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MTH644

Measure Theory

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FIRA NEEHSON

MTH644

Measure Theory

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Good Luck, Stay focused,

bring a cup of coffee to open yours eyes:)

and achieve your goals_ Fira

Topic #01

Introduction:

Mathematics is a place where you can do things which you can't do in the real world.

(Morris De Santoy)

Why the measure theory?

Do the measurements really need Abstraction and Concrete rigorous grounds?

yes!! Due to ∞ and Infinitesimals.

Srinivasa Ramanujan: the man who knew infinity!.

There are different ways to look at the size of a set.

$\{0, 3, 6, 9\}$ has order 4, but on real line,

these are "negligible".

For $\{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \dots, \{1\} \cup \{1, 2\} \cup \{1, 2, 3\} \cup \dots$
 $= 1 + 2 + 3 + \dots = ?$

But still on real line these are "Negligible".

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So a single point takes up no real width on the real line.

The mathematical concept of measure is the generalization of lengths (size) in $\mathbb{R}$, are in $\mathbb{R}^2$ values in $\mathbb{R}^3$ and hence in $\mathbb{R}^n$ and other spaces.

Starting from the lengths of intervals on $\mathbb{R}$.

Intuitively:

I) $|[0,1]| = 1$ and all $(0,1), (0,1), [0,1] \subset [0,1]$

$$\Rightarrow |[0,1]|, |(0,1)|, |[0,1]| \leq |[0,1]|$$

(Monotonicity: preserves with respect to measure in $\mathbb{R}$)

And, $|(-\infty, \infty)| = \infty$

MCQ

Intuitively, the measure is a monotone mapping from a subset of $\mathbb{R}$ to non-negative extended real numbers.

II) $[-\frac{1}{2}, \frac{1}{2}]$ is $[0,1]$ i.e. translated " $\frac{1}{2}$ " unit to the left. And both have length (measure) "1".
(Translation Invariant).

$$\text{III) } |[0,2]| = |[0,1] \cup [1,2]|$$

$$\Rightarrow 2 = 1 + 1$$

MCQ

Similarly, $(0,1) = (0,1) \cup \{1\} \Rightarrow 1 = 1 + 0$, means:

"the whole is the sum of its parts" (additivity)

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But for, $|(\emptyset, \Omega)| = |\bigcup_{i \in \mathbb{N}} \{x_i\}| = \aleph_0 = \aleph_0 + \aleph_0$

Therefore, we take the "whole is the sum of its parts", if a set is decomposed into a countable number of disjoint sets. (Countable Subadditivity)

Topic #02

Ring of Sets:

IMP

Let $X \neq \emptyset$, and let $\mathcal{A}$ be a class or collection of subsets of X . Then $\mathcal{A}$ is said to be a ring if:

$$\left. \begin{array}{l} \text{i) } A \cup B \in \mathcal{A} \\ \text{ii) } A - B \in \mathcal{A} \end{array} \right\} \forall A, B \in \mathcal{A}$$

For Examples:

(i) $X = \{a, b, c\}$ and $\mathcal{A} = \{\emptyset, X, \{a\}, \{a, c\}\}$
all are subset of X .

(ii) $X \neq \emptyset$, $P(X) = \{A : A \subseteq X\}$
 $\Rightarrow P(X)$ is a ring of X .

MCQ

(iii) N , $\mathcal{A} = \{A_i : A_i \in N \text{ such that } |A_i| < \infty\}$
 $\Rightarrow A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, $A - B \in \mathcal{A}$

Topic #3

Sigma Ring:

IMP

Let $X \neq \emptyset$, and $\mathcal{A}$ be collection of

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subsets of X . Then $\mathcal{A}$ is said to be a

σ -ring if:

i) $\forall A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$

ii) For any sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$

Examples:

$X \neq \emptyset, \mathcal{A} = \{\emptyset, X\}$ is σ -ring.

and $\mathcal{P}(X) = \{A : \forall A \subseteq X\}$ is σ -ring.

Topic #04

Ring vs Sigma Ring:

MCQ

S.Q

σ -ring is also a ring because σ -ring is satisfied the both properties of ring.

For $N, \mathcal{A} = \{A \subseteq N : |A| < \infty\}$ is ring.

$\Rightarrow A \setminus B \in \mathcal{A}$

Let $\{A_n\}, \{B_n\}, \{C_n\} \dots$ be a sequence finite subsets of N .

But $\bigcup_{n=1}^{\infty} \{C_n\} \neq \infty \Rightarrow \mathcal{A}$ is not σ -ring.

Topic #05

IMP

Boolean Algebra (Algebra of Sets):

Let $X \neq \emptyset$, and $\mathcal{A}$ is collection of

X . Then $\mathcal{A}$ is said to be a Boolean Algebra if

i) $\forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ ii) $\forall A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

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Examples:

MCQ'S

(i) $X = \{a, b, c\}$, $\mathcal{A} = \{\emptyset, X, \{a\}, \{b, c\}\}$

$\mathcal{A}$ is boolean algebra Ring+sigma Ring+ boolean algebra

(ii) $X = \emptyset$, $\mathcal{P}(X) = \{\emptyset, X\}$ is boolean algebra

(iii) let $X = [-1, 1] \subseteq \mathbb{R}$ ← $\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \}$
and $\mathcal{A} = \{\emptyset, X, [-1, 0], [0, 1]\}$ is boolean algebra

Topic #105

Theorems: let $\mathcal{A}$ be a ring of subsets of X then IMP (i) $\emptyset \in \mathcal{A}$, (ii) $A \Delta B \in \mathcal{A}$, (iii) $A \cap B \in \mathcal{A}$

Proof: (i) $\forall A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$

In particular if $A = B$

$A - B = A - A = \emptyset \in \mathcal{A}$: by definition of ring

(ii) $\forall A, B \in \mathcal{A}$

$\Rightarrow A - B, B - A \in \mathcal{A}$

$\Rightarrow (A - B) \cup (B - A) \in \mathcal{A}$

$\Rightarrow A \Delta B \in \mathcal{A}$

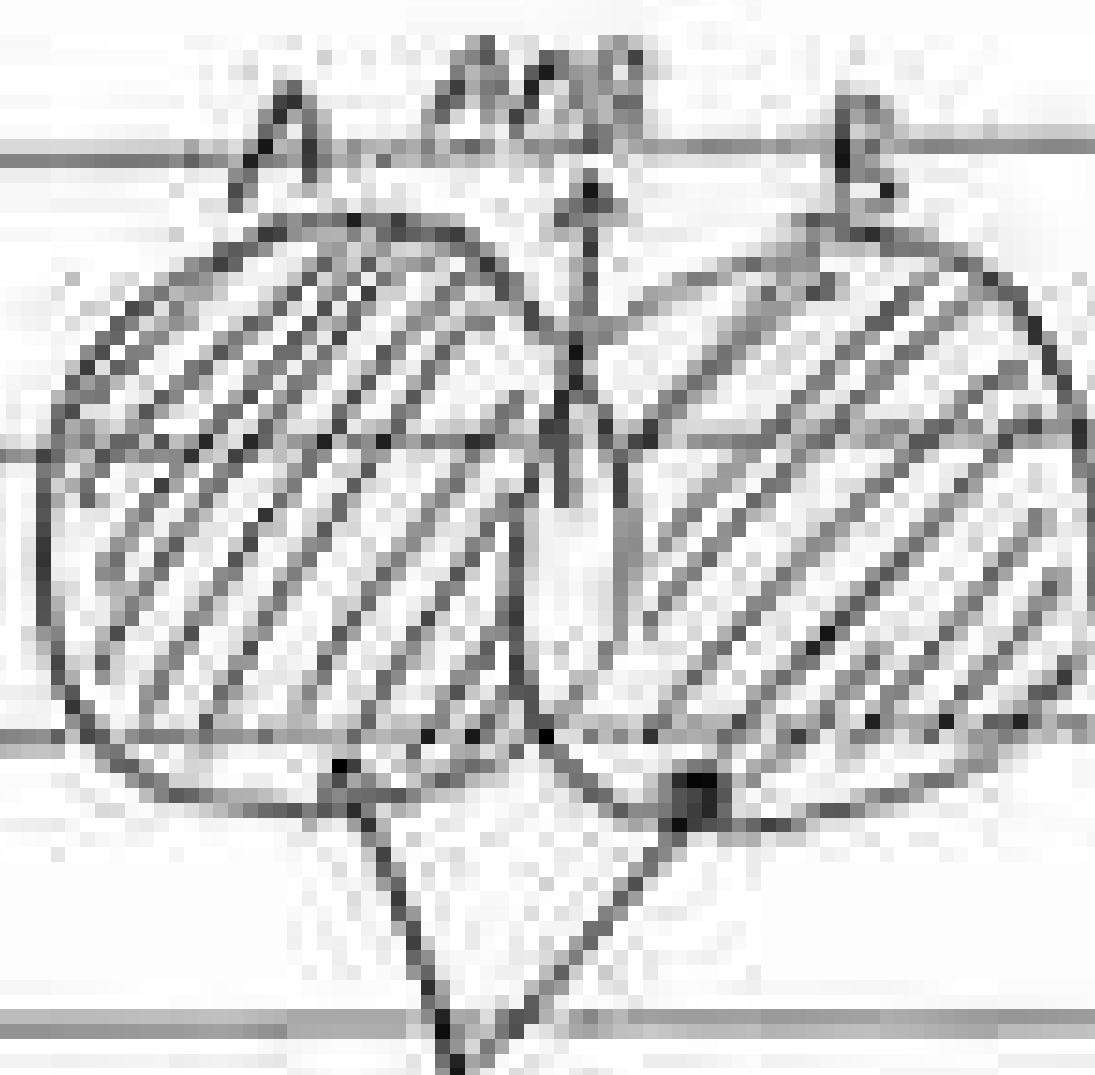
(iii) $\forall A, B \in \mathcal{A}$

$\Rightarrow A \cap B \in \mathcal{A}, A \cup B \in \mathcal{A}$

by definition of ring

MCQ

$(A - B) \cup (B - A) = A \Delta B$
"A symmetric B"



$(A - B) \cup (B - A) = A \Delta B$

Don't forget

$A \cup B = [(A - B) \cup (B - A)] \cup (A \cap B) = (A \Delta B) \cup (A \cap B)$ Formula

and $A \cap B = [(A \cup B) - (A \Delta B)] \in \mathcal{A}$

$\Rightarrow A \cap B \in \mathcal{A}$

Topic # 07

S.Q

MCQ

Theorem 1: Every Algebra is a Ring.

proof: By definition of Algebra $\mathcal{A}$ on set $X \neq \emptyset$:

$$\forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A} \text{ and } A \cap B \in \mathcal{A}$$

$$\therefore A \in \mathcal{A} \times A \in \mathcal{A} \Rightarrow (A \cup B) \in \mathcal{A}$$

$\Rightarrow$ By definition of Algebra

$$\Rightarrow (A \cup B)' \in \mathcal{A}$$

$$\Rightarrow A' \cap B' \in \mathcal{A} \quad \because \text{By De Morgan's law}$$

$$\Rightarrow A \cap B' \in \mathcal{A} \quad \because A' = A \times \text{set } A'$$

$$\Rightarrow A - B \in \mathcal{A} \quad \because A \cap B' = A - B \quad \forall A, B \in \mathcal{A}$$

$\Rightarrow \mathcal{A}$ is also a Ring.

Topic # 08

MCQ

Theorem 2: For a $X \neq \emptyset$, then ring $\mathcal{A}$ if subsets of X is also an algebra iff $X \in \mathcal{A}$.

proof: p: ring $\mathcal{A}$ is an algebra.

$$q: X \in \mathcal{A}$$

Given: "p" and to prove: "q".

$$\therefore \text{For } X \neq \emptyset, \mathcal{A} \text{ is a ring} \Rightarrow \forall A \in \mathcal{A}$$

$$\Rightarrow \emptyset \in \mathcal{A} \quad \because A - A = \emptyset \Rightarrow (A - A) \in \mathcal{A}$$

$$\Rightarrow \emptyset' \in \mathcal{A} \Rightarrow \emptyset^c = X \in \mathcal{A}$$

Now Given: "q" and to prove: "p".

$$\because X \in \mathcal{A} \text{ and } \mathcal{A} \text{ is a ring}$$

$$\Rightarrow \forall A \in \mathcal{A} \Rightarrow (X - A) \in \mathcal{A} \quad \because \text{by definition of ring}$$

$$\Rightarrow A \in \mathcal{A}$$

$$(ii) \forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$$

$\therefore$ (i) and (ii)

$\Rightarrow$ any $\mathcal{A}$ of subset of X is also an algebra.

Topic # 09

σ -Algebra of sets:

S.Q

Let $X \neq \emptyset$ and $\mathcal{A}$ be a class of subset of X . Then $\mathcal{A}$ is said to be a σ -algebra if

$$(i) \forall A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$$

$$(ii) \text{ whenever } \{A_1, A_2, A_3, \dots\} \subset \mathcal{A}$$

$$\text{or } \{A_n\}_{n=1}^{\infty} \subset \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$$

Example: (i) For any $X \neq \emptyset$, $\mathcal{A} = \{\emptyset, X\}$ is a σ -algebra.

(ii) For any $X \neq \emptyset$, $\mathcal{P}(X) = \{A : A \subset X\}$ is also a σ -algebra.

(iii) For $X = [-1, 1] \subset \mathbb{R}$

$\Rightarrow \mathcal{A} = \{\emptyset, X, [-1, 0], [0, 1]\}$ is also a σ -algebra.

(iv) For $X = \mathbb{N}$, $\mathcal{A} = \{A \subset \mathbb{N} : |A| < \infty \vee A \text{ is countable} \vee A^c \text{ is countable}\}$

$$A = E = \{2n : n \in \mathbb{N}\}, A^c = O = \{2n-1 : n \in \mathbb{N}\}$$

If $X = \mathbb{R}$ (uncountable)

$$\mathcal{A} = \{A \subset \mathbb{R} : |A| < \infty \vee A^c \text{ countable}\}$$

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$\mathbb{Z} \subset \mathbb{R}$ but $\mathbb{Z}$ is not countable.

(Not holds)

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Topic #10

Theorem: Let $X \neq \emptyset$ and $\mathcal{I}$ is a class of subsets of X , then $\mathcal{C} = \text{ring } \mathcal{I}$ is also a $\mathcal{C}$ -algebra iff $X \in \mathcal{I}$.

proof: See Topic #08 for proof.

MOST IMP

Theorem: Arbitrary intersection of $\mathcal{C}$ -algebras on $X \neq \emptyset$ is also a $\mathcal{C}$ -algebra.

proof: let $\mathcal{A} = \{A_i : i \in I\}$ be a class of $\mathcal{C}$ -algebras on $X \neq \emptyset$. clearly $\mathcal{A} \neq \emptyset$.

To prove $\bigcap_{i \in I} A_i$ is also a $\mathcal{C}$ -algebra.

(i) For any $A \in \mathcal{A}$, such that $A \in \bigcap_{i \in I} A_i$

$$\Rightarrow A \in A_i \quad \forall i \in I$$

But each A_i is a $\mathcal{C}$ -algebra.

$$\Rightarrow A \in \mathcal{C}_i \quad \forall i \in I$$

$$\Rightarrow A \in \bigcap_{i \in I} \mathcal{C}_i \rightarrow \textcircled{1}$$

If $x \in A \cap B$
then $x \in A$ and
 $x \in B$

(ii) let $\{A_i\}_{i \in I}^{\infty}$ be a sequence of subsets of $X \neq \emptyset$,

such that $\{A_i\}_{i \in I}^{\infty} \subseteq \bigcap_{i \in I} A_i \Rightarrow \{A_i\}_{i \in I}^{\infty} \subseteq A_i \quad \forall i \in I$

$\because$ each A_i is a $\mathcal{C}$ -algebra

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in A_i \quad \forall i \in I$$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap_{i \in I} A_i \rightarrow \textcircled{2}$$

from $\textcircled{1}$ and $\textcircled{2}$

$$\Rightarrow \bigcap_{i \in I} A_i \text{ is also a } \mathcal{C}\text{-algebra.}$$

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S.Q

Topic #11

problem: Give a counter example to show that union of two σ -algebra is not necessarily a σ -algebra.

