**Lesson 01**

## INTRODUCTION TO MATHEMATICAL ECONOMICS

### TOPIC 001: DEMYSTIFYING MATHEMATICS AND MATHEMATICAL ECONOMICS

**c t { –# r.mp#[ r#Jmt qy mıo–#] t q#[ xpq–'#.uqz p.G**

- Unity of Allah and His Hyyu$x$*Kalma* $okm oé$~é$k´r 2
- Inception of Life on Earth teaches us Math.
- Adam gave us 1st number.
- Eve gave us addition.
- Their children gave us multiplication.
- Total number of Prophets give us idea of large numbers.

**Origin of Mathematics**

L}yw$ Rk´x$ *ma* ´r k´$ é´kxné$ py}$ wk´r owk´sné$ kxn$ stands for mathematical.

**Age of Mathematics**

Sk´r owk´sné$é$ké$yn$ké$mkx2$(Stefan Banach)

**Universality of Mathematics**

Film is one of the three universal languages, the other two: mathematics and music. (Frank Capra)

The study of mathematics, like the Nile, begins in minuteness but ends in magnificence. (Charles Caleb Colton)

V∞}o$wk´r owk´sné$é$Ox$é$-k"0$r o$zyo´}" $yp$yqsmk$mké2 (Albert Einstein)

Mathematics is the music of reason. (James Joseph Sylvester)

God used beautiful mathematics in creating the world. (Paul Dirac)

All science requires mathematics. The knowledge of mathematical things is almost innate in us. This is the easiest of sciences, a fact which is obvious in that no one's brain rejects it; for laymen and people who are utterly illiterate know how to count and reckon. (Roger Bacon)

"Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers." (Shakuntala Devi)

Human beings have a certain number of eyes, ears, hands, arms, legs, feet. And there is also a certain number of skies i.e. seven skies.

**Precision via Mathematics**

My belief is that nothing that can be expressed by mathematics cannot be expressed by careful use of literary words. (Paul Samuelson)

Mathematics allows for no hypocrisy and no vagueness. (Stendhal)

I've always enjoyed mathematics. It is the most precise and concise way of expressing any idea. (N. R. Narayana Murthy)

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G$z}yl vow$–ow$é´k´on$é$k$z}yl vow$r kvp$éyv..on2$$(Charles Franklin Kettering)

**Applicability of Mathematics**

The true mathematician is not a juggler of numbers, but of concepts. (I. Stewart (1975))

**Mathematics as a Subject**

Despite, its primitive origin, students are usually afraid of mathematics.

Opinion of *Shakuntala Devi* ,k2u2k2$N∞wkx$l ywz∞´o}-0$a r”$ny$mr sn}ox$n}okn$wk´rowk´sméE

Because of the wrong approach, bomk∞éo$s´$é$yyuon$k´$ké$k$é∞l tom´2$(*Shakuntala Devi)*

Myyn$xo–é$py}$”y∞xq$zoyzvo$ké$´ro”$vsuo$kn..ox´∞}oé2$ Sk´rowk´sméSé$kx$kn..ox´∞}y∞é$kxn$ satisfying route ty$nysxq$´rsxqé2$K..o}”$wk´rowk´smkv$z}yl vow$é´k}´é$–sr$∞xmo}´ksx´”$yp$xy´$ having a definite solution but the solution, if obtained, gives a sense of fulfillment and the journey from the problem to the solution becomes an adventure.

A central figure in journey of scientific revolution is Galileo Galilei (a.k.a mathematical Platonist).

No$ék”é0$ Q$Q$–o}o$kqksx$l oqsxxsxq$w”$é´∞nsoé0$Q$–y∞vn$pywy–$´ro$kn..smo$yp$Vvk´y$kxn$é´k}´$–sr$ wk´rowk´sméE2

Jyx´$–y}}”$kl y∞´$”y∞}$nspm∞´vsoé$sx$mathematics. I can ass∞}o$”y∞$wsxo$k}o$é´sw$q}ok´o} . (Albert Einstein)

I've always enjoyed mathematics. It is the most precise and concise way of expressing any idea. (N. R. Narayana Murthy)

I've always been interested in using mathematics to make the world work better. (Alvin E. Roth)

This sums the learning and inspiration from the elders.

**Mathematics for Economics vs. Mathematical Economics**

No carving in the stone. However;

Ly}wo}$sé$kl y∞´$ Comprehension $yp$´ro$wk´rowk´smkv$´yyvé$kzzvson$´y$economic situations. [Algebra, matrices, differential calculus etc.]

Whereas, vk´´o}$nokvé$–sr$´ro$ Application $yp$mathematical tools to comprehend and solve of economic situations [e.g. application of algebra on market equilibrium, application of matrices on national income analysis, application of differentiation for marginal utility/cost/product etc].

**Mathematical Economics: An Approach to Economics rather than its Branch**

Can include problems from microeconomics, macroeconomics, public finance, international trade and other branches of economics.

e.g. Equilibrium for demand and supply curves, calculation of taxation revenue, local price elasticity of foreign demand of exports, among others.

**TOPIC 002: MATHEMATICS VS. NON-MATHEMATICAL ECONOMICS**

Non-mathematical economics can also be termed as literary economics.

Both remain approaches and should not fundamentally differ from each other.

However, there are two noteworthy differences:

- Former uses symbols/equations to denote assumptions and conclusions instead of words/sentences.

**An Assumption in Mathematical Economics**

Rather stating an assumption that a consumer can consume whole of any increase in his income, or a part of it or none of it is mentioned in mathematical fashion as follows:



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B Marginal Propensity to Consume.

**A Conclusion in Mathematical Economics**

Rather stating a conclusion that an increase in private investment has brought about 3 times of increase in national income can be expressed in mathematical expression as follows:

**E**— ————

**E** Investment multiplier

**=** Change (increase) in national income.

**=** Change (increase) in private investment.

**Theorems in Mathematical Economics**

**Theorem:** A rule in mathematics expressed in terms of symbols and formula.
**Envelope Theorem:** Applied to producer theory and auction theory.
**Roy's Identity:** Applied to Marshallian demand function.
**Shephard Lemma:** Used to explain relationship between expenditure (or cost) functions and Hicksian demand functions.

**Advantage(s) of Mathematical Economics**

- Mathematical economics utilizes the symbols that are more convenient to use in deductive reasoning.
- The use of symbols also increases conciseness and preciseness of statement.
- Use of mathematical functions allow to incorporate more than two economic variables in a situation without resorting to 3-dimensional or hyperspace graphs i.e. allows for number of variables.

**Practical Importance of Mathematical Economics**

- During current times, economics is said to be highly mathematized.
- Enables the comprehension of the professional articles one comes across in such periodicals as the American Economic Review, Quarterly Journal of Economics, Journal of Political Economy, Review of Economics and Statistics, and Economic Journal.

**TOPIC 003: MATHEMATICAL ECONOMICS VERSUS ECONOMETRICS**

Mathematical economics is often confused with econometrics.

- **Etymology of Econometrics:** G$zy}´wkx´ok∞$yp$–y$–y}né$Kmyxy $}yw$omyxywsné$kxn$ +wo´}sné$–rsmr$wzvsoé$woké∞}owox´2
- **In Jargonized language:** Econometrics is the study of empirical observations using statistical methods of estimation and hypothesis testing.
- Mathematical economics is, however, concerned with application of mathematical tools on economic theory without much concern to measurement of variables and errors in it.
- Thin line between a mathematical model and econometrics model:
- = + is a mathematical model while adding error term ( ) in it makes it look like:
- = + + which is regarded as econometric model.

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**Difference with an Example**
Consider law of demand and its mathematical and econometric models:

**Mathematical Model**
is a mathematical model.
Regardless of fact that many variables play a significant role in determining quantity demanded **( )**.
Zr oéo$∞x´kzzon$.k}skl voé$k}o$ésoxmon$∞ésxq$ ′ ′ L′A caveat is that other factors that can affect quantity demanded like income, price of related goods, tastes, weather, location etc. are not included in equation.

**Econometric Model**
While adding error term in it makes it look like:
which is regarded as econometric model.

- Here ´r o$kéé∞wz´syx$yp$ ′ is relaxed which gives rise to error term ( ).
- It makes a grab-bag of all other factors like income, price of related goods, tastes, weather, location etc. and includes all of them in the equation as a residual.

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**Lesson 02**

## INGREDIENTS OF A MATHEMATICAL MODEL

### TOPIC 004: INGREDIENTS OF A MATHEMATICAL MODEL: VARIABLES AND ECONOMIC VARIABLES

Mathematical economics resorts to models for economic analysis.

- Their building blocks are variables, constants/ coefficients and parameters.
- These building blocks are usually combined using algebra, trigonometry and other branches of mathematics to make a mathematical model.
- For example, law of demand uses algebra to connect building blocks:

- Solution of Price time path in dynamic analysis uses trigonometric ratios:

B B

**Variables**

- Etymology: (Vary + Able = Variable)
- A phenomenon that has ability to vary (usually over time).
- Life is full of specific variables: age, knowledge, weight, temperature, etc.
- However, mathematical variables are general in nature conventionally denoted by .

**Economic Variables**

Some instances of economic variables:

- Demand (D)
- Supply (S)
- Price (P)
- Individual Income (Y)
- Consumption (C)
- Investment (I)
- Savings (S)
- Revenue (R)
- Costs (C)
- Profit ( )
- Wage (W)
- Gross Domestic Product (GDP)
- Government Expenditure (G)
- Taxes (T)
- Supply of Money ($M_s$)
- Demand for Money ($M_d$)
- Interest Rate (i)
- Inflation ( )
- Exports (X)
- Imports (M)

**Freezing a Variable**

Allotting a certain value to a variable is freezing a variable.

**Microeconomic Examples**

- Excess supply in real market

$Q_d = 250$, $Q_s = 350$, $P = 50$.

Excess Supply (ES) = $Q_s - Q_d$

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Excess Supply (ES) = 350 - 250
**Excess Supply (ES) = 100 units**

**Profit of a Firm**
R = 5,00,000, C = 3,00,000, and = R C, then
= 5,00,000 - 3,00,000
= PKR 2,00,000.
Freezing a variable is useful in calculations of other economic variables.

**Macroeconomic Example**
If C = 20m, I = 5m, G = 0.2m, X = 0.1m, M = 0.5m, then national income in 4-sector economy is formulated as:
GDP = C + I + G + (X-M)
GDP = 20m + 5m + 0.2m + (0.1m - 0.5m)
**Gross Domestic Product (GDP) = PKR 24.8m**

**Solution Values**
Properly constructed economs m$wynové$qs.o$éyv∞´syx$.kv∞oé.
In superjacent examples. i.e.:
- Excess Supply (ES) = 100 units
- Profit ( ) = PKR 2,00,000.
- Gross Domestic Product (GDP) = PKR 24.8m

In addition, equilibrium D=S in goods market is a classic example of an economic model that gives solution values ($q^*$, $p^*$).

**Endogenous vs. Exogenous Variables**
**Endogenous Variables**
- **Etymology:** ’Kxny$,sx´o}xkv-$/$qoxy∞é$,qoxo}k´sxq-$C$Qx´o}xkw”$qoxo}k´on.
- Variables, solution values of which comes from the model.
- e.g. Market equilibrium model uses $Q_d$, $Q_s$ and P to give the values of $P^*$ and of $Q^*$.

**Exogenous Variables**.
- **Etymology:** ’K“y$,o“´o}xkv-$/$qoxy∞é$,qoxo}k´sxq-$C$K“´o}xkw”$qoxo}k´on.
- Variables, values of which come from outside the model.
- Their values are assumed to be given (not influenced by endogenous variables).
- Exogenous variables shift the relationship curves of endogenous variables.
- e.g. Income can shift the demand curve either inwards or outwards.

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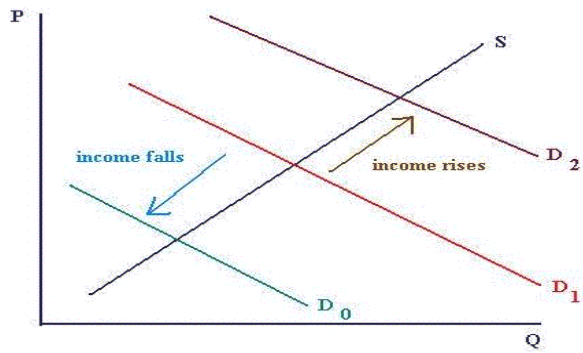


At macroeconomic level, political-will and policy variables like supply of money ($M_s$), tax rates (t) and government expenditure (G) are usually, but not always treated as exogenous variables.

## TOPIC 005: INGREDIENTS OF A MATHEMATICAL MODEL: CONSTANTS AND PARAMETERS

### Constants

- Etymology: Latin Origin (*Con* (with) + *Stare* (stand) = Constant)
- Lexically speaking; to stand or hold out against; resist or oppose, especially successfully.
- Óé an antithesis of a Variable.
- Daily life examples: total number of, God(s) = 1 and Prophets = 1,24,000.

### Constants in Mathematical/Economic Analysis

- In economic analysis, constants can appear in either numerical or symbolic manner.
- Range of numerical constants: e.g. 0.7, 7, -7000 etc.
- In symbolic manner, usually denoted by < or or .
- **Caveat:** G$myxé´kx´$sx$k$z}yn∞m´$–sr$k$.k}sklvo0$é$mkwon$cyopsmox´2$o2q$Qx$ and and are coefficients.
- Coefficients tend to amplify or compress the efficiency of variable - hence suitably called my-opsmox´2
- is times larger than and is 10 times smaller than .

### Parameters

- **Lexical Meaning:** D / or a boundary of some activity.
- Value of parameter can change but within a restriction. Therefore, it is also called zk}kwo´}sm$myxé´kx´2
- Y∞mr$}oé´}sm´yxé$mkx$lo$mkwon$ké$zk}kwo´}sm$}oé´}sm´yxé2
- e.g. value of MPC can vary but from 0 to 1.
- Mathematically, it is written as:
-
- Similarly, for Marginal Propensity to Save (MPS)
-
- Y∞wwk}s”0$–o$mkx$ék”$rk´$zk}kwo´o}sé$myxé´kx´$rk´$é$éywo–rk´$.k}sklvo2

### Exogenous Variables as Parameters

- For instance, in Keynes psychological law of consumption



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and are endogenous
is exogenous
is parameter.

- Some writers consider exogenous variables as parameters.
- However, no carving in stone, rather a matter of convention.

**TOPIC 006: A FEW ASPECTS OF LOGIC: PROPOSITIONS, IMPLICATIONS AND NECESSARY AND SUFFICIENT CONDITIONS**

**Logic**

- Etymology: Late Latin origin (*Logica* = Art of reason)
- In mathematical reasoning Logic is to be developed.
- Two major ingredients of logical reasoning: propositions and implications.

**Propositions**

- Etymology {Latin: *proponere* (propound/propose)}
- Assertions that are either true or false.
- Z}∞o$V}yzyéśsyx?$Gw$xns.sn∞kvé$–ry$l}ok´ro$k}o$kvs.o 2
- Lkvéo$V}yzyéśsyx?$kw$xns.sn∞kvé$–ry$l}ok´ro$k}o$rokvr” 2

**Propositions**

- Imprecise proposition hinders in developing logic.
- : ; $é$k$k}qo$x∞wlo}
- Toon$py}$nopxśsyx$yp$vk}qo$x∞wlo} 2
- Mathematics gives precise results using algebra, matrices and calculus etc.
- Therefore, mathematical economics helps to develop clear propositions and hence better logic.

**Implication(s)**

- lyxmk´oxk´oé$ro$z}yzyéśsyxé$x$yqsmkv$okéyxsxq$∞ésxq$swzvsmk´syx$k}}y– $ ).
- Let A and B be propositions and whenever *P* is true, *Q* is necessarily true.
- Owzvsmk´syx$k}}y– $myxnséov”$o“z}oééoé?$
- P Q
- *P* implies *Q* 0$y}$sp$P, then *Q* 0$y}$*Q* is a consequence of *P* 2$

**Implication(s)**

More examples:

-
-
- is a square is a rectangle

**Logical Equivalence**

- From last slide:
-
- Reverse is also true, as:
-
- Such is logical equivalence and uses equivalence arrow ( ):
-
- Similarly, for logical equivalence between propositions **A** and **B:**
-



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- Read as A is equivalent to B.

**Necessary and Sufficient Conditions**

- Literary economics extensively uses necessary and sufficient conditions.
- In mathematical economics, we use logical equivalence to show these conditions in a concise way.
- Logical equivalence means that is necessary and sufficient condition for .

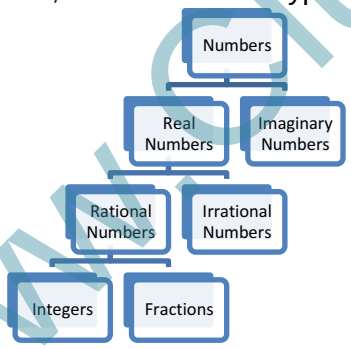
**Necessary and Sufficient Conditions**

- If MC cuts MR from below, then is maximum.
-
- **MC cuts MR** means MC = MR
- **From below** means
- Hence logical equivalence explains concisely.

**TOPIC 007: THE REAL-NUMBER SYSTEM**

**Number System**

- Economic variables adopt numerical values.
- Number system guides about types of numerical values.
- In this course, we deal with real numbers only.
- However, there is another type as well as shown in figure.




**Real vs. Imaginary Numbers**

Mostly, real number are used in economic analysis.

**A Digression**

- Differ from imaginary numbers: which include square root of (-1).
- This value is denoted by a Greek Letter *iota* .
- , so
- [illegible]2



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- However, in some economic situations, we have to deal with imaginary numbers. (Which is beyond the scope of this course).

**Rational vs. Irrational Numbers**
**Rational Numbers**

- Can be expressed in a ratio; where, .

Where, and are integers

- (undefined).
- Infinity is undefined and is hard to interpret just like timelessness.

**Examples**

same pattern repeats infinitely.

**Irrational Numbers**

- Other than rational
  - No repetitive pattern
  - Endless

  e.g.

