limits remain unchanged.

Then,

$$\text{Area}=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-v}\,\frac{dv}{\sqrt{2v}}=\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-v}v^{-1/2}dv=\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-v}v^{(1/2)-1}dv$$

$$=\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right)=\frac{1}{\sqrt{\pi}}.\sqrt{\pi}=1.$$

Thus the total area (probability) under the normal curve is **unity.**

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Topic No. 170

Derivation of the mean and a variance of the Normal Distribution

We know that the pdf of the normal distribution with the parameters μ and σ respectively is written as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]} \qquad -\infty \leq x \leq \infty$$

The **mean** the normal distribution is μ

Proof:

By definition,

$$\mu = E(X) = \int_{-\infty}^{\infty} xf\left(x\right)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x.e^{-[(x-\mu)^2/2\sigma^2]}dx$$

Let, $z = \frac{x-\mu}{\sigma}$. Then, $x = \mu + z\sigma$ and $dx = \sigma dx$.

{Limits: when $x = \infty, z = \infty$; *when* $x = -\infty, z = -\infty$}

$$\text{Therefore, } E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + z\sigma) e^{-z^2/2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz$$

The first integral presents μ times the area under a normal curve with zero mean and unit variance and hence is equal to μ. The integral being an odd function, equals zero.

Thus, E(X)= μ i.e. μ is the mean of the normal distribution.

And, now of find the variance:

$$Var(X) = E(X-\mu)^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \, f(x)dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \, e^{-[\{x-\mu)^2/2\sigma^2} dx$$

on putting $z = \dfrac{x-\mu}{\sigma}$

$$Var(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\sigma^2 . e^{-z^2/2} dz = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z.z\, e^{-z^2/2} dz,$$

To integrate by parts, we use the formula

$\int_{-\infty}^{\infty} u\,dv = uv - \int_{-\infty}^{\infty} v\,du$ and make the

substitution $dv = ze^{-z^2/2}dz$ and $u = z$

so that $v = -e^{-z^2/2}$ and $du = dz$.

Then,

$$Var(X) = \frac{\sigma^2}{\sqrt{2\pi}}\left[-z.e^{-z^2/2}\right]_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}dz$$

in first part applying L'Hospital rule,

$$= 0 + \sigma^2$$

Hence, $E(X) = \mu$ and $Var(X) = \sigma^2$.



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Topic No. 171

The Median and the Mode of the Normal distribution are each equal to μ,

the mean of the distribution

(i) The **median** , *a*, is obtained as the solution of

$$\int_{-\infty}^{M} f(x) = \frac{1}{2}.$$

m– the median of the distribution

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} e^{-[(x-\mu)^2/2\sigma^2]} dx = \frac{1}{2}$$

Putting $z = \frac{x-\mu}{\sigma} \Rightarrow x = \mu + \sigma z \Rightarrow \frac{dx}{dz} = 0 + \sigma = \sigma \Rightarrow dx = \sigma dz$

Limits

$x \to -\infty, \;\; z \to -\infty$

$x \to M, \;\; z \to \frac{M-\mu}{\sigma}$

Hence, the median is the solution of the equation

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-[(x-\mu)^2/2\sigma^2]} dx = \frac{1}{2}$$

or

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-z^2/2} \sigma dz = \frac{1}{2} \Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{M-\mu}{\sigma}} e^{-z^2/2} dz$$

But we know from the symmetry of the standard normal distribution that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-z^2/2}dz = \frac{1}{2},$$

$$\therefore \frac{M-\mu}{\sigma} = 0 \; or \; M = \mu,$$

i.e. μ is the median of the distribution.

ii) By definition, for any pdf $f(x)$, the **mode**, if any, is that value of x for which

$$f'(x) = 0$$

and

$$f''(x) < 0.$$

Now,

$$f'(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-[(x-\mu)^2/2\sigma^2]} \left(\frac{x-\mu}{\sigma^2} \right) \left(-\frac{2}{2} \right)$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}} (x-\mu) e^{-[(x-\mu)^2/2\sigma^2]}$$

Equating $f'(x) = 0$, we see that $x = \mu$.

Now,

Again differentiating, we obtain

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-(x-\mu)^2/2\sigma^2} - e^{-(x-\mu)^2/2\sigma^2}\ \frac{(x-\mu)^2}{\sigma^2} \right]$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \left[1 - e^{-(x-\mu)^2/2\sigma^2} \frac{(x-\mu)^2}{\sigma^2} \right]$$

Substituting $x = \mu$ in $f''(x)$, we see that $f''(x) < 0$.

Thus,

$x = \mu$ is the mode of the normal distribution.

Hence the median and mode are both equal to μ.

Since,

mean = median = mode,

the normal distribution is symmetrical and unimodal.

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Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 172

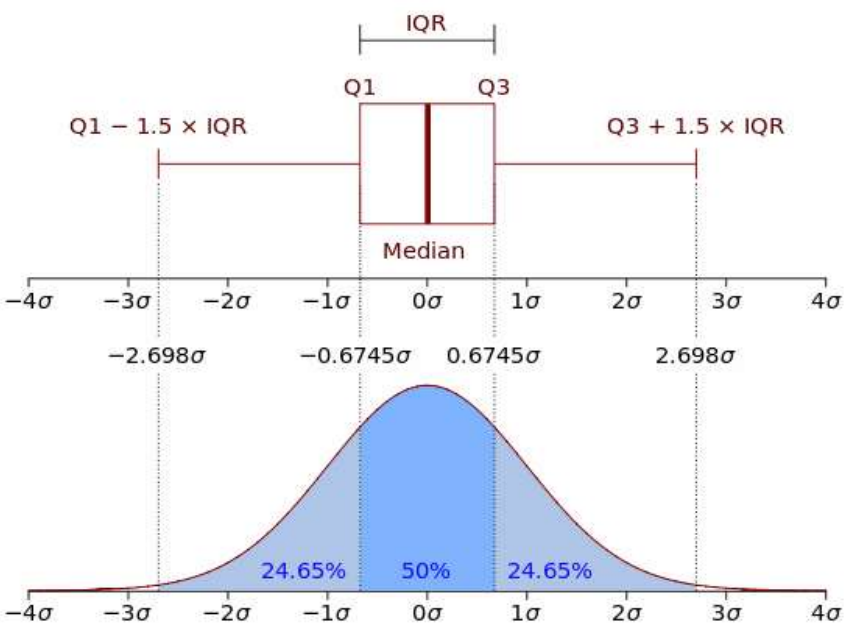
The Quartile Deviation of the Normal distribution is approximately $(2/3)^{rd}$ of its standard deviation

Explanation:

For any distribution, the **Semi-Inter-Quartile Range** or **Quartile Deviation**, Q, is given by

$$Q = \frac{Q_3 - Q_1}{2}$$

Inter-Quartile Range (IQR)



Now, for a **Symmetric** distribution,

$$Q_3 - Q_2 = Q_2 - Q_1 \Rightarrow Q_3 = 2Q_2 - Q_1$$

$$\Rightarrow Q = \frac{2Q_2 - Q_1 - Q_1}{2} = \frac{2Q_2 - 2Q_1}{2} = Q_2 - Q_1$$

and

$$Q = Q_3 - Q_2$$

But, for a **Symmetric** distribution,

$$Q_2 = \mu$$

implying that, for a symmetric distribution,

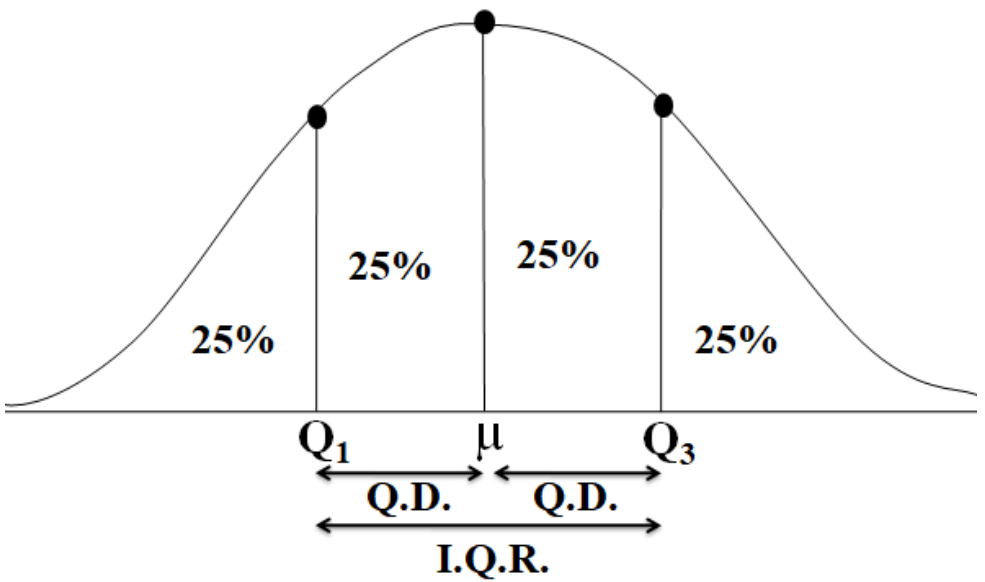
we can write

$$Q = \mu - Q_1$$

and also

$$Q = Q_3 - \mu$$

For any **symmetric** distribution,



As such, for the normal distribution, the following equation holds:

$$\int_{Q_1}^{Q_3} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2} \Rightarrow \int_{\mu-Q}^{\mu+Q} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

which can be re-written as

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu-Q}^{\mu+Q} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

We have

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{\mu-Q}^{\mu+Q} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

Putting, $z = \frac{x-\mu}{\sigma} \Rightarrow \sigma z = x-\mu \Rightarrow x = \mu + \sigma z$

$$\Rightarrow \frac{dx}{dz} = 0 + \sigma = \sigma \Rightarrow \sigma dz = dx$$

Change of limits:

$$\text{As } x \to \mu + Q \Rightarrow z \to \frac{\mu + Q - \mu}{\sigma} = \frac{Q}{\sigma}$$

$$\text{and as } x \to \mu - Q \Rightarrow z \to \frac{\mu - Q - \mu}{\sigma} = -\frac{Q}{\sigma}$$

So

substituting the values, we obtain

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-Q/\sigma}^{Q/\sigma} e^{-\frac{1}{2}(z)^2} .\sigma dz = \frac{1}{2}$$

Now, $e^{-\frac{1}{2}(z)^2}.\sigma$ is an even function

So

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-Q/\sigma}^{Q/\sigma} e^{-\frac{1}{2}(z)^2}.\sigma dz = \frac{1}{2}$$

can be re-written as

$$\frac{1}{\sigma\sqrt{2\pi}}2\int_{0}^{Q/\sigma} e^{-\frac{1}{2}(z)^2}.\sigma dz = \frac{1}{2} \Rightarrow \int_{0}^{Q/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \frac{1}{4}$$

Now, from the area table of the standard normal distribution, we find that

$$\int_{0}^{0.6745} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 0.2500$$

Hence, we can write

$$Q/\sigma = 0.6745$$

or

$$Q = 0.6745\ \sigma \qquad \ldots(1)$$

Therefore, using eq.(1), we obtain the values of the quartiles as follows:

$$Q_1 = \mu - 0.6745\sigma \quad \text{and} \quad Q_3 = \mu + 0.6745\sigma$$

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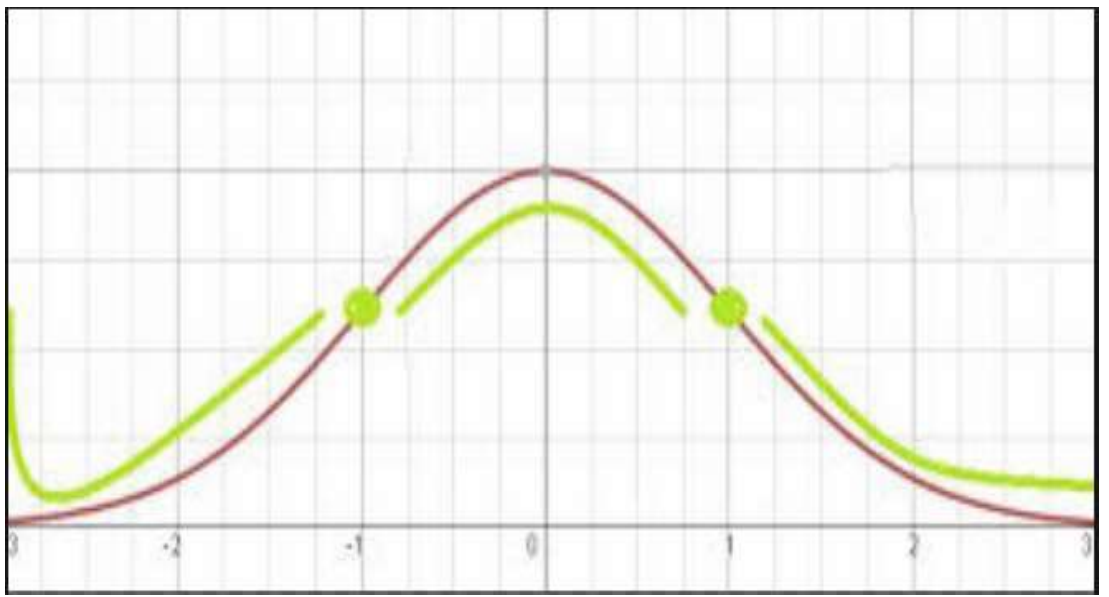
Topic No. 173

The normal curve has

Points of Inflection

which are equidistant from the mean

Definition: By a ‘point of inflection’ we mean a point at which the **concavity** of a curve changes**,**



From calculus, we know that such a point is obtained by solving the equation

$$f''(x) = 0$$

Taking the first derivative of the PDF of the normal distribution, we have

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \right) = \frac{1}{\sigma\sqrt{2\pi}} \frac{d}{dx}\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \right)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left\{ e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \frac{d}{dx}\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2 \right) \right\}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \left[\left(-\frac{1}{2} \right) . 2 . \left(\frac{x-\mu}{\sigma} \right) \left(\frac{1}{\sigma} \right) \right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \left[-\left(\frac{x-\mu}{\sigma^2} \right) \right] = -\frac{1}{\sigma^2\sqrt{2\pi}} (x-\mu) . e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

- To find the **points of inflection**, we take the **second** derivative:

$$f''(x) = \frac{d}{dx}\left(-\frac{1}{\sigma^3\sqrt{2\pi}}(x-\mu).e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \right) = -\frac{1}{\sigma^3\sqrt{2\pi}}\left[\frac{d}{dx}\left((x-\mu).e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \right) \right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \left(\frac{d}{dx}(x-\mu) \right) + (x-\mu)\left(\frac{d}{dx}\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \right) \right) \right]$$

or

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}(1) + (x-\mu)\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\frac{d}{dx}\left(-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2\right)\right)\right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} - (x-\mu)\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(\frac{1}{2}.2\left[\frac{x-\mu}{\sigma}\right]\left[\frac{d}{dx}\left[\frac{x-\mu}{\sigma}\right]\right]\right)\right)\right]$$