Solution: let $X = \{a, b, c\}$ and defining two σ -algebra
 $A_1 = \{\phi, X, \{a\}, \{a, c\}\}$ and

$$A_2 = \{\phi, X, \{c\}, \{a, b\}\}$$

Now taking

$$A_1 \cup A_2 = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$$

Taking finite (countable) sequence of subsets of

$A_1 \cup A_2$ i.e. $\{a\}, \{c\}$. Then

$$\{\{a\} \cup \{c\}\} = \{a, c\} \notin A_1 \cup A_2$$

$\Rightarrow A_1 \cup A_2$ is not a σ -algebra.

Topic #12

MCQ

Sigma($\mathcal{B}$) - Algebra Generated by a Class:

For $X \neq \phi$, $A \subseteq X$ and $B = \{A\}$. Then $\{A, X, A, A'\}$ is a σ -algebra generated by B .

Definition: let $X \neq \phi$ and B be a family of subsets of X . Then σ -algebra generated by B defined as the intersection of all σ -algebra containing B .

$\mathcal{A} = \{A_i : A_i \text{ is a } \sigma\text{-algebra, } B \subseteq A_i, \forall i \in I\}$

X



Smallest / Intersection

Topic #13

Theorem: Let B be a family of subsets of $X \neq \emptyset$. Then there exist a smallest σ -algebra containing B .

proof: let $\mathcal{O} = \{A_i : B \subseteq A_i, \forall i \in I\}$

$\therefore \rho(X) \cup$ a σ -algebra containing B

$\Rightarrow \mathcal{O} \neq \emptyset$

We will prove:

(i) Existence (ii) Minimality

(i) $G = \bigcap_{i \in I} A_i : B \subseteq A_i, \forall i \in I \Rightarrow \mathcal{O}$

$\therefore$ Arbitrary intersection of σ -algebra is a σ -algebra
and $B \subseteq A_i, \forall i \in I \Rightarrow B \subseteq \bigcap_{i \in I} A_i = G$

$\Rightarrow G$ is a σ -algebra containing B .

(ii) let J is a σ -algebra containing B

$\Rightarrow J \in \mathcal{O} \Rightarrow \bigcap_{i \in I} A_i \in J \therefore \bigcap_{i \in I} A_i \subseteq J$

$\Rightarrow G \subseteq J \Rightarrow G$ is a smallest σ -algebra.

MOST IMP

Topic #14

L.Q

S.Q

Theorem: let $\{E_n\}_{n=1}^{\infty}$ be a sequence of elements of σ -algebra $\mathcal{F}$ of subsets of $X \neq \emptyset$. Then there exist also a sequence $\{F_n\}_{n=1}^{\infty}$ of σ -algebra $\mathcal{F}$ such that

(i) $F_n \in \mathcal{F}, \forall n \in \mathbb{N}$, (ii) $F_n \subseteq E_n, \forall n \in \mathbb{N}$

(iii) $F_n \cap F_m = \emptyset, \forall m \neq n$, (iv) $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$

proof: (i) let us construct:

$F_1 \subseteq E_1, F_2 \subseteq E_2 - E_1, F_3 \subseteq E_3 - (E_1 \cup E_2)$

by induction: $F_n \subseteq E_n - (E_1 \cup E_2 \cup E_3 \cup \dots \cup E_{n-1})$

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$$\begin{aligned} \Rightarrow F_n &= E_n \cap (E_1 \cup E_2 \cup \dots \cup E_n)' : A \cap B = A \cap B' \\ &= E_n \cap (E_1' \cap E_2' \cap \dots \cap E_n') \text{ by De Morgan's law} \\ \because E_n \in \mathcal{A} \text{ and } \because E_1, E_2, \dots, E_n \in \mathcal{A} \\ \Rightarrow E_1', E_2', \dots, E_n' \in \mathcal{A} \\ \Rightarrow F_n &= E_n \cap (E_1' \cap E_2' \cap \dots \cap E_n') \in \mathcal{A} \rightarrow \textcircled{1} \end{aligned}$$

(ii) From (i)

$$\Rightarrow F_n \subseteq E_n \quad \forall n \in \mathbb{N} \rightarrow \textcircled{2}$$

(iii) $\forall m \neq n \Rightarrow m \neq n$ $A \cap B \subseteq A \cup B$

Say without loss of generality $m < n$.

$$\text{From } \textcircled{2} \quad F_m \subseteq E_m \Rightarrow F_m \cap F_n \subseteq E_m \cap E_n \rightarrow \textcircled{3}$$

$$\Rightarrow E_m \cap F_n = E_m \cap [E_n \cap (E_1' \cap E_2' \cap \dots \cap E_n')] \text{ } \textcircled{1}$$

$$E_m \cap F_n = (E_m \cap E_n) \cap (E_1' \cap E_2' \cap \dots \cap E_n')$$

$$E_m \cap F_n = \emptyset$$

$$\Rightarrow E_m \cap F_n \subseteq \emptyset \Rightarrow E_m \cap F_n = \emptyset \quad \forall m \neq n$$

(iv) $\because F_n \subseteq E_n \quad \forall n \in \mathbb{N}$

$$\Rightarrow \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \rightarrow \textcircled{3}$$

Let $y \in \bigcup_{n=1}^{\infty} F_n \Rightarrow y \in E_n$ for some $n \in \mathbb{N}$

But any y suppose it be a real positive integer

$y \in E_m$ but $y \notin E_1, E_2, \dots, E_m$ $y \in A \cap B$

$$\Rightarrow y \notin E_1 \cup E_2 \cup \dots \cup E_m \quad \text{or } y \in A \cap B$$

$$\Rightarrow y \in E_n = (E_1 \cup E_2 \cup \dots \cup E_m)'$$

$$\Rightarrow y \in E_n \Rightarrow y \in \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} E_n$$

$$\therefore \Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n$$

$$\Rightarrow \boxed{\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n}$$

MCQ

Topic #15

An Important Result of Analysis:
Theorem: Suppose that for any $\epsilon > 0$, if $a < b + \epsilon$,
then $a \leq b$.

Proof: $p: a < b + \epsilon$ and $q: a \leq b$

proof scheme: $\{p \rightarrow \text{true} \text{ and } \neg q \rightarrow \text{true}$
(and seek for contradiction)

$$p \Rightarrow a < b + \epsilon \text{ and } \neg q \Rightarrow a > b \\ \Rightarrow (a - b) > 0$$

$$\begin{cases} p, q \text{ if } p \rightarrow \text{true} \\ q, \neg p \text{ if } p \rightarrow \text{false} \end{cases}$$

But we have given from $p: a < b + \epsilon$

i.e. In particular taking $\epsilon = (a - b) > 0$

$$\therefore p \Rightarrow a < b + (a - b) \Rightarrow a < a$$

i.e. a contradiction

$$\therefore \boxed{a < b + \epsilon \Rightarrow a \leq b : \epsilon \rightarrow 0 \text{ arbitrarily}} \\ (\epsilon > 0)$$

MCQ

Topic # 16

Set Function:

Let $X \neq \emptyset$ and $I \subseteq \mathcal{P}(X)$. Let μ be a class of subsets of X , then a function defined on I as:

$$\mu: I \rightarrow \{a, \emptyset\} \cup \{\emptyset\} \text{ i.e. every}$$

element (set) is mapped to a non-negative

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Not extended real number.

For example: $[0, \infty) \subseteq \mathbb{R} \Rightarrow \mu([0, \infty)) = \infty$
 $(0, 1) \subseteq \mathbb{R} \Rightarrow \mu(0, 1) = 1 - 0 = 1$

No max point in this interval, the supremum is 1 & the infimum is 0

(iii) $\mu: P(N) \rightarrow [0, \infty) \cup \{\infty\} \quad \forall E \subseteq P(N)$
if $E = \{1, 2, 3\} \Rightarrow \mu(E) = \sigma(E) = 3$
 $E = \{1, 2, 3, \dots\} \Rightarrow \mu(E) = \sigma(E) = \infty$

Topic #17

Measure on a σ -Algebra:

Let $X \neq \emptyset$ and $\mathcal{A}$ be a σ -algebra on X , then a set function $\mu: \mathcal{A} \rightarrow [0, \infty) \cup \{\infty\}$ is said to be (i) **Finitely subadditive** if

$$\mu\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n \mu(E_i)$$

MCQ'S

(ii) **Countably subadditive** if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

S.Q

(iii) **Finitely additive** if

$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i) \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

(iv) **Countably additive** if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

(v) **Monotone** if

$$\mu(A) \leq \mu(B) \text{ whenever } A \subseteq B$$

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$\mu: \mathcal{A} \rightarrow [0, \infty) \cup \{\infty\}$
measure on $\mathcal{A}$ if

- ✓ (i) $\mu(\emptyset) = 0$
- ✓ (ii) $\mu(A) \geq 0 \quad \forall A \in \mathcal{A}$
- ✓ (iii) μ is ~~count~~ countably additive

MCQ'S

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \forall E_i \cap E_j = \emptyset \quad \forall i \neq j$$

Topic #18

Monotonicity of Measures

IMP

(i) If $\emptyset \neq A, B \in \mathcal{A}$ such that $B \subseteq A$, then

$$\mu(B) \leq \mu(A)$$

proof: $\because B \subseteq A \Rightarrow A = B \cup (A-B)$

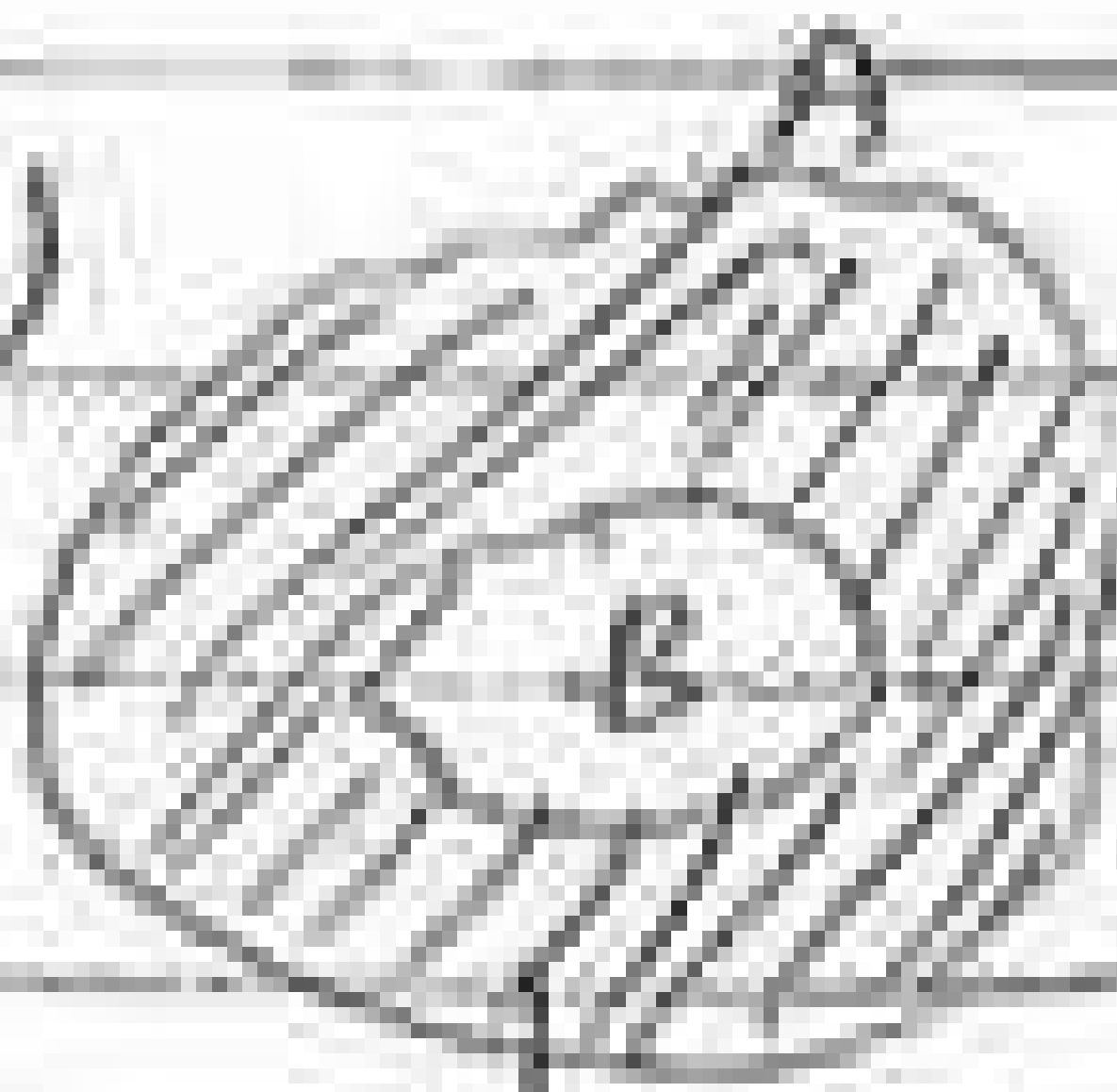
$$\Rightarrow \mu(A) = \mu[B \cup (A-B)]$$

$$\mu(A) = \mu(B) + \mu(A-B) \rightarrow \text{①}$$

$$\Rightarrow \mu(A) \geq \mu(B)$$

$$\Rightarrow \mu(B) \leq \mu(A)$$

$$\begin{cases} a \neq b + c \\ a \geq b, c > 0 \\ \Rightarrow a > b \end{cases} \quad \begin{array}{c} \text{A-B} \\ \downarrow \end{array}$$



Topic #19

(ii) If $\mu(B) < \infty$, then $\mu(A-B) = \mu(A) - \mu(B)$

proof: From eq ① $\Rightarrow \mu(A) - \mu(B) = \mu(A-B)$

$[0, \infty]$



subtraction

difference

operation on sets

Topic #20

Measurable Space:

For $X \neq \emptyset$, let $\mathcal{A}$ be the σ -algebra on X .
Then the tuple $(X, \mathcal{A}, \mu)$ is called measurable space and members of $\mathcal{A}$ are called measurable sets.

$\mathcal{A} \rightarrow$ class of subsets of X .