**Combining Rational & Irrational Numbers**

- We get real numbers
- Integers & fractions need some description.

**Integers vs Fractions**
**Integers**

- Combination of zero, natural numbers, and ve values of natural numbers.
-

**Fractions**

- Ty´$k$-r yvo$x∞wl o}$40$50$60$70$ -2

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**Lesson 03**

## USE OF SETS IN ECONOMICS

### TOPIC 008: USE OF SETS IN ECONOMICS: SET NOTATION

**Set Notation**

- Collection of distinct objects.
- e.g. Numbers, labor, firms, countries etc.
- Zr oéo$yI tomé$k}o$mk won$ovow ox´é $yp$éo´é
- Two ways to write sets: enumeration & description

**Enumeration**

B = {Al Baraka Bank Ltd., Barclays Bank PLC., Citibank, Deutsche Bank AG., Dubai Islamic Bank Ltd., Industrial and Commercial Bank of China Ltd., Standard Chartered Bank Ltd., The Bank of Tokyo-Mitsubishi UFJ Ltd}

**Description**

B B BB B B B B B

- Read as: "B is the set of all (banks in Pakistan) x, such that x is a foreign bank in Pakistan."
- B is the set of all foreign banks in Pakistan.
- Membership of an element is denoted by
- (Al Baraka Bank Ltd.) B

**Range in Set Notation**

Ranges can also be defined in set notation:

B B BB B B B B

**Relationship using Set Notation**

If A = {Pakistan, India, Afghanistan, Bangladesh, Bhutan, Maldives, Nepal, Sri Lanka}
B = {Pakistan, Bangladesh, Afghanistan}

- B A (B is contained in A) or A B (A includes B)
- U}$H$é$k$Y∞I éo´$yp$G2

### TOPIC 009: USE OF SETS IN ECONOMICS: OPERATIONS OF SETS

**Set Operations**

Three major operations:

- Union
- Intersection
- Complement

**Union**

- Similar to addition (+).
- Write all elements of all sets once only, regardless of their multiple presences in sets.
- J oxy´on$ ” $[

**Venn diagram of union**



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.

Green shaded area depicts union of set A and B

**A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}**

Union is denoted by .

**A B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}**

**A B = {1, 2, 3, 4, 5, 7}**

**In description, union of A and B:**

**A U B = {x | x A or x B}**

**Intersection**

- Write common elements of all sets.
- Xoéowl voé$mywwyx$pkmy} $x$pkmy}s k´syx2
- Denoted by
- **A B = {1, 3, 5}**
- In Description, Intersection of A and B:

**I # #J #E#? ## # A and x B}**

**Venn Diagram of Intersection**

Green shaded area depicts intersection of set A and B

- Complement
- Write uncommon elements of the universal set in comparison to set under process .
- Xoéowl voé$é∞l ´}kmsyx .
- Denoted by or
- Complement

**If U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9},**

**A = {1, 2, 3, 4, 5}, then**

- **or = U A**
- **or = {0, 6, 7, 8, 9}**
- **or = {x | x U and x A}**



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**Venn Diagram of Complement**

Green shaded area depicts complement of set A

**TOPIC 010: USE OF SETS IN ECONOMICS: LAWS OF OPERATIONS OF SETS**

**Laws of Operations**

Three major Laws:

- Commutative Law
- Associative Law
- Distributive Law

**Commutative Law of Union**

- Order does not matter.
- Of union: A U B = B U A

A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}

A B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}

**A B = {1, 2, 3, 4, 5, 7}**

H A = {1, 3, 5, 7} U {1, 2, 3, 4, 5}

**J A = {1, 2, 3, 4, 5, 7}**

Hence, **A U B = B U A**

**Commutative Law of Intersection**

A B = B A

A= {1, 2, 3, 4, 5} & B = {1, 3, 5, 7}

A B = {1, 2, 3, 4, 5} {1, 3, 5, 7}

**A B = {1, 3, 5}**

H A = {1, 3, 5, 7} {1, 2, 3, 4, 5}

**J A = {1, 3, 5}**

Hence, **A B = B A**

**Associative Law of Union**

- U}no}$yp$éovomsyx $nyoé$xy´$wk´´o}0$-r so$nokvsxq$-sr $wy}o$r kx$6$éo´é2
- Of union:

A U (BUC)= (AUB) U C

A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}

- **L.H.S**

A [ B = {1, 2, 3, 4, 5} U {1, 3, 5, 7}

A [ B = {1, 2, 3, 4, 5, 7}

(A [ B) U C = {1, 2, 3, 4, 5, 7} U {4, 6, 8, 10}

**(A a B) U C = {1, 2, 3, 4, 5, 6, 7, 8, 10}**



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- **R.H.S**

B [ C = {1, 3, 5, 7} U {4, 6, 8, 10}
B [ C = {1, 3, 4, 5, 6, 7, 8, 10}
A U (B [ C) = {1, 2, 3, 4, 5} U {1, 3, 4, 5, 6, 7, 8, 10}
**A a (B U C) = {1, 2, 3, 4, 5, 6, 7, 8, 10}**
Hence, **(A a B) U C = A a (B U C)**

**Associative Law of Intersection**
**A (B C) = (A B) C**
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
**L.H.S**
B C = {1, 3, 5, 7} {4, 6, 8, 10}
B C = {}
A (B C) = {1, 2, 3, 4, 5} {}
A (B C) **= {} =** [theta]
**R.H.S**
A B = {1, 2, 3, 4, 5} {1, 3, 5, 7}
A B = {1, 3, 5}
(A B) C = {1, 3, 5} {4, 6, 8, 10}
**(A B) C = {} =** [theta]

**Distributive Law of Union**
**A U (B C) = (AUB) (AUC)**
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
**L.H.S**
(B C) = {}
**A U (B C) = {1, 2, 3, 4, 5}**
**R.H.S**
AUB = {1, 2, 3, 4, 5, 7}
AUC = {1, 2, 3, 4, 5, 6, 8, 10}
**(AUB) (AUC) = {1, 2, 3, 4, 5}**
Hence, **L.H.S = R.H.S**

**Distributive Law of Intersection**
**A (BUC) = (A B) U (A C)**
A= {1, 2, 3, 4, 5}, B = {1, 3, 5, 7} & C = {4, 6, 8, 10}
**L.H.S**
(BUC) = {1, 2, 3, 4, 5, 6, 7, 8, 10}
**A (BUC) = {1, 2, 3, 4, 5}**
**R.H.S**
A B = {1, 3, 5}
A C = {2, 4}
**(A B) U (A C) = {1, 2, 3, 4, 5}**
Hence, **L.H.S = R.H.S**

**TOPIC 011: CARTESIAN COORDINATES**

- Named after d*escartes* (French philosopher, mathematician, and scientist).
- Intersection of 2 numbers lines ( and ) horizontally and vertically, respectively.



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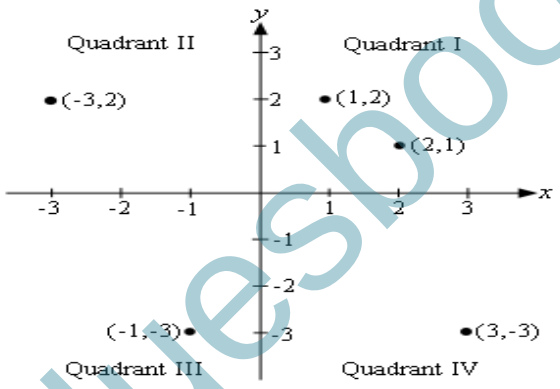
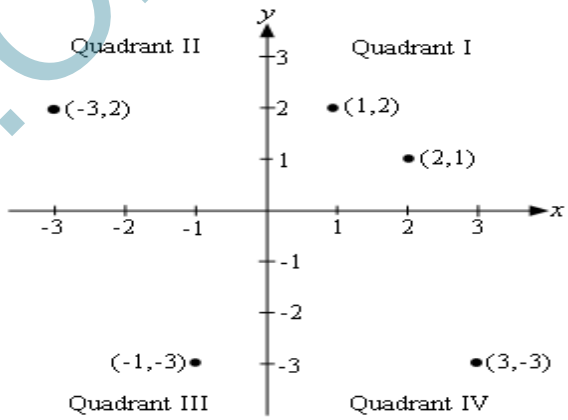
- 4 slices called quadrants (quarters of a circle).
- Ordered pairs are two variables (X & Y), in order, within parentheses:
    - 1st quadrant: (+ve, +ve)
    - 2nd quadrant: (-ve, +ve)
    - 3rd quadrant: (-ve, -ve)
    - 4th quadrant: (+ve, -ve)
- Most of economic variables lie in 1st quadrant as they are usually non-negative ( ).




- Order in the coordinates is not interchangeable .
- In quadrant I, , as they have different locations.




- For 3-dimensional (3D) graphs, ordered triples are used .
- Instead of lines, surfaces are generated.
- e.g. A cube is 3D.

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**Lesson 04**

## USE OF FUNCTIONS IN ECONOMICS

### TOPIC 012: WHAT ARE FUNCTIONS?

Vy–o}$y$km´$x$k$ézonspm$–k”2

- System to write the dependence of one variable ( ) on other ( ).
- is dependent variable.
- is independent variable.

- In addition to *f*, F, G, the Greek letters (phi) and (psi), and their capitals, are used to show functions.
- Yywo´swoé0$r o$nozoxnox´$.k}sklvo$é$éovp$~éon$xé´okn$yp$ .
  y = y(x), z = z(x) etc.
- It is suitable to use different symbols for multiple functions with same independent variable.
  y=f(x), z = g(x) etc.

**Plotting a Function**

Function (x=2) depicts a vertical line parallel to y-axis.

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- Function; y=f(x)=2 portrays a horizontal line parallel to x-axis.

- Functions; y=f(x)=2x, y=f(x)=x & y=f(x)=0.5x.

- Plotting a function gives a locus (route/way) of points hence called mapping.
- Function also converts values of independent variable into that of dependent variable hence called transformation.

**TOPIC 013: DOMAIN AND RANGE IN A FUNCTION**

- All permissible values of k}o$J ywksx
- All permissible values of k}o$Xkxqo 2



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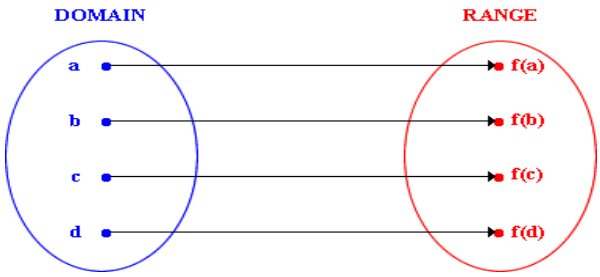


- Numerically speaking:

- Ordered pairs show the mapping of function.

Numerically speaking:
- Domain=
- Range=

**Economic Example**
- Total cost/day 'C' of a firm depends on its daily output
- Its functional form is:

C = 150 + 7Q.
- Capacity limit = 100 units/day.

- Extreme values nyx ´$xomoéék}s" $ymm~}0$r y–o..o}2

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| Daily Output | Cost/Day | |
|---|---|---|
| 0 | 150 | |
| 10 | 220 | |
| 20 | 290 | |
| 30 | 360 | |
| 40 | 430 | |
| 50 | 500 | |
| 60 | 570 | |
| 70 | 640 | |
| 80 | 710 | |
| 90 | 780 | |
| 100 | 850 | |

**TOPIC 014: DIFFERENCE BETWEEN FUNCTIONS AND RELATIONS**

Slight but noteworthy difference.
Formation of ordered pairs

**Function:** (1,2),(2,4),(3,6)
**Relation:** (**1**,**3**),(2,5),(**1**,**7**).

- One value of dependent variable (**1**) has multiple corresponding values of independent variable (**3**, **7**).
- Formerly a.k.a. single-valued function and multi-valued function, respectively.

**Vertical Line Test**
**Function** passes vertical line test while **relation** does not.

- A vertical line cuts a function on one point only.
- While it cuts a relation on multiple points.
- Case no. 5 is classic example of relation.

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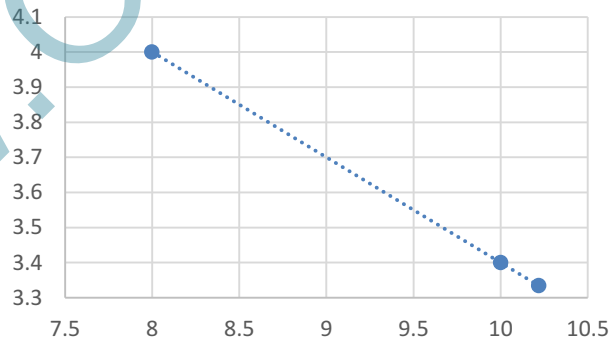
**Mapping of Function (Formerly)**

**Mapping of Relation**

## TOPIC 015: ECONOMIC EXAMPLE OF SCHULTZ DEMAND FUNCTION

- H. Schultz estimated US cotton demand for the period 1915_1919.
-
- If then
- If then
- If then
- As expected, negatively sloped demand curve is generated.




- Conversely, we can find price of cotton ( ), if quantity demanded ( ) is given:
-
- If then

___

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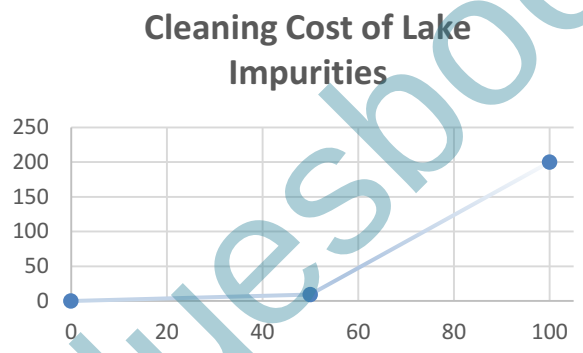
**TOPIC 016: ECONOMIC EXAMPLE OF COST FUNCTION OF CLEANING IMPURITIES FROM A LAKE:**

Cleaning cost of p% of impurities in a lake:

> [ _____

- If then _____
- If then _____
- If then _____

- The increase in cleaning cost of lake impurities has an increasing trend.
- However, it does not grow as a straight line.




Additional cost of additional cleaning (h%) above p% can be written as follows:

> [ { > [ _____ _____

If then _____ _____

Additional cost of cleaning above 50% is 22.09 Units.

**TOPIC 017: ECONOMIC EXAMPLE OF FUNCTION: LAFFER CURVE**

Laffer curve shows the theoretical relationship between rates of taxation and the corresponding levels of government revenue.

- In functional form:

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B B

- Zsw$´r o$zysx´$yp$ revenue mk“swś k´syx 0$´r o$}ovk´syxérsz$l o´–oox$tax rates and tax revenue is positive/direct.

B B

- After this point, the relationship becomes negative/inverse.

B

B

- The tax revenue maximizing point in the Laffer curve can be found using *Rolle's* theorem.
- It is a calculus-based theorem but is beyond the scope of this course.

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**Lesson 05**

## CONSTANT FUNCTIONS AND LINEAR FUNCTIONS

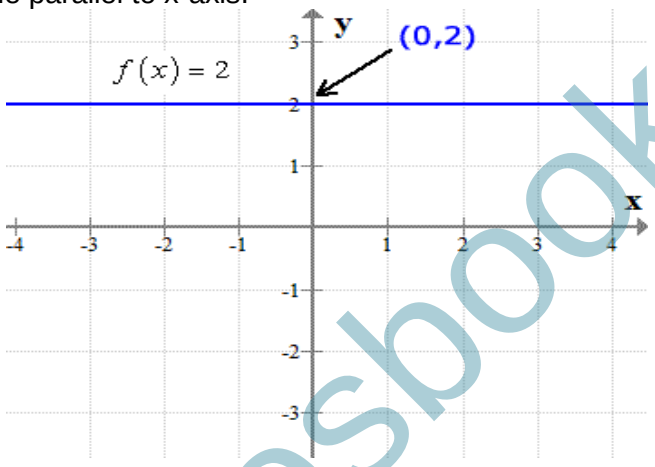
### TOPIC 018: TYPES OF FUNCTIONS: CONSTANT FUNCTIONS

Functions where the power of independent variable is zero (0).
e.g.

Where, can be any constant value.

- The graphs of such constant functions appear as horizontal lines.

- In constant function, , let a = 2.
- Then, .
- Its graph is a line parallel to x-axis.




**Autonomous Consumption ( )**

- Consumption at .
- .
- Consumption independent of disposable income.

**Autonomous Investment ( )**

- Investment at .
- .

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**Autonomous Government Spending ( )**

- Government spending is usually based on political will (independent of Y)
- .

**Autonomous Tax ( )**

- Taxes independent of level of income (Y).
- e.g. Lump-sum taxes.
- .

**Supply of Money ( )**

- In the short run, supply of money is usually independent of interest rate ( ).
- Determined by central monetary authority (e.g. SBP, FED etc.).
- .

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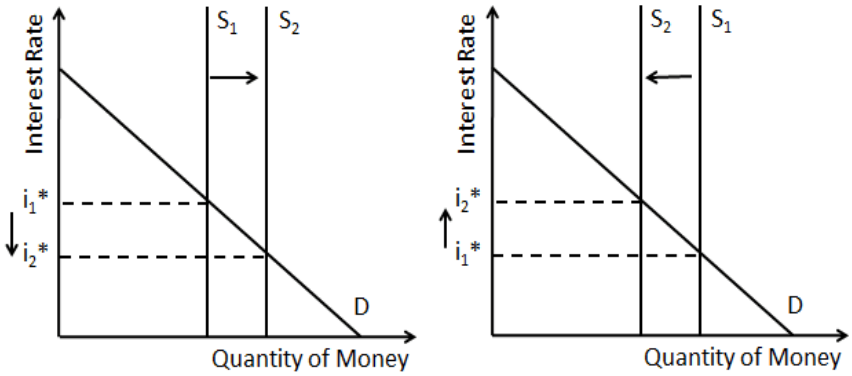


## TOPIC 019: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: LINEAR FUNCTIONS

**Polynomial Functions**

K´"wywyq"?$, zyv" $wkx"-$ $xywskv$zk}´é-2

**Linear Functions**

- Different values of give different types of functions.
- Linear function
- Degree of linear equation = .
- If , then , which gives a y-intercept (vertical intercept).
- It holds true for both positive and negative sloped linear function.
- If , +ve slope.
- If , -ve slope.