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} - (x-\mu)\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left(\frac{1}{2}.2\left[\frac{x-\mu}{\sigma}\right]\left[\frac{1}{\sigma}\right]\right)\right)\right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} - (x-\mu)\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}.\left(\frac{x-\mu}{\sigma^2}\right)\right)\right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}\left[e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} - \frac{(x-\mu)^2}{\sigma^2}\left(e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\right)\right]$$

$$= -\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

so that, finally, we have:

$$f''(x) = -\frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \left[1 - \frac{(x-\mu)^2}{\sigma^2}\right]$$

Now, equating the second derivative to zero, we have

$$-\frac{1}{\sigma^3\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}\left[1-\frac{(x-\mu)^2}{\sigma^2}\right]=0$$

$$\Rightarrow\left[1-\frac{(x-\mu)^2}{\sigma^2}\right]=0\times\left(-\frac{\sigma^3\sqrt{2\pi}}{e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}}\right)\Rightarrow 1-\frac{(x-\mu)^2}{\sigma^2}=0$$

$$\Rightarrow\frac{(x-\mu)^2}{\sigma^2}=1\Rightarrow(x-\mu)^2=\sigma^2$$

or $\quad x-\mu=\pm\sigma\Rightarrow x=\mu\pm\sigma$

or $\quad x=\mu+\sigma,\quad x=\mu-\sigma.$

At these two points, the value of the function $f(x)$ is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\mu \pm \sigma - \mu}{\sigma}\right]^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\pm\sigma}{\sigma}\right]^2} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}[\pm 1]^2}$$

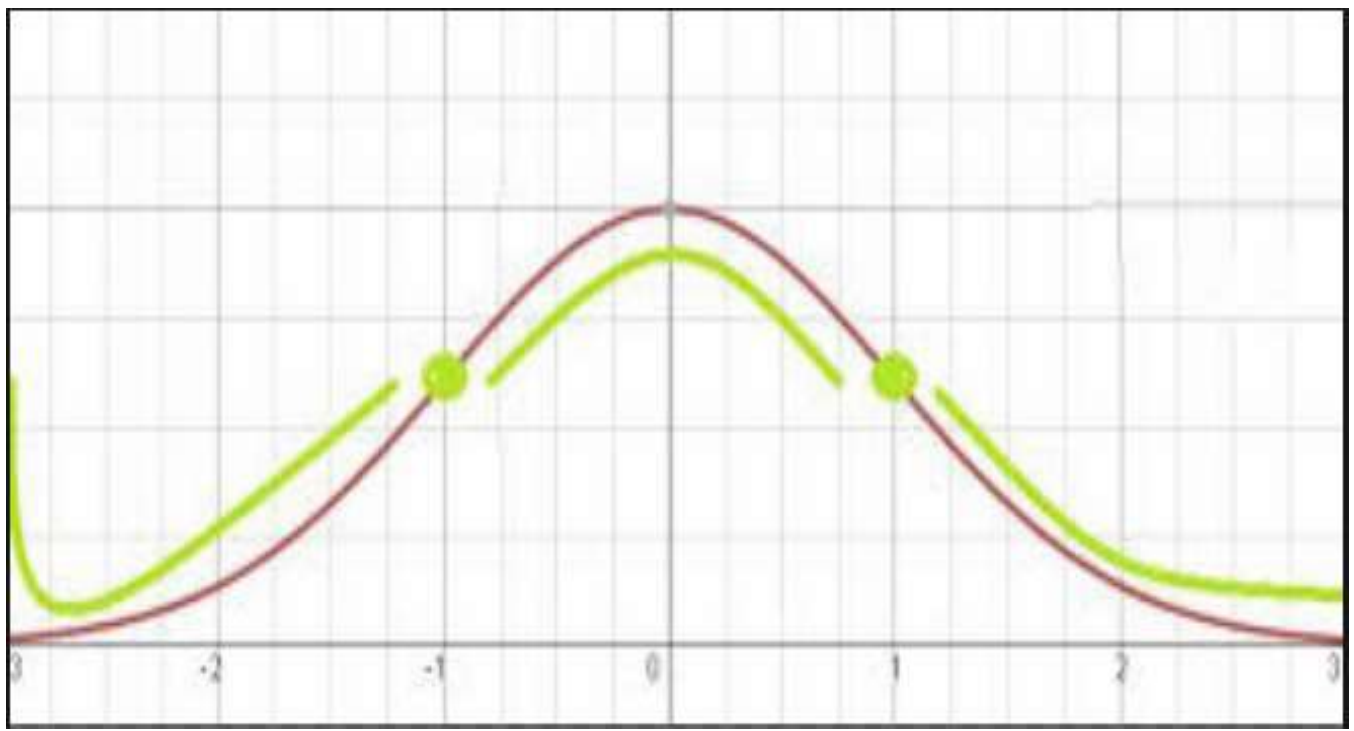
$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} = \frac{1}{\sigma\sqrt{2\pi} e^{\frac{1}{2}}} = \frac{1}{\sigma\sqrt{2\pi e}}.$$

Hence, the two points of inflection of normal curve are

$$\left[\mu-\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right] \textit{and} \left[\mu+\sigma,\frac{1}{\sigma\sqrt{2\pi e}}\right].$$

In other words, the points of inflection occur on the **right** and on the **left** of the mean **at a distance equal to standard deviation** and thus the graph of the normal curve is **bell-shaped**.

The ‘bell curve’



Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 174

For the normal distribution,

the odd order moments about the mean are All zero

and

the even order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)(2n-5)\ldots 5.3.1\,\sigma^{2n}$$

Theorem:

For the normal distribution, the **odd** order moments about the mean are all zero and the **even** order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)\ldots 5.3.1\,\sigma^{2n}$$

Proof:

The **odd order moments** about the mean are given by

$$\mu_{2n+1} = E\left(X-\mu\right)^{2n+1} = \int_{-\infty}^{\infty}\left(x-\mu\right)^{2n+1}.\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$$

where $n \in \square$.

Substituting, $z = \dfrac{x-\mu}{\sigma} \Rightarrow \sigma z = x-\mu \Rightarrow x = \mu+\sigma z$

$$\Rightarrow \frac{dx}{dz} = \sigma \Rightarrow \sigma dz = dx$$

Limits remain unchanged.

& we get,

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (\sigma z)^{2n+1} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \sigma dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} e^{-\frac{1}{2}(z)^2} dz$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{z^2}{2}} dz = 0 \text{ as } z^{2n+1} e^{-\frac{z^2}{2}} \text{ is an odd function of } z.$$

Thus,

$$\mu_1 = \mu_3 = \mu_5 = \cdots = 0$$

$$\mu_{2n+1} = \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{-\frac{z^2}{2}} dz = 0$$

as $z^{2n+1}e^{-\frac{z^2}{2}}$ is an odd function of z.

Thus,

$$\mu_1 = \mu_3 = \mu_5 = \cdots = 0$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 175

Derivation of
the Variance of
the Exponential Distribution

A very interesting property:

The **mean and standard deviation** of the exponential distribution are equal.

We know that, for the exponential distribution given by

$$f(x) = \lambda e^{-\lambda x}, \qquad x \geq 0, \lambda > 0$$

the mean is

$$E(X) = \frac{1}{\lambda}$$

1. Derivation of the **Variance:**

According to the short-cut formula

$$\mathrm{Var}\left(X\right)=E\left(X^{2}\right)-\left[E\left(X\right)\right]^{2},$$

Now

$$E\left(X^2\right)=\int_0^\infty x^2.f\left(x\right)dx=\int_0^\infty x^2.\lambda e^{-\lambda x}dx$$

First of all we will apply the formula of integration by parts:

$$\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

Here, we let $u = x^2$ and $v = \lambda e^{-\lambda x} = -(-\lambda e^{-\lambda x})$

$$\text{so that } \int vdx = \int -(-\lambda e^{-\lambda x})dx = -\left(e^{-\lambda x}\right)$$

$$\text{and } \frac{du}{dx} = \frac{d}{dx}\left(x^2\right) = 2x.$$

Hence, we have

$$\int x^2.\lambda e^{-\lambda x}dx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

$$= x^2(-e^{-\lambda x}) - \int 2x.\left(-e^{-\lambda x}\right)dx = -x^2e^{-\lambda x} - 2\int\left(-xe^{-\lambda x}\right)dx$$

$$= -x^2e^{-\lambda x} - \frac{2}{\lambda}\int\left(\lambda\left(-xe^{-\lambda x}\right)\right)dx$$

$$= -x^2e^{-\lambda x} - \frac{2}{\lambda}\int x\left(-\lambda e^{-\lambda x}\right)dx$$

Hence, we have

$$\int x^2.\lambda e^{-\lambda x} dx = -x^2 e^{-\lambda x} + \frac{2}{\lambda}\int x\left(\lambda e^{-\lambda x}\right) dx$$

implying that

$$E\left(X^2\right) = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \frac{2}{\lambda} E\left(X\right)$$

or

$$E\left(X^2\right) = -\left[\frac{x^2}{e^{\lambda x}}\right]_0^{\infty} + \frac{2}{\lambda} E\left(X\right)$$

In order to apply the upper limit to $\left[\frac{x^2}{e^{\lambda x}}\right]$, we need to utilize the L'Hospital Rule.

We have $\frac{x^2}{e^{\lambda x}}$

Now, $\lim_{x\to\infty}\frac{x^2}{e^{\lambda x}} = \lim_{x\to\infty}\frac{2x}{\lambda e^{\lambda x}} = \lim_{x\to\infty}\frac{2}{\lambda^2 e^{\lambda x}} = \frac{2}{\lambda^2 e^{\lambda(\infty)}} = \frac{2}{\infty} = 0.$

On the other hand, $\lim_{x\to 0}\frac{x^2}{e^{\lambda x}} = \frac{(0)^2}{e^{\lambda(0)}} = \frac{0}{e^0} = \frac{0}{1} = 0.$

Hence, we have

$$E\left(X^2\right) = -\left[\frac{x^2}{e^{\lambda x}}\right]_0^{\infty} + \frac{2}{\lambda}E\left(X\right)$$

$$= -\left[0-0\right] + \frac{2}{\lambda}E\left(X\right)$$

$$= -0 + \frac{2}{\lambda}E\left(X\right) = \frac{2}{\lambda}E\left(X\right) = \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}$$

Hence,

$$\mathrm{Var}(X) = E(X^2) - \left[E(X)\right]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{2-1}{\lambda^2} = \frac{1}{\lambda^2}$$

implying that

$$S.D.(X) = \sigma = \frac{1}{\lambda}.$$

Hence we have the interesting property that the **mean and standard deviation** of the exponential distribution are equal.

This, in turn, means that **the Coefficient of Variation of the Exponential distribution is always equal to 1.**

Explanation: The coefficient of variation is defined as $CV = \frac{\sigma}{\mu}$.

Hence, for the exponential distribution given by

$$f(x) = \lambda e^{-\lambda x}, 0 < x < \infty$$

the coefficient of variation is

$$CV = \frac{\sigma}{\mu} = \frac{1/\lambda}{1/\lambda} = \frac{\lambda}{\lambda} = 1.$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 176

Obtaining the first two Raw Moments of the Normal Distribution from its MGF

The first two Raw Moments of the Normal Distribution are given by

$$E\left(X\right) \ \& \ E\left(X^2\right).$$

We know that, the **mgf** of Normal Distribution is as

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}, \qquad \text{for} \quad -\infty < t < \infty.$$

The first two derivatives of $M_X(t)$ are easily derived as,

$$M_X'(t) = \frac{d}{dt}\left[e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \frac{d}{dt}\left[\mu t + \frac{1}{2}\sigma^2 t^2 \right]$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left[\mu + \frac{1}{2}\sigma^2 2t \right] = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left[\mu + \sigma^2 t \right]$$

Putting $t = 0$, we have

$$M_X'(0) = e^{\mu(0)+\frac{1}{2}\sigma^2(0^2)}\left[\mu + \sigma^2(0)\right]$$

$$= e^{0+0}(\mu + 0) = e^0(\mu) = (1)\mu = \mu.$$

But, we know that

$$M_X'(0) = E(X)$$

Therefore

$$E(X) = \mu.$$

And now taking the **second derivative** of $M_X(t)$ as follows:

$$M_X''(t)$$

$$= \frac{d}{dt}\left[M_X'(t) \right] = \frac{d}{dt}\left[e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left[\mu + \sigma^2 t \right] \right] = \frac{d}{dt}\left[\mu e^{\mu t + \frac{1}{2}\sigma^2 t^2} + \sigma^2 t\, e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right]$$

$$= \left[\mu e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left(\mu + \frac{1}{2}\sigma^2 2t \right) \right] + \left[\sigma^2 \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2} (1) + t \left(e^{\mu t + \frac{1}{2}\sigma^2 t^2} \left(\mu + \frac{1}{2}\sigma^2 2t \right) \right) \right) \right]$$

$$= \left[\mu^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} + \mu\sigma^2 t e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right] + \left[\left(\sigma^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} + \sigma^2 \mu t e^{\mu t + \frac{1}{2}\sigma^2 t^2} + \sigma^4 t^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right) \right]$$

Putting t = 0, we have,

$$M_X''(0) = \left[\mu^2 e^{\mu(0)+\frac{1}{2}\sigma^2(0)} + \mu\sigma^2(0) e^{\mu t+\frac{1}{2}\sigma^2(0)} \right]$$

$$+\left[\left(\sigma^2 e^{\mu(0)+\frac{1}{2}\sigma^2(0)} + \sigma^2\mu(0) e^{\mu(0)+\frac{1}{2}\sigma^2(0)} + \sigma^4(0) e^{\mu(0)+\frac{1}{2}\sigma^2(0)} \right) \right]$$

$$= \left[\mu^2 e^0 + 0 \right] + \left[\sigma^2 e^0 + 0 + 0 \right]$$

$$= \mu^2(1) + \sigma^2(1) = \mu^2 + \sigma^2.$$

But, we know that

$$M_X''(0) = E\left(X^2\right)$$

Therefore

$$E\left(X^2\right) = \mu^2 + \sigma^2.$$

So, we find the **variance** as

$$\begin{aligned}\mathrm{Var}(X) &= E\left(X^2\right) - \left[E(X)\right]^2 \\ &= \mu^2 + \sigma^2 - \mu^2 = \sigma^2.\end{aligned}$$

NOTE:

All the higher moments can be derived in a similar manner.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 177

Cumulant Generating Function

of the $N(\mu,\sigma^2)$ distribution

and

its utilization for finding its cumulants

Definition of Cumulant Generating Function

For any distribution for which the mgf exists, the cumulant-generating function $K(t)$ is the natural logarithm of the moment generating function i.e.

$$K\left(t\right) = \ln E\left(e^{tX}\right)$$

For the **normal distribution** with expected value μ and variance σ^2, the cumulant generating function is

$$K(t) = \mu t + \frac{\sigma^2 t^2}{2}.$$

- The first derivative of the cumulant generating function is obtained as follows:

$$K'(t) = \frac{d}{dt}\left(\mu t + \frac{\sigma^2 t^2}{2}\right) = \left(\mu \frac{d}{dt} t + \frac{d}{dt}\frac{\sigma^2 t^2}{2}\right)$$

$$= \left(\mu(1) + \frac{\sigma^2}{2}(2t)\right) = \left(\mu + \sigma^2(t)\right) = \mu + \sigma^2 t$$

The second derivative of the cumulant generating function is obtained as follows:

$$K''(t) = \frac{d}{dt}\left(K'(t)\right) = \frac{d}{dt}\left(\mu + \sigma^2 t\right)$$

$$= \left(\frac{d}{dt}(\mu) + \sigma^2 \frac{d}{dt}(t)\right) = 0 + \sigma^2 (1) = \sigma^2.$$

The third derivative of the cumulant generating function is given by:

$$K'''(t) = \frac{d}{dt}\left(K''(t)\right) = \frac{d}{dt}\left(\sigma^2\right) = 0$$

implying that all higher derivatives will be zero.