Topic #21

imp

Theorem:- Let $(X, \mathcal{A}, \mu)$ be a measure space.
Then if $\exists$ a $\phi \in \mathcal{A}$ such that $\mu(A) < \infty$
then $\mu(\phi) = 0$.

proof: $\because A = A \cup \phi$

$$\Rightarrow \mu(A) = \mu(A \cup \phi) = \mu(A) + \mu(\phi)$$

By countable additivity of μ

$$\Rightarrow \mu(\phi) = 0$$

Topic #22

Theorem: Let $\{A_n\}$ is a sequence of pairwise disjoint subsets of X in σ -algebra $\mathcal{A}$ then for any $T \in \mathcal{A}$,
 $\mu(T) = \sum_{n=1}^{\infty} \mu(A_n \cap T) = \mu(T \cap (\bigcup_{n=1}^{\infty} A_n)^c)$.

proof: $\because$ for any $T \in \mathcal{A} \Rightarrow T = T \cap X$

$$\Rightarrow T = T \cap (\bigcup_{n=1}^{\infty} A_n \cup (\bigcup_{n=1}^{\infty} A_n)^c)$$



$$\Rightarrow T = [T \cap (\bigcup_{n=1}^{\infty} A_n)] \cup [T \cap (\bigcup_{n=1}^{\infty} A_n)^c]$$

$A \cup A^c = X$
 $A \cap A^c = \emptyset$
 $\mu(A \cup A^c) = \mu(A) + \mu(A^c)$

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$$\begin{cases} T \cap (\bigcup_{n=1}^{\infty} A_n) = T \cap (A_1 \cup A_2 \cup A_3 \cup \dots) \\ = (T \cap A_1) \cup (T \cap A_2) \cup \dots \\ = \bigcup_{n=1}^{\infty} (T \cap A_n) \end{cases}$$

$$\Rightarrow T = \left[\bigcup_{n=1}^{\infty} (T \cap A_n) \right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$

$$\Rightarrow \mu(T) = \mu \left[\left(\bigcup_{n=1}^{\infty} (T \cap A_n) \right) \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right] \right]$$

By additive property of μ

$$\Rightarrow \mu(T) = \mu \left[\bigcup_{n=1}^{\infty} (T \cap A_n) \right] + \mu \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$

$\therefore$ All $T \cap A_n$'s are disjoint

$\therefore$ by additive property of μ

$$\mu \left(\bigcup_{n=1}^{\infty} B_n \right) = \mu(B_1 \cup B_2 \cup \dots)$$

$$= \mu(B_1) + \mu(B_2) + \dots$$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

$$\Rightarrow \mu(I) = \sum_{n=1}^{\infty} \mu(T \cap A_n) + \mu \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n \right)^c \right]$$



MCQ

MOST IMP

Topic # 23

L.Q

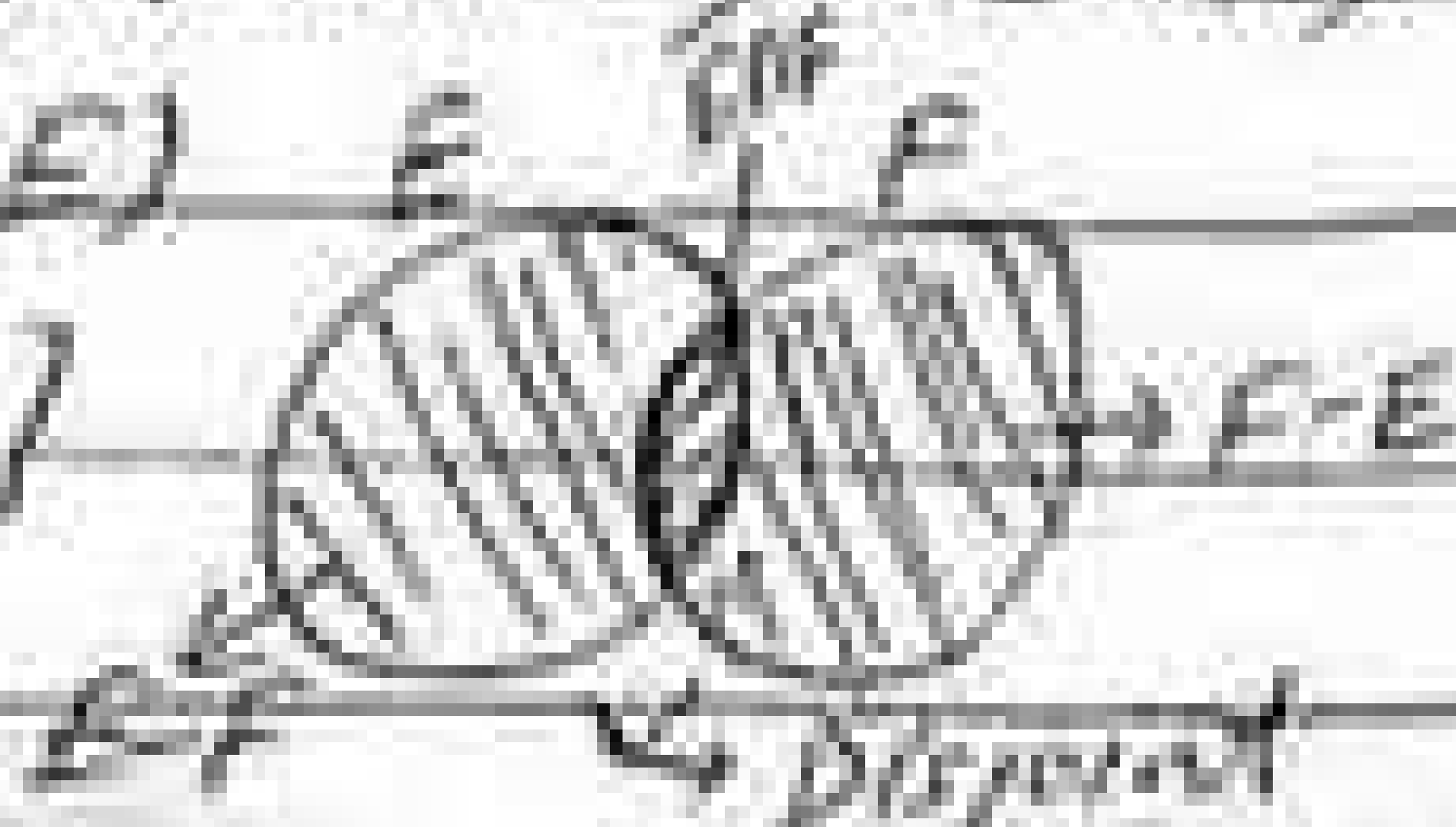
S.Q

Theorem: In a measure space $(X, \mathcal{A}, \mu)$, if

$$\mu(E \cap F) = 0, \text{ then } \mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$$

$$\text{Proof: } \because E \cap F = (E - F) \cup (F - E)$$

$$\Rightarrow \mu(E \cap F) = \mu(E - F) + \mu(F - E)$$



TRA NEEHSON

$$\Rightarrow \mu(E \cup F) = \mu(E \setminus F) + \mu(F \setminus E)$$

Given that $\mu(E \cup F) = 0$

$$\Rightarrow 0 = \mu(E \setminus F) + \mu(F \setminus E)$$

$$\Rightarrow \mu(E \setminus F) = 0 \text{ and } \mu(F \setminus E) = 0$$

Note that $F = (F \setminus E) \cup (F \cap E)$

$\therefore F \setminus E$ and $F \cap E$ are disjoint

$$\Rightarrow \mu(F) = \mu[(F \setminus E) \cup (F \cap E)]$$

$$\Rightarrow \mu(F) = \mu(F \setminus E) + \mu(F \cap E)$$

$$\Rightarrow \mu(F) = 0 + \mu(F \cap E)$$

$$\Rightarrow \mu(F) = \mu(F \cap E) \quad \text{--- (1)}$$

Also $E \cup F = (E \cap F) \cup (E \setminus F)$

$$\Rightarrow \mu(E \cup F) = \mu[(E \cap F) \cup (E \setminus F)]$$

$$\Rightarrow \mu(E \cup F) = \mu(E \cap F) + \mu(E \setminus F)$$

$$\Rightarrow \mu(E \cup F) = 0 + \mu(E \cap F)$$

$$\Rightarrow \mu(E \cup F) = \mu(E \cap F) \quad \text{--- (2)}$$

Now $E = (E \setminus F) \cup (E \cap F)$

$$\Rightarrow \mu(E) = \mu[(E \setminus F) \cup (E \cap F)]$$

$$\Rightarrow \mu(E) = \mu(E \setminus F) + \mu(E \cap F)$$

$$\Rightarrow \mu(E) = 0 + \mu(E \cap F)$$

$$\Rightarrow \mu(E) = \mu(E \cap F) \quad \text{--- (3)}$$

From (1), (2) and (3) $\Rightarrow \mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$

$$a+b=0$$

$$a, b \geq 0$$

$$\Rightarrow a=0$$

$$\text{and } b=0$$



S.Q

Finite Measure:

S.Q

For $\phi \neq X$ the triplet $(X, \mathcal{A}, \mu)$ is said to be finite measure space if $\mu(X) < \infty$ and μ is called finite measure.

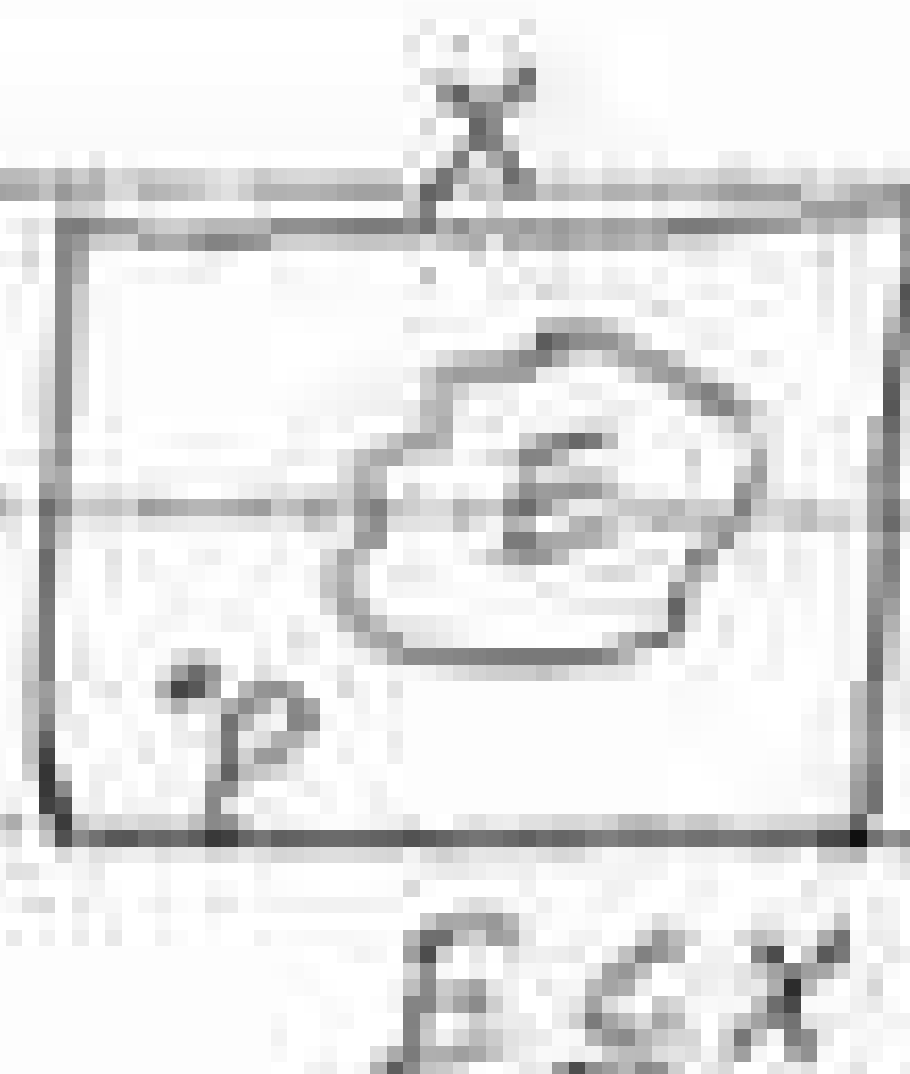
Sigma(σ)-Finite Measures:

For $\phi \neq X$, the tuple $(X, \mathcal{A}, \mu)$ is σ -finite measure if and only if $\mu(X) < \infty$ and $\exists$ a sequence $\{A_n\}$ of subsets of X , such that $\bigcup_{n=1}^{\infty} A_n = X$ and $\mu(A_n) < \infty \forall n \in \mathbb{N}$.

For example:

$$(i) \quad X \neq \phi, \mu: \mathcal{P}(X) \rightarrow [0, \infty]$$

$$\mu(E) = \begin{cases} 0 & \text{if } p \notin E \\ 1 & \text{if } p \in E \end{cases}$$



for any fixed point 'p' in 'X'.

$$(ii) \quad Z = \{a, \pm 1, \pm 2, \dots\}$$

$$\mu: \mathcal{P}(Z) \rightarrow [0, \infty]$$

$$\mu(A) = \begin{cases} \sum_{n \in A} 1 = \#(A) & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite.} \end{cases}$$

counting measure.

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Topic #25

For $\phi \subseteq X$, defining: $\mu: P(X) \rightarrow [0, \infty]$, as

$$\mu(A) = \begin{cases} 0 & ; p \notin A \\ 1 & ; p \in A \end{cases} \quad \text{for fixed point } p \in X. \text{ Then}$$

Show that μ is a finite measure on X .

proof: (i) $p \in \phi \Rightarrow \mu(\phi) = 0$

by given definition of μ

(ii) for any $A \subseteq X, \forall A \in \mathcal{A}$; by definition

$$\mu(A) \geq 0$$

(iii) let $\{A_n\}$ be the sequence of pairwise disjoint subset of X in $\mathcal{A}$.

μ is σ -algebra on X

$$i) \mu(\phi) = 0$$

$$ii) \mu(A) \geq 0$$

$$iii) \{A_n\} \rightarrow \text{disjoint} \\ \in \mathcal{A}$$

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

$$\text{Finite } \mu \\ \mu(X) < \infty$$

$$\text{Case I: let } p \in (\bigcup_{n=1}^{\infty} A_n) \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = 1$$

$$\Rightarrow p \in A_k \text{ for some fixed } k \in \mathbb{N}$$

$$\Rightarrow \mu(A_k) = 1 \text{ and } \mu(A_n) = 0 \text{ for } k \neq n$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 1 = \mu(\bigcup_{n=1}^{\infty} A_n)$$

$$\text{Case II: } p \notin (\bigcup_{n=1}^{\infty} A_n) \Rightarrow p \notin A_n \forall n$$

$$\Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) = 0 \text{ and } \mu(A_n) = 0 \forall n \in \mathbb{N}$$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 0$$

$$\text{So } \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) = 0$$

$$\text{Also, Trivially } p \in X \Rightarrow \mu(X) = 1$$

$$\Rightarrow \mu(X) < \infty$$

Topic #26

Closed Operations of Measures:-

(i) Sum of Measures:

let μ and η are measures on σ -algebra of $X + \phi$, then

$$(i) \mu(\phi) = 0, \eta(\phi) = 0$$

$$(ii) \mu(A) \geq 0, \eta(A) \geq 0, \forall A \in \mathcal{A}$$

(iii) For any sequence $\{A_n\}$ in $\mathcal{A}$ of disjoint pairs

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

and

$$\eta\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \eta(A_n)$$

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$$(1) (\mu + \eta)(\phi) = \mu(\phi) + \eta(\phi) = 0 + 0 = 0$$

$$(2) (\mu + \eta)(A) = \mu(A) + \eta(A) \geq 0 \because \mu(A), \eta(A) \geq 0$$

$$(3) (\mu + \eta)\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) + \eta\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \eta(A_n)$$

$$= \sum_{n=1}^{\infty} (\mu(A_n) + \eta(A_n))$$

$$= \sum_{n=1}^{\infty} (\mu + \eta)(A_n)$$

For any $\alpha \in \mathbb{R}$

$$\alpha \mu(A) = \alpha(\mu A) \quad \text{scalar composition}$$

Counter case: For $A \in \mathcal{A}$

$$\mu(A) \geq 0, \eta(A) \geq 0$$

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But $\mu(A) - \eta(A) < 0$

if $\eta(A) > \mu(A)$

$\Rightarrow \mu - \eta^+$ is not defined

Topic # 27

Outer Measure:

For $X \neq \emptyset$, a function $\mu^*: P(X) \rightarrow [0, \infty]$

is said to be outer measure if:

(i) $\mu^*(\emptyset) = 0$

MCQ'S

(ii) whenever $A \subseteq B$, $\mu^*(A) \leq \mu^*(B)$

(iii) If $\{A_n\}$ be a sequence of subsets of X s.t. $\{A_n\} \subseteq P(X)$. Then

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

For Example $X \neq \emptyset$: define $\mu^*: P(X) \rightarrow [0, \infty]$

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{if } A \neq \emptyset \end{cases}$$

Sol: (i) If $\emptyset = A \Rightarrow \mu^*(A) = 0$ trivially true.

(ii) If $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(a) $A = B = \emptyset \Rightarrow \mu^*(A) = 0 = \mu^*(B)$

(b) $A = \emptyset, B \neq \emptyset \Rightarrow \mu^*(A) = 0 < \mu^*(B) = 1$

(c) $A \neq B \neq \emptyset \Rightarrow \mu^*(A) = 1 = \mu^*(B)$

In all three cases $\mu^*(A) \leq \mu^*(B)$

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(iii) Let $\{A_n\}$ be the sequence of subsets in X .