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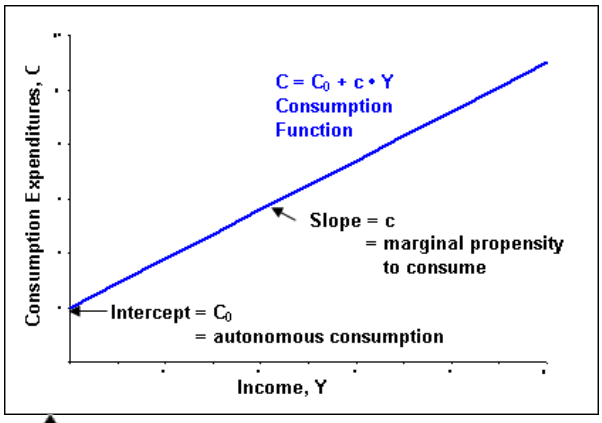


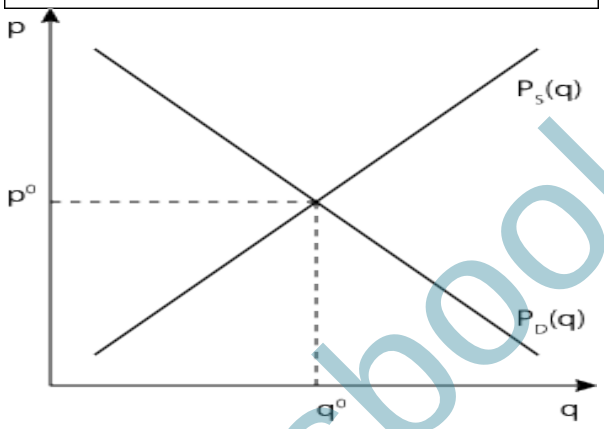


**TOPIC 020: INTERPRETING LINEAR ECONOMIC FUNCTIONS**

Estimated cost function for the US Steel Corp. (1917 1938).

**Interpreting linear Function**

Compare with slope intercept form of a straight line:

**Where, & .**

- Here, slope = 55.73,
- If production increases by 1 ton, then the cost increases by $55.73.
- Estimated annual demand function for rice in India for the period 1949 1964.

**Interpretation**

- Similar to last example.
- Zr o$éy zo$é$ 42592



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- If price increases by one Indian rupee, then the quantity demanded decreases by 0.15 units.

**Calculating Slope of a Straight Line**

- Take two points e.g. & .
- Draw perpendicular from higher point and horizontal line from lower point to get a triangleB .
-

**TOPIC 021: APPLICATIONS OF LINEAR FUNCTIONS: POPULATION AND CONSUMPTION FUNCTIONS**

**Population Function**

European population was 641 million in 1960, and 705 million in 1970.

- Let (in millions) & (in years).
- for 1960 & for 1961 and so on.
- Linear function:
- Given points: and .
- Using point-point formula of slope of straight line:

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- Back-cast and forecast

**Consumption Function**
Haavelmo estimated for US economy (1929 1941):

is autonomous consumption.
- Geometrically, it shows the intercept of the linear function.
.
- About 71.2% of increase in income was being spent in US.
- ***MPC*** also shows the slope of consumption function.

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**Lesson 06**

## QUADRATIC FUNCTIONS AND CUBIC FUNCTIONS

### TOPIC 022: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: QUADRATIC FUNCTIONS

**Polynomial Functions**

**Quadratic Functions**

Or standard form of quadratic equation is:

- Degree of quadratic equation = .
- Ms.oé$k$zk}kl yvk

- **Etymology:** {*para* l oésno $ $ k$r }y– –2
- Curve with **a single** bump or wiggle either in a valley or hill.

- First graph shows a happy/U-shaped/open upwards parabola (valley), as .
- Whereas, second is sad/Inverted U-shaped/open downwards parabola (hill), as .

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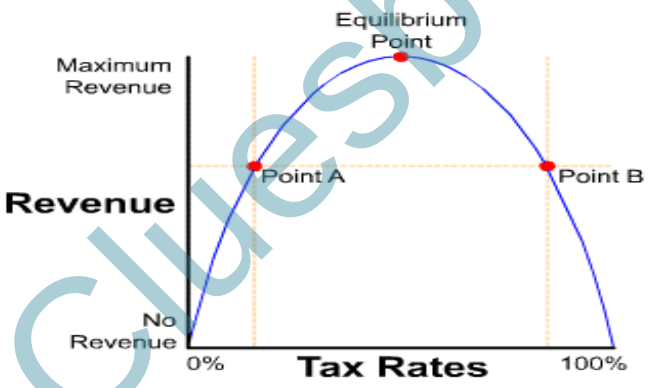


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## TOPIC 023: QUADRATIC COST FUNCTION AND PROFIT FUNCTION OF A MONOPOLY

**Quadratic Cost Function**

Condition for maximization of quadratic function/at parabola

**Quadratic Profit Function**

Condition for maximization of quadratic function/at parabola

- Here, for quadratic formula:

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**TOPIC 024: QUADRATIC FUNCTION AND PRODUCTION POSSIBILITIES FRONTIER**

Where, and are two goods.
Let ;

Quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, **let ;**

Combining the 3 intercepts;

- Only intercepts cannot give a precise production possibilities frontier.
- Other points on curve are also required where both coordinates are non-zero.

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**TOPIC 025: TYPES OF FUNCTIONS: POLYNOMIAL FUNCTION: CUBIC FUNCTIONS**

**Cubic Functions**

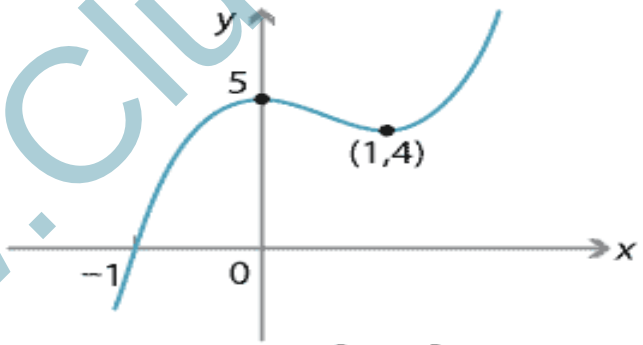
**Cubic Functions**

Or standard form of cubic equation is:

are constants

- , as is the requisite for a cubic function

- Degree of cubic equation = .




Graph of $f(x) = 2x^3 - 3x^2 + 5$.

- Curve with **two** bumps or wiggles.

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## TOPIC 026: CUBIC COST FUNCTIONS

**Cubic Cost Functions**

- Then

**Example**

- Then should hold.
- Here .

  Which is true.

**Electric Power Generating Plant Cubic Cost Function**

Here 0.

- Testing the validity of .

, Which is against the condition.



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**Components of Cost Function (FC & VC)**

Here, .

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**Lesson 07**

## RATIONAL FUNCTIONS AND EXPONENTIAL FUNCTIONS

### TOPIC 027: RATIONAL FUNCTIONS

**Rational Functions**

Ratio of two polynomial functions.

- and are not necessarily same.
- It shows that it is not necessary to have same degree of equation in numerator and denominator.
- , else the rational function will become undefined.

, for to be defined.

- Here n = 1 and m = 2.
- Ratio of a linear function to a quadratic function.

**Asymptote** is a line or curve that approaches a given curve arbitrarily closely.
**Vertical asymptote** is at such value of **x** that turns **y** into infinity.
**Horizontal asymptote** is at such value of **y** that turns **x** into infinity.

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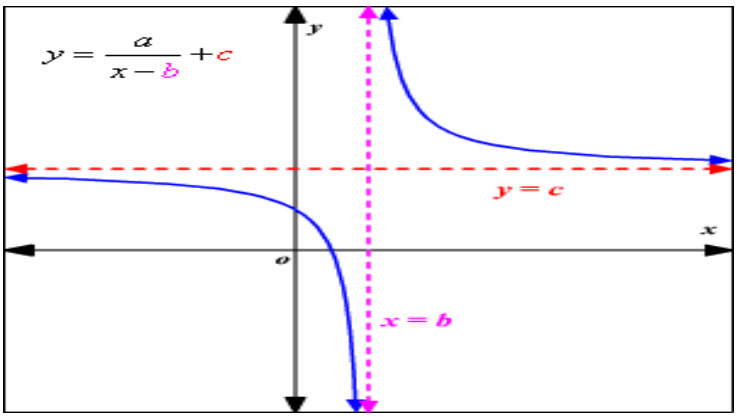


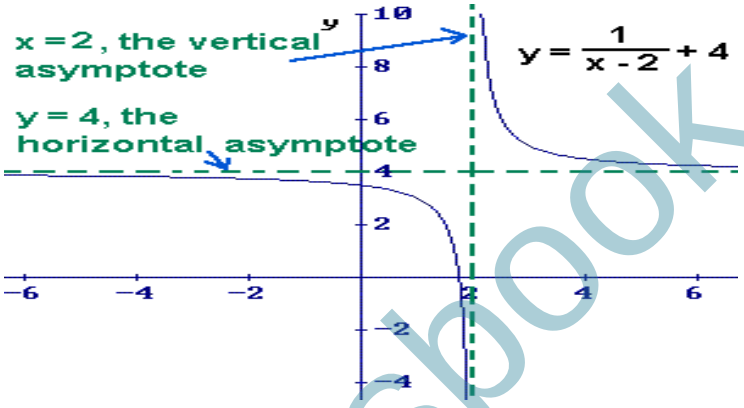


**TOPIC 028: OTHER TYPES: RECTANGULAR HYPERBOLIC FUNCTIONS**

**Rectangular Hyperbolic Functions**

**Hyperbola: Etymology:** {Hyper (Beyond)+Bola (Throw)}.

- Similar to parabola but not the same.
- Usually wider than parabola.

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- Rectangular hyperbolic functions have x and y asymptotes.
- Economic examples include aggregate demand (AD) curve.

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**TOPIC 029: OTHER TYPES: NON-ALGEBRAIC EXPONENTIAL FUNCTION**

**Etymology:** Latin origin; *k* . In english expound: explain in detail.

- Independent variable occurs as a root or power.
  where, >0 and .
- Non-linear graphs.

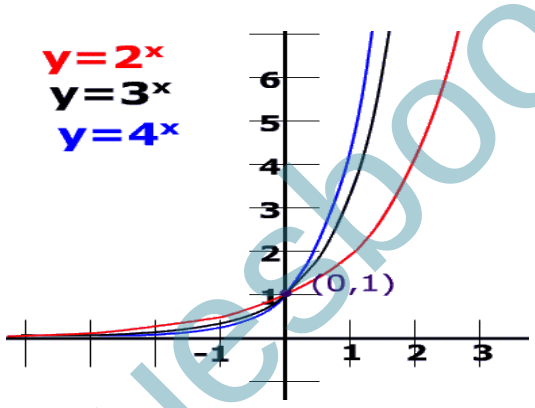
- For ,

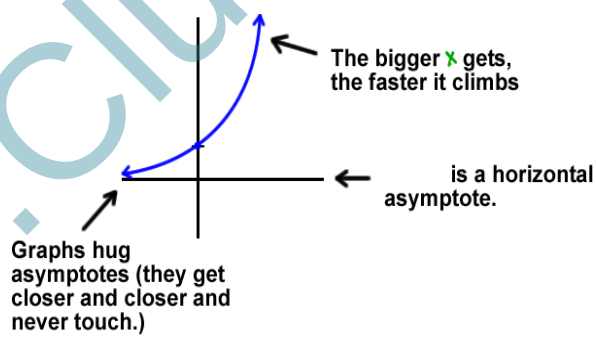
- Exponential function reduces to a constant function if .
- Caveat: unlike power functions which have variables in base instead of exponent.

e.g.

- Exponential function can either have growth or decay.

**Case of Exponential Growth**




**Case of Exponential Decay**

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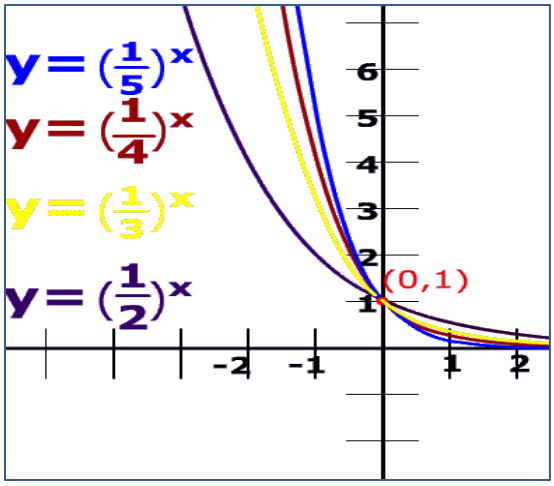
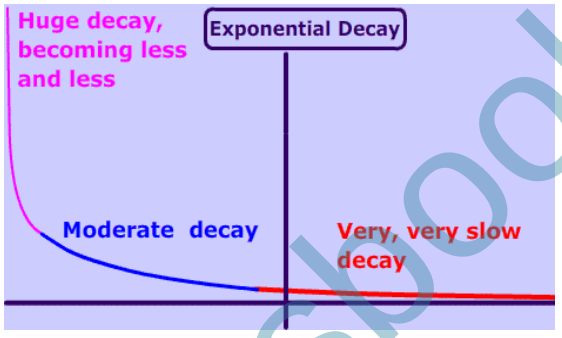


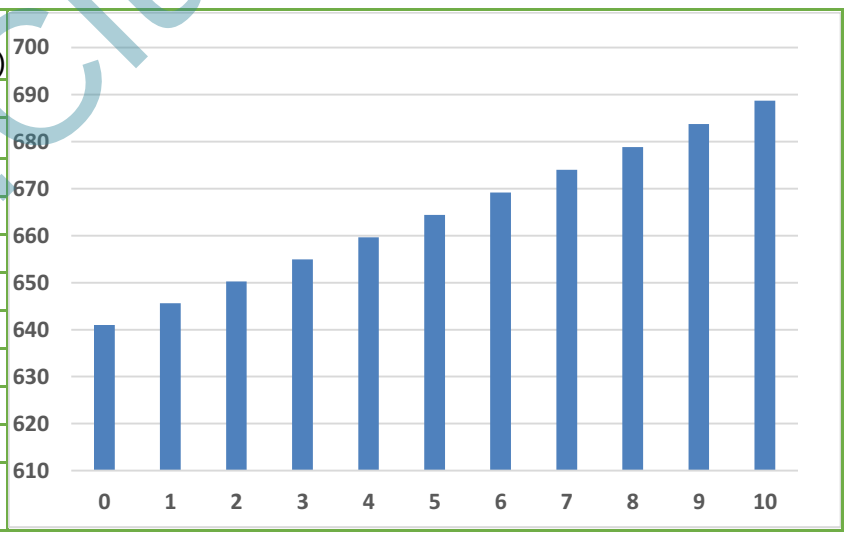




**TOPIC 030: POPULATION GROWTH USING GENERAL EXPONENTIAL FUNCTIONS**

Europe population function function:

- is population in millions,
- is time in years

| Time (t) | Population (P) |
|---|---|
| 0 | 641 |
| 1 | 645.6152 |
| 2 | 650.2636 |
| 3 | 654.9455 |
| 4 | 659.6611 |
| 5 | 664.4107 |
| 6 | 669.1945 |
| 7 | 674.0127 |
| 8 | 678.8655 |
| 9 | 683.7534 |
| 10 | 688.6764 |




For , year 2000

million

- Whereas, D/]] <} million.



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- Overestimation of 126 million could be due to poor projection.
- Time required for certain level of population ]<{ }}[

  years.
- Detailed steps

**TOPIC 031: OTHER TYPES: NON-ALGEBRAIC NATURAL EXPONENTIAL FUNCTIONS**

Special type of exponential functions

- Represented by using in the base of the exponent.
- exponential
- It has a constant value: **2.718**.
- Gé y$uxy– x$ké$wkqsnkv$x∞wl o} 2

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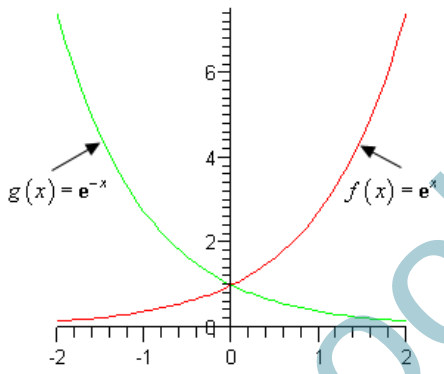


**TOPIC 032: POPULATION GROWTH USING NATURAL EXPONENTIAL FUNCTIONS**

**Population growth of a region**

- 
- in year 2000
- in year 2005
- Substituting:

- 2.02%

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**Detailed solution**

---

**Forecast for Population Growth after 10 years**

- t = 10 year 2010.
-
-
-

**Time needed for doubling of Population**

Let be that time.

- Double population =

-
-
- 657 years

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**Lesson 08**

## LOGARITHMIC FUNCTIONS AND INVERSE FUNCTIONS

### TOPIC 033: OTHER TYPES: NON-ALGEBRAIC LOGARITHMIC FUNCTIONS

K“zyxox´é$x$nséq∞séo 2

- Output is an exponent.
- If is index/exponent form then
- is its logarithmic form.
- Numerically:
  , Where, , and .
- Then, .

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**TOPIC 034: OTHER TYPES: NON-ALGEBRAIC NATURAL LOGARITHMIC FUNCTIONS**

A wyqk}śr w$y$r o$ kéo$b$62 5=6=$ -22

- Represented using instead of .
- Base of natural log is omitted as it is understood.
- Laws of natural logarithm are similar to that of common logarithm.