Now, the first cumulant is obtained as follows:

$$K'(0) = \left(\mu + \sigma^2(t)\right)_{t=0} = \mu + \sigma^2(0) = \mu + 0 = \mu$$

Similarly, the second cumulant is:

$$K''(0) = \left(\sigma^2\right)_{t=0} = \sigma^2.$$

And the third cumulant is:

$$K'''(0) = (0)_{t=0} = 0.$$

Similarly, all the higher cumulants will be zero.

Thus, in a nutshell, the cumulants of the normal distribution with expected value μ and variance σ^2 are

$$K_1 = \mu,$$

$$K_2 = \sigma^2,$$

and

$$K_3 = K_4 = \ldots = 0.$$

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Probability Distributions

by

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Topic No. 178

For the normal distribution,
the odd order moments about the mean are All zero
and
the even order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)(2n-5)\ldots 5.3.1\,\sigma^{2n}$$

Theorem:

For the normal distribution, the **odd** order moments about the mean are all zero and the **even** order moments are given by

$$\mu_{2n} = (2n-1)(2n-3)\ldots 5.3.1\,\sigma^{2n}$$

Proof:

The **even order moments** about the mean are obtained as follows:

$$\mu_{2n} = E\left(X-\mu\right)^{2n}$$

$$= \int_{-\infty}^{\infty} \left(x-\mu\right)^{2n} . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Substituting, $z = \frac{x-\mu}{\sigma} \Rightarrow \sigma z = x-\mu \Rightarrow x = \mu + \sigma z$

$\Rightarrow \frac{dx}{dz} = \sigma \Rightarrow \sigma dz = dx,$ Limits will remain **unchanged**.

Therefore,

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

can be re-written as

$$\mu_{2n} = \int_{-\infty}^{\infty} (\sigma z)^{2n} . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \sigma dz$$

And, Now, as we know that, for any **even function** g i.e. a function for which $g(-x) = g(x)\ \forall x \in \mathbb{R}$

the integral $\int_{-\infty}^{\infty} g(x)\,dx$ is equal to $2\int_{0}^{\infty} g(x)\,dx$.

Therefore, we have

$$\mu_{2n} = 2\int_0^{\infty} (\sigma z)^{2n} . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \sigma dz = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} (\sigma z)^{2n} e^{-\frac{1}{2}(z)^2} dz$$

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} \left(z^2\right)^n e^{-\frac{z^2}{2}} dz,$$

Let $y = \frac{z^2}{2} \Rightarrow 2y = z^2 \Rightarrow \sqrt{2y} = z$

Also, $y = \frac{z^2}{2} \Rightarrow \frac{dy}{dz} = \frac{2z}{2} = z \Rightarrow dy = zdz \Rightarrow \frac{dy}{z} = dz \Rightarrow \frac{dy}{\sqrt{2y}} = dz$

Limits remain unchanged.

Therefore, $\mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^\infty (2y)^n e^{-y} \frac{dy}{\sqrt{2y}} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^\infty (2y)^{n-\frac{1}{2}} e^{-y} dy$

$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^\infty 2^{n-\frac{1}{2}} (y)^{n-\frac{1}{2}} e^{-y} dy = \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_0^\infty 2^{n+1-\frac{1}{2}} (y)^{n-1/2} e^{-y} dy$

or

$$\mu_{2n} = \frac{\sigma^{2n}}{\sqrt{2\pi}} 2^{n+\frac{1}{2}} \int_0^\infty y^{n-1/2} e^{-y} dy = \frac{\sigma^{2n}}{\sqrt{\pi}} .2^{-\frac{1}{2}} .2^{n+\frac{1}{2}} \int_0^\infty y^{n-1/2} e^{-y} dy$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty y^{n-1/2-1+1} e^{-y} dy = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty y^{n+1/2-1} e^{-y} dy$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^\infty y^{(n+1/2)-1} e^{-y} dy$$

$$\mu_{2n} = \frac{2^n \sigma^{2n} \Gamma\left(n+1/2\right)}{\sqrt{\pi}}$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \cdot \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

$$\left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\mu_{2n} = \left(2n-1\right)\left(2n-3\right)\ldots 5.3.1\ \sigma^{2n}$$

Putting $n = 1$ *and* 2, *we get* $\mu_2 = \sigma^2$ *and* $\mu_4 = 3\sigma^4$.

Hence $\beta_1=0$ meaning that skewness is zero,

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3,$$

i.e. the normal curve has zero kurtosis.

Virtual University of Pakistan
Probability Distributions

by

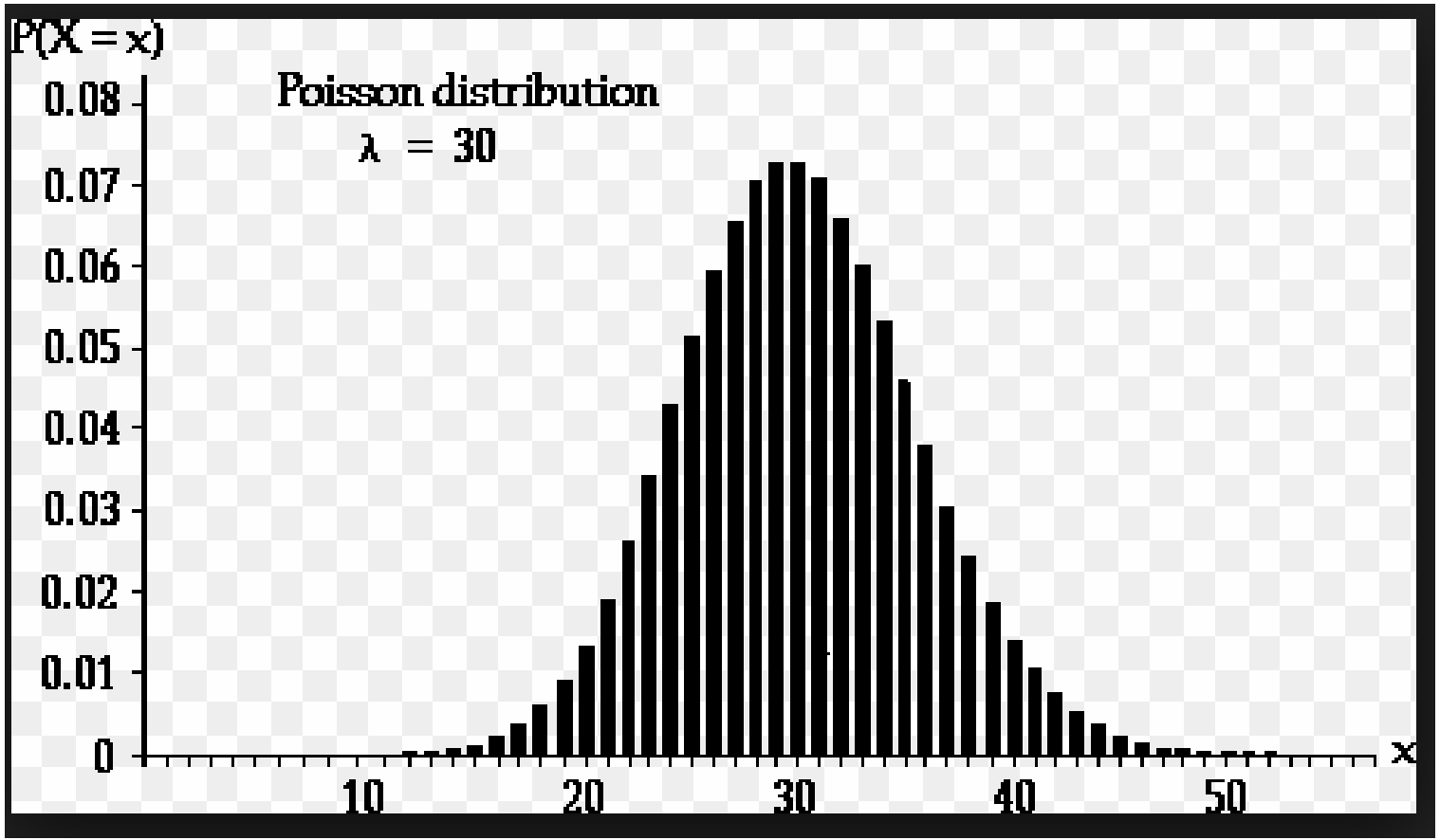
Dr. Saleha Naghmi Habibullah

Topic No. 178

An example of
the
Normal Approximation to the Poisson distribution

Normal Approximation to the Poisson distribution

- It has been mathematically proved that, in the case of a Poisson Process leading to a Poisson distribution with mean=variance= λ, if λ tends to infinity, then the Poisson distribution tends to the Poisson distribution.


P(X = x)
Poisson distribution
λ = 30
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
10
20
30
40
50
x

- In practice, for sufficiently large values of λ, (say $\lambda>1{,}000$), the **Normal**($\mu = \lambda, \sigma^2 = \lambda$) **Distribution** is an excellent **approximation to the Poisson**(λ) **Distribution**.

- If λ is greater than about 10, then the **Normal Distribution** is a good **approximation** if an appropriate continuity correction is performed.

Example:

The number of calls received by an office switch board per hour follows a Poisson distribution with parameter 25.

Find the probabilities that in one hour

(a) There are between 23 and 26 calls (inclusive),

(b) more than 30 calls, using the **normal approximation** to the Poisson distribution.

Let X be the random variable, the number of calls received in one hour.

Then, X is $P(x\ ;\ 25)$.

We require (a) $P(23 \leq X \leq 26)$ and (b) $P(X>30)$.

Using the normal approximation, X is $N(25,25)$

(a) P(23≤X≤26) becomes on continuous scale P(22.5≤X≤26.5) , X is N(25,25).

The z-values are: At, $x = 22.5,\ z = \dfrac{22.5 - 25}{5} = -0.5,$ and

at $x = 26.5,\ z = \dfrac{26.5 - 25}{5} = 0.3,$

$$\therefore \quad P\left(22.5 \le X \le 26.5\right) = P\left(-0.5 \le Z \le 0.3\right)$$

$$= P\left(0 < Z < 0.5\right) + P\left(0 < Z < 0.3\right) = 0.1915 + 0.1179$$

$$= 0.3094 = 30.94\%.$$

(b) P(X >30) becomes on continuous scale P(X >29.5)

$$\therefore \quad P\left(X > 29.5\right) = P\left(\frac{X-25}{5} > \frac{29.5-25}{5}\right)$$

$$= P\left(Z > \frac{4.5}{5}\right) = P\left(Z > 0.9\right) = 1 - P\left(Z < 0.9\right)$$

$$= 1 - 0.8159 = 0.1841 = 18.41\%.$$

I would like to encourage you to compute the exact probabilities by using the pmf of the Poisson distribution

and then compare the exact probabilities with those obtained through the Normal Approximation to the Poisson distribution.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 180

Bivariate Normal Distribution

(PDF and Shape
of the distribution
&
a few real-life examples)

Definition

A pair of random variables X_1 and X_2 have a bivariate normal distribution and they are referred to as **jointly normally distributed** random variables **if and only if** their joint probability density is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}}$$

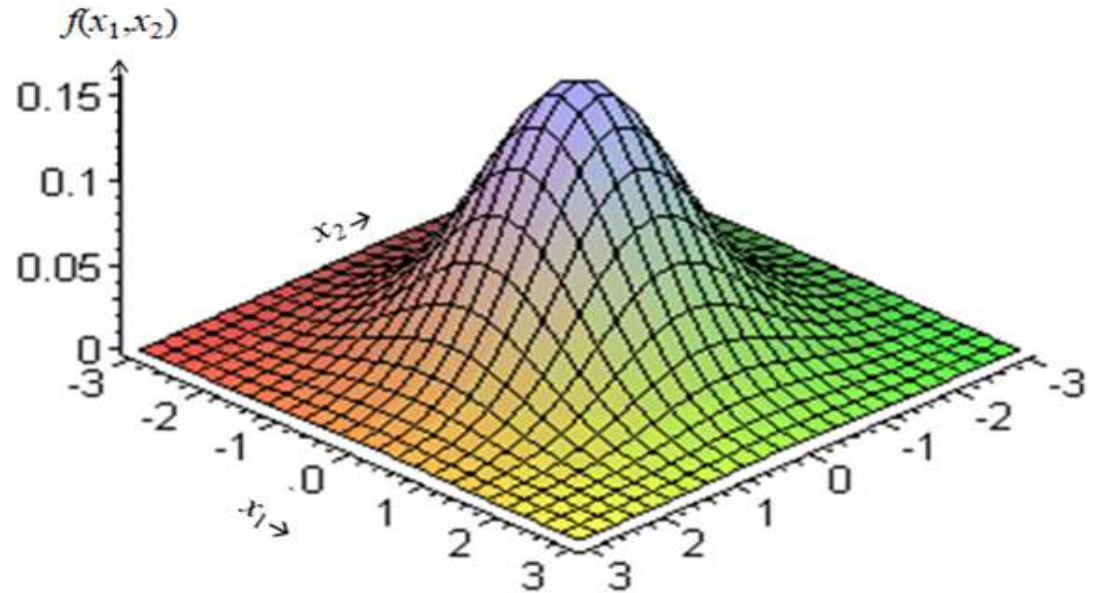
for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$,

where $\mu_1 \in \square$, $\mu_2 \in \square$, $\sigma_1 > 0, \sigma_2 > 0$, and $-1 \le \rho \le 1$.

This is referred to as the $N\left(\mu_x,\ \mu_y,\ \sigma_x^2, \sigma_y^2, \rho\right)$ distribution.