(a) If all $A_n = \emptyset \quad \forall n \in \mathbb{N} \Rightarrow (\bigcup_{n=1}^{\infty} A_n) = \emptyset$
 $\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = 0 = \mu^*(A_n) = \sum_{n=1}^{\infty} \mu^*(A_n)$

(b) If $A_n \neq \emptyset$ for some $n \in \mathbb{N}$

$\Rightarrow \mu^*(A_n) = 1$ for some $n \in \mathbb{N}$

$\Rightarrow \sum_{n=1}^{\infty} \mu^*(A_n) \geq 1 \rightarrow \textcircled{1}$

MCQ'S

and if $A_n \neq \emptyset \Rightarrow \bigcup_{n=1}^{\infty} A_n \neq \emptyset$

$\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) = 1 \rightarrow \textcircled{2}$

from $\textcircled{1}$ and $\textcircled{2}$

$\Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Topic # 28

Lebesgue Outer Measure:

MOST IMP

For $p, t \in \mathbb{R}$, the Lebesgue measure of $\mathbb{R}^n$ defined on $\mathcal{P}(\mathbb{R})$ as $m^*: \mathcal{P}(\mathbb{R}) \rightarrow [0, \infty] \cup \{\infty\}$ and given by:

$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n \text{ where } \{I_n\} \text{ is a sequence of open intervals covering } E \right\}$

MCQ

Here $l(I_n) \rightarrow$ length of n th interval $\{I_n\}$

Topic # 29

Non-negativity and Monotonicity of Lebesgue Outer Measure:

Since $I(I_n) \geq 0, \forall n \in \mathbb{N}$

$$\Rightarrow \sum_{n=1}^{\infty} I(I_n) \geq 0 \quad \because E \subseteq \bigcup_{n=1}^{\infty} I_n$$

$\Rightarrow \inf \left\{ \sum_{n=1}^{\infty} I(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \text{ be the sequence of interval covering } E \right\} \geq 0$

$$\Rightarrow m^*(E) \geq 0$$

Let $A, B \in \mathcal{P}(\mathbb{R})$ such that $A \subseteq B$ and $\{I_n\}$ be a sequence of intervals covering B in $\mathbb{R}$.

$$\Rightarrow B \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow A \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(A) \leq \sum_{n=1}^{\infty} I(I_n) \quad \because B \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(A) \text{ is lower bound of } \left\{ \sum_{n=1}^{\infty} I(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\Rightarrow m^*(A) \leq \inf \left\{ \sum_{n=1}^{\infty} I(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

$$\Rightarrow m^*(A) \leq m^*(B)$$

$$\{x\} \subseteq \mathbb{R}, \forall x \in \mathbb{R}$$

$$\{x\} \subseteq (x-\epsilon, x+\epsilon)$$

$$\Rightarrow m^*\{x\} \leq \frac{x+\epsilon}{2} - \frac{x-\epsilon}{2} = \epsilon$$

$$\left| \begin{array}{l} A \subseteq B \\ m^*(A) \leq m^*(B) \end{array} \right.$$

$$\Rightarrow m^*\{x\} = 0 \text{ where } \epsilon > 0 \text{ arbitrary}$$

Countable Subadditivity of Lebesgue

Outer Measure:

Theorem: For any arbitrary seq $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathbb{R})$
 $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ i.e. m^* is countably subadditive.

proof: (i) For fixed $n \in \mathbb{N}$ if $m^*(A_n) < \infty$
 $\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) = \infty = m^*(A_n) = \sum_{n=1}^{\infty} m^*(A_n)$

(ii) For $m^*(A_n) < \infty, \forall n \in \mathbb{N}$

Let $m^*(A_n) = \inf \left\{ \sum_{k=1}^{\infty} l(I_{nk}) : A_n \subseteq \bigcup_{k=1}^{\infty} I_{nk} \right\}$

where $\{I_{nk}\}_{k=1}^{\infty}$ is open covering of A_n

For $\epsilon > 0, \Rightarrow \frac{\epsilon}{2^n} > 0$

$\Rightarrow$ there exists a sequence $\{I_{nm}\}_{m=1}^{\infty}$

$$\sum_{m=1}^{\infty} l(I_{nm}) < m^*(A_n) + \frac{\epsilon}{2^n} \rightarrow 0$$

$$\therefore A_n \subseteq \bigcup_{m=1}^{\infty} I_{nm} \Rightarrow \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right)$$

Here, countable union $\bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right)$ is countable
 sets: $\bigcup_{m=1}^{\infty} I_{nm}$ is countable.

$$\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) = \inf \left\{ \sum_{m=1}^{\infty} l(I_{nm}) : \right.$$

$$\left. \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \left(\bigcup_{m=1}^{\infty} I_{nm} \right) \right\}$$

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where $\{I_m\}$ is sequence of open intervals covering $\bigcup_{n=1}^{\infty} A_n$

$$m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} l(I_m)$$

$$\text{From (1)} \Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} (m^*(A_n) + \frac{\epsilon}{2^n})$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{Geometric Series}$$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \epsilon$$

$$\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$S = \frac{a}{1-r}$$

$$S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

Topic #31

Lebesgue Outer Measure of Countable set:

Since every countable set can be expressed in the form of sequence i.e. $1, 2, 3, \dots, n, \dots$

$$\Rightarrow A = \{x_1, x_2, x_3, \dots, x_n, \dots\} = \bigcup_{n=1}^{\infty} \{x_n\}$$

$$0 \leq m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} \{x_n\}\right) \leq \sum_{n=1}^{\infty} m^*\{x_n\} = 0$$

$\therefore$ Lebesgue Outer Measure of A is 0

$$\Rightarrow m^*(A) = 0$$

MCQ

Topic # 38

Theorem: For any $A \subseteq \mathbb{R}$ and any $\epsilon > 0$, there is an open set U such that $A \subseteq U$ and $m^*(U) < m^*(A) + \epsilon$

proof: By definition

$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$ where $\{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals covering A in $\mathbb{R}$ and $l(I_n)$ is the length of interval I_n .

For any $\epsilon > 0$, $\exists \{I_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that $m^*(A) + \epsilon > \sum_{n=1}^{\infty} l(I_n)$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon \rightarrow (1)$$

Union of open interval in $\mathbb{R}$ is an open set $\Rightarrow$ there exists an open set

$$\bigcup_{n=1}^{\infty} I_n = U \text{ (open set)}$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = m^*(U)$$

$$\Rightarrow m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \rightarrow (2)$$

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} m^*(I_n)$$

$$\text{From (2)} \Rightarrow m^*(U) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} l(I_n)$$

$$\text{From (1)} \Rightarrow m^*(U) < m^*(A) + \epsilon$$

Topic #33

~~The~~ Lebesgue Outer Measure of an Interval:

Theorem: Lebesgue outer measure of an interval is its length. i.e. $m^*(a, b) = m^*[a, b] = m^*(a, b] = m^*[a, b) = b - a$

MCQ'S

proof: Case 1: Let $I = [a, b]$ in $\mathbb{R}$. By definition $m^*(I) = \inf \left\{ \sum_{i=1}^{\infty} l(I_i) : I \subseteq \bigcup_{i=1}^{\infty} I_i, \text{ where } \{I_i\}_{i=1}^{\infty} \text{ are open intervals covering } I \text{ in } \mathbb{R} \text{ and } l(I_i) \text{ is length of } i\text{th interval} \right\} \rightarrow \text{①}$

In particular for $\epsilon > 0$, $(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$ will also serve as covering $I = [a, b]$.

$$\text{i.e. } [a, b] \subset \left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right)$$

By definition ① $\Rightarrow m^*(I) \leq l\left(a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2}\right) = b - a + \epsilon$
 $\Rightarrow m^*(I) \leq b - a, \quad \epsilon > 0 \text{ and } \epsilon \rightarrow 0 \text{ arbitrary}$
 $\rightarrow \text{②}$

For proving $m^*(I) \geq b - a$, let $\{I_k\}_{k=1}^{\infty}$ be a countable family of open intervals covering $I = [a, b]$

$\because [a, b]$ is bounded and closed $\therefore [a, b] \subseteq \bigcup_{k=1}^{\infty} I_k$

$\Rightarrow$ By Heine Borel: there exist

a finite sub cover (collection) $\{I_j\}_{j=1}^n$ s.t. $[a, b] \subseteq \bigcup_{j=1}^n I_j$

$$\Rightarrow l[a, b] \leq \sum_{j=1}^n l(I_j)$$

$$\Rightarrow \sum_{j=1}^n l(I_j) \geq b - a \rightarrow \text{③}$$

is true for all families of sequence of open intervals covering $I = [a, b]$.

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$$\rightarrow \sum_{i=1}^n l(I_i) \leq \sum_{i=1}^n l(I_i)$$

Taking Infimum in ③

Let $\left\{ \sum_{i=1}^n l(I_i) : I \subseteq \bigcup I_i, \text{ whenever } \{I_i\}_{i=1}^n \text{ is a covering of } I \right\} \geq b-a$

$$m^*(I) \geq b-a \quad \text{④}$$

$$\text{From ③ and ④} \Rightarrow \boxed{m^*([a,b]) = b-a} \rightarrow \text{⑤}$$

$$\text{Case I: If } I = (a,b) \Rightarrow (a,b) \subset [a,b]$$

$$\Rightarrow m^*(a,b) \leq m^*[a,b] \text{ by monotonicity of } m^* \\ \text{for } \epsilon > 0; \left[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2} \right] \subset (a,b)$$

$$\Rightarrow m^*\left[a + \frac{\epsilon}{2}, b - \frac{\epsilon}{2}\right] \leq m^*(a,b)$$

$$\Rightarrow (b-a) - \epsilon \leq m^*(a,b)$$

$$\Rightarrow b-a \leq m^*(a,b) \rightarrow \text{⑥}$$

$$\text{From ⑤, ⑥ and ④} \Rightarrow \boxed{m^*(a,b) = b-a} \rightarrow \text{⑦}$$

$$\text{Case II: If } I = [a,b] \text{ or } [a,b)$$

$$\Rightarrow (a,b) \subset I \subset [a,b]$$

$$\Rightarrow m^*(a,b) \leq m^*(I) \leq m^*[a,b]$$

$$\Rightarrow b-a \leq m^*(I) \leq b-a \quad \text{From ⑤ and ⑥}$$

$$\Rightarrow m^*(I) = b-a$$

$$\Rightarrow \boxed{m^*(a,b) = m^*[a,b] = b-a}$$

MCQ

It's also length formula

Determining Uncountability by Lebesgue Outer Measure:

MCQ

Theorem: Intervals in $\mathbb{R}$ are uncountable.

proof: Let $[a, b]$ be an interval and it is countable.

$\therefore m^*[a, b] = 0$ $\because$ it is countable

But $m^*[a, b] = l[a, b] = b - a > 0$, if $a < b$
 is contradiction.

$\Rightarrow [a, b]$ in $\mathbb{R}$ is uncountable.

Topic #35

Translation Invariance of Lebesgue Outer Measure:

Theorem: Show that m^* is translation invariant.

proof: Let $p \in \mathbb{R}$ and

$x \in E \Rightarrow p + x \in p + E = \{p + x, x \in E\}$.

It is translation of E by p units.



By definition:

$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \text{ where } \{I_n\}_{n=1}^{\infty} \text{ is sequence of intervals covering } E \right\}$

Let $I_n = (a_n, b_n) \Rightarrow l(I_n) = b_n - a_n$ $\rightarrow$ ①

If $E \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow p + E \subseteq \bigcup_{n=1}^{\infty} J_n$ such that

$J_n = (p + a_n, p + b_n) \Rightarrow l(J_n) = b_n - a_n = l(I_n)$

$\Rightarrow \sum_{n=1}^{\infty} l(J_n) = \sum_{n=1}^{\infty} l(I_n) \rightarrow$ ②

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From ① and ②

$$\Rightarrow m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : p \in E \subseteq \bigcup_{n=1}^{\infty} I_n \text{ where } \{I_n\}_{n=1}^{\infty} \text{ sequence of open intervals covering } p \in E \right\} = m^*(E \cup p).$$

Topic #36

Least Length of Open Covering of All Rationals in an Interval:

Theorem: If $E = \{p \in \mathbb{Q} : 0 < p < 1\}$ and $\{I_n\}_{n=1}^{\infty}$ is the finite sequence of open intervals such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$, then $\sum_{n=1}^{\infty} l(I_n) \geq 1$ **MCQ**

proof:

$$\because A \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow (0,1) \subseteq \bigcup_{n=1}^{\infty} I_n$$

$$\Rightarrow m^*(0,1) \leq m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \quad \text{by monotonicity of } m^*$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n) \quad \because m^* \text{ is sub-additive}$$

$$m^*(0,1) \leq \sum_{n=1}^{\infty} l(I_n) \rightarrow \text{①}$$

$$\text{But } m^*(0,1) = l(0,1) = 1 - 0 = 1$$

$$\text{From ① } \Rightarrow 1 \leq \sum_{n=1}^{\infty} l(I_n)$$

$$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \geq 1$$

Topic # 37

MOST IMP

Lebesgue Measurable Sets:

A $\phi \neq E \subseteq \mathbb{R}$ is said to be Lebesgue measurable or simply measurable if $\forall A \subseteq \mathbb{R}$, we have,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (1)$$

Note: $A = A \cap \mathbb{R} = A \cap (E \cup E^c)$

MCQ

$$A = (A \cap E) \cup (A \cap E^c)$$

$$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E^c)]$$

$$\Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (2)$$

In order to prove (1), we will go for

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \rightarrow (3)$$

Topic # 38

MCQ

Lebesgue Measurability of Complement of E set:

Theorem: A set E is measurable if and only if its complement E^c is measurable.

Proof: $\because E \subseteq \mathbb{R}$ is measurable if $\forall A \subseteq \mathbb{R}$

$$(1) m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$$(2) m^*(A) = m^*(A \cap E^c) + m^*(A \cap E)$$

$$(3) \text{ if } E^c \text{ is measurable.}$$

Topic # 39

MOST IMP

Theorem: ϕ and $\mathbb{R}$ are measurable.

$$\text{Proof: } m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \phi)$$

Lebesgue measure of empty set is zero

MCQ

$$\therefore m^*(\phi) = 0$$

$$\phi^c = \mathbb{R}$$

$$\mathbb{R}^c = \phi$$

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$$\Rightarrow m^*(A) = m^*(A \cap R) + m^*(A \cap \phi)$$

$$\Rightarrow m^*(A) = m^*(A \cap R) + m^*(A \cap R^c)$$

$\Rightarrow R$ is measurable.

$\Rightarrow R^c = \phi$ is also measurable.

$\therefore$ {from previous theorem}

Topic #40

Theorem: If $m^*(E) = 0$, then $E \cap R$ is measurable.

proof: For any $\phi \in \mathcal{A} \subseteq \mathcal{R}$, $E \cap A \subseteq E$

$$\Rightarrow m^*(A \cap E) \leq m^*(E) \quad \because m^* \text{ is monotone}$$

$$\Rightarrow m^*(A \cap E) \leq 0 \rightarrow \textcircled{1} \quad \because \text{given}$$

But $m^*(A \cap E) \geq 0 \rightarrow \textcircled{2}$ By definition

$$\text{in } \textcircled{1} \text{ and } \textcircled{2} \Rightarrow m^*(A \cap E) = 0 \text{ or } 0 = m^*(A \cap E)$$

Similarly: $A \cap E^c \subseteq A$

$$\Rightarrow m^*(A \cap E^c) \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E^c)$$

$\textcircled{3}$

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

$$m^*(A) \geq$$

$\therefore$ From $\textcircled{3}$ and $\textcircled{4} \Rightarrow$

$$m^*(A) + 0 \geq m^*(A \cap E) + m^*(A \cap E^c)$$

$\Rightarrow E$ is measurable.