**TOPIC 035: RATE OF GROWTH OF GNP USING LOGARITHMIC FUNCTIONS**

-
- Rate of growth of GNP of China ]
-
- Rate of growth of GNP of UnD
- If the GNP of each country continued to grow exponentially, when would the GNP of the two nations be the same?

**TOPIC 036: INVERSE FUNCTIONS**

A reciprocal of a function.

- If original function is:

  ( ,

  Then (



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Or (

- Kmyxyw sm$kzzvsmk´syxé$yp$x..o}éo$p∞xm´syx$xmv∞no$Q..o}éo$nowkxn$p∞xm´syx

**TOPIC 037: FUNCTIONS WITH TWO OR MORE INDEPENDENT VARIABLES**

B

- A given pair of x and y give value of z.
- e.g. Linear specification:
- e.g. Quadratic specification:

- Ordered triples .

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- More than two independent variables can also exist.
- e.g. Utility function: B
- Ordered quadruples ( p .
- Hypersurface non-graphable.

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- Generally speaking, number of independent variables.
- e.g. Utility function:

p .

**TOPIC 038: SURFACES AND DISTANCE IN GRAPHS OF TWO OR MORE INDEPENDENT VARIABLES**

**Surfaces**

- makes a point in graph.
- { makes a surface in graph.

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**Lesson 09**

## EQUATIONS AND TYPES OF EQUATIONS

### TOPIC 039: EQUATIONS AND IDENTITIES

**Equations**

Mathematical expression with equality. e.g.

- True for certain values of .
- is true only for .
- Not for other values.

**Economic Examples of Equations**

- Demand function

B

- Production function
- Cost function
- Optimization condition
- Market equilibrium condition

**Identities**

**Etymology:** Latin: *Idem* (same).

- Mathematical equalities that are true for all values of .
- e.g.

] /[]

- is used to represent an identity.

] /[] ] /[] B ] /[] B

**Economic Example of Identity**

Equation of profit:

- Deducting cost from revenue will always give profit, therefore:
-

### TOPIC 040: TYPES OF EQUATIONS IN ECONOMICS: DEFINITIONAL EQUATIONS

Reflection of a definition in an equation.

- An Identity having exactly same meaning.

**Economic Examples**

Revenue Function:

- [illegible] revenue and cost 2
- However, is also used.
- Here



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- Regardless of the positivity/negatively of answer.

- Identity of profit function remains intact even if answer is negative (loss) or zero (breakeven).

**National Income Accounting Identity**

- Is read as: Sum of expenditures by consumers, investors & government, and net exports is equation to national xmywo 0$~xno}$expenditure approach.

**TOPIC 041: FISCAL SURPLUS AND FISCAL DEFICIT USING EQUATIONS**

**Fiscal Deficit/Surplus**

**Fiscal Deficit**

B B B

- Is read as: Fiscal deficit is identically equal to the difference of government revenue and government expenditure 0$-r so$y}wo}$é$ewkwo}$r kx$k´´o}2

B

- e.g.
- Government is suffering from 50M of fiscal deficit.

**Fiscal Surplus**

B B

- Is read as: Fiscal Surplus is identically equal to the difference of government revenue and government expenditure 0$-r so$y}wo}$é$}ok´o}$r kx$atter.

- e.g.
- Government is 50M of fiscal surplus.

**Neither Fiscal Deficit nor Surplus**

B

- Is read as: Tosr o}$fiscal deficit nor surplus exist when there is no difference b/w government revenue and government expenditure 2$

- e.g.
- Government is neither having any fiscal deficit nor fiscal surplus.

- Fiscal deficit/surplus using tax revenue function.

If

If ,

Budget surplus = 80.



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- If G□ by 60, T□ due to □in government spending multiplier ).
- New G= Old Y .
-
- New Y= Old u .
- New T= .
- New budget surplus (compared with 80).

**TOPIC 042: TYPES OF EQUATIONS IN ECONOMICS: BEHAVIORAL EQUATIONS**

Specifies the manner in which dependent variables behaves in response to changes in independent variable(s).

- Can include technological and legal aspects.
- Such behavior can be either human or non-human.

- Human behavior: Aggregate consumption in relation to national income.

- Non-human behavior: Total Cost in relation to output.

- Consider two cost functions:

- Fixed cost(FC) =
- FC are 75 and 110, respectively.
- First function has linear relationship.
- While second has quadratic.

- Consider two cost functions:

- Fixed cost(FC) =
- FC are 75 and 110, respectively.
- First function has linear relationship.
- While second has quadratic.

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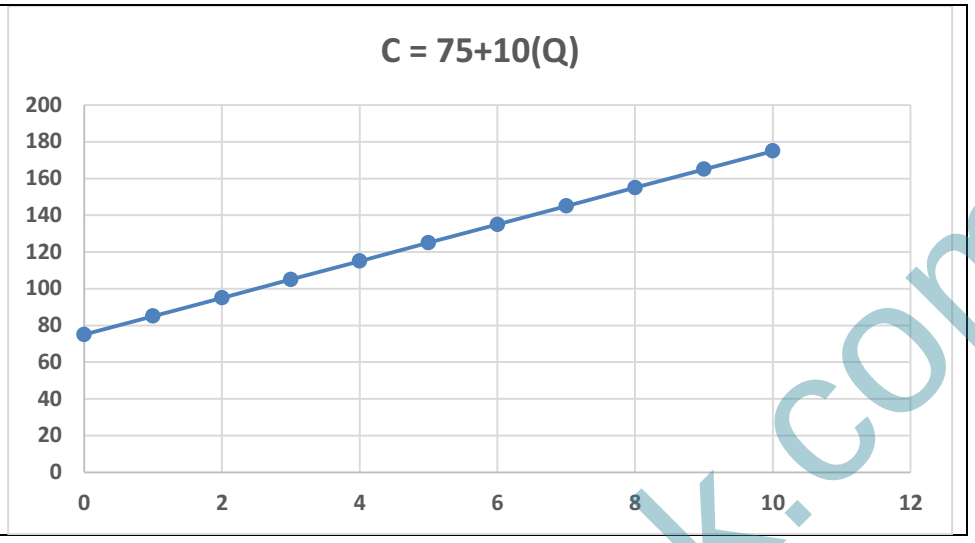
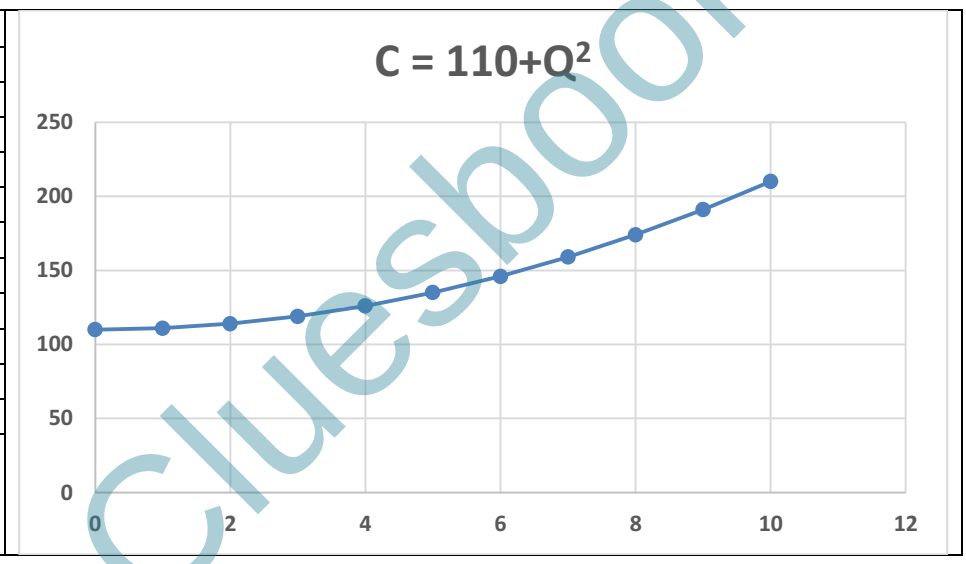
| C=75+10Q | |
|---|---|
| Q | C |
| 0 | 75 |
| 1 | 85 |
| 2 | 95 |
| 3 | 105 |
| 4 | 115 |
| 5 | 125 |
| 6 | 135 |
| 7 | 145 |
| 8 | 155 |
| 9 | 165 |
| 10 | 175 |




| C=110+Q$^2$ | |
|---|---|
| Q | C |
| 0 | 110 |
| 1 | 111 |
| 2 | 114 |
| 3 | 119 |
| 4 | 126 |
| 5 | 135 |
| 6 | 146 |
| 7 | 159 |
| 8 | 174 |
| 9 | 191 |
| 10 | 210 |




**TOPIC 043: TYPES OF EQUATIONS IN ECONOMICS: CONDITIONAL EQUATIONS**

Specifies a requirement to be satisfied.

- To specify equilibrium, an equilibrium condition should be specified.
- Two famous equilibrium conditions are:

Quantity demanded = Quantity supplied .

Desired savings = Desired investment .

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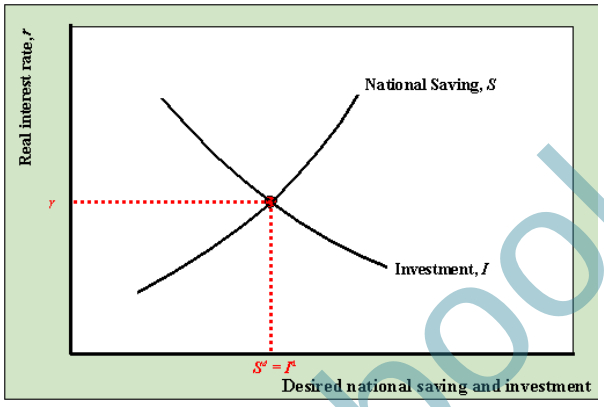


- Optimization conditions is also an example:
  - Marginal cost = Marginal revenue .

**TOPIC 044: STRUCTURAL AND REDUCED FORM EQUATIONS**

Assume following equations:

,5-

,6-

National income

Consumption

Autonomous investment

and are parameters.

Equation (1) & (2) are structural form equations.

- Putting value of C in eq. (1).



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- Value of Y (endogenous variable) in terms of exogenous variable ( ) and parameters ( >).
- Reduced form equation
- Value of other endogenous variable ( ):

= 

Numerical results:

- If , and , then

**3000**

B

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**Lesson 10**

## PARTIAL LINEAR MARKET EQUILIBRIUM

### TOPIC 045: CONSTRUCTING A PARTIAL LINEAR MARKET EQUILIBRIUM

- Three variables; , and :
- Specify conditional equation for equilibrium:

2,5-

Or

- Excess demand is equal to zero.

- Behavioral equations of and .

,6-

2 ,7-

Where

- Where, is the intercept of . is the slope of which is negative.
- is the intercept of . is the slope of which is positive.

- has horizontal intercept at which is reservation price lowest **price** at which a seller is willing to sell.
- Contrary to convention where is plotted on y-axis and , on x-axis.
- Convention is based on inverse demand function.
- Mathematically justified.

### TOPIC 046: SOLVING PLMM USING ELIMINATION OF VARIABLE METHOD

Equate behavioral equations of and by assuming

.

,6 -

2 ,7 -



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**Equilibrium price**

;

- Substitute in (or ).

,

OR for

- **In set notation**

**TOPIC 047: SHIFTS IN DEMAND IN MARKET EQUILIBRIUM**

Assume demand & supply functions:

&

- Equilibrium price and equilibrium output are 30 and 70, respectively.
- Any exogenous factor (e.g. increased income) increases the demand shifting its curve to the right side:

- New equilibrium price and equilibrium output are 33.3 and 76.7, respectively.

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**TOPIC 048: SHIFTS IN SUPPLY IN MARKET EQUILIBRIUM**

Assume demand & supply functions:

&

- Equilibrium price and equilibrium output are 30 and 70, respectively.
- Any exogenous factor (e.g. improved technology) increases the supply shifting its curve to the right side:

- Equilibrium price and equilibrium output are 28 and 72, respectively.

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**TOPIC 049: EFFECT OF TAX ON PRODUCER ON PARTIAL MARKET EQUILIBRIUM**

Assume demand & supply functions:

- &

_

_____

_

_____

- Rs. 2 per unit tax on producer:
- Ro´$r o$k“$ o$

_

_____

_____

- Revenue of producer before tax and after tax :

_____

_____

- Change in producer revenue = _____

**TOPIC 050: EFFECT OF TAX ON CONSUMER ON PARTIAL MARKET EQUILIBRIUM**

Assume demand & supply functions:

- &

_

_____

_

_____

- Rs. 2 per unit tax on consumer:



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**Lesson 11**

## GENERAL EQUILIBRIUM AND NATIONAL INCOME EQUILIBRIUM

### TOPIC 051: PARTIAL MARKET EQUILIBRIUM-A NON-LINEAR MODEL

Assume demand & supply functions:

&

B

Either

Or

B B

- is the standard form of quadratic (non-linear) function.
- It can also be solved using quadratic formula:
-

- Here, , , and .



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- Other nonlinear models may be cubic, quartile etc.

**TOPIC 052: GENERAL MARKET EQUILIBRIUM: GENERAL FORM OF TWO GOOD CASE**

Assume demand & supply functions:

**Good 1**

$

OR

(No excess demand of Good 1)

**Good 2**

$

OR

(No excess demand of Good 2)

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- The equilibrium outputs of Good -1 and Good -2 can be found by putting equilibrium values of their prices i.e and .

**TOPIC 053: GENERAL MARKET EQUILIBRIUM: NUMERICAL SOLUTION OF TWO GOOD CASE**

Assume demand & supply functions:

**Good 1**

**Good 2**

- Solving for Good 1:

- Solving for Good 2:

Solving underlined equations, simultaneously:

&

Put in for :

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Put in for :

, 0

**TOPIC 054: GENERAL MARKET EQUILIBRIUM: N-GOOD CASE**

Transition from partial equilibrium analysis to general-equilibrium analysis.

- Two goods case can be generalized to -number of goods.
-

&

Where, ( .

- Excess demands for - goods (complete market).

-
- Let
- Then
- Solved simultaneously, these equations can determine the equilibrium pricesB .

**TOPIC 055: NATIONAL INCOME EQUILIBRIUM**

Application of algebra to macroeconomic analysis.

- Simple Keynesian model.

B

- Endogenous variables: and represent national Income & consumption respectively.
- Exogenous variables: and represent autonomous investment & government expenditure respectively.
- Second equation is a behavioral equation showing the behavior of consumption with respect to income.

- Substituting consumption function in national income equation.

- Solution is in terms of exogenous variables and parameters.
  , for to be defined.
- Equilibrium level of the other endogenous variable (consumption).
- Substitute in consumption function.



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- , for to be defined.

**TOPIC 056: NATIONAL INCOME EQUILIBRIUM WITH INDUCED AND AUTONOMOUS TAX**

- Macroeconomic analysis with autonomous tax & induced tax .
- Keynesian model.

- Endogenous variables: and represent national income, consumption & taxes respectively.
- Exogenous variables: and represent autonomous investment & government expenditure respectively.
- 2nd and 3rd equations are behavioral equations showing the behavior of consumption and taxes with respect to income.

- Substituting consumption function and tax function in national income equation.

- Equilibrium level of taxes.

- Equilibrium level of consumption.

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**TOPIC 057: NATIONAL INCOME EQUILIBRIUM WITH PROPORTION OF GOVERNMENT EXPENDITURE**

Macroeconomic analysis with proportion of government expenditure.

- Keynesian model.

- Endogenous variables: , and represent national income, consumption and government expenditure, respectively.
- Exogenous variables: and represent autonomous investment & autonomous tax respectively.
- 2nd and 3rd equations are behavioral equations showing the behavior of consumption and government expenditure w.r.t income.

- It implies the government expenditure is a ratio of national income.
- Substituting consumption function and government expenditure function in national income equation.

- Parametric restriction for avoiding undefined value of national income.



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**Lesson 12**

## USE OF MATRICES IN ECONOMICS

### TOPIC 058: MATRICES AND VECTORS

Matrix is a rectangular array of numbers considered as one mathematical object.

- Bold capital letters such as **A**, **B**, **C**
- When there are rows and columns in the matrix, it is a matrix (written as ).
- A matrix is of the form:

- are the elements/entries of the row of matrix.
- are the elements/entries of the row of matrix.
- are the elements/entries of the column of matrix.
- are the elements/entries of the column of matrix.

- Alternative ways of writing a matrix are:
- or
- In more simpler notation:
- or

- **Zero Matrices:** Four examples w.r.t matrix order

- If matrix appears with either a single row or column, it is known as vector.
- If and then it is a row vector.
- If and then it is a column vector.

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**TOPIC 059: MATRICES OPERATIONS**

Like algebraic expressions, matrices can also be operated upon using arithmetic operators .

**Caveat I:** S k´}smoé$mkx ´$ o$ns.snon$ .

**Caveat II:** Matrices can only be solved in linear algebra.

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**TOPIC 060: USING PRODUCT OF MATRICES TO CALCULATE TOTAL COST**

There are three coffee shops in a small college town.

Kkmr $mypoo$ér yz$éowé$py∞}$ voxné$ypmypoo?$Kéz}oééy 0$l kzz∞mmsxy 0$l vkéésm$kxn$L}oxmr 2$

The cost of coffee for a cup of each blend is:

| Cost of Coffee per cup [Cost (C)] | |
|---|---|
| Expresso | $1.50 |
| Cappuccino | $0.75 |
| Classic | $0.50 |
| French | $1.00 |

| Number of Cups of Coffee Sold Per Week [Quantity Produced (Q)] | | | | |
|---|---|---|---|---|
| | Expresso | Cappuccino | Classic | French |
| Coffee Shop 1 | 112 | 100 | 80 | 35 |
| Coffee Shop 2 | 182 | 160 | 110 | 58 |
| Coffee Shop 3 | 206 | 192 | 130 | 76 |



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**TOPIC 061: USING PRODUCT OF MATRICES TO CALCULATE TOTAL REVENUE AND PROFIT**

There are three coffee shops in a small college town.