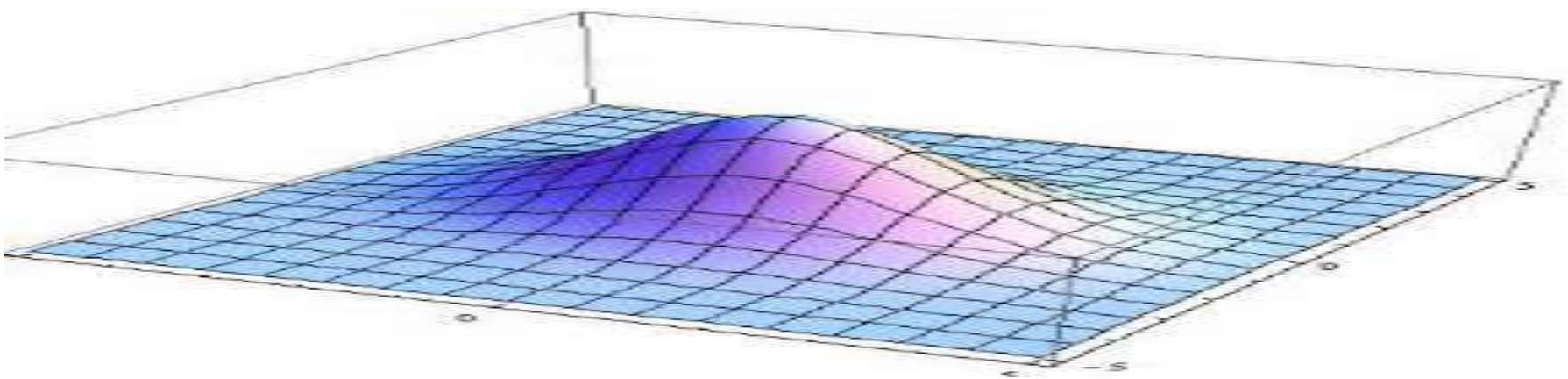
This joint p.d.f., has a bell-type shape.

Bivariate Normal Distribution with $\rho = 0$ and $\sigma_1 = \sigma_2 = \sigma$:



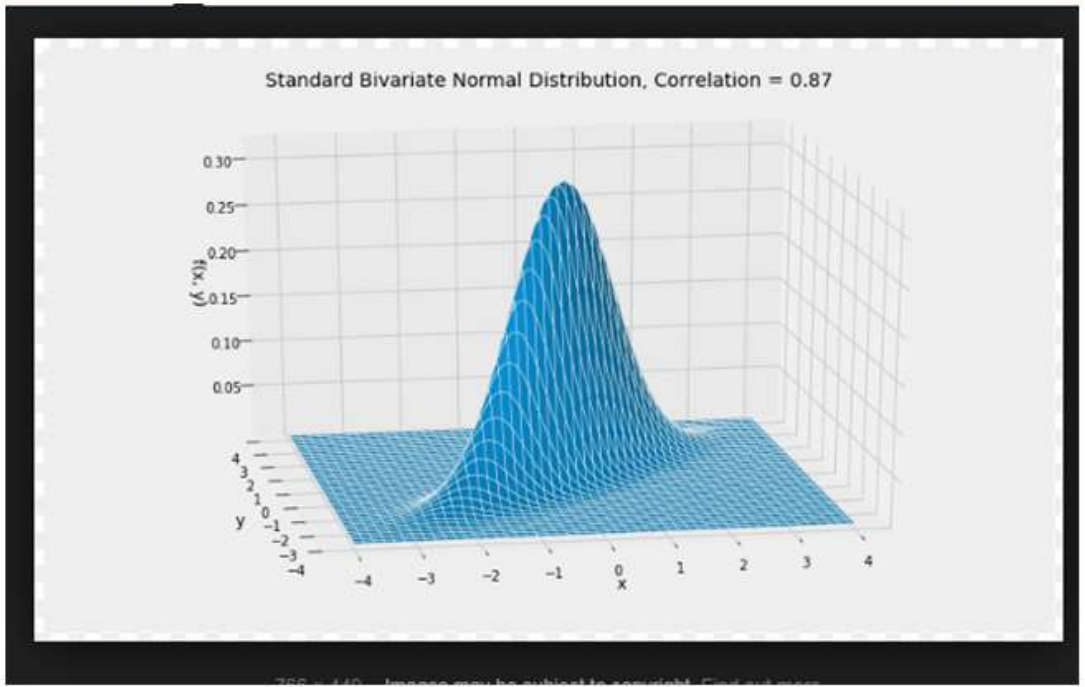
An <u>exact</u>, <u>beautiful</u> bell-shaped surface

Bivariate Normal Distribution with $\rho = 0$ and $\sigma_1 \neq \sigma_2$:



Even then, it is <u>almost</u> a bell-type surface

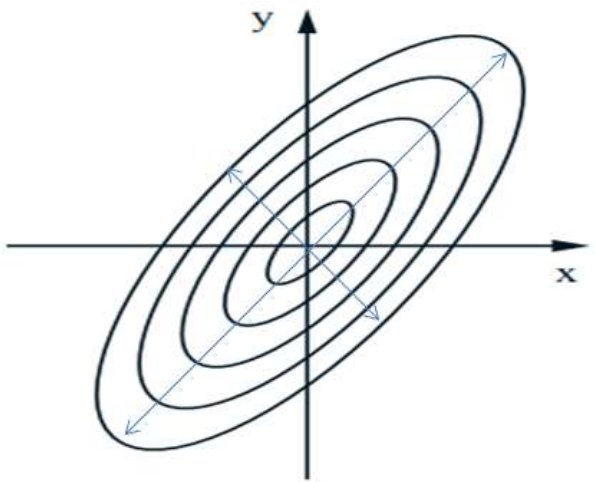
Bivariate Normal Distribution with $\rho = 0.87$



https://www.google.com/search?q=bivariate+normal+distribution&rlz=1C1CHBD_enPK785PK785&source=lnms&tbm=isch&sa=X&ved=0ahUKEwjN7IqN-ODhAhUQGuwKHeq3DgcQ_AUIDigB&biw=1280&bih=657#imgrc=j_DR8twmIeWiwM:

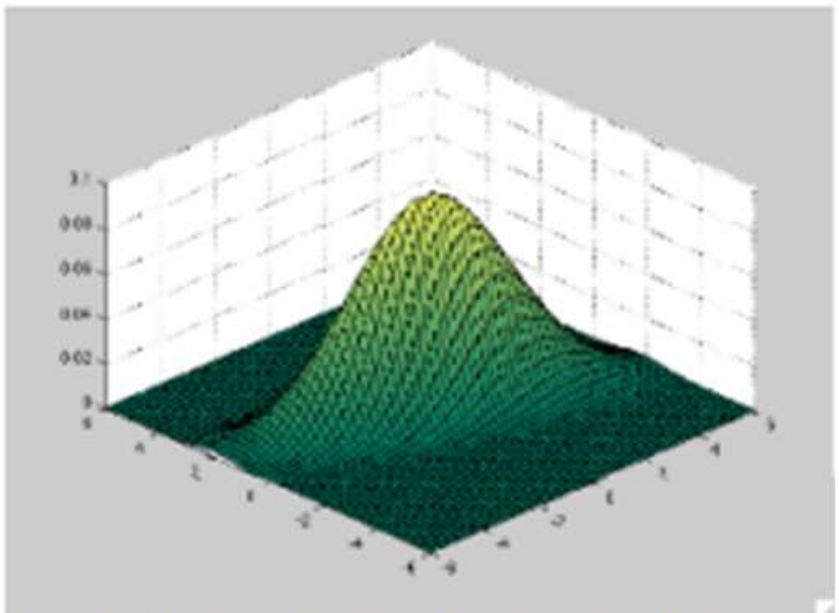
‘Base’

of the Bivariate Normal Distribution with ρ = 0.87



The diagram above needs to be ACCORDING to the bell-type shape in the previous graph (should be “thinner”).

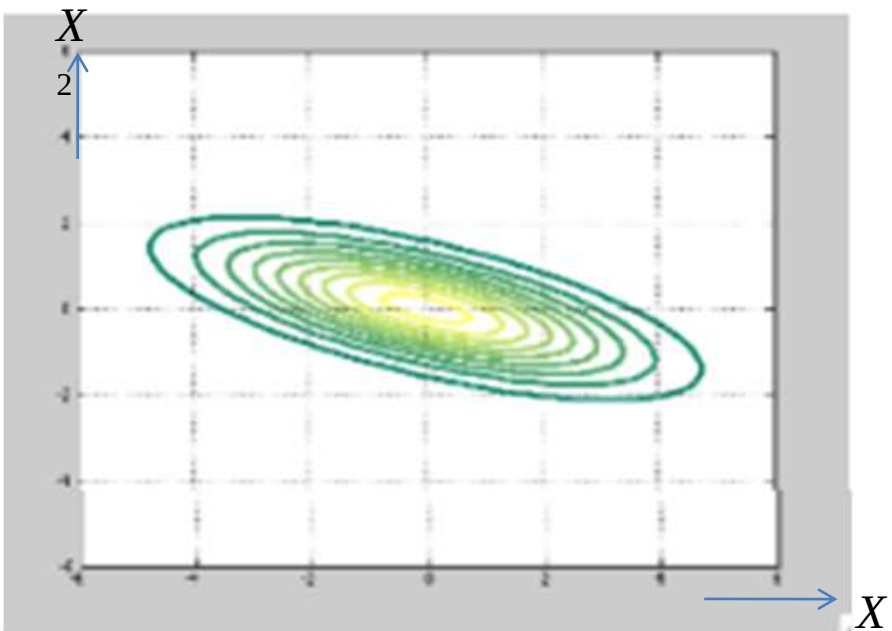
Bivariate Normal Distribution with ρ = - 0.67



https://cedar.buffalo.edu/~srihari/CSE674/Chap7/7.1-MultiGauss.pdf

‘Base’

of the Bivariate Normal Distribution with ρ = - 0.67



https://cedar.buffalo.edu/~srihari/CSE674/Chap7/7.1-MultiGauss.pdf

Explanation

Conceive it:

as if we have a ‘bell-type’
that is placed on the
<u>FLOOR</u>.

The concept of

the ‘X_1X_2 floor’

What is meant by the 'X_1X_2 floor'?

Consider the three-dimensional space for example the room in which you are sitting right now.

- Now focus on the floor of the room.
- One 'edge' of the floor will be called the X_1 axis
- The other 'edge' of the floor will be called the X_2 axis

And, as such, the floor will be called the 'X_1X_2 floor'

(actually it is the X_1X_2-plane)

Next:

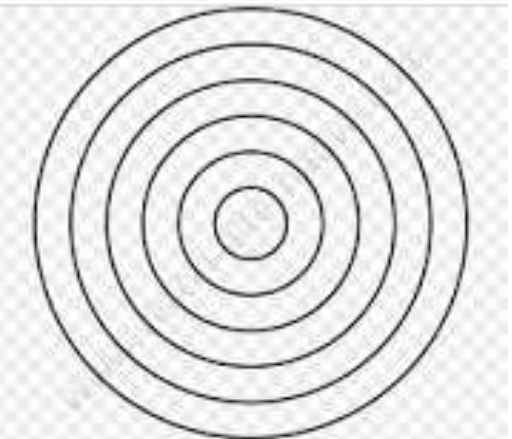
The concept of

Contours

By the term **‘contour’**, we mean the **set of points on the X_1X_2 floor** corresponding to which **the ordinates** are of **equal height.**

Explanation: Consider the bivariate normal distribution for which $\rho = 0$ and $\sigma_1 = \sigma_2 = \sigma$.

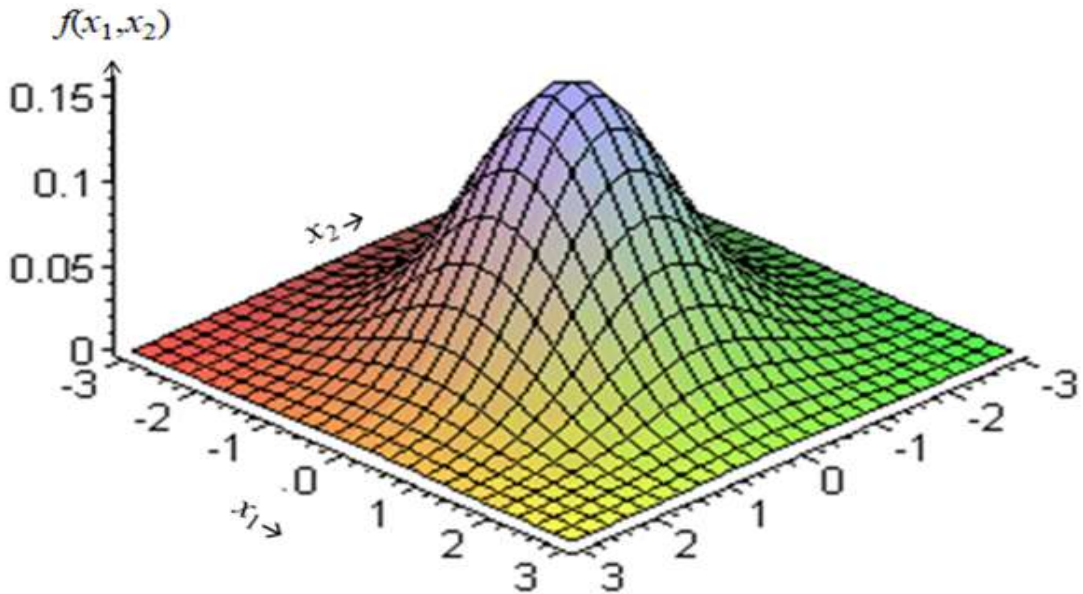
In this case the **contours** of the distribution will be **concentric circles** because, against **each** of these **circles** (on the X_1X_2 floor), the ordinates are of **equal height**.

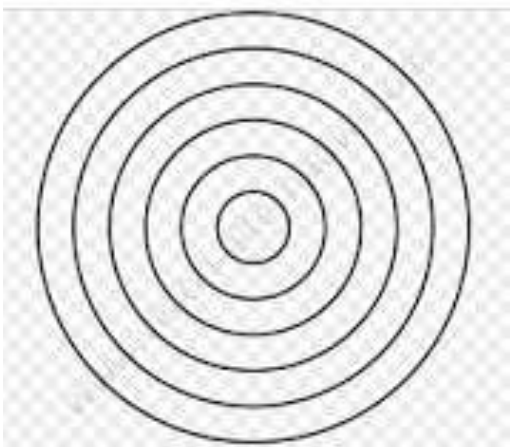


- The ordinates against the outer circle are very short;
- The ordinates against the Second circle (inner circle) are a little taller than the ones against the outer circle;
- The ordinates against the Third (inner) circle are taller than the ones against the Second circle;
- The ordinates against the Fourth (inner) circle are taller than the ones against the Third circle;

And so on.

The **circular contour** corresponding to a lower height is **wider** than the one corresponding to a greater height.