Topic #41

MOST IMP

Theorem: Union and Intersection of two measurable sets is measurable.

proof: $\forall A \in \mathcal{R}, E \rightarrow \text{measurable}$

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$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

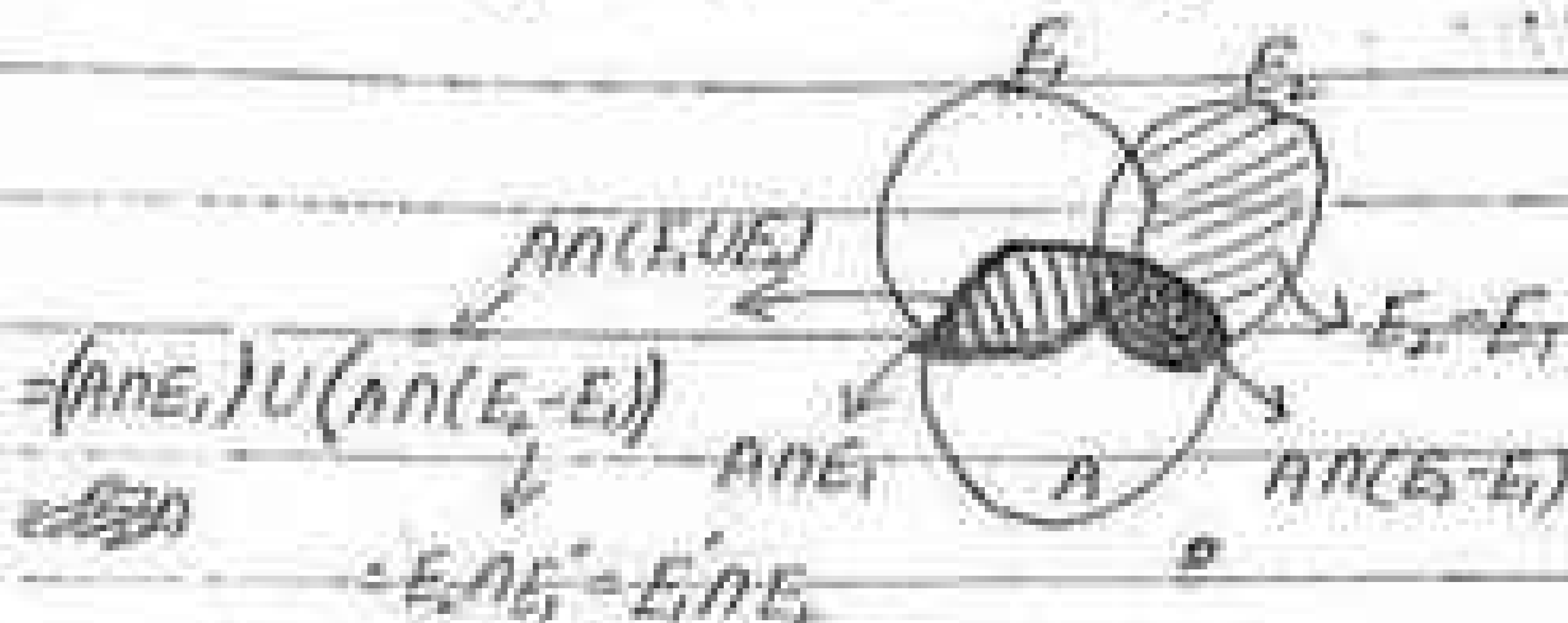
Let E_1, E_2 are two measurable sets.

To prove $E_1 \cup E_2$ is measurable.

$$m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

QED

$$RHS = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$



$$RHS = m^*[(A \cap E_1) \cup (A \cap (E_1 \cup E_2)^c)] + m^*[A \cap (E_1 \cup E_2)^c]$$

By sub-additivity of m^*

$$RHS \leq m^*(A \cap E_1) + m^*(A \cap (E_1 \cup E_2)^c) + m^*(A \cap (E_1 \cup E_2)^c)$$

$\therefore E_1$ is measurable and using last set (1) (2)

$$\Rightarrow m^*(A \cap E_1) = m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \cap E_2^c)$$

$$\therefore RHS \leq m^*(A \cap E_1) + m^*(A \cap E_2^c)$$

Since again E_2 is measurable

$$RHS \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$

$\therefore E_1$ and E_2 are measurable.

$\Rightarrow E_1^c$ and E_2^c are measurable.

$\Rightarrow (E_1^c \cup E_2^c)$ is also measurable.

QED

$\Rightarrow (E \cup E')' = E \cap E'$ is measurable

Final Term Syllabus

MCQ

Topic # 4

Theorem: For any $\mu \in \mathcal{R}$ and $\{E_i\}_{i=1}^{\infty}$ be a family of pairwise disjoint measurable sets then

$$\mu^*\left[\mu\left(\bigcup_{i=1}^{\infty} E_i\right)\right] = \sum_{i=1}^{\infty} \mu^*(\mu E_i)$$

Proof: We will prove this by mathematical induction. For $n=1$, result is self-evident.

Suppose $\textcircled{A}$ is true for $n=k$.

$$\textcircled{A} \Rightarrow \mu^*\left[\mu\left(\bigcup_{i=1}^k E_i\right)\right] = \sum_{i=1}^k \mu^*(\mu E_i) \quad \textcircled{B}$$

$$\text{Consider: } \mu\left(\bigcup_{i=1}^k E_i\right) \cap E_{k+1} = \mu\left[\left(\bigcup_{i=1}^k E_i\right) \cap E_{k+1}\right]$$

$$= \mu[(E_1 \cup E_2 \cup \dots \cup E_k) \cap E_{k+1}]$$

$$\mu\left(\bigcup_{i=1}^k E_i\right) \cap E_{k+1} = \mu[(E_1 \cap E_{k+1}) \cup (E_2 \cap E_{k+1}) \cup \dots \cup (E_k \cap E_{k+1})]$$

$$\because E_i \cap E_{k+1} = \emptyset \quad \forall i \in \{1, 2, \dots, k\}$$

$$\Rightarrow \mu\left(\bigcup_{i=1}^k E_i\right) \cap E_{k+1} = \mu E_{k+1} \Rightarrow \textcircled{C}$$

Now

$$\mu\left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+2} = \mu\left[\left(\bigcup_{i=1}^{k+1} E_i\right) \cap E_{k+2}\right]$$

$$= \mu[(E_1 \cup E_2 \cup \dots \cup E_{k+1}) \cap E_{k+2}]$$

$$= \mu[(E_1 \cap E_{k+2}) \cup (E_2 \cap E_{k+2}) \cup \dots \cup (E_{k+1} \cap E_{k+2})]$$

$\uparrow$

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$$\Rightarrow A \cap \left(\bigcup_{i=1}^{\infty} E_i \right) \cap E'_0$$

$$= A \cap (E_1 \cup E_2 \cup \dots \cup E_{n+1})$$

$$\Rightarrow A \cap \left(\bigcup_{i=1}^n E_i \right) \cap E'_0 = A \cap \left(\bigcup_{i=1}^{n+1} E_i \right) \rightarrow \textcircled{2}$$



$$E_i \subseteq E'_0$$

$$\Rightarrow E_i \cap E'_0 = E_i$$

Taking " $A \cap \left(\bigcup_{i=1}^{\infty} E_i \right)$ " as test set and using Lebesgue measurability of " E_k ":

$$m^* \left[A \cap \left(\bigcup_{i=1}^{\infty} E_i \right) \right] = m^* \left[A \cap \left(\bigcup_{i=1}^n E_i \right) \cap E'_0 \right]$$

$$+ m^* \left[A \cap \left(\bigcup_{i=1}^n E_i \right) \cap E'_0 \right] \text{ by (1)}$$

$$m^* \left[A \cap \left(\bigcup_{i=1}^{\infty} E_i \right) \right] = m^* (A \cap E'_0) + m^* \left[A \cap \left(\bigcup_{i=1}^n E_i \right) \right]$$

$$= m^* (A \cap E'_0) + \sum_{i=1}^n m^* (A \cap E_i)$$

$$m^* \left[A \cap \left(\bigcup_{i=1}^{\infty} E_i \right) \right] = \sum_{i=1}^{\infty} m^* (A \cap E_i)$$

$\therefore$ Truth of $n=k-1$ also implies its truth for $n=k$. This proves the given theorem.

Topic #43

Theorem: The class " M " of all measurable sets is a σ -algebra.

MCQ'S

proof: (i) If E is measurable

$\Rightarrow E$ is also measurable

$\Rightarrow E \in M$

$$q = x_i \quad 1 \leq p(n)$$

since $q = x_i$ is a subset of A

(ii) Let $A \subseteq B$ and

$$\bigcup_{i=1}^{\infty} E_i \subseteq A \Rightarrow \bigcup_{i=1}^{\infty} E_i \subseteq B$$

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(ii) If E_1, E_2 are measurable $\Rightarrow E_1 \cup E_2$ is also measurable.

$$\Rightarrow E, \cup E_2 \in M$$

(iii) Let $\{E_i\}_{i=1}^{\infty}$ be the sequence of measurable sets in M , then we will show

$$\bigcup_{i=1}^{\infty} E_i = E \in M$$

Constructing: $E_1 = E_1, E_2 = E_2 - E_1 \Rightarrow E_2 \cap E_1 = \emptyset$
 $E_3 = E_3 - (E_1 \cup E_2) \Rightarrow E_3 \cap E_2 = \emptyset$

$$\vdots$$

$$E_n = E_n - (E_1 \cup E_2 \cup \dots \cup E_{n-1}) = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right)$$

$\Rightarrow \{E_i\}_{i=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets

$$\Rightarrow \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_i \rightarrow \text{①}$$

Let $H_n = \bigcup_{i=1}^n E_i$ is a measurable as well being finite union of measurable sets.

$$\Rightarrow H_n = \bigcup_{i=1}^n E_i \subseteq \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_i = E$$

$$\Rightarrow H'_n \supseteq E' \Rightarrow \forall A \subseteq R, A \cap H'_n \supseteq A \cap E'$$

$$\Rightarrow m^*(A \cap H'_n) \geq m^*(A \cap E') \rightarrow \text{②}$$

$\therefore H'_n$ is measurable and taking

$\neq \emptyset A \subseteq R$ as test set:

$$\Rightarrow m^*(A) = m^*(A \cap H'_n) + m^*(A \cap H'_n)^c \text{ by defn}$$

$A \subseteq B$
 $A' \supseteq B'$

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$$\Rightarrow m^*(A) \geq m^*(A \cap H_n) + m^*(A \cap E')$$

$$m^*(A) \geq m^*[A \cap (\bigcup_{j=1}^n F_j)] + m^*(A \cap E')$$

$$m^*(A) \geq \sum_{j=1}^n m^*(A \cap F_j) + m^*(A \cap E')$$

$\therefore$ LHS is independent of 'n', so for $n \rightarrow \infty$

$$\therefore m^*(A) \geq \sum_{j=1}^{\infty} m^*(A \cap F_j) + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (\bigcup_{j=1}^{\infty} F_j)] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (\bigcup_{j=1}^{\infty} E_j)] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$$

$\Rightarrow$ "E" is also measurable

$\Rightarrow E \in M$

$\Rightarrow$ "M" is a σ -algebra.

Topic # 44

IMP

Theorem: If E is measurable set and

$m^*(E \Delta G) = 0$ then G is measurable set.

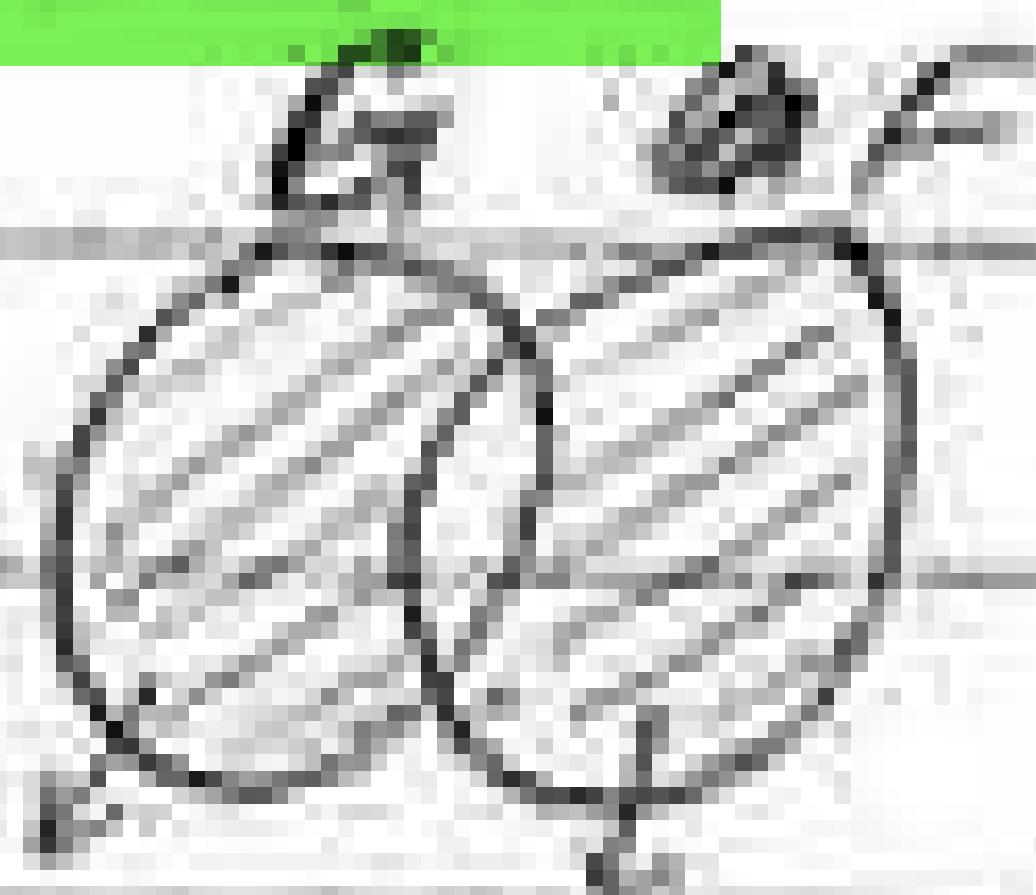
Proof: $G - F \subseteq (E \Delta G)$

$$\Rightarrow m^*(G - F) \leq m^*(E \Delta G) = 0$$

$$\Rightarrow m^*(G - F) = 0$$

$\therefore m^*$ is non-negative

$$E \Delta G = (G - F) \cup (F - G)$$

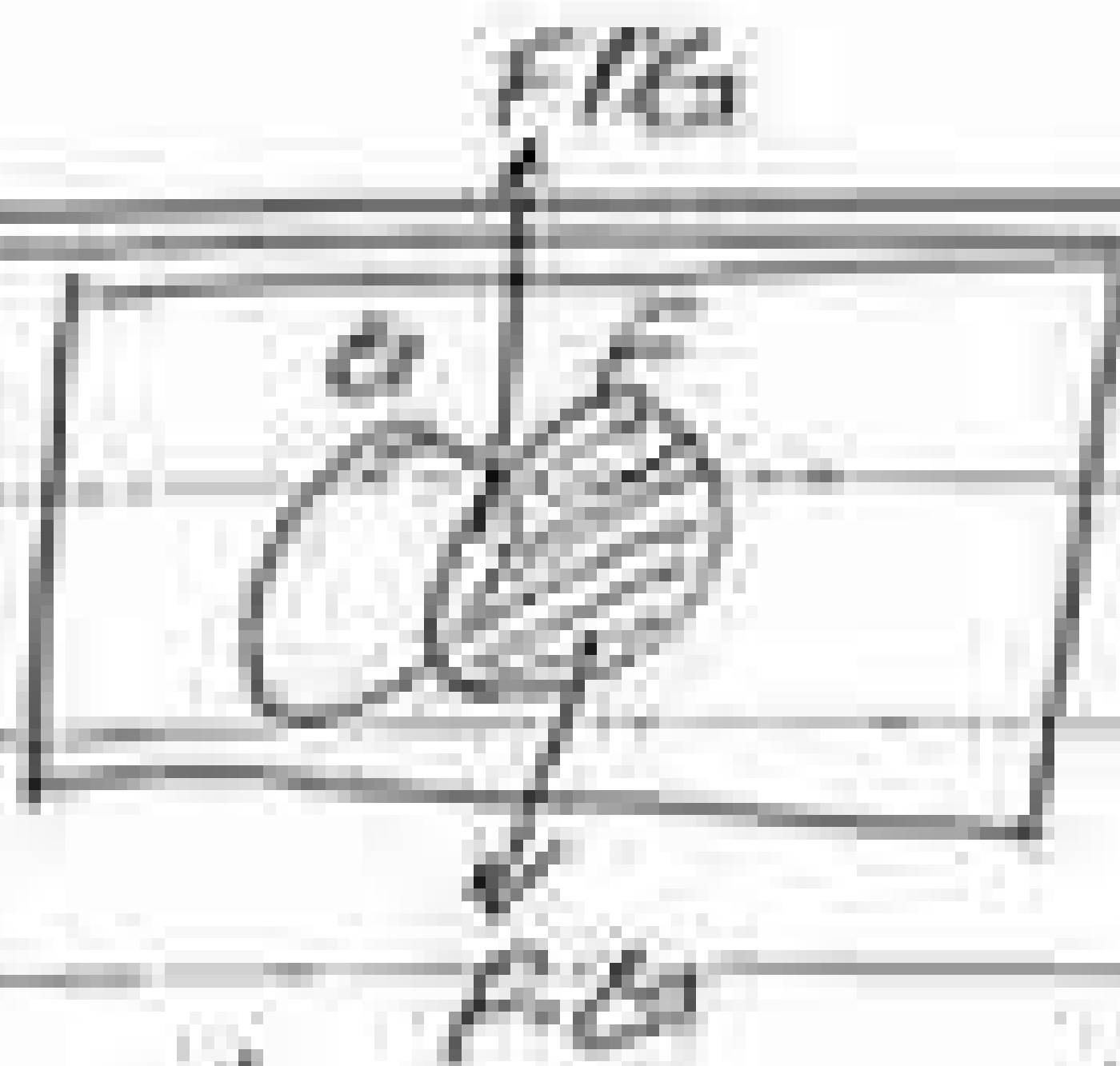


$\Rightarrow (G - F)$ is measurable.