Kkmr $mypoo$ér yz$éowé$py∞}$l voxné$ypmypoo?$Kéz}oéé y 0$l kzz∞mmsxy 0$l vkéésm$kxn$L}oxmr 2$

The cost of coffee for a cup of each blend is:

| Cost of Coffee per cup [Cost (C)] | |
|---|---|
| Expresso | $1.50 |
| Cappuccino | $0.75 |
| Classic | $0.50 |
| French | $1.00 |

| Number of Cups of Coffee Sold Per Week [Quantity Sold (Q)] | | | | |
|---|---|---|---|---|
| | **Expresso** | **Cappuccino** | **Classic** | **French** |
| Coffee Shop 1 | 112 | 100 | 80 | 35 |
| Coffee Shop 2 | 182 | 160 | 110 | 58 |
| Coffee Shop 3 | 206 | 192 | 130 | 76 |



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| Retail Price ($) of Each Blend at Each Shop [Price (P)] | | | | |
|---|---|---|---|---|
| | **Expresso** | **Cappuccino** | **Classic** | **French** |
| Coffee Shop 1 | 8 | 4 | 3 | 6 |
| Coffee Shop 2 | 6 | 2 | 2 | 5 |
| Coffee Shop 3 | 5 | 3 | 1 | 3 |

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**TOPIC 062: QUESTION OF MATRIX DIVISION**

Subject to the conformability conditions, matrices, like numbers, can undergo the operations of addition, subtraction, and multiplication.

- However, division of two matrices is not possible i.e. is not possible
- For two numbers is defined when .
- Alternative representations are or .
- where shows the reciprocal of .
- If is defined then there is no assurance if also defined (due to multiplication conformity condition).
- Even if and are both defined, still they may not be necessarily equal.
- and are two distinct products.
- It is necessary to distinguish between them while specifying.

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**Lesson 13**

## LAWS OF OPERATIONS OF MATRICES

### TOPIC 063: COMMUTATIVE, ASSOCIATIVE, AND DISTRIBUTIVE LAWS

**Commutative Law** in case of addition holds but not in case of multiplication of matrices:

-
-

**Associative Law** holds both in case of addition and multiplication of matrices:

**Distributive Law** holds in case of Matrices operation:

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**TOPIC 064: VECTOR OPERATIONS**

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## TOPIC 065: TRANSPOSE OF A MATRIX

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**TOPIC 066: COFACTORS OF A MATRIX**

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**TOPIC 067: ADJOINT OF A MATRIX**

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**Lesson 14**

# DETERMINANT AND INVERSE OF MATRICES

## TOPIC 068: DETERMINANT OF A MATRIX

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**TOPIC 069: SARRUS'S RULE FOR 3X3 ORDER DETERMINANT OF A MATRIX**

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}\,a_{22}a_{31}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}\,a_{22}a_{31}$$

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**TOPIC 070: INVERSE OF A MATRIX**

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3\times8-4\times6}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{24-24}\begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6 + 7\times-0.2 & 4\times-0.7 + 7\times0.4 \\ 2\times0.6 + 6\times-0.2 & 2\times-0.7 + 6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**TOPIC 071: CONDITION(S) FOR NON-SINGULARITY**

Assuming a system of equation.

Writing in matrix form

, t G

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(A is non-singular matrix & )

Solving equations simultaneously

**[Consistent, Independent]**

Assume

, t G

(A is a singular matrix & )

Solving equations simultaneously

**(Undefined: Consistent & Dependent)**

No solution for x and y.

Assume

Matrix form



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, t G

(A is a Singular matrix & )

Solving equations simultaneously

**[Inconsistent, Independent]**

No solution for x and y.

Summarizing conditions:

- **Necessary condition:** Coefficients matrix should be square [Rows= Columns].
- **Sufficient condition:** Determinant of coefficients matrix 0. Equations should be intersecting. Neither coincident nor parallel.

**TOPIC 072: EXPRESSION OF NATIONAL INCOME USING MATRIX FORM**

Simple national income model in two endogenous variables Y and C is:

Re-arranging:

Writing in the equation in matrix form.

,

Following that:

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Corresponding elements are equal which is equivalent to original equations.

**TOPIC 073: MINORS AND COFACTORS**

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**Lesson 15**

**MATRIX INVERSION METHOD**

**TOPIC 074: MARKET MODEL ANALYSIS USING MATRIX INVERSION METHOD**

$$Q_d = Q_s$$

$$Q_d = a - bP$$

$$Q_s = -c + dP$$

$$Ax = b \Leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & d \end{bmatrix} \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

$$det \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & d \end{bmatrix} = -(d+b) \neq 0$$

$$x = A^{-1}b$$

$$x = A^{-1}b \Leftrightarrow \frac{1}{-(d+b)} \begin{bmatrix} -b & -d & -b \\ d & -d & -b \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

$$\frac{1}{-(d+b)} \begin{bmatrix} -(ad-bc) \\ -(ad-bc) \\ -(a+c) \end{bmatrix} = \begin{bmatrix} Q_d \\ Q_s \\ P \end{bmatrix}$$

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## TOPIC 075: NATIONAL INCOME ANALYSIS USING MATRIX INVERSION METHOD

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## TOPIC 076: EQUILIBRIUM PRICES USING MATRIX INVERSION METHOD

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**Lesson 16**

## CRAMER'S RULE IN MATRICES

### TOPIC 077: SOLVING MARKET MODEL USING CRAMER'S RULE

Consider a linear partial market model:

### TOPIC 078: EQUILIBRIUM PRICES USING CRAMER'S RULE

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**TOPIC 79: NATIONAL INCOME DETERMINATION USING CRAMER'S RULE**

2-sector national income model:

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**Lesson 17**

## INPUT OUTPUT ANALYSIS USING MATRICES

### TOPIC 080: INPUT COEFFICIENT MATRIX

Attributed to Wassely Leontief (1951)

- Y´k´sn$..o}ésyx?$ a r k´$vo..ov$yp$y∞´z∞´$ér y∞vn$okm r $yp$´r o$ industries in an economy z}yn∞mo0$sx$y}no}$r k´$s´$-sv$t∞é´$l o$é∞psmsox´$y$ék´sép" $r o$y´kv$now kxn$py}$r k´$z}yn∞mÍE
- Input-Output: output of one industry is needed as input in another industry & vice versa Inter-industry dependence.
- Input-output analysis can be of great use economic planning.
- Toon$py}$my}}omí$vo..ov$yp$y∞´z∞´$l kéon$yx$technical input-output, rather than market equilibrium conditions.
- Mathematically speaking, s´é the solution of simultaneous equations.
- Normally has large number of industries.

**Assumptions**

- Each industry has homogeneous output or jointly product in fixed proportions.
- Each industry has fixed input ratio for output.
- CRS in each industry: **k-r{ xp# #z #inputs k-r{ xp# #z #output.**
- **Input coefficient matrix** contains **input coefficient.**

B

B B

B B

BB B

BB B

0.35 PKR input **8 #7nd input)** is needed in producing 1 PKR of **7 #7nd output).**

| Input | Output I | II | III | · | · | · | N |
|---|---|---|---|---|---|---|---|
| I | 11 | 12 | 13 | · | · | · | 1 |
| II | 21 | 22 | 23 | · | · | · | 2 |
| III | 31 | 32 | 33 | · | · | · | 3 |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| N | 1 | 2 | 3 | · | · | · | |

- Number of input coefficients in input coefficient matrix are:
-
- Zero elements in principal diagonal:

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| | Output | | | | | | |
|---|---|---|---|---|---|---|---|
| **Input** | I | II | III | . | . | . | N |
| I | 11 | 12 | 13 | . | . | . | 1 |
| II | 21 | 22 | 23 | . | . | . | 2 |
| III | 31 | 32 | 33 | . | . | . | 3 |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| N | 1 | 2 | 3 | . | . | . | |

**TOPIC 081: ECONOMIC MEANING OF HAWKINS-SIMON CONDITION**

Attributed to David Hawkins and Herbert A. Simon,

- Guarantees the existence of a non-negative output vector.
- Assume a 2-sector economy:

- **Two conditions**

B B

- **First condition**

implies that amount of first commodity used in production of 1st commodity is less than PKR 1.

- **Second condition**

B



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can be omitted without affecting the inequality.

B B B B

- Practicability and viability in production via Hawkins-Simon condition.

**TOPIC 082: INPUT-OUTPUT ANALYSIS IN CASE OF OPEN ECONOMY**

Assume a 3-sector economy (Agriculture, Industry & Services).
Its input coefficient matrix is:

| | Agri | Indu | Serv |
|---|---|---|---|
| Agri | | | |
| Indu | | | |
| Serv | | | |

- Agriculture uses PKR 0.2 of agricultural output in producing PKR 1.
- Agriculture uses PKR 0.4 & PKR 0.1 of industrial & services outputs, respectively, in producing PKR 1.
- In addition to mutual demand, consumer demand also exists.
- Demands (in billion PKR) are in demand vector.

- Forming a system of equations:

- Total demand = Mutual demand + Consumer demand

B

B

B

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(Leontief Matrix)

B

**Interpretation:** To suffice for mutual and consumer demand.

- Agricultural output = 24.84 billion PKR.
- Industrial output = 20.68 billion PKR.
- Services output = 18.36 billion PKR.

**TOPIC 083: INPUT-OUTPUT ANALYSIS IN CASE OF CLOSED ECONOMY**

Assume a 3-sector economy (Agriculture, Industry & Services).
Its input coefficient matrix is:

| | Agri | Indu | Serv |
|---|---|---|---|
| Agri | | | |
| Indu | | | |
| Serv | | | |

- No consumer demand:

S

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- All outputs are consumed in process of production.
- All columns sum to 1.

Total demand = Inter-industry demand

B

B

B

Writing in matrix form:

However, the homogeneous system of equations can be solved simultaneously:

— B —

— B —

In ratio form (in terms of z):

— —

Converting to whole numbers:

For the closed model of 3 industries, we get a guiding ratio.

**TOPIC 084: THE NEED AND NATURE OF COMPARATIVE STATICS**

If all variables are at rest, an equilibrium is often called a static.
Comparing equilibria is, therefore, called comparative statics.
How would be the new equilibrium compared with the old?
Qualitative (in which direction) and quantitative (by how much) aspects.

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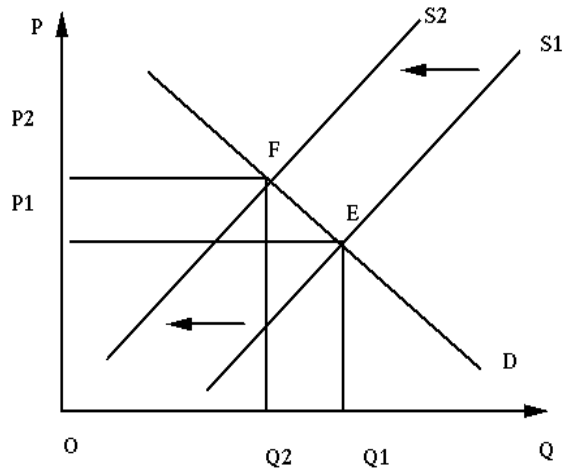


**Two Assumptions**

- System returns to equilibrium instantaneously merely compare the initial (pre-change) equilibrium with the final (post-change) equilibrium .
- Stable equilibrium.

- Xk´o$ypm kxqo$ypr o$equilibrium value of an endogenous variable w.r.t the change in a particular parameter or exogenous variable Toon$y}$no}s.k´s.oé2
- Functionally speaking:



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= Equilibrium value of endogenous variable
= parameter/exogenous variable.

**Difference Quotient**

- Let, = change in from to .
- Then, .
  = old value of
  = new value of
- Then function changes from to .
- Change in per unit of change in : —

— —

**Numerical**

B B B B B B — —

B B B

— B

If , and — exists. Expression — is derivative of the function.

— —

— —

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**Lesson 18**

## CONCEPT OF DERIVATIVE AND RULES OF DIFFERENTIATION

### TOPIC 085: CONCEPT OF LIMIT AND CONTINUITY

**Limit:** A point y}$o..ov$ o" yxn$–r sm $éywo´r sxq$nyoé$xy´$y}$wk" $xy´$o" ´oxn$y}$zkéé2

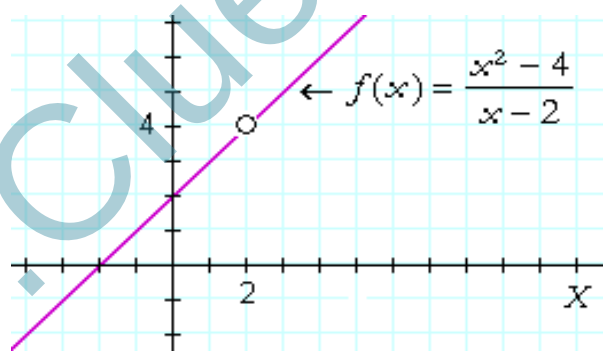
Let ( ——

Clearly, ( at ( . i.e. ( .

Before and after ( 2, ( is defined.

| x | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|
| f(x) | 3.9 | 3.99 | 3.999 | | 4.001 | 4.01 | 4.1 |

Observe: x approaches to 2, f(x) approaches close to 4. i.e. , .

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$\}$ ——

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Let

- For ( or ( , denominator=0
- Function is discontinuous at ( and ( .

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**TOPIC 086: RATE OF CHANGE, SLOPE & DERIVATIVE**

For any continuous function , rate of change can be calculated using differentiation.

- Differentiation is process of calculating derivatives.
- Derivative is also the slope of the function.
- Derivative of function = — a.k.a Leibniz's notation of derivative

  }oz}oéox´é$r o$m kxqo$ 2

  Rate of change is function is —.

  Therefore;— —

  i.e. Derivative of function represents the rate of change.
- Slope also shows the rate of change.

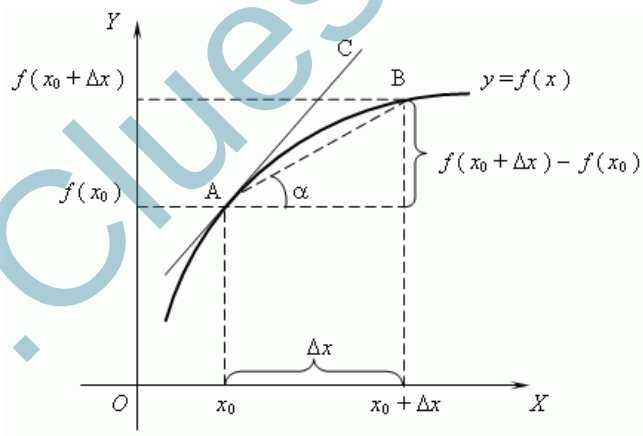
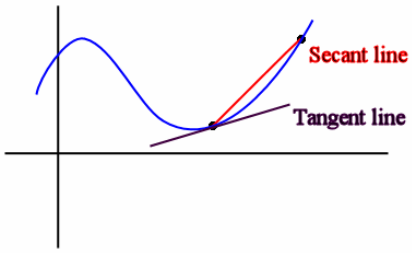



Fig. 1



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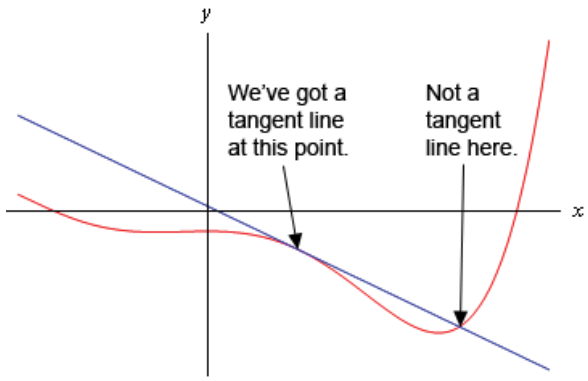


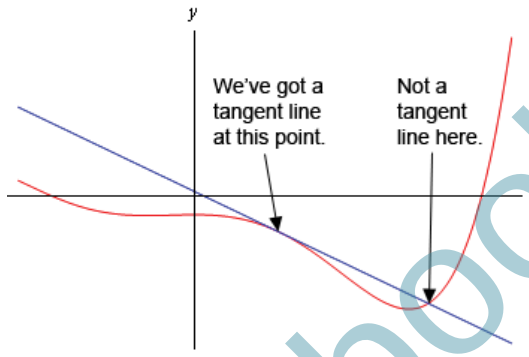


Therefore, derivative, rate of change and slope can be used synonymously.

— — B B

- In economics, taking derivative is equal to calculating the marginal function of the original function.

e.g. Derivative of total cost function gives marginal cost (rate of change of total cost).

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**Lesson 19**

## PRODUCT RULE AND QUOTIENT RULE OF DIFFERENTIATION

### TOPIC 087: DIFFERENTIATION RULES FOR SINGLE VARIABLE FUNCTIONS: CONSTANT FUNCTION RULE AND POWER FUNCTION RULE

Rules of differentiation are necessary for calculating derivatives.

- Most commonly used are:
    - Constant function rule
    - Power rule
    - Sum-difference rule
    - Product rule
    - Quotient rule

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## TOPIC 088: SUM-DIFFERENCE RULE OF DIFFERENTIATION

## TOPIC 089: SUM-DIFFERENCE RULE: NUMERICAL ANALYSIS OF COST FUNCTION

- Is a cost function in cubic form.
- To calculate marginal cost, we can take its derivate with respect to .