Next:

A circle can be regarded as

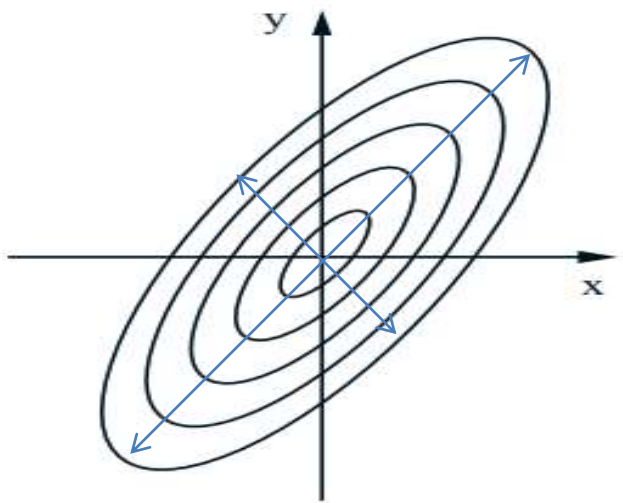
a <u>special case</u>

of an **<u>ELLIPSE.</u>**

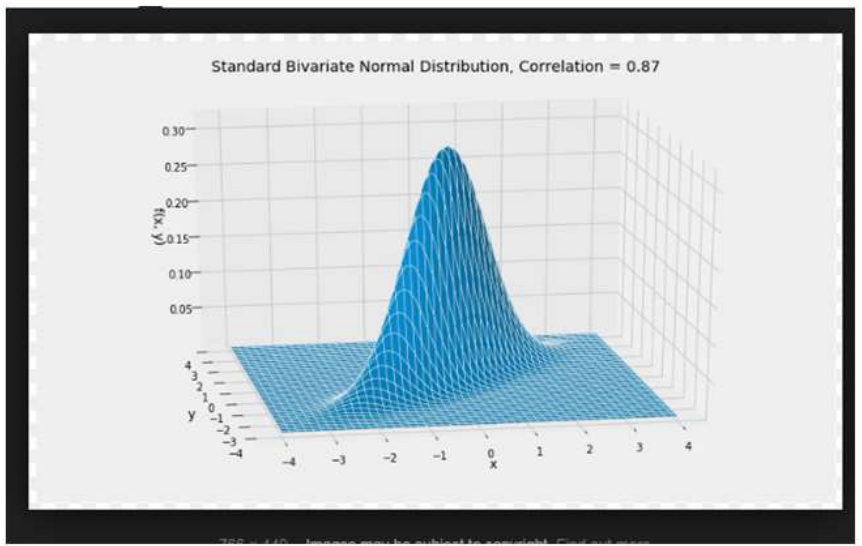
General Concept of Ellipse:

We may assume that **one** of the many diameters of the circle is being **‘elongated’**.

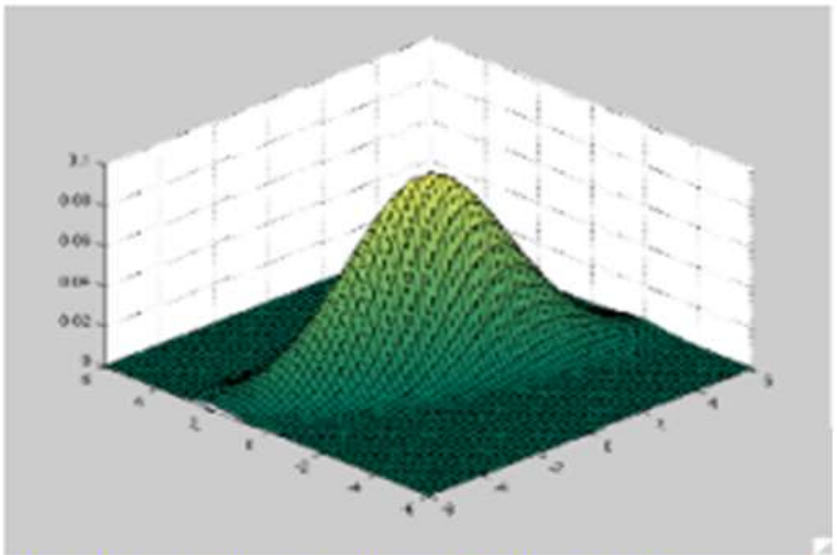
As soon as we ‘elongate’ one of the diameters, we get a **major axis** as well as a **minor axis**.



Bivariate Normal Distributions with ρ = 0.87 (to the left) and ρ = - 0.67 (to the right)

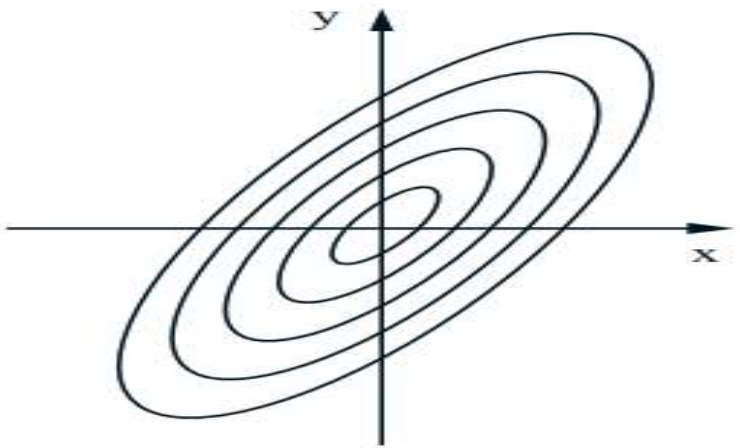


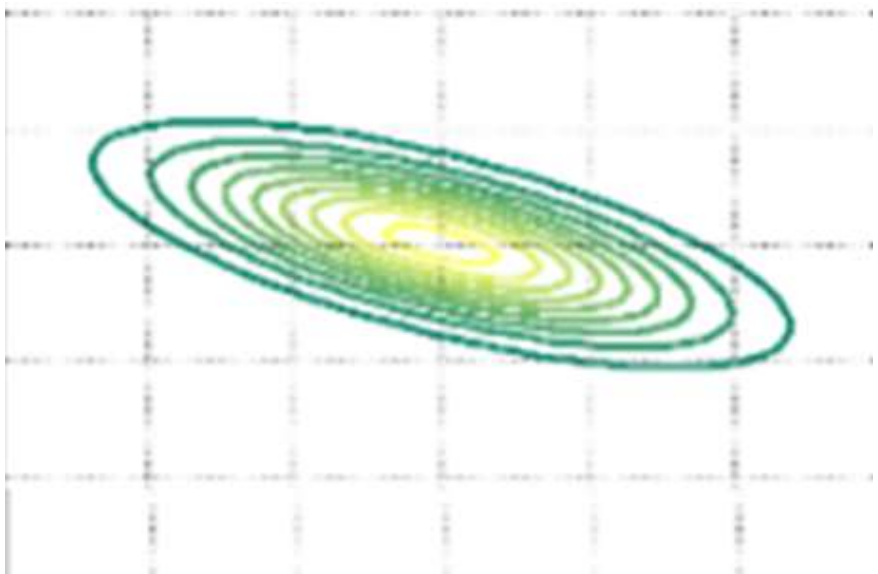
https://www.google.com/search?q=bivariate+normal+distribution&rlz=1C1CHBD_enPK785PK785&source=lnms&tbm=isch&sa=X&ved=0ahUKEwjN7IqN-ODhAhUQGuwKHeq3DgcQ_AUIDigB&biw=1280&bih=657#imgrc=j_DR8twmIeWiwM:



https://cedar.buffalo.edu/~srihari/CSE674/Chap7/7.1-MultiGauss.pdf

Bivariate Normal Distributions with ρ = 0.87 (to the left) and ρ = - 0.67 (to the right)





https://cedar.buffalo.edu/~srihari/CSE674/Chap7/7.1-MultiGauss.pdf

A few real-life examples

- Heights and weights
- Weights and Blood Pressures
- Marks in Mathematics & Marks in Statistics

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 181

Bivariate Normal Distribution

Detailed discussion of the Shape of the distribution when $\rho = 0$)

RECALL Definition

Joint probability density is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}}$$

for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$,

where $\mu_1 \in \square$, $\mu_2 \in \square$, $\sigma_1 > 0, \sigma_2 > 0$, and $-1 \le \rho \le 1$.

PDF of the bivariate normal distribution for the case $\rho = 0$.

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)\right\}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-0^2}} e^{-\frac{1}{2\left(1-0^2\right)}\left[\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-2(0)\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right]}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1}} e^{-\frac{1}{2(1)}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-0+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)} = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)}$$

Now, since $f(x_1,x_2)$ represents the **ordinate** of the distribution against the point (x_1,x_2) on the X_1X_2-floor, therefore, for any particular **contour**, we can write:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \underline{\text{constant}} = c$$

$$\text{Now } \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \underline{\text{constant}} = c$$

$$\Rightarrow e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \left(2\pi\sigma_1\sigma_2\right).c = \text{a new constant}$$

$$\Rightarrow \ln e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \ln[\left(2\pi\sigma_1\sigma_2\right).c] = \text{another new constant} = e$$

$$\text{or } -\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right) = e$$

Further,

$$-\frac{1}{2}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=e \Rightarrow -\frac{1}{2e}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=1$$

$$\Rightarrow \frac{1}{k}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=1 \text{ where } k=-2e$$

Hence, we have

$$\frac{\left(x_1-\mu_1\right)^2}{k\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{k\sigma_2^2}=1 \qquad \ldots(2)$$

RECALL

the Mathematical Equation of An Ellipse

The equation of an ellipse with major axis parallel to the X_1-axis and minor axis parallel to the X_2-axis and having its center at (c_1,c_2) is as follows:

$$\frac{(x_1-c_1)^2}{a^2}+\frac{(x_2-c_2)^2}{b^2}=1 \qquad \ldots(1)$$

Next:

The Shape of the distribution when

$\rho = 0$ **AND** $\sigma_1 = \sigma_2$

In the special case $a = b$, equation (1) reduces to

$$\frac{(x_1 - c_1)^2}{a^2} + \frac{(x_2 - c_2)^2}{a^2} = 1 \Rightarrow (x_1 - c_1)^2 + (x_2 - c_2)^2 = a^2$$

and we **already** know that

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 = a^2$$

is the equation of **<u>that</u> particular circle** with center at the point (c_1,c_2) and with radius a.

Hence, if we put $\sigma_1 = \sigma_2 = \sigma$ in equation (2), we obtain

$$\frac{(x_1 - \mu_1)^2}{k\sigma^2} + \frac{(x_2 - \mu_2)^2}{k\sigma^2} = 1$$

$$\Rightarrow (x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 = k\sigma^2 \qquad \ldots(3)$$

which is the equation of a <u>circle</u>

with center at (μ_1, μ_2) and with radius = $k\sigma$.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 187

Conditional distributions
in the case of
Bivariate Normal Distribution
(Graphical Interpretation)

Graphical interpretation

of the concept of

conditional

distributions

associated with

the

bivariate normal distribution.

So, let us start with the pdf of the bivariate normal distribution:

We know that the **joint p.d.f.** of Bivariate Normal distribution (X, Y) is

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

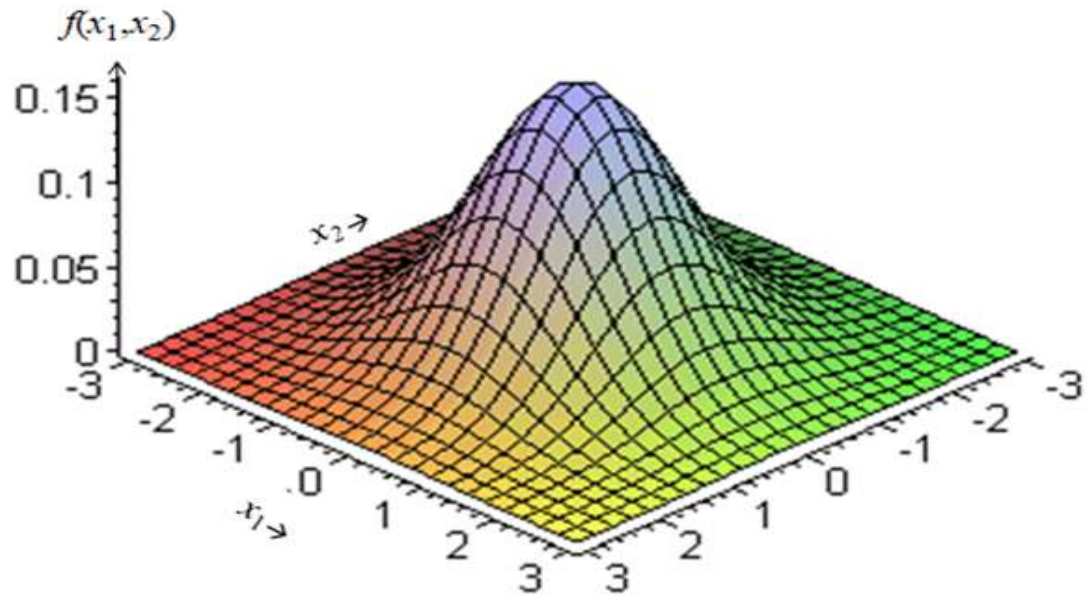
where $-\infty < x, y < \infty$ and the parameters are such that

$-\infty < \mu_x, \mu_y < \infty;$

$\sigma_x, \sigma_y > 0;$

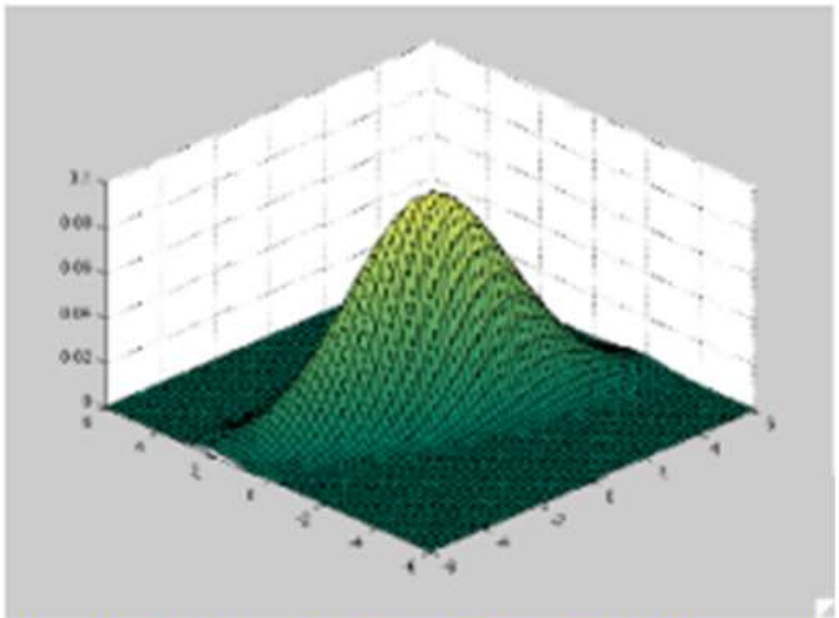
$-1 < \rho < 1.$

Bivariate Normal Distribution with $\rho = 0$ and $\sigma_1 = \sigma_2 = \sigma$:



An <u>exact</u>, <u>beautiful</u> bell-shaped surface

Bivariate Normal Distribution with ρ = - 0.67



https://cedar.buffalo.edu/~srihari/CSE674/Chap7/7.1-MultiGauss.pdf

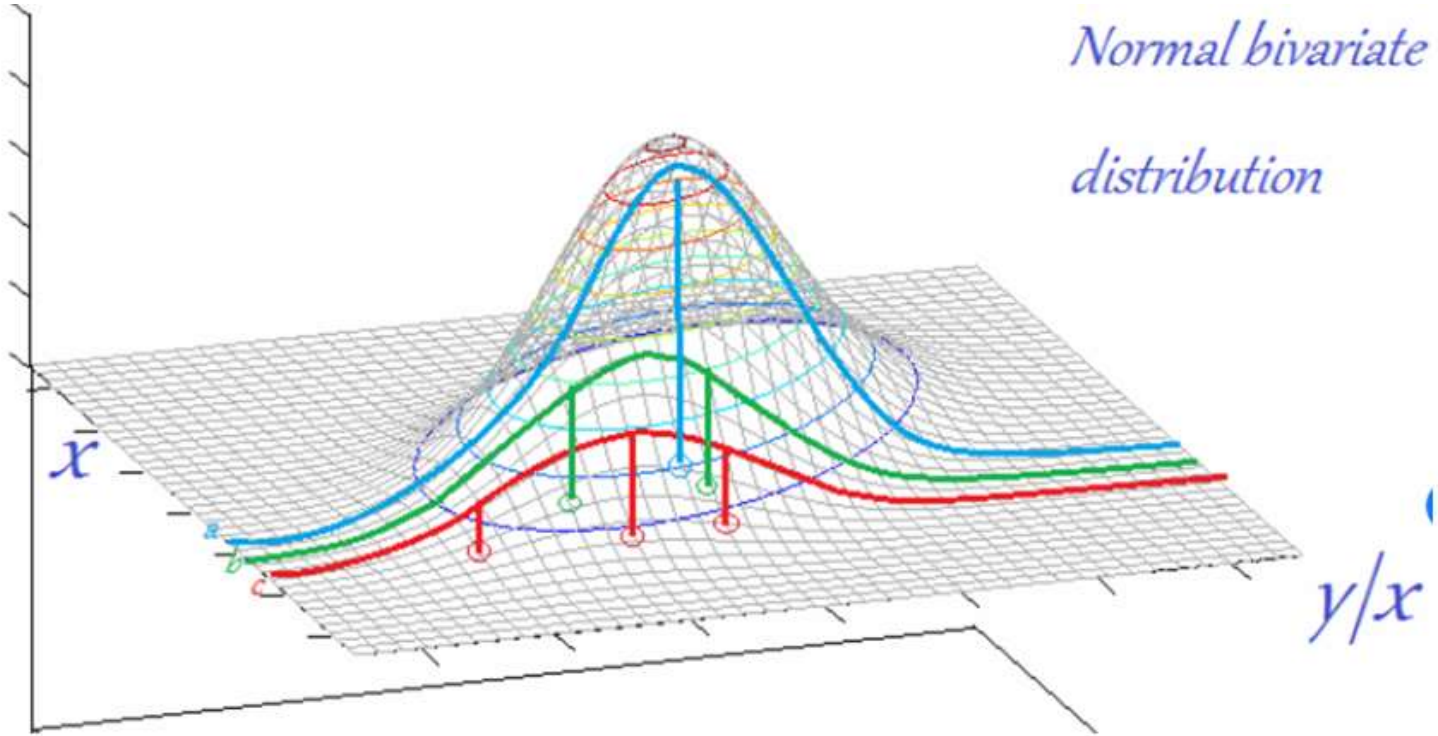
As far as the

Graphical interpretation

of the conditional distributions

associated with the bivariate normal distribution are concerned,

please consider the following diagram:



https://www.researchgate.net/figure/Stage-wise-conditional-distributions-sitting-on-the-tree-Gaussian-case_fig11_299488305

Important Note:

In equation (1) i.e.

$$f_{Y|X}(y|X=x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+(x-\mu_x)\left(\frac{\sigma_y}{\sigma_x}\right)\rho\right]\right)^2}$$

we are **talking about**

considering the distribution of Y against a **given** value of X which we are denoting by x.

In other words, the '*x*' occurring in the **exponent**

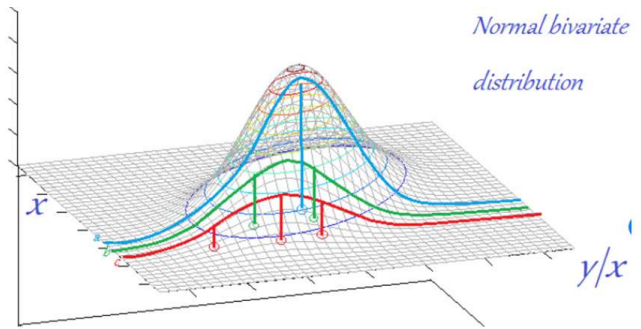
$$f_{Y \mid X}(y \mid X = x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+(x-\mu_x)\left(\frac{\sigma_y}{\sigma_x}\right)\rho\right]\right)^2}$$

is a **<u>particular</u> value** of *X*, i.e. it is a **<u>constant</u>.**

Because of the above, we can see that:

As far as the **mean** of the random variable Y corresponding to any particular value of X is concerned,

$$\tilde{\mu}_y = \mu_y + (x - \mu_x)\left(\frac{\sigma_y}{\sigma_x}\right)\rho$$



https://www.researchgate.net/figure/Stage-wise-conditional-distributions-sitting-on-the-tree-Gaussian-case_fig11_299488305

represents the mean value of y against some **particular value** of x.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 185

In the case of a Bivariate Normal Distribution,
the random variables X and Y are
independent
IF AND ONLY IF
the Correlation Coefficient
$\boldsymbol{\rho = 0}$

Now we will attempt to prove that

In the case of a Bivariate Normal Distribution,
the random variables X and Y are

independent

IF AND ONLY IF

the Correlation Coefficient

$\rho = 0$

Theorem:

If X and Y have a bivariate normal distribution with correlation coefficient ρ. Then X and Y are independent **if and only if** $\rho = 0$.

The statement "if and only if" means:

1) If X and Y are independent, then $\rho = 0$,

&

2) If $\rho = 0$, then X and Y are independent.

Note that, **in <u>general</u>**, if X and Y are **independent** random variables, **then** their correlation coefficient is 0.

(In other words, it is **<u>ALWAYS</u>** true that in the case of two **<u>independent</u>** random variables X and Y, the correlation coefficient ρ is equal to **zero**.)

As such, there is **no real need** to prove the **first** part of the statement **“if and only if”.**

The **actual goal** is to prove the **second** part of the statement
"if and only if"

i.e.

In the case of a **bivariate normal distribution** $N\left(\mu_x,\ \mu_y,\ \sigma_x^2, \sigma_y^2, \rho\right)$,

if $\rho = 0$, **then** X and Y are **independent**.

the basic condition by which two jointly distributed random variables *X* and *Y* can be regarded as being statistically independent.

Definition of Independence:

Two **random** variables X and Y are **independent,**
statistically independent or **stochastically independent** if

$$f_{XY}(x, y) = f_X(x) f_Y(y) \qquad \text{...(1)}$$

i.e.

if the **joint** density $f_{XY}(x, y)$ is equal to the product of the **marginal** density $f_X(x)$ and the **marginal** density $f_Y(y)$.

Note that, due to equation (1), if X and Y are **independent**, then the conditional distribution of Y given $X = x$ i.e. $f_{Y|X}(y \mid X = x)$ can be written as

$$f_{Y|X}(y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{f_X(x) f_Y(y)}{f_X(x)} = f_Y(y).$$

So, in the case of independent random variables (X and Y), **conditioning on $X = x$ does not change the distribution of Y. (i.e. the conditional distribution of Y is the SAME as the unconditional distribution of Y)**

The **proof** of the **second** part of the statement **"if and only if"**

i.e.

In the case of a **bivariate <u>normal</u> distribution**

<u>if</u> $\rho = 0$, **<u>then</u>** X and Y are **independent**. $N\left(\mu_x,\ \mu_y,\ \sigma_x^2, \sigma_y^2, \rho\right)$,

Proof: In order to prove that if X and Y have the bivariate normal distribution with **zero** correlation, **then** X and Y are **independent,** we need to show that the bivariate normal density function

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

factorizes into the **product** of the marginal p.d.f of X and the marginal p.d.f. of Y.

Substituting $\rho = 0$ in the pdf of the bivariate normal distribution, we have

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-0^2}} \mathrm{e}^{-\frac{1}{2(1-0^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2(0)\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

which reduces to

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \mathrm{e}^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

Further simplifying, we have:

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]} \times e^{-\frac{1}{2}\left[\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \times \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} = f_X(x)\, f_Y(y)$$

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 181

Bivariate Normal Distribution

Detailed discussion of the Shape of the distribution when $\rho = 0$)

RECALL Definition

Joint probability density is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)\right\}}$$

for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$,

where $\mu_1 \in \square$, $\mu_2 \in \square$, $\sigma_1 > 0, \sigma_2 > 0$, and $-1 \le \rho \le 1$.

PDF of the bivariate normal distribution for the case $\rho = 0$.

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)\right\}}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-0^2}} e^{-\frac{1}{2\left(1-0^2\right)}\left[\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-2(0)\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right]}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1}} e^{-\frac{1}{2(1)}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-0+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)} = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)}$$

Now, since $f(x_1,x_2)$ represents the **ordinate** of the distribution against the point (x_1,x_2) on the X_1X_2-floor, therefore, for any particular **contour**, we can write:

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \underline{\text{constant}} = c$$

$$\text{Now } \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \underline{\text{constant}} = c$$

$$\Rightarrow e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \left(2\pi\sigma_1\sigma_2\right).c = \text{a new constant}$$

$$\Rightarrow \ln e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} = \ln[\left(2\pi\sigma_1\sigma_2\right).c] = \text{another new constant} = e$$

$$\text{or } -\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right) = e$$

Further,

$$-\frac{1}{2}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=e \Rightarrow -\frac{1}{2e}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=1$$

$$\Rightarrow \frac{1}{k}\left(\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right)=1 \text{ where } k=-2e$$

Hence, we have

$$\frac{\left(x_1-\mu_1\right)^2}{k\sigma_1^2}+\frac{\left(x_2-\mu_2\right)^2}{k\sigma_2^2}=1 \qquad \ldots(2)$$

RECALL

the Mathematical Equation of An Ellipse

The equation of an ellipse with major axis parallel to the X_1-axis and minor axis parallel to the X_2-axis and having its center at (c_1,c_2) is as follows:

$$\frac{(x_1-c_1)^2}{a^2}+\frac{(x_2-c_2)^2}{b^2}=1 \qquad \ldots(1)$$

Next:

The Shape of the distribution when

$$\rho = 0 \quad \textbf{AND} \quad \sigma_1 = \sigma_2$$

In the special case $a = b$, equation (1) reduces to

$$\frac{(x_1 - c_1)^2}{a^2} + \frac{(x_2 - c_2)^2}{a^2} = 1 \Rightarrow (x_1 - c_1)^2 + (x_2 - c_2)^2 = a^2$$

and we **already** know that

$$(x_1 - c_1)^2 + (x_2 - c_2)^2 = a^2$$

is the equation of **<u>that</u> particular circle** with center at the point (c_1,c_2) and with radius a.

Hence, if we put $\sigma_1 = \sigma_2 = \sigma$ in equation (2), we obtain

$$\frac{(x_1 - \mu_1)^2}{k\sigma^2} + \frac{(x_2 - \mu_2)^2}{k\sigma^2} = 1$$

$$\Rightarrow (x_1 - \mu_1)^2 + (x_2 - \mu_2)^2 = k\sigma^2 \qquad \ldots(3)$$

which is the equation of a <u>circle</u>

with center at (μ_1, μ_2) and with radius = $k\sigma$.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 182

Computation of Probabilities

in the case of
Bivariate Normal Distribution
explained through an
EXAMPLE

Example:

A statistics class takes two exams, Exam 1 (Midterm Exam) and Exam 2 (Final Exam), and let us suppose that the marks of the two exams (to be called X and Y respectively) follow a **bivariate normal distribution** with parameters:

$\mu_x = 70$ and $\mu_y = 60$ (which can be called the marginal means)

$\sigma_x = 10$ and $\sigma_y = 15$ (which can be called the marginal standard deviations)

and $\rho = 0.6$ (which is the correlation coefficient).

Suppose that we select a student **at random.**

1) What is the probability that the student scores **over 75** on **Exam 1 (Midterm Exam)**?
2) What is the probability that the student scores over 85 on Exam 2 (Final Exam) **given that** he/she scored 75 in Exam 1 (Midterm Exam)?

Solution of Part (1):

Note that marks obtained in Exam 1 (denoted by X) follow a **univariate normal distribution** with

mean = 70 and variance = 100 i.e. $X \sim N(70, 100)$.

Hence, the probability that a randomly selected student scores over 75 in Exam 1 is given by

$$P(X > 75) = P\left(Z > \frac{x-\mu_x}{\sigma_x}\right) = P\left(Z > \frac{75-70}{10}\right) = P(Z > 0.5)$$

$$= 1 - P(Z < 0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085 = 30.85\% \text{ or } 31\%$$

--- a little less than one-third.

Next,

Part (2) of the question is as follows:

Suppose that we select a student **at random.**

What is the probability that the student scores over 85 on Exam 2 (Final Exam) **given that** he/she scored 75 in Exam 1 (Midterm Exam)?

Solution of Part (2):

Note that the distribution of marks obtained in Exam 2 (denoted by Y) given that the marks in Exam 1 were 75 is the **conditional distribution** of Y given $X=x=75$.

The pdf of the conditional distribution of Y given $X=x$ is

$$f_{Y|X}(y \mid X = x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+\rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)\right]\right)^2}$$

It is easy to see that the conditional distribution of Y given x i.e. $f_{Y|X}(y|x)$ is **normal** with mean $\mu_y + \rho\frac{\sigma_y}{\sigma_x}(x-\mu_x)$ and variance $\sigma_y^2\left(1-\rho^2\right)$.