Similarly, $(F - G)$ is also measurable.

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$\therefore F \cap G = F \cap (F \cup G)'$
is measurable being
intersection of two measurable sets.



$\therefore G = (F \cap G) \cup (G - F)$ is also measurable

Topic # 45

G_δ -Delta and F_σ -Sigma Sets:

IMP

G_δ -Set

MCQ'S

F_σ -set

A $\phi \neq A \subseteq R$ is said to be
 G_δ -set if it can be
written as **countable intersection**
of open sets.

For Example:

$$\bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}) = (a, b)$$

A $\phi \neq A \subseteq R$ is said to be
 F_σ -set if it can be
written as **countable union**
of closed sets.

For Example:

$$[a, b] = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n} \right]$$

(a, b) is trivially G_δ -set.

$[a, b]$ is also F_σ -set.

Notes: G_δ -set is measurable
being the **countable intersection**
of measurable sets.

Note: F_σ -set is measurable
being the **countable union**
of measurable sets.

Note: $G_\delta' = F_\sigma$ and $F_\sigma' = G_\delta$

FRANKESSON

Topic #46

Theorem: Given any $A \subseteq \mathbb{R}$ and $\epsilon > 0$, there is a G_δ set G with $A \subseteq G$ and $m^*(A) = m^*(G)$.

Proof: $\because \forall A \subseteq \mathbb{R}$ and for any $\epsilon > 0$, there exist an open set U such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \epsilon$

In particular for $\epsilon = \frac{1}{n}$ let U_n be open set such that $A \subseteq U_n$

$$\Rightarrow m^*(A) \leq m^*(U_n) \rightarrow (1)$$

$$\text{and } m^*(U_n) \leq m^*(A) + \frac{1}{n} \rightarrow (2)$$

$$\text{Since } A \subseteq U_n \Rightarrow A \subseteq \bigcap_{n=1}^{\infty} U_n = G \text{ (say)} \rightarrow (3)$$

i.e. G -set being intersection of measurable sets (open sets)

$$\Rightarrow m^*(G) \leq m^*\left(\bigcap_{n=1}^{\infty} U_n\right) \leq m^*(U_n)$$

$$\Rightarrow m^*(G) \leq m^*(A) + \frac{1}{n} \text{ (From (2))}$$

$$\text{As for } n \rightarrow \infty \Rightarrow \frac{1}{n} \rightarrow 0$$

$$\therefore m^*(G) \leq m^*(A) \rightarrow (4)$$

$$\text{From (1)} \Rightarrow A \subseteq G \Rightarrow m^*(A) \leq m^*(G) \rightarrow (5)$$

$$\text{a. From (4) and (5)} \Rightarrow m^*(A) = m^*(G)$$

$$A \subseteq B$$

$$m^*(A) \leq m^*(B)$$

$$A \subseteq U$$

$$\Rightarrow m^*(A) \leq m^*(U)$$

$$m^*(A) + \epsilon \geq m^*(U)$$

$$m^*(U) \leq m^*(A) + \epsilon$$

$$\text{where } \epsilon > 0$$

$$A \cap B \subseteq A$$

$$m^*(A \cap B) \leq m^*(A)$$

Topic # 47

Theorem: Interval (a, ∞) is measurable $\forall a \in \mathbb{R}$.

Proof:

Taking test set as $\mathbb{R}$ $\left| \begin{array}{l} m^*(A) = m^*(A \cap (-\infty, a]) + m^*(A \cap (a, \infty)) \\ \text{and we only need } m^*(a, \infty) = \mu(a, \infty) = \infty \end{array} \right.$
to prove:

$$m^*(A) \geq m^*[A \cap (a, \infty)] + m^*[A \cap (-\infty, a)] \quad \text{--- (1)}$$

let $A_n = A \cap (a, a)$ and $A_p = A \cap (-\infty, a)$

$$\text{in (1)} \Rightarrow m^*(A) \geq m^*(A_n) + m^*(A_p) \quad \text{--- (2)}$$

case 1: $m^*(A) = \infty$, then (1) is obviously

case 2: if $m^*(A) < \infty$ and for $a \in \mathbb{R}$ I a seq. $\{I_n\}_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} I_n$ --- (3)

$\forall \epsilon > 0$. Then there exists another such sequence $\{I_n\}$ such that

$$\text{or } m^*(A) < m^*(\bigcup_{n=1}^{\infty} I_n) + \epsilon$$

$$A \subset \bigcup_{n=1}^{\infty} I_n \text{ and } m^*(A + \epsilon) > m^*(\bigcup_{n=1}^{\infty} I_n) + \epsilon \quad \text{--- (4)}$$

$$\therefore \text{ on } \mathbb{R} : m^*(A) = \mu(A)$$

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \stackrel{\text{subadditivity}}{=} \sum_{n=1}^{\infty} J(I_n) \rightarrow \textcircled{2}$$

$$\textcircled{2} \Rightarrow \sum_{n=1}^{\infty} J(I_n) \leq m^*(A) + \epsilon \rightarrow \textcircled{3}$$

$$\text{Let } M_n = I_n \cap (a, \infty) \text{ and } N_n = I_n \cap (-\infty, a]$$

$$M_n \cap N_n = I_n \cap (a, \infty) \cap \{I_n \cap (-\infty, a]\}$$

$$= I_n \cap \{(a, \infty) \cap (-\infty, a]\}$$

$$M_n \cap N_n = I_n \cap \emptyset = \emptyset$$

$$\text{Now } M_n \cup N_n = \{I_n \cap (a, \infty)\} \cup \{I_n \cap (-\infty, a]\}$$

$$= I_n \cap \{(a, \infty) \cup (-\infty, a]\}$$

$$= I_n \cap \mathbb{R}$$

$$M_n \cup N_n = I_n$$

$$\Rightarrow J(I_n) = J(M_n) + J(N_n) \rightarrow \textcircled{4}$$

$$\because A_n = A \cap (a, \infty) \text{ and } A_n \subseteq \bigcup_{n=1}^{\infty} I_n \quad \left| \begin{array}{l} \because M_n \cap N_n = \emptyset \\ \forall n \in \mathbb{N} \end{array} \right.$$

$$\Rightarrow A_n \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (a, \infty)$$

$$\Rightarrow A_n \subseteq (I_1 \cup I_2 \cup \dots) \cap (a, \infty)$$

$$\Rightarrow A_n \subseteq \{I_1 \cap (a, \infty)\} \cup \{I_2 \cap (a, \infty)\} \cup \dots$$

$$\Rightarrow A_n \subseteq \bigcup_{n=1}^{\infty} \{I_n \cap (a, \infty)\}$$

$$\Rightarrow A_n \subseteq \bigcup_{n=1}^{\infty} M_n$$

$$\Rightarrow m(A_n) \leq m^*\left(\bigcup_{n=1}^{\infty} M_n\right) \rightarrow \textcircled{5}$$

$$\text{Similarly } A_p \subseteq \bigcup_{n=1}^{\infty} N_n$$

$$\Rightarrow m(A_p) \leq m^*\left(\bigcup_{n=1}^{\infty} N_n\right) \rightarrow \textcircled{6}$$

Adding $\textcircled{5}$ and $\textcircled{6}$

$$\begin{aligned} m(A_n) + m(A_p) &\leq m^*\left(\bigcup_{n=1}^{\infty} M_n\right) + m^*\left(\bigcup_{n=1}^{\infty} N_n\right) \\ &\leq \sum_{n=1}^{\infty} m^*(M_n) + \sum_{n=1}^{\infty} m^*(N_n) \end{aligned} \quad \textcircled{7}$$

$$m^*(A_n) + m^*(A_p) \leq \sum_{n=1}^{\infty} l(M_n) + \sum_{n=1}^{\infty} l(N_n) \\ \leq \sum_{n=1}^{\infty} (l(M_n) + l(N_n))$$

$$m^*(A) + m^*(A_p) \leq \sum_{n=1}^{\infty} l(I_n) \quad \text{From ①}$$

$$\Rightarrow m^*(A_n) + m^*(A_p) \leq m^*(A) + \epsilon \quad \text{From ②}$$

$\therefore \epsilon \rightarrow 0$ arbitrary,

$$\therefore m^*(A_n) + m^*(A_p) \leq m^*(A)$$

$$\text{or } m^*(A) \geq m^*(A_n) + m^*(A_p)$$

Hence proved: $(0, \infty)$ is measurable.

IMP

Topic # 48

Theorem: Open interval $(a, b) \subset \mathbb{R}$ is measurable.

proof: $\because (a, \infty)$ is measurable

$\Rightarrow (a, \infty)^c = (-\infty, a]$ is also measurable.

And $(-\infty, b] = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$ is also measurable $\forall n \in \mathbb{N}$ as $(-\infty, b - \frac{1}{n}]$ is measurable.

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$ is also measurable because being countable union of measurable set.

$\Rightarrow (a, \infty) \cap (-\infty, b) = (a, b)$ is also measurable.

being intersection of two measurable set.

Open set in $\mathbb{R}$:

Any open set in $\mathbb{R}$ say U can be written as countable union of disjoint open intervals.

ie $U = \bigcup_{n=1}^{\infty} I_n$ and each open interval is

measurable and countable union of measurable set is measurable.

Topic # 49

Theorem: A subset $\phi \neq E \subseteq \mathbb{R}$ is measurable $\Leftrightarrow$ for given $\epsilon > 0$, $\exists$ an open set $U \supseteq E$ with $m^*(U - E) < \epsilon$.

Proof: Suppose E is measurable MCQ

Case 1: If $m^*(E) < \infty$, $\forall \phi \neq E \subseteq \mathbb{R}$.

then exists an open set U such that $E \subseteq U$ and for any $\epsilon > 0$, $m^*(U) < m^*(E) + \epsilon \rightarrow$ (A)

$$E \subseteq U \Rightarrow U = E \cup (U - E)$$

$$\Rightarrow m^*(U) = m^*(E) + m^*(U - E)$$

$$\Rightarrow m^*(U - E) = m^*(U) - m^*(E) < \epsilon \text{ for } \epsilon > 0$$



Case 2: If $m^*(E) = \infty$

$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where I_n is a sequence of disjoint intervals $I_j \cap I_k = \phi$, $j \neq k$

put $E_n = E \cap I_n \Rightarrow E_n \subseteq I_n \forall n \in \mathbb{N}$

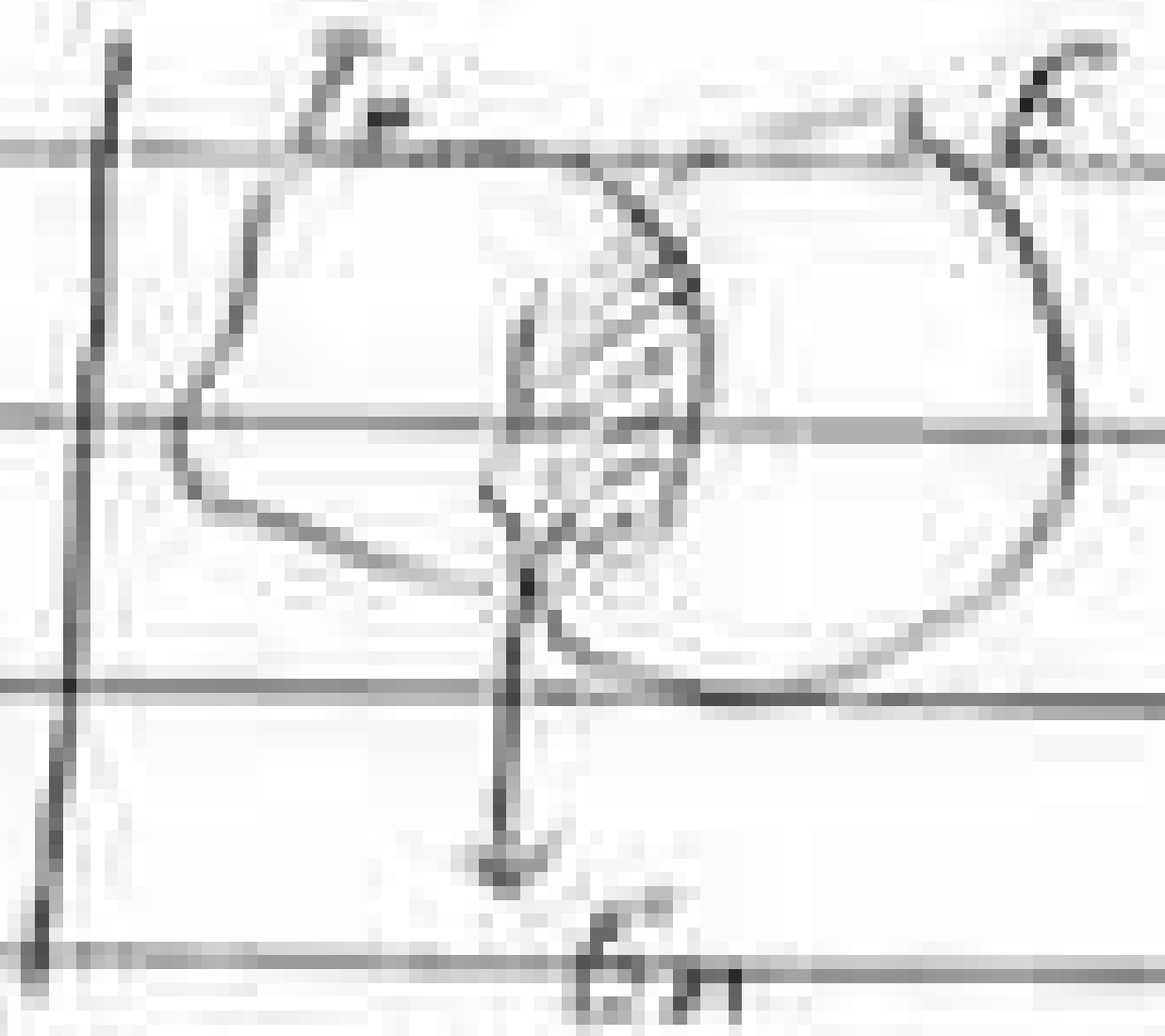
$$\Rightarrow m^*(E_n) \leq m^*(I_n) = l(I_n) < \infty$$

By case 1 $\forall E_n \exists$ an open set say

" V_n " such that $E_n \subseteq V_n$, $\mu \in \mathbb{R}$

$$\text{and } m^*(V_n - E_n) < \frac{\epsilon}{2^n} \rightarrow (B)$$

Also $V = \bigcup_{n=1}^{\infty} V_n$ is an open set.