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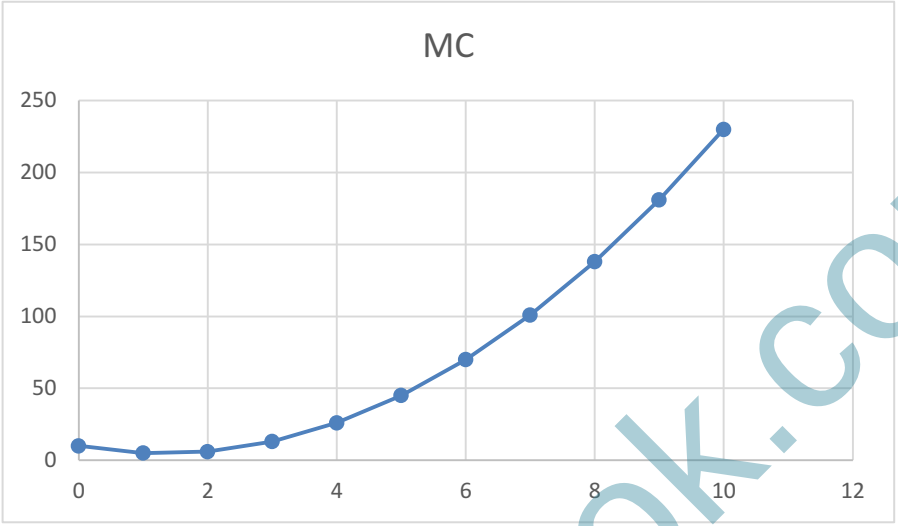
**Slope of Cost Curve, Rate of Change in Cost Function, Marginal Cost**

| Q | MC |
|---|---|
| 0 | 10 |
| 1 | 5 |
| 2 | 6 |
| 3 | 13 |
| 4 | 26 |
| 5 | 45 |
| 6 | 70 |
| 7 | 101 |
| 8 | 138 |
| 9 | 181 |
| 10 | 230 |




**TOPIC 090: UNDERSTANDING GRAPHS OF FUNCTION AND THEIR DERIVATIVES**

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**Lesson 20**

## COST AND REVENUE ANALYSIS USING DIFFERENTIATION

### TOPIC 091: PRODUCT RULE OF DIFFERENTIATION

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**TOPIC 092: RELATIONSHIP BETWEEN AVERAGE REVENUE AND MARGINAL REVENUE USING PRODUCT RULE**

Given an average revenue function:

To find marginal revenue function, one needs to find total revenue first.

Differentiating the total revenue function w.r.t Q, we get marginal revenue.

Symbolic treatment of revenue function:

*[as AR depends on quantity sold]*

Differentiating w.r.t Q using product rule

Here **+** **and**

**TOPIC 093: QUOTIENT RULE OF DIFFERENTIATION**

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**TOPIC 094: MARGINAL PROPENSITY TO CONSUME VIA DIFFERENTIATION WITH AND WITHOUT TAX**

Given a consumption function:

Numerically speaking

Differentiating consumption function w.r.t. Y.

Which is marginal propensity to consume.

Now, assume imposition of tax.

0.8 is the marginal propensity to consume in presence of tax



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**TOPIC 095: RELATIONSHIP BETWEEN MARGINAL-COST AND AVERAGE-COST FUNCTIONS USING QUOTIENT RULE**

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**Lesson 21**

## CHAIN RULE AND INVERSE FUNCTION RULE OF DIFFERENTIATION

### TOPIC 096: VARIABLE AND FIXED COST COMPONENTS IN TOTAL COST FUNCTION

In short run, some costs do not change (cost of land, equipment and rent) Fixed costs (FC)

- However, in long run all costs become variables.
- Other costs vary with output (cost of raw material, components, energy and unskilled labor) Variable costs (VC).
  Total variable costs: =( ).

**Total Costs**

**Average Costs**

**If FC = 1000, VC = 4, then:**

- **Average Cost**



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**TOPIC 097: OBTAINING MARGINAL COST FUNCTION FROM AVERAGE COST FUNCTION**

Given average cost function:

**Total cost function**

**Marginal cost function**

**Long run or Short run?**

Consider the total cost function.

Since all term involve (variable).

All components can vary, which is possible in long run Long run cost function.

**TOPIC 098: MARGINAL COST ANALYSIS**

Given total cost function:

**Marginal cost function**



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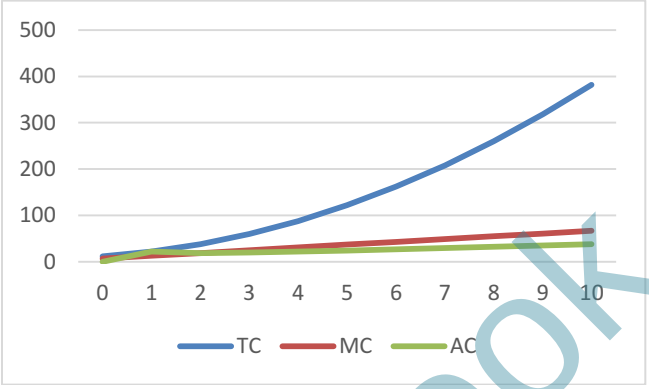
**Average cost function**B

B–




**TOPIC 099: MARGINAL REVENUE ANALYSIS**

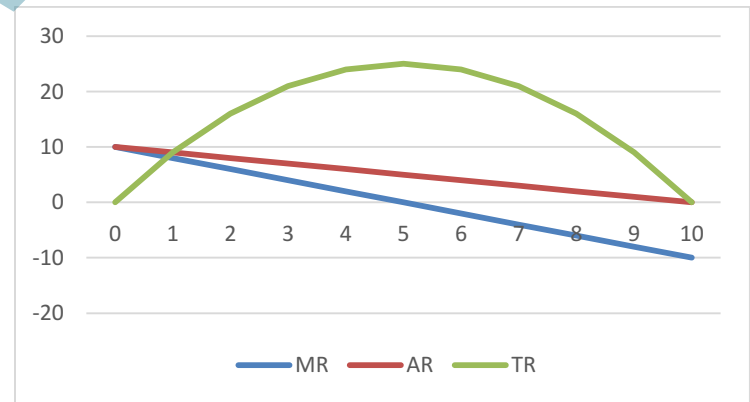
Given total revenue function:

**Marginal revenue function**

B–

**Average revenue function**

B–

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**TOPIC 100: MARGINAL PRODUCT ANALYSIS**

Given total product function:

Where,

**Marginal product (of labor) function**

B——

**Average product function**

B–

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**Lesson 22**

## USE OF PARTIAL DIFFERENTIATION IN ECONOMICS

**TOPIC 101: RULES OF DIFFERENTIATION FUNCTIONS WITH DIFFERENT VARIABLES CHAIN RULE**

Possibility of indirect dependence of one variable on other.

where

- Dependence of on via .

— — —

- Hence chain reaction.

Possibility of more than 2 functions.

Where, &

Dependence of on via and , respectively.

— — — —

3-variables & 4-functions in a chain rule.

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**TOPIC 102: MARGINAL REVENUE PRODUCT OF LABOR (MRPL) ANALYSIS**

Given total revenue function of a firm:

Where, output Q is further a function of labor input , or

Dependence of on via .
Situation of chain rule.

Revenue due to labor = Marginal revenue Product due to labor

Marginal revenue product of labor = Marginal revenue Marginal physical product of labor

**TOPIC 103: MARGINAL ANALYSIS OF FISHERY PRODUCTION FUNCTION**

Estimated production function for a certain lobster fishery:

; E AE7Ri;

Where **S = Stock of lobsters**,
**E = Effort**, and **the catch**.

Marginal Product w.r.t Stock of lobsters.

E ; E A — 7Ri; B

Marginal product w.r.t effort.

E ; E A — 7Ri; B

Knowledge of values of and can give rise to numerical values of and that will be more interpretable.

**TOPIC 104: INVERSE FUNCTION RULE**

Given a function:

It can be reciprocal can be written as:

Xokn$ké?$“$é$kx$x..o}éo$p~xḿsyx$yp$” 2

is function related to original function similar to .

**Numerical example**

B



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**Numerical example**

Finding — is tricky.

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**Lesson 23**

## MARKET MODEL ANALYSIS USING PARTIAL DERIVATIVES

### TOPIC 105: PARTIAL DIFFERENTIATION: THE CONCEPT

There can be more than one independent variables in a function.
For example, number of independent variable case.

Partial differentiation is suitable in such situation.
Delta is used to represent change.

is partial derivative of function w.r.t. independent variable .

Further derivatives will be .

So, the number of partial derivatives shall be equal to the number of independent variables.

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**TOPIC 106: MARGINAL PHYSICAL PRODUCT OF LABOR AND CAPITAL USING PARTIAL DERIVATIVES**

Specific Cobb-Douglas production function:

***; E AEX;***

Where, **K = Stock of capital**,

**L = Labor**, and **Q(K,L) the output**.

Marginal physical product w.r.t stock of capital.

**E** ***; E A*** — ***XT***B

***XT***B

—

Specific Cobb-Douglas production function:

***; E AEX;*** Where,

Where, **K = Stock of capital**,

**L = Labor**, and **Q(K,L) the output**.

Marginal product w.r.t of labor.

**E** ***; E A*** — ***XT***B

***XT***B

—

Marginal physical products of capital and labor:

—

—

Knowledge of values of and can give rise to numerical values of and that will be more interpretable.

**TOPIC 107: MARGINAL UTILITY FUNCTIONS USING PARTIAL DERIVATIVES**



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Specific utility function:

***; E AE***

where **= 1st Good**,

**= 2nd Good**, and ***; E A*** **is the total utility**.

Marginal utility w.r.t 1st good.

**E** ***; E A***

Marginal utility w.r.t 2nd good.

**E** ***; E A***

Marginal utilities w.r.t. 1st and 2nd Goods, respectively:

; knowledge of values of and gave rise to numerical value of which is more interpretable.

**TOPIC 108: OUTPUT ELASTICITY OF LABOR AND CAPITAL USING PARTIAL DERIVATIVES**

General, Cobb-Douglas production function:

***; E AE6***

Where, **K = Stock of capital**,

**L = Labor**, and **Q(K, L) the output**.

While, = Output elasticity of capital ( ) and = Output elasticity of labor ( ).

**Numerator**

**E** ***; E A*** 6 B

6

6 6

6

6

**As**

6 6

6

6

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**D.I.Y for**

**TOPIC 109: MONEY MARKET ANALYSIS USING PARTIAL DERIVATIVES**

Money supply has two components: cash holdings and bank deposits .

Assume a constant ratio – Where, B .

High-powered money is the sum of cash holdings held by the public and the reserves held by the banks.

Bank reserves are a fraction of bank deposits, determined by the reserve ratio. –

Where .

Using given information following tasks can be done:

- Money supply expressed in terms of high-powered money.
- Impact of reserve ratio and money supply
- Impact of cash-deposit ratio and money supply.

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**Lesson 24**

## SECOND AND HIGHER ORDER DERIVATIVES

### TOPIC 110: PARTIAL MARKET MODEL USING PARTIAL DIFFERENTIATION

Money supply has two components: Cash holdings and bank deposits .

Assume a constant ratio – Where, .

High-powered money is the sum of cash holdings held by the public and the reserves held by the banks.

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## TOPIC 111: NATIONAL-INCOME MODEL USING PARTIAL DIFFERENTIATION

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**TOPIC 112: SECOND AND HIGHER ORDER DERIVATIVES**

Let, first derivative be of a function .
Higher-order derivatives can also be calculated.
They are also called higher derivatives.
e.g. Second order derivative is twice differentiation of .
Also denoted by —.

Further, (higher than second) derivatives can be expressed as follows:

Or by using notation:



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**Numerical example**

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## TOPIC 113: ECONOMIC APPLICATIONS OF SECOND DERIVATIVE: PROFIT MAXIMIZATION CONDITION

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**TOPIC 114: NUMERICAL EXAMPLE OF PROFIT MAXIMIZATION CONDITION USING SECOND DERIVATIVE**

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**Lesson 25**

**PARTIAL DERIVATIVES APPLICATION ON ELASTICITY AND PRODUCTION FUNCTIONS**

**TOPIC 115: YOUNG'S THEOREM**

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## TOPIC 116: DEMAND FOR MONEY FUNCTION ANALYSIS USING PARTIAL DERIVATIVES

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**TOPIC 117: INCOME ELASTICITY OF DEMAND USING PARTIAL DERIVATIVES**

Income elasticity of demand shows the percentage change in demand of a good w.r.t percentage change in income .

Where, = Marginal demand function w.r.t income and = Average demand function w.r.t income.

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**TOPIC 118: CROSS PRICE ELASTICITY OF DEMAND USING PARTIAL DERIVATIVES**

Cross price elasticity of demand shows the percentage change in **demand of one good** w.r.t percentage change in **price of other** .



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Where, = Marginal demand function of Good-1 w.r.t price of Good-2. = Average demand function of Good-1 w.r.t price of Good-2.

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**TOPIC 119: PARTIAL DERIVATIVES APPLICATION ON HERRING PRODUCTION FUNCTION**

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**TOPIC 120: PARTIAL DERIVATIVES APPLICATION ON THREE INPUT PRODUCTION FUNCTION**

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**Lesson 26**

## PARTIAL DERIVATIVES APPLICATION ON CONSUMER AND PRODUCER THEORIES

### TOPIC 121: ENVELOPE THEOREM

A mathematical theorem based on calculus.
Multiple applications in economics including producer theory.
Here, consumer theory is being subjected.

Objective function

Constraint function

Lagrangian function

Optimized utility function:

Where, are the Marshallian demand functions.
Detailed optimized utility function:

are variables and are parameters.
Variables in is now parametrized by .
If marginal utility of money is to be found, — (a total derivative) needs to be found at optimum points.

Which can be a tedious calculation.

However, Envelope theorem suggests that a partial derivative — can be found instead.
Therefore;

**Envelope Theorem:** Partial derivative of Lagrangian w.r.t a given parameter of variables equals ´r o$y´kv$ho}s k´s.o$ypyl tom´s.o$p∞xm´syx$o.kv∞k´on$k´$é$.k}skl voé$yz´sw∞w$zysx´é2
**Numerical Exercise:** Choose a utility function and a budget constraint and evaluate.

### TOPIC 677 BZ We [ #Q.MV] Qe

Attributed to for French economist René Roy.
One of the two ways to calculate Marshallian demand functions.
(Other being constrained optimization of utility function & budget constraint).i.e.

**Objective function**

**Constraint function**



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Uses ratio of partial derivatives of indirect utility function with respect to price of good (under consideration) and income, respectively.

For 2 Goods case:

and

, are needed.

**e.g.:**

D.I.Y, similarly for

**] WXQ&#678B#PW] MTTQYO [ #TMUUI**

Attributed to Harold Hotelling.

Mathematical result used to related supply of a good with z}yn∞mo} é$z}yps´2

The change in profits from a change in price is equal to the quantity produced.

Specifically speaking for a maximized profit function:



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Numerically speaking.

Assume

Profit

Price of output

Quantity of output produced

Price of capital input

Quantity of capital input employed

Let , , , .

After change

, , , .

in profits due to in price is 20 equals output produced.

**TOPIC 124: SHEPHARD LEMMA**

Attributed to Ronald Shephard

Mathematical result used in consumer & producer theory.

Demand for a particular good for a given level of utility and prices , equals the derivative of the expenditure function with respect to the price of the relevant good:

**K{ z–' y q.-- point of view**

is the Hicksian demand function for good .

is the expenditure function.

**Producers point of view**

is the conditional factor demand function for input .

is the cost function.

**Consumer Theory**

If maximized expenditure function:

Hicksian demand function for and are:



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Similarly, producer theory kvéy$×éoé$Yr ozr k}n é$Rowwk2
D.I.Y by using a cost objective function and an output constraint.

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**Lesson 27**

## USE OF DIFFERENTIALS IN ECONOMICS

### TOPIC 125: DIFFERENTIALS VERSUS DERIVATIVES

Derivative — of : a single entity.

Ratio of two quantities, and differentials.

Derivative as the ratio of differential of function by the differential of variable.

If — is the derivative of , then rearranging:

B

Differential of function.

Differentiation Differentials .

Differentiation w.r.t. Derivatives — .

**Rules of Differentials**

- **Sum-Difference Rule(s)**
- **Product Rule**
- **Quotient Rule**

— _____

Where

**Examples**

Since,

Since,

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## TOPIC 126: POINT ELASTICITY USING DIFFERENTIALS

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## TOPIC 127: ELASTICITY OF RECTANGULAR HYPERBOLIC DEMAND CURVE

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## TOPIC 128: INCOME AND PRICE ELASTICITY OF DEMAND USING DIFFERENTIALS



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## TOPIC 129: INCOME ELASTICITY OF CONSUMPTION USING DIFFERENTIALS

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## TOPIC 130: INCOME AND PRICE ELASTICITY OF IMPORT FUNCTION USING DIFFERENTIALS

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## TOPIC 131: OUTPUT ELASTICITY OF COST

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**Lesson 28**

## USE OF TOTAL DIFFERENTIALS IN ECONOMICS

### TOPIC 132: CONCEPT OF TOTAL DIFFERENTIALS

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### TOPIC 133: SAVINGS FUNCTION AND TOTAL DIFFERENTIALS

If

Savings, National income, Interest rate.

Synonymous to:



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Where,

**Total Differential:** Sum of partial differentials of the function w.r.t to each independent variable.

**Example**

If

: Change in savings due to change in income is half of it.

: Change in savings due to change in interest rate is thrice of it.

**TOPIC 134: GENERAL UTILITY FUNCTION AND TOTAL DIFFERENTIALS**

In real world, utility depends on multiple goods instead of 2.
Case of n-goods utility function.

B

Total differential :

Total differential is composed of partial differentials.
In case of -goods utility function , there will be -partial differentials:

Borrowing the partial derivative expressions.

**1 1.**

Multiple partial elasticities can be developed from these:

General form of partial elasticities of utility functions:



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## TOPIC 135: SPECIFIC UTILITY FUNCTION AND TOTAL DIFFERENTIALS

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## TOPIC 136: PRICE AND RAIN ELASTICITY OF SUPPLY USING TOTAL DIFFERENTIALS

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**TOPIC 137: LOCAL PRICE ELASTICITY OF FOREIGN DEMAND OF EXPORTS USING TOTAL DIFFERENTIALS**

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**Lesson 29**

## CONCEPT OF TOTAL DERIVATIVES

### TOPIC 138: CONCEPT OF TOTAL DERIVATIVE

Assume Where, .
Channel map.

Combining the two functions

: Ultimate source of change.

Two channels: Direct & Indirect.

- Indirect: via function .
- Direct: via function .

- Indirect:
- Direct:

Total (Direct +Indirect):

B B B

Total Differentiation of w.r.t. .