Here, $\mu_x = 70$, $\mu_y = 60$, $\sigma_x = 10$, $\sigma_y = 15$ and $\rho = 0.6$ Hence, we have

$$f_{Y|X}(y \mid X = x)$$

$$= \frac{1}{\sqrt{2\pi}(15)\sqrt{1-(0.6)^2}} e^{-\frac{1}{2(15)^2(1-(0.6)^2)}\left(y-\left[60+(0.6)\frac{15}{10}(x-70)\right]\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}(15)\sqrt{1-0.36}} e^{-\frac{1}{2(15)^2(1-0.36)}\left(y-\left[60+(0.6)(1.5)(x-70)\right]\right)^2}$$

or

$$f_{Y|X}(y \mid X = x) = \frac{1}{\sqrt{2\pi}(15)\sqrt{0.64}} \mathrm{e}^{-\frac{1}{2(15)^2(0.64)}\left(y-\left[60+(0.9)(x-70)\right]\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}(15)(0.8)} \mathrm{e}^{-\frac{1}{2(225)(0.64)}\left(y-\left[60+(0.9)(x-70)\right]\right)^2}$$

$$= \frac{1}{(12)\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2(144)}\left(y-\left[60+(0.9)(x-70)\right]\right)^2}$$

Now, since the **given value** of X is x**=75**, therefore we have

$$f_{Y|X}(y \mid X = 75) = \frac{1}{(12)\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2(12)^2}\left(y-\left[60+(0.9)(75-70)\right]\right)^2}$$

$$= \frac{1}{(12)\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2(12)^2}\left(y-\left[60+(0.9)(5)\right]\right)^2} = \frac{1}{(12)\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2(12)^2}(y-64.5)^2}$$

which is the pdf of the **univariate normal distribution** with mean and standard deviation

$\tilde{\mu}_y = 64.5$ $\tilde{\sigma}_y = 12.$

Hence, the probability that a randomly selected student scores **over 85** on Exam 2 (Final Exam) **given that** he/she scored 75 in Exam 1 (Midterm Exam) is given by

$$P(Y > 85) = P\left(Z > \frac{y-\tilde{\mu}_y}{\tilde{\sigma}_y}\right) = P\left(Z > \frac{85-64.5}{12}\right) = P\left(Z > \frac{20.5}{12}\right)$$

$$= P(Z > 1.71) = 1 - P(Z \le 1.71) = 1 - \Phi(1.71)$$

$$= 1 - 0.9564 = 0.0436 = 4.36\%$$

--- a little less than 5%.

Virtual University of Pakistan
Probability Distributions

by

Dr. Saleha Naghmi Habibullah

Topic No. 183 is at the end.

Topic No. 184

Marginals of the Bivariate Normal Distribution are themselves Normal

We know that the **joint p.d.f.** of Bivariate Normal distribution (X, Y) is

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]}$$

where $-\infty < x, y < \infty$ and the parameters are such that

$-\infty < \mu_x, \mu_y < \infty;$

$\sigma_x, \sigma_y > 0;$

$-1 < \rho < 1.$

Let us now **derive** the **Marginal distribution** of X:

For any bivariate distribution $f(x,y)$, the **marginal distribution of X** is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad -\infty < x < \infty.$$

Hence, in the case of the **bivariate <u>normal</u>**, the **<u>marginal</u>** distribution of X will be given by

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]} dy$$

Let us now do some algebraic manipulation in the expression of the exponent:

The <u>exponent</u> is given by:

$$-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]$$

Dividing and multiplying the above expression by $\sigma_x^2\sigma_y^2$, we have

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\frac{\sigma_x^2\sigma_y^2(x-\mu_x)^2}{\sigma_x^2}+\frac{\sigma_x^2\sigma_y^2(y-\mu_y)^2}{\sigma_y^2}-\frac{2\sigma_x^2\sigma_y^2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]$$

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\sigma_y^2(x-\mu_x)^2+\sigma_x^2(y-\mu_y)^2-2\rho\sigma_x\sigma_y(x-\mu_x)(y-\mu_y)\right]$$

Completing the square in the exponent by Adding and Subtracting $\left(x-\mu_x\right)^2\sigma_y^2\rho^2$

in the square brackets:

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\times$$

$$\left[\sigma_y^2(x-\mu_x)^2+\sigma_x^2(y-\mu_y)^2-2\rho\sigma_x\sigma_y(x-\mu_x)\left(y-\mu_y\right)+\left(x-\mu_x\right)^2\sigma_y^2\rho^2-\left(x-\mu_x\right)^2\sigma_y^2\rho^2\right]$$

Combining (i) the second, third and fourth terms, and (ii) the first and fifth terms, we have

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\times$$

$$\left[\left[\left(x-\mu_x\right)^2\sigma_y^2\rho^2-2\rho\sigma_x\sigma_y(x-\mu_x\ )\left(y-\mu_y\right)+\sigma_x^2(y-\mu_y)^2\right]+\left[\sigma_y^2(x-\mu_x)^2-\left(x-\mu_x\right)^2\sigma_y^2\rho^2\right]\right]$$

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\left[\sigma_x(y-\mu_y)-\left(x-\mu_x\right)\sigma_y\rho\right]^2+\sigma_y^2(x-\mu_x)^2\left[1-\rho^2\right]\right]$$

which can be re-written as

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\sigma_x(y-\mu_y)-\left(x-\mu_x\right)\sigma_y\rho\right]^2-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\sigma_y^2(x-\mu_x)^2\left[1-\rho^2\right]$$

$$=-\frac{1}{2(1-\rho^2)}\left(\frac{\sigma_x(y-\mu_y)}{\sigma_x\sigma_y}-\frac{\left(x-\mu_x\right)\sigma_y\rho}{\sigma_x\sigma_y}\right)^2-\frac{(x-\mu_x)^2}{2\sigma_x^2}$$

$$=-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{\left(x-\mu_x\right)\rho}{\sigma_x}\right)^2-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2$$

Coming back to the overall expression of the marginal pdf:

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]} dy$$

Substituting the newly obtained expression of the exponent in the above pdf, we have

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2} \times \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} dy$$

Now, let us shift our attention to the first part of the expression inside the integral sign (the one that does NOT involve the exponent):

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} = \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{2\pi}\left(\sigma_y\sqrt{1-\rho^2}\right)}$$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\left(\sigma_y\sqrt{1-\rho^2}\right)} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2} dy \quad ...(1)$$

Multiplying and dividing the exponent inside the integral sign by σ_y^2, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\sigma_y^2}{2\sigma_y^2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(\frac{(y-\mu_y)\sigma_y}{\sigma_y}-\frac{(x-\mu_x)\rho\sigma_y}{\sigma_x}\right)^2} dy$$

Now, let $\tilde{\sigma}_y^2 = \sigma_y^2\left(1-\rho^2\right) \Rightarrow \tilde{\sigma}_y = \sigma_y\sqrt{1-\rho^2}$,

therefore

$$f_X\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma_x}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\tilde{\sigma}_y}\mathrm{e}^{-\frac{1}{2\tilde{\sigma}_y^2}\left(\frac{(y-\mu_y)\sigma_y}{\sigma_y}-\frac{(x-\mu_x)\rho\sigma_y}{\sigma_x}\right)^2}dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\tilde{\sigma}_y}\mathrm{e}^{-\frac{1}{2\tilde{\sigma}_y^2}\left(\left(y-\mu_y\right)-\left(x-\mu_x\right)\frac{\sigma_y}{\sigma_x}\rho\right)^2}dy$$

Now

$$f_X\left(x\right)=\frac{1}{\sqrt{2\pi}\sigma_x}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\tilde{\sigma}_y}\mathrm{e}^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\mu_y-\left(x-\mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho\right)^2}dy$$

can be re-written as

$$=\frac{1}{\sqrt{2\pi}\sigma_x}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\tilde{\sigma}_y}e^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\left[\mu_y+\left(x-\mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho\right]\right)^2}dy$$

Letting $\tilde{\mu}_y = \mu_y + \left(x - \mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho,$

we have $f_X\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\tilde{\sigma}_y} e^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\tilde{\mu}_y\right)^2} dy$

Hence, the integrand is the pdf of a normal density of the variable Y with:

$E\left(Y\right) = \tilde{\mu}_y = \mu_y + \left(x - \mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho,\ \mathrm{Var}\left(Y\right) = \tilde{\sigma}_y^2 = \left(1-\rho^2\right)\sigma_y^2$

Now, it is obvious that, for any density function, the total area under the curve is unity.

Hence, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\tilde{\sigma}_y} e^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\tilde{\mu}_y\right)^2} dy = 1$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} (1) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

Therefore, the marginal distribution of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_1}\right)^2}, \quad -\infty < x < \infty$$

i.e. the normal distribution with mean μ_x and variance σ_x^2.

i.e.

$$X \sim N\left(\mu_x, \sigma_x^2\right) \text{ with } \mathrm{E}\left(X\right) = \mu_x,\ \mathrm{Var}\left(X\right) = \sigma_x^2,$$

Similarly,

$$Y \sim N\left(\mu_y, \sigma_y^2\right) \text{ with } \mathrm{E}\left(Y\right) = \mu_y,\ \mathrm{Var}\left(Y\right) = \sigma_y^2.$$

Important Note:

It is possible to have a joint p.d.f. which has marginal p.d.f.s which are Normal, yet which is NOT bivariate normal.

Virtual University of Pakistan
Probability Distributions

by

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Topic No.186

Conditional distributions in the case of Bivariate Normal Distribution

the Conditional Distributions
related to
the Bivariate Normal Distribution
and,
through
DERIVATION,
I will show that the Conditional Distributions
related to the bivariate normal distribution
are themselves Normal.

So, let us start with the pdf of the bivariate normal distribution:

We know that the **joint p.d.f.** of Bivariate Normal distribution (X, Y) is

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \mathrm{e}^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

where $-\infty < x, y < \infty$ and the parameters are such that

$-\infty < \mu_x, \mu_y < \infty;$

$\sigma_x, \sigma_y > 0;$

$-1 < \rho < 1.$

Let us now attempt to derive the pdf of the conditional distributions

associated with the bivariate normal distribution:

In general, we know that, in the case of **any** bivariate distribution $f(x,y)$, the **conditional distribution** of Y given $X = x$ is expressed as follows:

$$f_{Y|X}(y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

where $f_{XY}(x, y)$ is the joint pdf of X and Y and $f_X(x)$ is the marginal pdf of X.

Now, we know that, in the case of bivariate normal distribution, the **marginal** distribution of X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}, \quad -\infty < x < \infty$$

i.e. the normal distribution with mean μ_x and variance σ_x^2.

Therefore, in the case of the **bivariate normal** distribution, we will have:

$$f_{Y|X}(y \mid X = x) = \frac{f_{XY}(x,\ y)}{f_X(x)} = \frac{\left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right) e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}}{\left(\frac{1}{\sqrt{2\pi}\sigma_x}\right) e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}$$

Let us now do some algebraic manipulation in the expression of the

exponent

in the

numerator:

The <u>exponent</u> is given by:

$$-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}+\frac{(y-\mu_y)^2}{\sigma_y^2}-\frac{2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right]$$

Dividing and multiplying the above expression by $\sigma_x^2\sigma_y^2$, we have

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\frac{\sigma_x^2\sigma_y^2(x-\mu_x)^2}{\sigma_x^2}+\frac{\sigma_x^2\sigma_y^2(y-\mu_y)^2}{\sigma_y^2}-\frac{2\sigma_x^2\sigma_y^2\rho(x-\mu_x)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right]$$

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\sigma_y^2(x-\mu_x)^2+\sigma_x^2(y-\mu_y)^2-2\rho\sigma_x\sigma_y(x-\mu_x)\left(y-\mu_y\right)\right]$$

Completing the square in the exponent by Adding and Subtracting $\left(x-\mu_x\right)^2\sigma_y^2\rho^2$ in the square brackets:

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\times$$

$$\left[\sigma_y^2(x-\mu_x)^2+\sigma_x^2(y-\mu_y)^2-2\rho\sigma_x\sigma_y(x-\mu_x)\left(y-\mu_y\right)+\left(x-\mu_x\right)^2\sigma_y^2\rho^2-\left(x-\mu_x\right)^2\sigma_y^2\rho^2\right]$$

Combining (i) the second, third and fourth terms, and (ii) the first and fifth terms, we have

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\times$$

$$\left[\left[\left(x-\mu_x\right)^2\sigma_y^2\rho^2-2\rho\sigma_x\sigma_y(x-\mu_x)\left(y-\mu_y\right)+\sigma_x^2(y-\mu_y)^2\right]+\left[\sigma_y^2(x-\mu_x)^2-\left(x-\mu_x\right)^2\sigma_y^2\rho^2\right]\right]$$

$$=-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\left[\sigma_x(y-\mu_y)-\left(x-\mu_x\right)\sigma_y\rho\right]^2+\sigma_y^2(x-\mu_x)^2\left[1-\rho^2\right]\right]$$

which can be re-written as

$$-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\left[\sigma_x(y-\mu_y)-(x-\mu_x)\sigma_y\rho\right]^2-\frac{1}{2(1-\rho^2)\sigma_x^2\sigma_y^2}\sigma_y^2(x-\mu_x)^2\left[1-\rho^2\right]$$

$$=-\frac{1}{2(1-\rho^2)}\left(\frac{\sigma_x(y-\mu_y)}{\sigma_x\sigma_y}-\frac{(x-\mu_x)\sigma_y\rho}{\sigma_x\sigma_y}\right)^2-\frac{(x-\mu_x)^2}{2\sigma_x^2}$$

$$=-\frac{1}{2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2$$

Multiplying and dividing the first part of the exponent by σ_y^2, we have

$$-\frac{\sigma_y^2}{2\sigma_y^2(1-\rho^2)}\left(\frac{(y-\mu_y)}{\sigma_y}-\frac{(x-\mu_x)\rho}{\sigma_x}\right)^2-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2$$

$$=-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(\frac{(y-\mu_y)\sigma_y}{\sigma_y}-\frac{(x-\mu_x)\rho\sigma_y}{\sigma_x}\right)^2-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2$$

Now, let us revert back to the expression of the **conditional** distribution:

$$f_{Y|X}(y \mid X = x)$$

$$= \frac{f_{XY}\left(x,\, y\right)}{f_X\left(x\right)} = \frac{\left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right) e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}}{\left(\frac{1}{\sqrt{2\pi}\sigma_x}\right) e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}$$

This can be re-written as:

$$f_{Y|X}(y \mid X = x) = \frac{\left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right) e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left((y-\mu_y)-(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right)^2 - \frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\left(\frac{1}{\sqrt{2\pi}\sigma_x}\right) e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}$$

$$= \frac{\left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right) e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left((y-\mu_y)-(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right)^2} \times e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}{\left(\frac{1}{\sqrt{2\pi}\sigma_x}\right) e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}}$$

Cancelling out the term $e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$ in the numerator with the same term in the denominator, we have

$$f_{Y|X}(y \mid X = x) = \frac{\left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right)}{\left(\frac{1}{\sqrt{2\pi}\sigma_x}\right)} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left((y-\mu_y)-(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right)^2}$$

$$= \frac{\sqrt{2\pi}\sigma_x}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right]\right)^2}$$

Now,

$$f_{Y|X}(y \mid X = x)$$

$$= \frac{\sqrt{2\pi}\sigma_x}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right]\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2\sigma_y^2(1-\rho^2)}\left(y-\left[\mu_y+(x-\mu_x)\frac{\sigma_y}{\sigma_x}\rho\right]\right)^2} \quad \ldots(1)$$