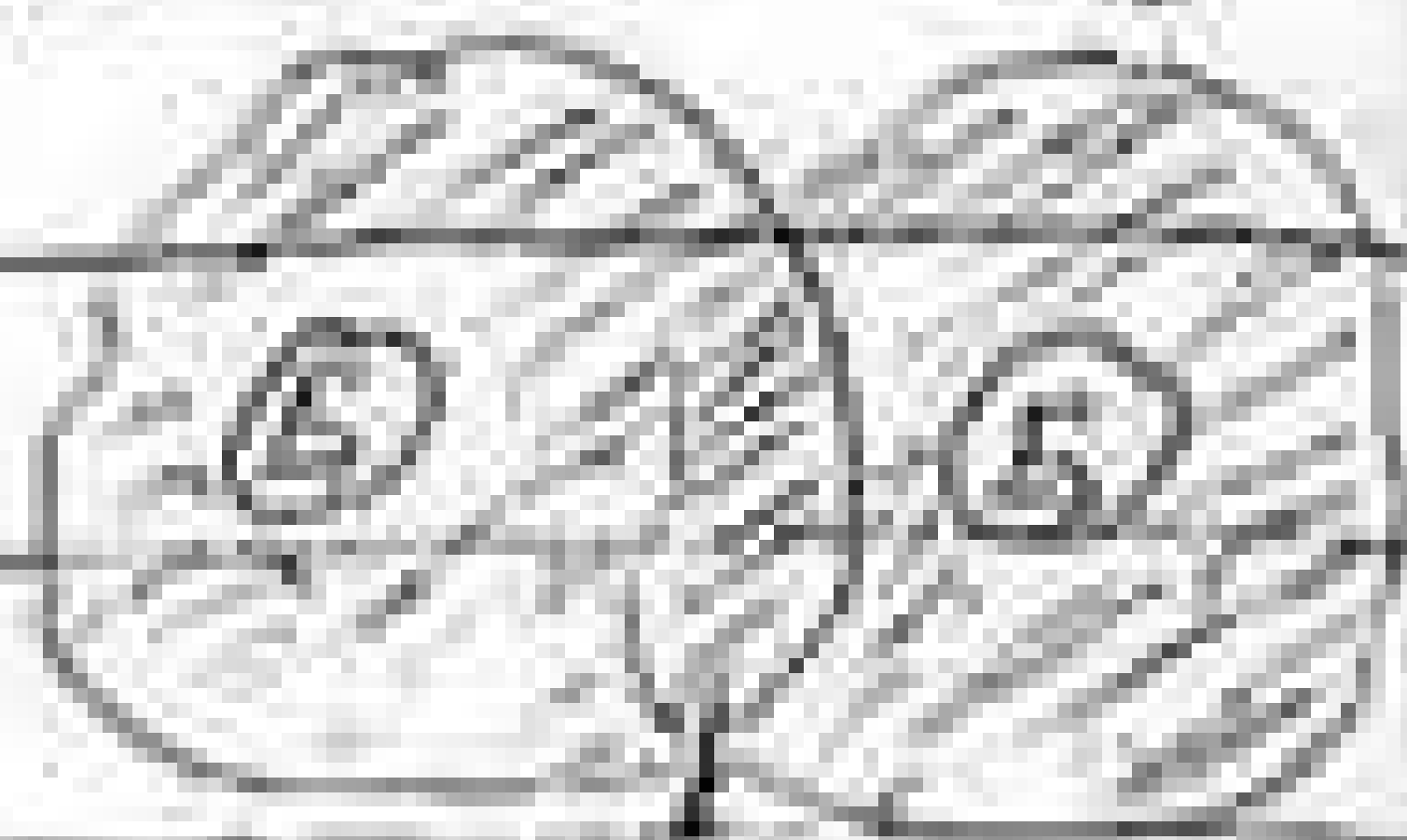
$$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n) = E \cap \left(\bigcup_{n=1}^{\infty} I_n \right) = E \cap \mathbb{R} = E$$

FRANKESSON

$$E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} V_n = V \quad \because E_n \subseteq V_n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow V - E = \bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (V_n - E_n)$$

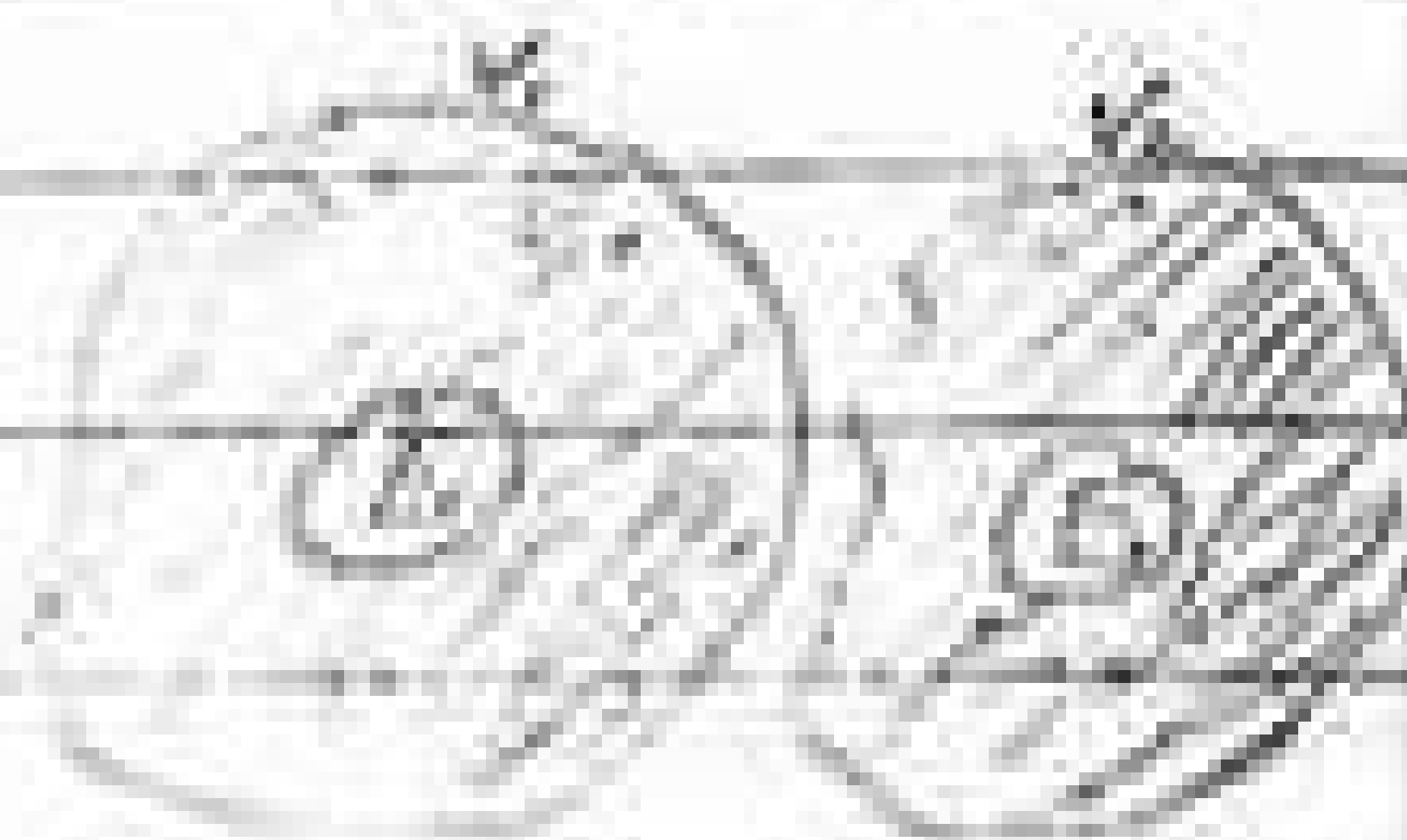
$$\Rightarrow m^*(V - E) \leq \sum_{n=1}^{\infty} m^*(V_n - E_n)$$



$$\text{From (1)} \Rightarrow m^*(V - E) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^n}$$

$$(V, \mathcal{O}_V) = (E, \mathcal{O}_E)$$

$$m^*(V - E) \leq \epsilon \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \right) = \epsilon(1)$$



$$m^*(V - E) < \epsilon$$

$$V - E \subseteq \bigcup_{n=1}^{\infty} (V_n - E_n)$$

Conversely, let $\epsilon > 0$ be any real

R.I. on $\mathcal{O}_V$ and $\mathcal{O}_E$ such that $E \in \mathcal{O}_V$ and $m^*(V - E) < \epsilon$

$$\text{and } m^*(V - E) < \epsilon \Rightarrow \text{②}$$

We will prove E' is measurable

put $G = \bigcup_{n=1}^{\infty} G_n \Rightarrow G$ is measurable being countable intersection of measurable open set.

$$E \subseteq G_n \quad \forall n \text{ (given)}$$

$$\text{and } E \subseteq \bigcup_{n=1}^{\infty} G_n = G$$

$$G - E \subseteq G_n - E$$

$$\Rightarrow m^*(G - E) \leq m^*(G_n - E) < \frac{\epsilon}{2^n} = \epsilon$$

$$\because \epsilon \text{ is arbitrary } m^*(G - E) = 0$$

$$\Rightarrow G - E \text{ is measurable } \Rightarrow (G - E)' \text{ is also measurable}$$

$\therefore E = (A - (A - E)) = (A \cap (A - E)') \cap E$ is measurable
being intersection of two measurable set.

Topic #50

Theorem: For a subset E' of R and given $\epsilon > 0$
 $\exists$ an open set $O \supset E$ and $m^*(O - E) < \epsilon \iff \exists$
a closed set $F \subset E$ and $m^*(E - F) < \epsilon$.

proof: For $\phi \neq E \subset R \Rightarrow E' \subset R$ $\exists$ for given $\epsilon > 0$
 $\exists$ an open set $O \supset E$ and
 $m^*(O - E) < \epsilon \rightarrow \textcircled{1}$



$O \supset E' \Rightarrow E' \supset O' = F$ (say)

is closed set

or

$\Rightarrow E \supset F \Rightarrow m^*(E - F) = m^*(E \cap F')$

$m^*(E - F) = m^*(E' \cap E)$

$m^*(E - F) = m^*(O - E') < \epsilon$ from $\textcircled{1}$

$\therefore m^*(E - F) < \epsilon$

Conversely For the set $E' \subset R$, $\exists$ a closed set F'
and $\epsilon > 0$ such that $F \subset E'$ and $m^*(E' - F) < \epsilon \rightarrow \textcircled{2}$ (since

$F \subset E' \Rightarrow E = E' \cap E = F \cap E = O$ (open set) say

$\Rightarrow m^*(O - E) = m^*(O \cap E') = m^*(F' \cap O)$

$\Rightarrow m^*(O - E) = m^*(E' - F) = m^*(E' - F) < \epsilon$ from $\textcircled{2}$

$m^*(O - E) < \epsilon$ (Hence proved)

Topic # 51

Theorem: let $p, E \subseteq R$, then the following statements are equivalent

- (i) There is G_0 set G with $E \subseteq G$, $m^*(G-E) = 0$ (to give)
- (ii) There is a E_0 set F with $E \subseteq F$, $m^*(F-E) = 0$ (to prove)

proof: (i) $\Rightarrow$ (ii) $\because p \subseteq E \subseteq R \Rightarrow p \subseteq E' \subseteq R$

MCQ

$\therefore$ For $p \subseteq E'$ give $E' \subseteq G$, $m^*(G-E') = 0$

$\therefore 0 = m^*(G-E') = m^*(G \cap E') = m^*(G \cap E)$

$= m^*(E \cap G) = m^*(E \cap G') = m^*(E-G')$

$= m^*(E-F)$, where $G_0 = G = \bigcup_{n=1}^{\infty} G_n$

$\Rightarrow G_0 = G = \left(\bigcup_{n=1}^{\infty} G_n \right)'$

$\Rightarrow G_0' = \bigcap_{n=1}^{\infty} G_n'$

$\Rightarrow G_0' = \bigcap_{n=1}^{\infty} F_n' = F_0'$

(ii) $\Rightarrow$ (i)

(analogously, again for $E' \subseteq R$ with $E \subseteq E'$

$m^*(E'-E) = 0$ (given))

$\therefore 0 = m^*(E'-E) = m^*(E' \cap E') = m^*(E' \cap E)$

$= m^*(E-E) = m^*(G-E)$

where $G_0 = E' = \left(\bigcap_{n=1}^{\infty} F_n \right)' = \bigcup_{n=1}^{\infty} F_n' = \bigcup_{n=1}^{\infty} G_n = G = G_0$

(proved)

Topic # 52

Lebesgue Measure: IMP

let $\mathcal{M}$ be the class of measurable sets, which is σ -algebra. Then the Lebesgue measure of any set $E \in \mathcal{M}$ is defined to be its Lebesgue outer measure

$$m^* \equiv m; \{A, m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)\}$$

Topic #53

Theorem: let $\{E_i\}_{i=1}^{\infty}$ be a sequence $\mathcal{A}$ in measurable space M . then m is

- (i) countably subadditive MCQ'S
- (ii) finitely additive provided that $E_i \cap E_j = \emptyset, \forall i \neq j$
- (iii) countably additive if E_i are pairwise disjoint
- (iv) m is measure

proof: (i) All E_i are measurable.

$\Rightarrow \bigcup_{i=1}^{\infty} E_i$ is also measurable. $\therefore M$ is space of Lebesgue measurable sets MCQ'S

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) = \sum_{i=1}^{\infty} m(E_i)$$

(ii) $\forall A \in \mathcal{A}$ and for the sequence $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint measurable sets

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

$$\text{put } A = R \Rightarrow A \cap \left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} E_i \quad A \cap E_i = R \cap E_i$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

(iii) $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

finely additive

if Monotonicity, let $p \in A \subseteq M$, such that $B \subseteq A$. **IMP**

$$\Rightarrow A = B \cup (A - B)$$

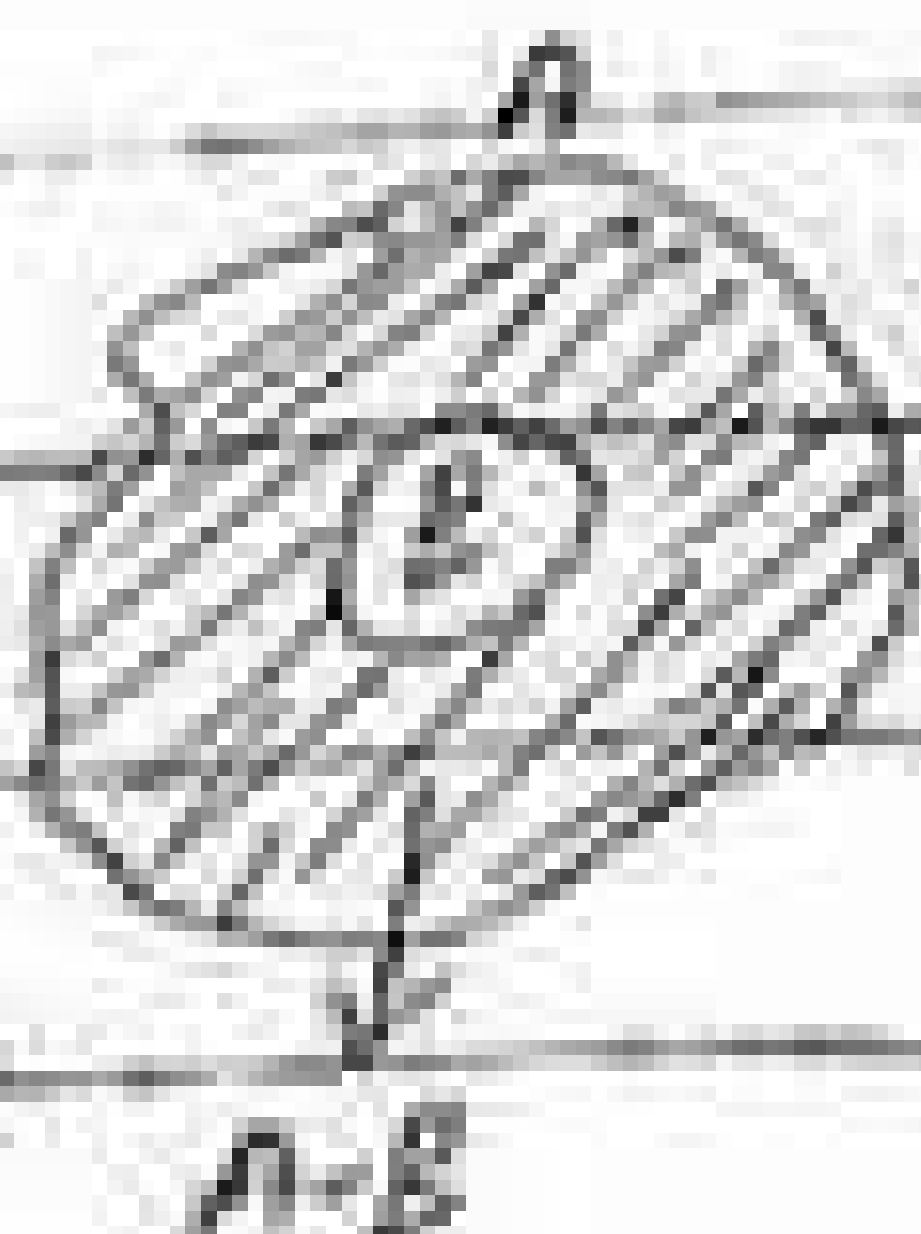
$\therefore m^*$ is finitely additive

$$\Rightarrow m^*(A) = m^*(B) + m^*(A - B)$$

$$\Rightarrow m^*(A) \geq m^*(B) \quad \because m^*(A - B) \geq 0$$

$$\Rightarrow m^*(B) \leq m^*(A)$$

MCQ



$$\because B \cup (A - B) = A$$

Topic # 54

Theorem: Lebesgue measure m is translation invariant
 proof: Given that say $p \in E \in M$ is measurable

$\therefore E$ is measurable for for given $\epsilon > 0$, $\exists$ an open set $U \supseteq E$, such that $m^*(U - E) < \epsilon$



"U" is open set in $\mathbb{R} \Rightarrow "u+y"$ is also open in $\mathbb{R}$, $\forall y \in \mathbb{R}$

$$E \subseteq U \Rightarrow E + y \subseteq U + y$$

$$\Rightarrow (U + y) - (E + y) = (U - E) + y$$

$$\Rightarrow m^*[(U + y) - (E + y)] = m^*[(U - E) + y]$$



$\therefore$ Lebesgue outer measure is translation invariant

$$\Rightarrow m^*(U - E) < \epsilon$$

$$\text{but } m^* \circ m \Rightarrow m^*[(U + y) - (E + y)] = m^*(U - E) < \epsilon$$

$\Rightarrow "E+y"$ is also measurable $\therefore m$ is translation invariant

Topic # 55

Theorem: Suppose E_1 and E_2 are two measurable sets, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

proof: If $m(E_1) = m(E_2) = \infty$, then the result is obvious.