**Caveat:**

**Example**

**Verification**



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Substituting:

Differentiating w.r.t.

— — —

— (Same as that in Total Derivative formula).

**TOPIC 139: COMPLEMENTARITY BETWEEN COFFEE AND SUGAR USING TOTAL DERIVATIVE**

Assume

Where and are coffee and sugar, respectively.

Furthermore; .

This implies complementarity between sugar and coffee (Financial Times newspaper).

Channel map:

Main variable causing change is :

— — — —

— —

**Example**

B

— B

— B

— —

— —

—

—

Verify using substitution.

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## TOPIC 140: GENERAL PRODUCTION FUNCTION WITH TIME-DEPENDENT LABOR AND CAPITAL

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## TOPIC 141: SPECIFIC PRODUCTION FUNCTION WITH TIME-DEPENDENT LABOR AND CAPITAL

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**Lesson 30**

## CONCEPT OF IMPLICIT DIFFERENTIATION AND THEIR ECONOMIC APPLICATIONS

### TOPIC 142: CONCEPT OF IMPLICIT DIFFERENTIATION

Deals with implicit functions:

- 
- 
- 

To find — of implicit function:

- Differentiate each side of the equation w.r.t. x, considering y as a function of x.
- Solve the resulting equation for —.

**Example**

B

— — —

— — — —

— — — —

— — B

— — B

— —

— —

—

— ———

Implicit derivative of the given implicit function.

### TOPIC 143: PRODUCTION FUNCTION ANALYSIS USING IMPLICIT DIFFERENTIATION

Assume

Implicit function version.

Marginal physical products of labor & capital & .

These are partial derivatives — **&** — .

— — —

Resorting to implicit differentiation: —

— — —



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Marginal physical product of capital expressed in relation to the function .

Similarly,

Marginal physical product of labor expressed in relation to the function .

Also can be found using implicit differentiation.

Marginal rate of technical substitution expressed in relation to the function .

**TOPIC 144: MARGINAL RATE OF TECHNICAL SUBSTITUTION USING IMPLICIT DIFFERENTIATION**

Assume

An implicit function that can yield a production function:

Marginal rate of technical substitution,

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**: Implicit Differentiation**

Marginal rate of technical substitution expressed in relation to the function .

**TOPIC 145: MARGINAL UTILITIES AND MARGINAL RATE OF SUBSTITUTION USING IMPLICIT DIFFERENTIATION**

Assume an implicit equation:

Utility function can be extracted:

For —, assume *ceteris paribus* .

Re-arranging

For —, assume

, *ceteris paribus*

Marginal utility due to an additional unit of ( good).

For B—, assume , *ceteris paribus*



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For —, Assume

, *ceteris paribus*

**TOPIC 146: NERLOVE-RINGSTAD PRODUCTION FUNCTION USING IMPLICIT DIFFERENTIATION**

Attributed to Nerlove (1963) and Ringstad (1967).

B

Where, **,** and **.**

is not in explicit function form.

Implicit differentiation w.r.t.

**Writing in simpler notation.**

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**Implicit differentiation w.r.t. .**

**Writing in simpler notation.**

**TOPIC 147: MARGINAL PRODUCTS OF THREE INPUT LOGARITHMIC PRODUCTION FUNCTION**

Assume an endogenous growth model:

B

: Human capital

Becker (1964): Skills and adequate motivation to apply them.

Logarithmically, linearizing the production function.

B B B

For , & , partially differentiation w.r.t logarithms of , , & .



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(Labor elasticity of output)

B B

For :

(Capital elasticity of output)

(Human capital elasticity of output)

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**Lesson 31**

## EXPONENTIAL FUNCTIONS AND GROWTH

### TOPIC 148: EXPONENTIAL FUNCTIONS AND GROWTH

A quantity that increases (or decreases) by a fixed factor per unit of time is said to increase (or decrease) exponentially.

Where, is a factor by which increases when increases by 1.

Each base gives a different value of . E.g. etc.
In calculus, most of exponential functions appear with base

An irrational number.
Represented by .
Its function is a.k.a. Natural exponential function.

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Exemplifies growth of a sum of money capital over time.
Can also represent growth of population, wealth, or real capital etc.

**TOPIC 149: INSTANTANEOUS RATE OF GROWTH**

Given a value function of continuous interest compounding

:Points in time, : Principal amount, Natural base, : Interest rate, **:** Value at point in time.

**{— }= { ( )}=**

**Instantaneous Rate of Change**

For Instantaneous Growth Rate.

- We find the rate of change of value in relative (%age) terms:
- 
- **Instantaneous Rate of Growth**

Time dependent.
Constant (simplicity sake).

**Numerical Example**

- 

Instantaneous Rate of Growth of is a constant.

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**TOPIC 150: NUMERICAL EXAMPLES OF INSTANTANEOUS RATE OF GROWTH**

Given:

Interest compounding:

: Principal amount = 1

: Instantaneous growth rate

B B

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Given:

Interest compounding:

: Principal amount = 0.03,
: Instantaneous growth rate

B B

**TOPIC 151: CONTINUOUS VS DISCRETE GROWTH**

Usually in economic situations, growth does not always take place on a continuous basis.
Not even in interest compounding.
However, for discrete growth, where changes occur only once per period rather than from instant to instant, the assumption of continuous exponential growth function can be justified.

B

Exponents are time periods covered in compounding, : Principal amount, : Interest rate.
Series can be considered an exponential expression as .
That is . even for as (%age terms).
To convert base to natural exponent, compare with standard form of natural exponential function :

**#**

That's why natural exponential functions are extensively applied in economic analysis despite not all growth patterns may actually be (purely) continuous.



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**TOPIC 152: DISCOUNTING AND NEGATIVE GROWTH**

From interest compounding to discounting.

Compound-interest problem: future value **V** (Principal + Interest) from a given present value **A** (Initial principal).

Discounting problem: Present value **A** of a given sum **V** which will be available after **t** years.

- **Discrete case:**

Cross-multiplying:

Reversed roles (as in intro.) of ***A*** and ***V.***

- **Continuous Case:**

Principal **A** grows into after ***t*** years of continuous compounding at the rate ***r*** .

Cross multiplying:

is also referred as discounting factor.

Reversed roles (like in discrete case) of ***A*** and ***V.***

From growth to negative-growth.

**TOPIC: 153: APPLICATIONS OF CONTINUOUS COMPOUNDING**

Suppose, the sum of PKR 5000 is invested in an account earning interest at an annual rate of 9%. What will be the balance after 8 years if interest is compounded continuously?

Formula for continuous compounding:



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Here, , , .
Substituting values:

B

B

**Interpretation:** Principal will increase from PKR 5000 to PKR 10272.17 during after 8 years, if the annual interest is 9%, and there is continuous compounding of interest.

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**Lesson 32**

## USE OF LOGARITHMS IN ECONOMICS

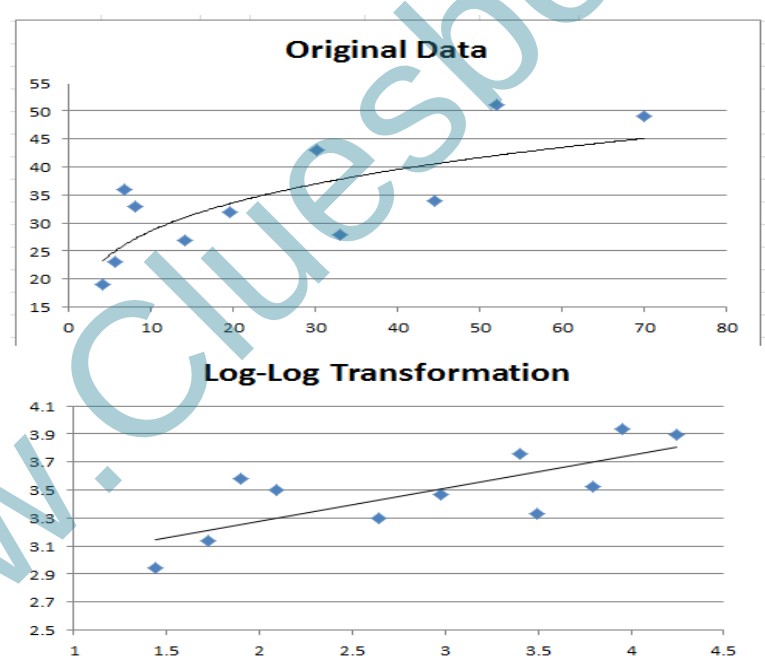
### TOPIC 154: LOGARITHMS MEANING AND TYPES

Attributed to a Scottish mathematician John Napier (1550-1617)
K´" wywq" ?$M}oou$*logos* $}k´sy$ $ 'number.
Inverse of exponentiation.
Helps to find the power of a number result of which is known.

Can have base different than 10:
}[{ , }[{
Also 62, 5= ,K∞vo} é$x∞wl o}-.

is the natural Logarithm of .
Contrary to natural exponential function (
and B are inverse functions of each other.

Order independence in inverse.




### TOPIC 155: LAWS OF LOGARITHMS

**1 - Logarithm of a Product**

**Or**

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**Example**

Using calculator

**Economic Instance**

**2 - Logarithm of a Quotient**

OR

**Example**

Using calculator

**Economic Instance**

**3 - Logarithm of a Power**

OR

**Example**

Using calculator

**Economic Instance**

Log-linearized form of Cobb-Douglas production function.

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**TOPIC 156: LAWS OF LOGARITHMS FOR TRANSFORMATION OF CES PRODUCTION FUNCTION**

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**Lesson 33**

## RULES OF DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

### TOPIC 157: RULES OF DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Possibility of exponential functions in differentiation.

Example:
**Here,**

Deploying the rule:

Possibility of logarithmic functions in differentiation.
Where **0**

Example:
**Here**

Deploying the rule:

### TOPIC 158: OPTIMAL TIMING: A PROBLEM OF WINE STORAGE

Sell wine today for $ K or store and sell at higher price.
Higher value (V):.

Sell now, implies:
=

**Assumptions**

- Storage costs = 0 (container)
- Sunk costs as wine is possessed in prior and not being produced currently.

Therefore:
reduces to as
As , value of is known.

[ : ]



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**F.o.C implies:**

OR

If ,
Then
Storage should be done for 25 years before selling.

**For Rate of Growth of V**

**Second Order Condition**

implies

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**TOPIC 159: OPTIMAL TIMING: A PROBLEM OF TIMBER CUTTING**

Value of timber already planted in $1000.

Harvest now, implies:

:

**Assumptions**

- Upkeep costs = 0
- Sunk costs as plants are already grown not being grown currently.

Therefore:

reduces to as

As , value of is known.

**First Order Condition**

B

B

B

B

**Or**

Optimum number of years of growth.

Greater the discount rate, earlier should the timber be cut.

Numerically: if



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,

**Or**

Present value at

(thousands)

Or

If ,

Then

Storage should be done for 25 years before selling.

**For Rate of Growth of V**

**Second Order Condition**

implies maximum

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**TOPIC 160: OPTIMAL TIMING: LAND PURCHASE FOR SPECULATION**

Land purchased for speculation.

Value is increasing:

5444

Sell now, implies:

:

**Assumptions**

- Upkeep Costs = 0
- Sunk Costs as plants are already grown not being grown currently.

As , value of is known.

**Given that**

B

**First Order Condition**

B

B

B

B B

B

B

Optimum number of years.

Land should be held for 7.127 years before sale.

**Second Order Condition**

=



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implies maximum

**TOPIC 161: OPTIMAL TIMING: ART COLLECTION**

Estimated value of art collection of a deceased painter:

644444

Sell now, implies:

644444

:

As , value of is known.

B

**Given that**

**First Order Condition**

years

Art collection should be sold after 15.24 years.

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**TOPIC 162: OPTIMAL TIMING: DIAMOND PURCHASE**

Estimated value of diamond, bought for investment purpose, is:

694444

Sell now, implies:

694444

:

As , value of is known.

B

**Given that**

**First Order Condition**

years

Diamond should be sold after 2.52 years.

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**Lesson 34**

## FINDING THE RATE OF GROWTH USING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

### TOPIC 163: FINDING THE RATE OF GROWTH USING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Assume
depends on (time).
Instantaneous Growth Rate:

Example:

**Growth Rate of .**

Taking derivative w.r.t **.**

**(Rate of growth of V is r)**

### TOPIC 164: GROWTH OF EXPORTS OF A COUNTRY

A country export both goods and services .
Both depend on time i.e. and .
Total exports: .
Rate of growth of exports

Substituting in

**#**

+

+

/

/ ,G

Given – and –



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Equation (A) becomes:

– – +– –

**TOPIC 165: FINDING THE POINT ELASTICITY**

Using the usual formula of price elasticity of demand:

Assuming a specific demand function:

Logarithmically differentiating w.r.t .

(Unitary elastic demand curve)

**TOPIC 166: RATES OF GROWTH OF POPULATION, CONSUMPTION, AND PER CAPITA CONSUMPTION**

Growth rate of consumption is .

Growth rate of population is .

Consumption per capita .

=

Given and

Rate of growth of a quotient variable is equal to difference of individual growth rates.



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**TOPIC 167: RATE OF GROWTH OF PER CAPITA EMPLOYMENT**

Growth rate of employment opportunities is .
Growth rate of population is .
Employment per capita .

Given and

If and

Growth rate of a per capita employment is equal to difference of growth rate of employment and population, respectively.

**TOPIC 168: RATE OF GROWTH OF EXPORT EARNINGS OF A COUNTRY**

Two exports of a country **C** and **B.**
, and .
grows at 10% and grows at 20%.

Differentiate w.r.t.

and

Imply:
and

Given and .

**TOPIC 169: RATE OF GROWTH OF SALES**

Sales function is:



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Sales function is:

Assuming ,

Sales shall grow at 12.5% if 4 years are allowed.

**TOPIC 170: RATE OF GROWTH OF PROFIT**

In addition to maximization of profit, rate of growth of profit can also be of interest for a firm.
Profit function is:

Growth rate of profit function is:

Assuming ,

B

Profit shall grow at 10% if 8 years are allowed.

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**Lesson 35**

## CONCEPT OF OPTIMIZATION

### TOPIC 171: CONCEPT OF OPTIMIZATION

Etymology optimum: Latin *u* l oé´
Process of maximizing favorable variables and minimizing unfavorable variables.
Assume a function:

In cubic specification:

Two wiggles.
One upwards and other downwards.

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Extremum vs Extrema
Maximum vs Maxima
Minimum vs Minima

**TOPIC 172: CALCULUS APPROACH TO OPTIMIZATION: 1ST ORDER TEST**

Considering a function .
Which is continuous and can have extremum/extrema.
Finding the critical value at which function is optimized.

Since slope at maximum or a minimum is zero.

**Slope =** — .

a.k.a First order condition (F.o.C).



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a.k.a Necessary condition for optimization.

**Instance:**

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**TOPIC 173: AVERAGE COST ANALYSIS**



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**TOPIC 174: CALCULUS APPROACH TO OPTIMIZATION: 2ND ORDER TEST**

Considering a function .

Which is continuous and can have extremum/extrema.

Having found the critical value(s) at which function is optimized, one needs to confirm if function s é$k m´∞ k w” $w k “ s w s´ on $y} $w s x s w s´ on $y} $t ∞ é´ $k $l o x n $x $m ∞} . . o $, s x p o m´ s y x-2

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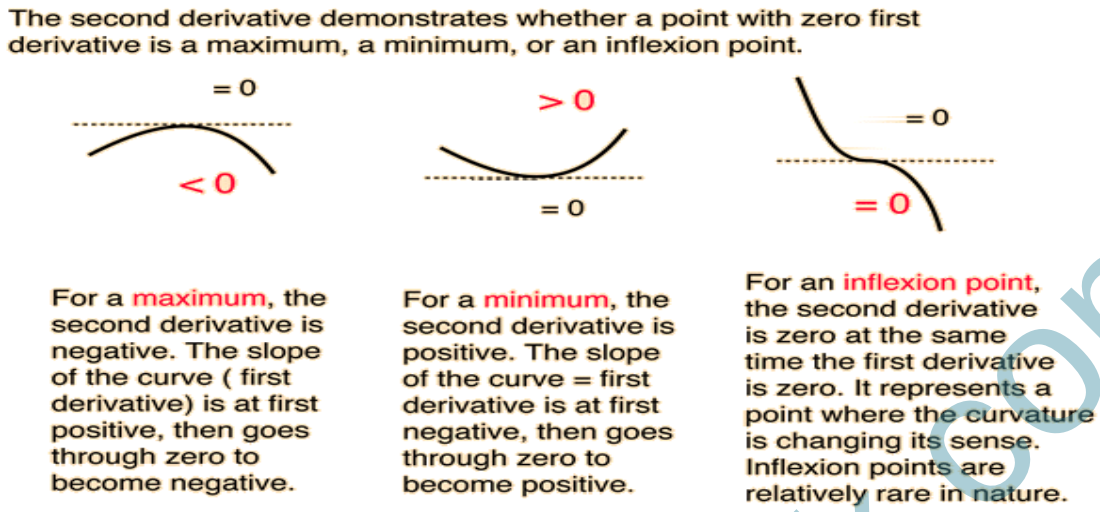


Rate of change of slope.

**Rate of change of Slope =** ——— .

a.k.a Second order condition (S.o.C)
a.k.a Sufficient condition for optimization

**Instance:**

B

**For**

(maximum at )

**For**

(minimum at )

**For** , , (inflection)

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**TOPIC 175: MATRIX APPROACH TO OPTIMIZATION: 2ND ORDER TEST HESSIAN**

Hessian *n* is a determinant with all the second-order partial derivatives.
2nd order direct partials on the principal diagonal
2nd order cross partials off the principal diagonal.

Where, ,cy∞xq é$theorem)
Principal minors = 2
1st Principal minor: *n*

2nd Principal minor: *n*

If both *n* and *n* are positive then minimum will be evident.
**Minimum**
If *n* and *n* have alternative signs then minimum will be evident.
**,** **Maximum**

If

Since both *n* and *n* are positive, there exists a minimum.