Now, let $\tilde{\sigma}_y^2 = \sigma_y^2\left(1-\rho^2\right) \Rightarrow \tilde{\sigma}_y = \sigma_y\sqrt{1-\rho^2}$

and let $\tilde{\mu}_y = \mu_y + \left(x-\mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho,$

Then, equation (1) can be re-written as

$$f_{Y|X}(y \mid X=x) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}_y} e^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\tilde{\mu}_y\right)^2}$$

Now, $f_{Y|X}(y|X=x) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}_y} e^{-\frac{1}{2\tilde{\sigma}_y^2}\left(y-\tilde{\mu}_y\right)^2}$

is the pdf of a normal distribution of the variable Y with:

$$E(Y) = \tilde{\mu}_y = \mu_y + \left(x-\mu_x\right)\left(\frac{\sigma_y}{\sigma_x}\right)\rho,$$

$$\mathrm{Var}(Y) = \tilde{\sigma}_y^2 = \left(1-\rho^2\right)\sigma_y^2.$$

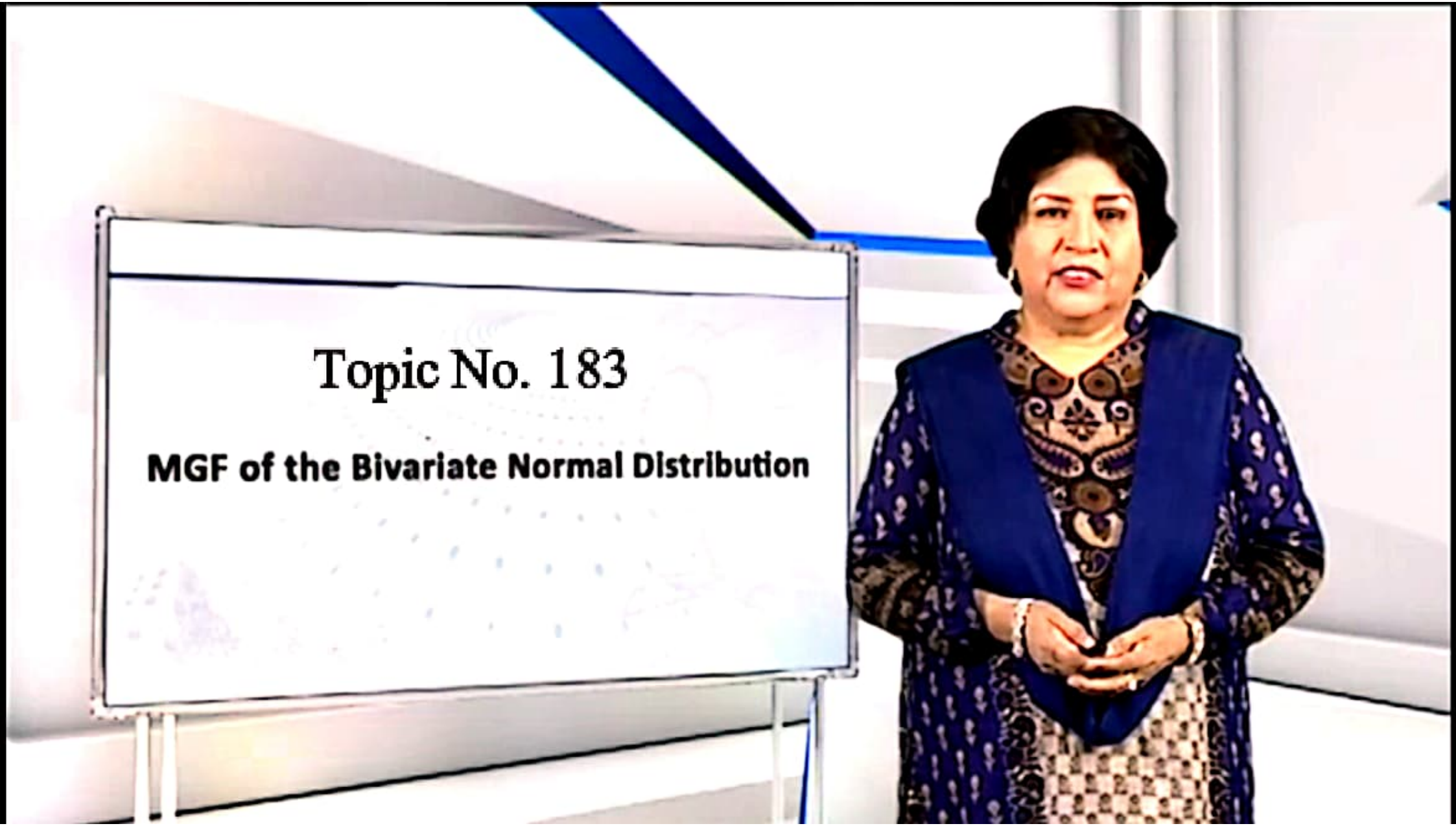
The conditional distribution of Y given x can be expressed as follows:

$$N\left(\mu_y + \rho\frac{\sigma_y}{\sigma_x}\left(x-\mu_x\right),\ \sigma_y^2\left(1-\rho^2\right)\right) = N\left(\tilde{\mu}_y,\ \tilde{\sigma}_y^2\right)$$

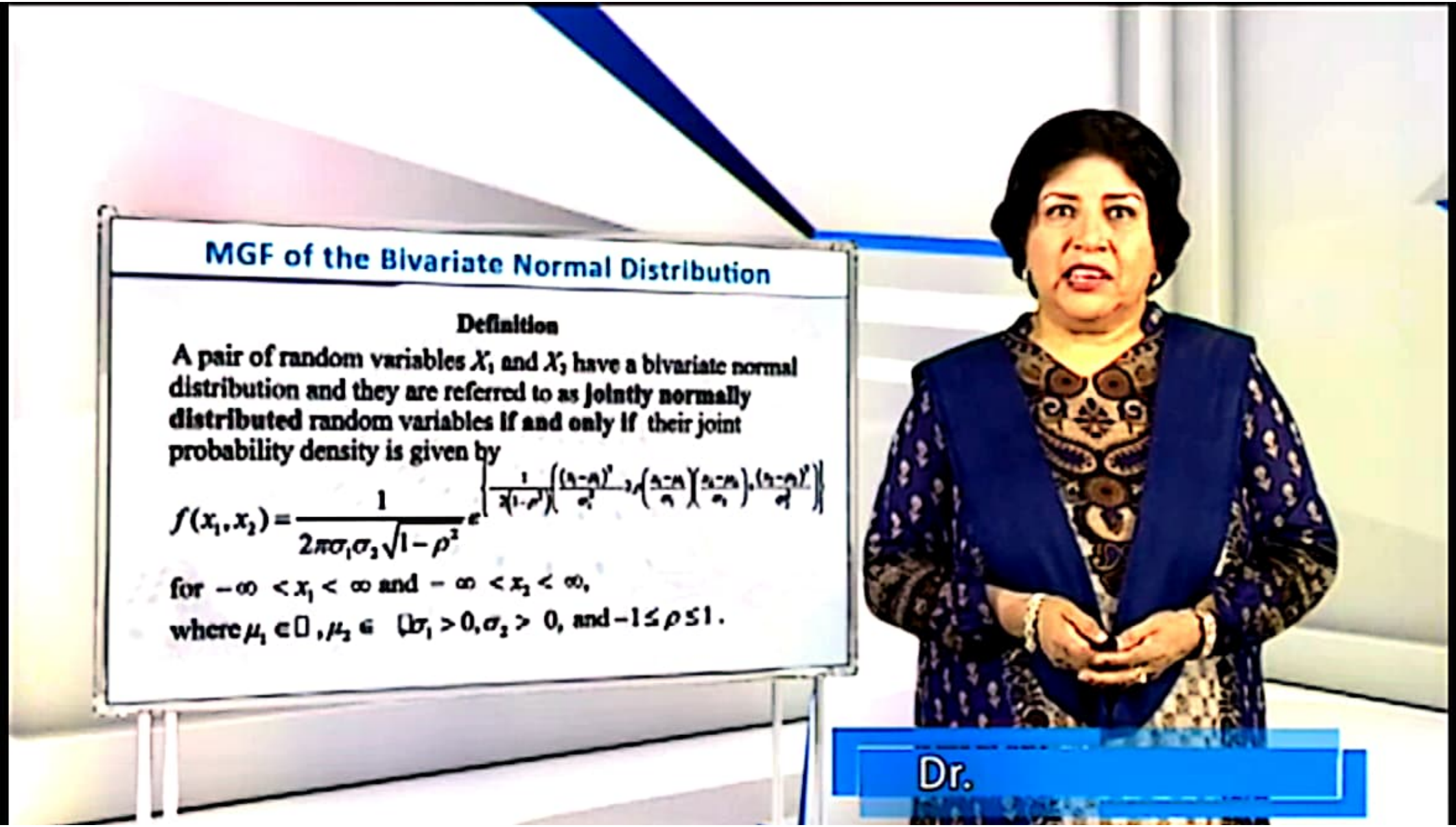
تمام دوست و احباب دعاؤں میں یاد رکھیے گا۔

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MGF of the Bivariate Normal Distribution

NOW

Let us consider the Moment Generating Function of the Bivariate Normal distribution.

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MGF of the Bivariate Normal Distribution

For this purpose, we begin with the definition of the mgf of a bivariate distribution (in general):

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Definition of Moment Generating Function of a Random Vector having 2 components:

Let $\underline{X} = (X_1, X_2)$ be a random vector.

If $E\left(e^{t_1X_1+t_2X_2}\right)$ exists for $|t_1| < h_1, |t_2| < h_2$
where h_1, h_2 are some positive constants,

then $E\left(e^{t_1X_1+t_2X_2}\right)$ will be denoted by $M_{X_1,X_2}(t_1, t_2)$
and will be called the moment generating function (mgf)
of the random vector X.

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MGF of the Bivariate Normal Distribution

As far as the moment generating function of the bivariate **normal** distribution is concerned, it can be shown that:

$$M_{\underline{X}}(t) = M_{\underline{X}}(t_1, t_2) = E\left(e^{\sum_{i=1}^{2} t_i X_i}\right)$$

$$= e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}\left(\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2\right)}, \quad -\infty < t_1 < \infty; -\infty < t_2 < \infty$$

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MGF of the Bivariate Normal Distribution

For this, RECALL:

The general method of obtaining Product Moments and simple Moments from the MGF of a random vector

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MGF of the Bivariate Normal Distribution

RECALL:

Let (X_1, X_2) be a random vector. The joint moment generating function of (X_1, X_2) is

$$M_{X_1,X_2}(t_1, t_2) = E\left(e^{t_1X_1+t_2X_2}\right)$$

Provided that the expected value exists on $-h_1 < t_1 < h_1$ and $-h_2 < t_2 < h_2$ for some positive constant h_1 and h_2.

$$E(X_1) = \frac{\partial M(0,0)}{\partial t_1}, \qquad E(X_2) = \frac{\partial M(0,0)}{\partial t_2}$$

$$E(X_1^2) = \frac{\partial^2 M(0,0)}{\partial t_1^2}, \qquad E(X_2^2) = \frac{\partial^2 M(0,0)}{\partial t_2^2}$$

$$E(X_1X_2) = \frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2}$$

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MGF of the Bivariate Normal Distribution

Here: $M(t_1,t_2)=e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2+\sigma_2^2 t_2^2\right)}$

Therefore

$$E(X_1)=\frac{\partial M(0,0)}{\partial t_1}=\left.\frac{\partial\left(e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2+\sigma_2^2 t_2^2\right)}\right)}{\partial t_1}\right|_{t_1=t_2=0}$$

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MGF of the Bivariate Normal Distribution

Now:

$$\frac{\partial\left(e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2+\sigma_2^2 t_2^2\right)}\right)}{\partial t_1}=\frac{\partial\left(e^{\mu_1 t_1+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2\right)}e^{\mu_2 t_2+\frac{1}{2}\left(\sigma_2^2 t_2^2\right)}\right)}{\partial t_1}$$

$$=e^{\mu_2 t_2+\frac{1}{2}\left(\sigma_2^2 t_2^2\right)}\frac{\partial\left(e^{\mu_1 t_1+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2\right)}\right)}{\partial t_1}$$

$$=e^{\mu_2 t_2+\frac{1}{2}\left(\sigma_2^2 t_2^2\right)}\left(\mu_1+\sigma_1^2 t_1+2\rho\sigma_1\sigma_2 t_2\right)e^{\mu_1 t_1+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2\right)}$$

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MGF of the Bivariate Normal Distribution

$$\text{Therefore } E(X_1)=\frac{\partial M(0,0)}{\partial t_1}=\frac{\partial\left(e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2+\sigma_2^2 t_2^2\right)}\right)}{\partial t_1}\Bigg|_{t_1=t_2=0}$$

$$=e^{\mu_2 t_2+\frac{1}{2}\left(\sigma_2^2 t_2^2\right)}\left(\mu_1+\sigma_1^2 t_1+2\rho\sigma_1\sigma_2 t_2\right)e^{\mu_1 t_1+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2\right)}\Bigg|_{t_1=t_2=0}$$

$$=e^{\mu_2(0)+\frac{1}{2}\left(\sigma_2^2(0)^2\right)}\left(\mu_1+\sigma_1^2(0)+2\rho\sigma_1\sigma_2(0)\right)e^{\mu_1(0)+\frac{1}{2}\left(\sigma_1^2(0)^2+2\rho\sigma_1\sigma_2(0)(0)\right)}$$

$$=e^0\left(\mu_1+0+0\right)e^{0+\frac{1}{2}(0+0)}=e^0\left(\mu_1\right)e^0=(1)\times(\mu_1)\times(1)=\mu_1$$

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MGF of the Bivariate Normal Distribution

Similarly, it can be proved that

$$E(X_2)=\frac{\partial M(0,0)}{\partial t_2}=\frac{\partial\left(e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}\left(\sigma_1^2 t_1^2+2\rho\sigma_1\sigma_2 t_1 t_2+\sigma_2^2 t_2^2\right)}\right)}{\partial t_2}\Bigg|_{t_1=t_2=0}=\mu_2$$

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MGF of the Bivariate Normal Distribution

Second moments about zero i.e. $E(X_1)^2$ & $E(X_2)^2$ can be found through the formulae

$$E(X_1^2) = \frac{\partial^2 M(0,0)}{\partial t_1^2}, \qquad E(X_2^2) = \frac{\partial^2 M(0,0)}{\partial t_2^2}$$

And, once they are obtained, the variances can be found very quickly.

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MGF of the Bivariate Normal Distribution

The product moment $E(X_1X_2)$ can be found through the formula

$$E(X_1X_2)=\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2}$$

And, once this is obtained, the covariance and the correlation coefficient can be found easily.

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