If , Evaluate 2nd order condition using Hessian Determinant.

For

implies a maximum.
implies a minimum.

N-number of variables can be dealt with using Hessian determinant.

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**Lesson 36**

# PROFIT MAXIMIZATION ANALYSIS

## TOPIC 176: PROFIT MAXIMIZATION ANALYSIS

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## TOPIC 177: NUMERICAL EXAMPLE OF PROFIT MAXIMIZATION

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## TOPIC 178: PROFIT MAXIMIZATION OF TECHNICALLY RELATED GOODS

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**TOPIC 179: PROFIT MAXIMIZATION OF MONOPOLISTIC FIRM PRODUCING RELATED GOODS**

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## TOPIC 180: PROFIT MAXIMIZATION OF FIRM PRODUCING SUBSTITUTE GOODS

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# PROFIT MAXIMIZATION ANALYSIS (CONTINUED 1)

## TOPIC 181: MARGINAL AND AVERAGE REVENUE ANALYSIS

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## TOPIC 182: SHORT RUN PRODUCTION FUNCTION ANALYSIS



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## TOPIC 183: TOTAL COST, TOTAL REVENUE AND PROFIT MAXIMIZATION

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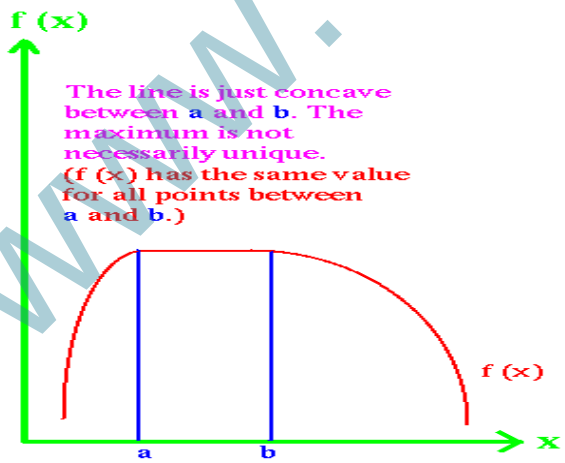
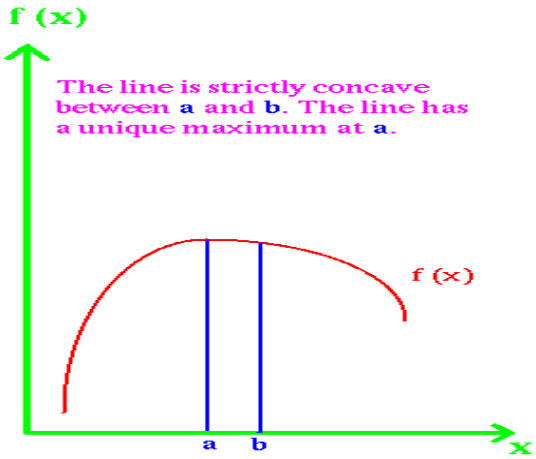
**Lesson 38**

## PROFIT MAXIMIZATION ANALYSIS (CONTINUED 2)

### TOPIC 184: QUADRATIC PROFIT FUNCTION ANALYSIS

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## TOPIC 185: OPTIMIZATION OF EXPONENTIAL REVENUE FUNCTION

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**TOPIC 186: OPTIMIZATION OF MORE THAN ONE CHOICE VARIABLE**

Two (or more) independent variables can exist.
For instance,
Surface (or hypersurface).
Peaks of domes & bottoms bowls can exist.

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| Conditions for Relative Extremum: | | |
|---|---|---|
| **Condition** | **Maximum** | **Minimum** |
| **1st Order (Necessary)** | | |
| **2nd Order(Necessary)** | B | B |
| **2nd Order (Sufficient)** | | |
| **Conditions for Inflection and Saddle Points:** | | |
| | **Inflection Point** | **Saddle Point** |
| **2nd Order(Necessary)** | **OR** | |
| **2nd Order (Sufficient)** | | |
| **Conditions for Inconclusiveness of the Test:** | | |
| **2nd Order (Sufficient)** | | N.A. |
| **Note:** By c y∞xq é$Zr oy}ow? | | |

- **Numerical Example**

**1st order conditions:**

B

**Critical Points:**

B

**A.**
**B.**
**C.**
**D.**

**POINT - I:**

B

Different signs of & **Saddle** pt.

**POINT - II:**

B

& Maximum/inflection



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At **Maximum** is confirmed.

**POINT - III:**

BB

& :Minimum/inflection

At **Minimum** is confirmed.

**POINT - IV:**

BB

Different signs of & **Saddle** pt.

**D.I.Y**

**1.**

**2.**

**TOPIC 187: ECONOMIC APPLICATION ON MULTI-PRODUCT FIRM**

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## TOPIC 188: ECONOMIC APPLICATION ON MULTI-PLANT FIRM



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**Lesson 39**

## PROFIT MAXIMIZATION ANALYSIS (CONTINUED 3)

### TOPIC 189: PRICE DISCRIMINATION BY MONOPOLY

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## TOPIC 190: PRICE DISCRIMINATION BY MONOPSONY

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**TOPIC 191: INPUT DECISION OF A FIRM**

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**TOPIC 192: PROFIT MAXIMIZATION OF TWO-PRODUCT FIRM**

Given that:

, , , ,

,

+

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**Lesson 40**

## CONSTRAINED OPTIMIZATION ANALYSIS

### TOPIC 193: COMPARATIVE-STATIC ASPECTS OF OPTIMIZATION

**Example # 1 Reduced form Solutions**

B B

Need for comparative static analysis.

Partial differentiation of w.r.t. & .

- Technical relation in production

### TOPIC 194: RATIONALE FOR CONSTRAINED OPTIMIZATION

Constraints can be considered as old as scarcity.
a.k.a. Restraint, side relation, or subsidiary condition.
Narrows the domain and hence range of objective function (say ).

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Free optimum is higher than constrained optimum.
Sometimes both can be the same.
However, constrained optiw∞w $mkx ´$ o$r sqr o}$r kx$free optimum.

**TOPIC 195: FINDING STATIONARY VALUES USING SUBSTITUTION/ELIMINATION METHOD**

Assume an objective function:

A constraint function is:

Use constraint:

Substitute in objective function.

**First order condition**

Critical values: and
Substitute and in objective function:

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**TOPIC 196: FINDING STATIONARY VALUES USING METHOD OF LAGRANGE MULTIPLIER**

Assume an objective function:

A constraint function is:

Forming Lagrange function

First of conditions:

_____

_____

_____

Rewriting:

#

#

##$

Solving first two equation for .

— & – equation we get:

——— – B

Substituting in 3rd F.o.C.

Critical values: and

Substitute and in objective function:

**TOPIC 197: INTERPRETATION OF THE LAGRANGE MULTIPLIER**

Introduced by Joseph-Louis Lagrange.

What if we want to predict the impact of per unit change in constraint on objective function.

**Answer:** Value of Lagrange multiplier.

a.k.a Shadow prices.

Considering example:

↑ **E 9 0 8² ³ 0 ; ³** Subject to:



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1-unit increase (decrease) in the constant of the constraint (56) would cause ‘ to increase (decrease) by approximately 348 units.

**9² 0 8² ³ 0 ; ³**

**9² 0 8² ³ 0 ; ³**

**E9 0#8 0#, B**

Increasing the constant of constraint by 1 unit.

↑ **E 9 0 8² ³ 0 ; ³** Subject to:

**9² 0#8² ³ 0#, ³**

**E9 0 8 0 ;**

**close to as predicted by .**

**Economics Instances:**

In utility maximization subject to a budget constraint: sb. to , will estimate the marginal utility of an extra PKR of income.

In output maximization subject to a cost constraint: sb. to , will estimate the marginal product of an extra PKR of cost.

**TOPIC 198: SECOND ORDER CONDITION: THE BORDERED HESSIAN**

Attributed to German mathematician, *Ludwig Otto Hesse.*

In addition to algebra, determinant of matrices can also be used for 2nd order condition in constrained optimization.

sb. to , is tested with 2nd order condition:

The standard form.

Order of a bordered principal minor equals the order of the principal minor being bordered (here, ).

represents a second bordered principal minor

In algebraic form:



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For a function with variables , subject to :

Where , because of the principal minor being bordered.
For maximum:
For maximum:
: Explained.
For even value of , final bordered principal minor:

For odd value of , final bordered principal minor:

helps to retain alternative signs.

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B B

Since, same signs of bordered principal minors occur, a minimum is evident.
**Economic Instance:** Two period model of utility.

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**Lesson 41**

# UTILITY MAXIMIZATION ANALYSIS

## TOPIC 199: TWO PERIOD MODEL OF UTILITY

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**TOPIC 200: CONVEXITY AND CONCAVITY USING SECOND ORDER DERIVATIVE**

Convexity/concavity are closely related to minimum/maximum.

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is continuous in the interval and twice differentiable:
**is convex on for all in**
**is concave on for all in**

Numerical:
;

Single input production function:

**Case I:**

**(Concave production function)**



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**Case II:**

**(Convex production function)**

**TOPIC 201: UTILITY MAXIMIZATION AND CONSUMER DEMAND: FIRST ORDER CONDITION**

Hypothetical consumer with 2 goods , with continuous, +ve marginal utility .

Market-determined prices - exogenous

Budget constraint:

Utility (objective) function:

B

**Lagrangian function:**

**First order conditions:**

—

—

Extracting value of from 1st & 2nd F.o.C and equating:— —

— —

Interpretation:

— — is actually law of equi-marginal utility.



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Marginal utility of money spend on each goods is equal.
can be termed as the marginal utility of (budget) money when utility is maximized.

— — — —

Alternatively, — **&** — **:**

— B B

— —

— — **(IC has ve slope)**

—

B B

— — — —

— — B

**TOPIC 202: UTILITY MAXIMIZATION AND CONSUMER DEMAND: SECOND ORDER CONDITION**

After developing and analyzing the 1st order condition, we develop 2nd order condition.

***;***
***;***

B B B

Diagrammatically the shape of IC should be convex to origin.

Algebraically: —



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Similarly,

— [F.o.C.]

;

**Interpretation:** Since, both, and are positive — , fulfilling the condition for convexity (to origin).



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Both S.o.C (Bordered Hessian ) & convexity condition — are verified and inter-related.

**TOPIC 203: NUMERICAL EXAMPLE OF UTILITY MAXIMIZATION**

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## TOPIC 204: LAW OF EQUI-MARGINAL UTILITY USING LAGRANGIAN MULTIPLIER



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**Lesson 42**

**HOMOGENOUS PRODUCTION FUNCTION**

**TOPIC 205: ECONOMIC APPLICATION OF PRODUCTION FUNCTION MAXIMIZATION**

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**TOPIC 206: ECONOMIC APPLICATION ON LOGARITHMICALLY TRANSFORMED PRODUCTION FUNCTION**

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**TOPIC 207: HOMOGENEOUS FUNCTIONS**

**Etymology:** Greek : *homos* ékwo $ $ *genos* }knoGusxn 2

Here, *genos* refers to degree of term.

A function is considered homogenous if each term involved has same degree.



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Polynomial of degree :

Each term has degree .

**Example:**
To check, introduce scalar (say ) on both side.

Homogenous of degree 3 in variables and
**D.I.Y.:** Check, if functions are homogenous or not.
If yes, then of what degree.

**TOPIC 208: HOMOGENEOUS PRODUCTION FUNCTION, AVERAGE PRODUCTS AND CAPITAL-LABOR RATIO**

Consider Cobb-Douglas production function:

Testing homogeneity:

Degree of homogeneity is .

For a Cobb-Douglas production functionB : Linearly homogeneous or shows linear homogeneity.
Caveat: Misleading terms (linear homogeneous, linear and homogeneous)
Economic implication: Linear homogeneity implies that if all independent variables (inputs) are increased by same proportion , the dependent variable (output) shall increase by same proportion Constant returns to scale.

B B

Average product of labor :

Since,

Relationship between average product of labor and capital-labor ratio.
Degree of homogeneity of average product of labor :



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Average Product of Labor :

Relationship between average product of capital and capital-labor ratio.

Degree of homogeneity of :

Interpretation of degree of homogeneity & : both are homogeneous of degree zero in the variables and ,

Since equal proportionate changes in and (maintaining a constant – **will not change** the magnitudes of the average products.

**TOPIC 209: HOMOGENEOUS PRODUCTION FUNCTION, MARGINAL PRODUCTS AND CAPITAL-LABOR RATIO**

Marginal product of capital :

Relationship between marginal product of labor and capital-labor ratio.

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Degree of homogeneity of :

Marginal product of labor :

Since

Marginal product of capital and capital-labor ratio are related
Degree of homogeneity of :

Interpretation of degree of homogeneity & : both are homogeneous of degree zero in the variables and ,
Since equal proportionate changes in and (maintaining a constant – **will not change** the magnitudes of the marginal products.



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**TOPIC 210: HOMOGENEOUS PRODUCTION FUNCTION AND EULER'S THEOREM**

K∞v} é$heorem: Leonhard Euler, a Swiss mathematician

A property of with degree of homogeneous:

B B B

In production context:

In another notation:

For linearly homogeneous production function :

Restoring –

**Interpretation:** Under CRS, if input factor is paid the amount of its ***MP***, the ***TP*** will be exhausted by the distributive shares for all the input factors.

Or, pure economic profit will be zero.

**Caveat:** K∞v} é$r oy}ow $holds if perfect competition holds in factors market.

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**Lesson 43**

# COBB DOUGLAS PRODUCTION FUNCTION

## TOPIC 211: COBB-DOUGLAS PRODUCTION FUNCTION AND RETURNS TO SCALE

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**TOPIC 212: HOMOGENEITY AND RETURNS TO SCALE OF THREE INPUT PRODUCTION FUNCTION**

3 input production function.

Where,
= Capital, = Labor, = Material

a+b+c is the degree of homogeneity.
Introducing on both sides.
And Euler Theorem requires:

B B
B B

is completely factored out Homogenous production function
is the exponent showing degree of homogeneity.
If , then CRS prevails.
If , then IRS and if , then DRS, respectively.

**TOPIC 213: LEAST-COST COMBINATION IN COBB-DOUGLAS PRODUCTION FUNCTION**

CES production function.

+

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Forming Lagrangian function:

2#

2#

Solving for in 1st and 2nd L2U2 é$kxn$b{ ∞k´sxq2

Digressing to marginal products.

Reverting to previous equation.

—— —— OR — —— (Least cost input ratio).

— ——— =—— —

**TOPIC 214: EXPANSION PATH USING FIRST ORDER CONDITION**

Comparative-static aspect of producer equilibrium.

With fixed ratio of and i.e. — , postulate successive increases of (ascent to higher and higher isoquants.

Trace the effect on the least-cost combination —.

Each shift of the isoquant, gives new point of tangency, with a higher Iso-cost.



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Locus of such points of tangency is expansion path of the firm, serves to describe the least-cost combinations required to produce varying levels of

**1st order Condition**

**As**

Substituting in condition.

At equilibrium:

Answer will be numerical as are all constants.
Therefore, all points on expansion path shall show a fixed input ratio.
Or, expansion path shall be a straight line starting from origin.
A homogeneous production function gives rise to a straight line of expansion path.

**TOPIC 215: HOMOTHETIC FUNCTIONS**

**Etymology:** Ancient Greek (*homo-*[illegible] *thésis*[illegible]
Attributed Shepard (1953).
Diagrammatically speaking, homothetic functions generate radial expansions, preserving both angles and ratios of distances.

Consider a production function :

Its monotonic transformation generates a new production (composite) function :

Homogeneity may disappear:



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.

has a homogeneous of degree , however, is not homogeneous.
Dependence of monotone function on original function:

Expansion path of is linear like that .
As observed in diagram slope of Isoquants will remain the same, at given :

**TOPIC 216: HOMOTHETICITY OF COBB-DOUGLAS PRODUCTION FUNCTION**

Consider Cobb-Douglas production function :

Its homogeneous.
Monotonic transformation (squaring) generates a new production (composite) function :

Slope of Isoquant =

With given , slope of will be constant.
is homogeneous of degree .
is also homogeneous, but of degree
However, a homothetic function is not necessarily homogeneous.

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**Lesson 44**

## CES PRODUCTION FUNCTION

### TOPIC 217: INTRODUCING CES PRODUCTION FUNCTION

Relatively new.
Attributed to Solow, Minhas, Arrow and Chenery.
a.k.a SMAC production function
A more general form of production functions, while Cobb-Douglas can be a specific case of it.
Standard form.

Where,

B

Distribution parameter
Efficiency parameter
Substitution parameter
Degree of homogeneity

Generates a constant (not variable) value of elasticity of substitution between capital and labor.
Hence called constant elasticity of substitution (CES) production function.
Elasticity of substitution.

**1-**

CRS.

As

**= 0.71 1 less elastic** lower substitution between capital and labor.

### TOPIC 218: HOMOGENEITY OF CES PRODUCTION FUNCTION

CES production function.

For simplicity, let

Introducing on both sides.

B

B B

B B B

B



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As is degree of homogeneity. CES production function is homogenous of degree .
However, the value of can be any positive number. i.e.

If , it implies decreasing returns to scale.

B

If , it implies constant returns to scale.

B

If , it implies increasing returns to scale.

B

**TOPIC 219: MARGINAL PRODUCTS OF CES PRODUCTION FUNCTION**

CES production function.

Where,

B

Marginal product of capital — .

Marginal product of labor — .

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**Lesson 45**

## CES PRODUCTION FUNCTION (CONTINUED)

### TOPIC 220: SHARE OF LABOR AND CAPITAL IN CES PRODUCTION FUNCTION

CES production function.

Where,

B

**Let**

Capital share of output _____ _____ .

Labor share of output _____ _____

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**TOPIC 221: CES PRODUCTION FUNCTION AND EULER'S THEOREM**

CES production function.

Where,

is the degree of homogeneity.

And Euler theorem requires:

+

Where:

—

—

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## TOPIC 222: NUMERICAL CES PRODUCTION FUNCTION CALCULATION

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