Case 2: If $m(E_1) < \infty$, $m(E_2) < \infty$

$\therefore$ if E_1 is measurable, then for any $A \subseteq R$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c) \quad \text{--- (1)}$$

In particular if $A \subseteq E_2$

$$\text{--- (2)} \quad m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1^c) \quad \text{--- (2)}$$

Similarly $E_2 \subseteq E_1$

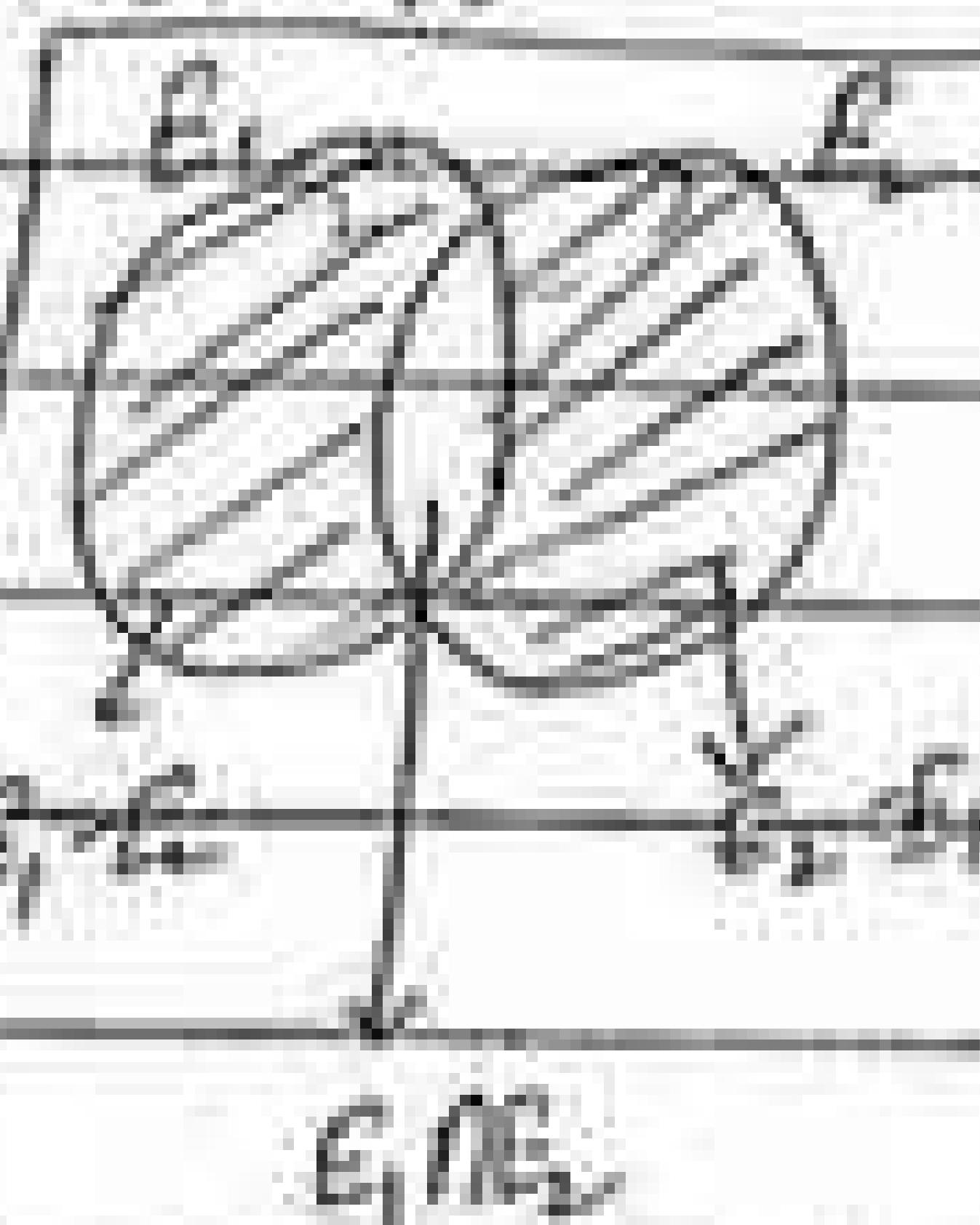
$$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2^c) \quad \text{--- (3)}$$

Adding (2) and (3)

($m^* = m$)

$$m(E_1) + m(E_2) = m(E_2 \cap E_1) + m(E_2 \cap E_1^c) + m^*(E_1 \cap E_2) + m(E_1 \cap E_2^c) \quad \text{--- (4)}$$

$$E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2^c) \\ = (E_1 \cap E_2) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2^c)$$



$$\Rightarrow m(E_1 \cup E_2) = m[(E_1 \cap E_2^c) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2)]$$

" ∞ " is finitely additive

$$m(E_1 \cup E_2) = m(E_1 \cap E_2^c) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2)$$

$$\text{--- (5)} \quad m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$$

From question statement we require intersection not minus so we change minus with intersection and complement of second element

proved

What distinguishes the cantor from its positive length:
Its measure is being zero(Correct option)

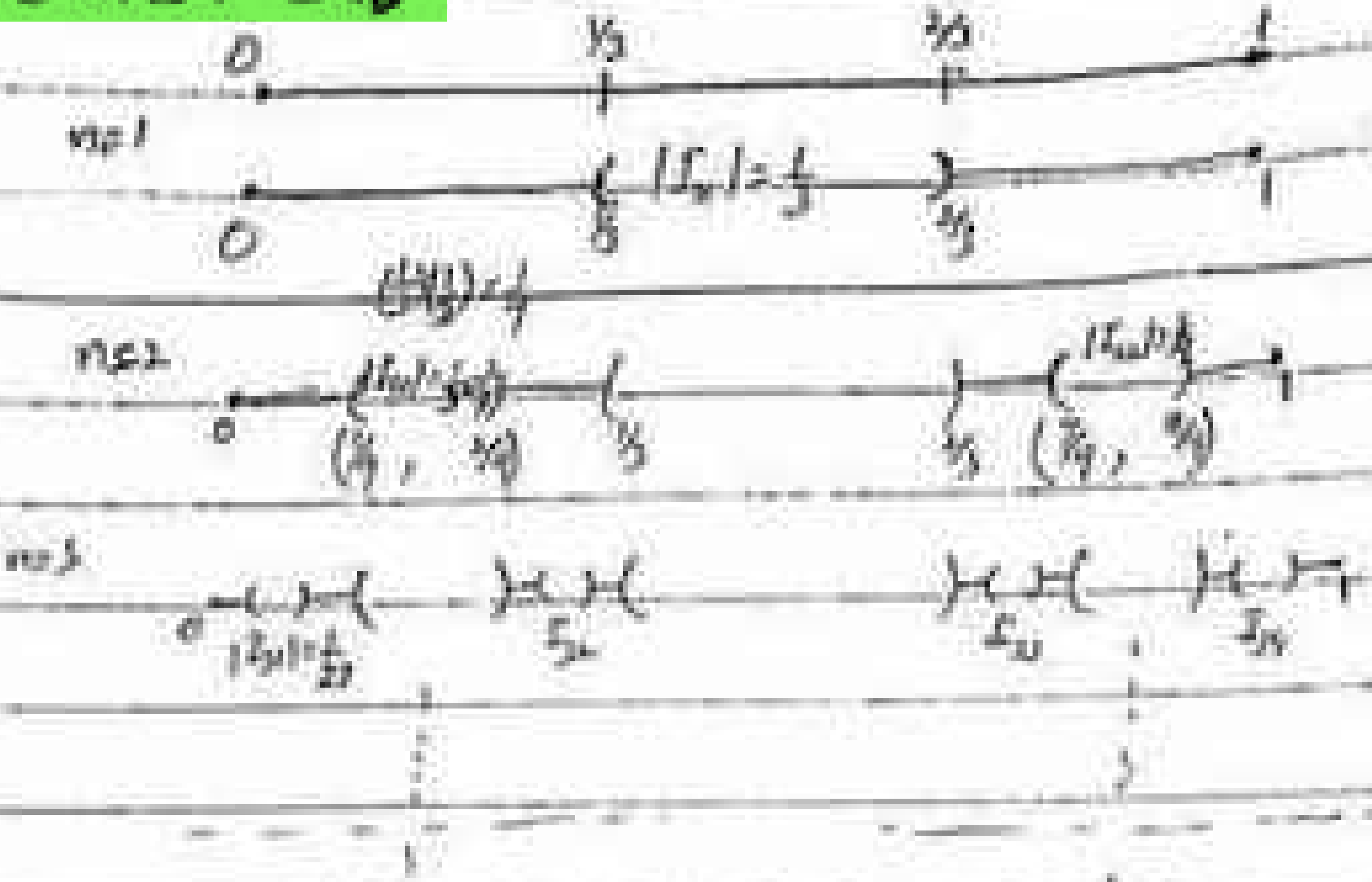
MCQ

JIRA NEEHSON

Topic # 56

Cantor Set:

R, R^1, R^2



Cantor set

# Step	# Dropped Interval	Length	Interval
no 1	1 interval	$ I_1 = \frac{1}{3}$	$(\frac{1}{3}, \frac{2}{3})$
no 2	2 interval	$ I_2 = I_3 = \frac{1}{3}$	$(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})$
no 3	4 interval	$ I_4 = \frac{1}{9}$	
...	...	...	...
n	$n = 2^n$	$ I_n = \frac{1}{3^n}$	

MCQ'S

Dropped Interval - $G = \bigcup_{n=1}^{\infty} I_n = \bigcup_{n=1}^{\infty} (a_n, b_n)$, where $a_1 = \frac{1}{3}, b_1 = \frac{2}{3}, a_2 = \frac{1}{9}, b_2 = \frac{2}{9}, a_3 = \frac{7}{9}, b_3 = \frac{8}{9}, \dots$

$i.e. G = \{ (a_n, b_n) \}_{n=1}^{\infty}$

$$\Rightarrow G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} I_{nm} \quad \text{open set}$$

$$\therefore \text{Cantor Set} = C = [0,1] - G = G'$$

Topic #57

Measure of Cantor Set:

"G" is open - being countable union of disjoint open intervals $\Rightarrow$ "G" is measurable. MCQ

"C" = "G" is closed and also measurable.

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} I_{nm}\right) \quad \because m \text{ is countably additive}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(I_{nm}) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} l(I_{nm})$$

depending pretty by "n"

$$= \sum_{n=1}^{\infty} l(I_{n1}) + \sum_{n=1}^{\infty} l(I_{n2}) + \sum_{n=1}^{\infty} l(I_{n3}) + \dots$$

$$= l(I_{11}) + l(I_{12}) + l(I_{13}) + l(I_{21}) + l(I_{22}) + l(I_{23}) + l(I_{31}) + l(I_{32}) + l(I_{33}) + \dots$$

$$= \frac{1}{3} \left(\frac{1}{3} + \frac{1}{3} \right) + \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) + \dots$$

$$m(G) = \frac{1}{3} + \frac{2}{3} + \frac{4}{3} + \dots \quad \left\{ a = \frac{1}{3}, r = \frac{2}{3} \right\}$$

MCQ

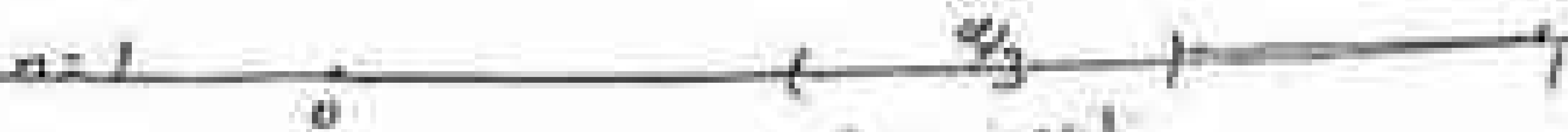
$$S_{\infty} = \frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

$$m(G) = 1$$

$$\therefore [0,1] - G = C \Rightarrow m(C) = m([0,1]) - m(G) = 1 - 1 = 0$$

Topic #5B

Generalized Cantor Set and its Measure:



$$m(A) = \frac{2}{3} + (\frac{2}{3} + \frac{2}{3}) + (\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}) + \dots$$

$$m(A) = \frac{2}{3} + \frac{2 \cdot 2}{3} + \frac{4 \cdot 2}{3} + \dots$$

$$\begin{cases} a = \frac{2}{3} & r = \frac{2}{3} \end{cases}$$

MCQ'S

$$S_{\infty} = \frac{a}{1-r} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{3}} = 2$$

$$m(A) = \infty$$

$$[0, 1] = C \cup C^c$$

MCQ'S

$$\Rightarrow m([0, 1]) = m(C) + m(C^c)$$

$$1 = \infty + m(C)$$

$$\Rightarrow m(C) = 1 - \infty$$

Cantor set constructed from 0,1 is uncountable.
it can be divided into 3 parts
it length is 1/3

digit

if $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$
 $E_1 = E_2 = E_3 = \dots$ then E_n is JIRA NEEHSON

Topic # 59

Theorem: let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then

MCQ

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m^*(E_n)$$

proof: Given that $\{E_n\}_{n=1}^{\infty}$, such that
 $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$ of measurable sets
 Defining: $F_1 = E_1, F_2 = E_2 - E_1, F_3 = E_3 - E_2, \dots$
 $F_n = E_n - E_{n-1}$, where $\{F_n\}_{n=1}^{\infty}$ is a seq
 of disjoint measurable sets

$$\Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$$

$$\therefore m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) \quad \text{disjoint measurable.}$$

$$= \sum_{k=1}^{\infty} m(F_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m(F_k)$$

$$= \lim_{n \rightarrow \infty} m\left(\bigcup_{k=1}^n F_k\right) = \lim_{n \rightarrow \infty} m\left(\bigcup_{k=1}^n E_k\right)$$

$$= \lim_{n \rightarrow \infty} m^*(E_n) \quad (\text{proved})$$

Topic # 60 if $E_3 \subseteq E_2 \subseteq E_1$

Theorem: let $\{E_n\}_{n=1}^{\infty}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$ for some $k \in \mathbb{N}$. Then

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m^*(E_n)$$

proof: let $m(E_1) < \infty$, say (without loss of generality)

$$\therefore \{E_n\} \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

$$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \dots E_1 - E_n \subseteq$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} (E_1 - E_n)\right) = \lim_{n \rightarrow \infty} m^*(E_1 - E_n)$$

$$\bigcup_{n=1}^{\infty} (E_1 - E_n) = \bigcap_{n=1}^{\infty} (E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_1 \cap E_2') \cup \dots$$

$$= E_1 \cap [E_1' \cup E_2' \cup \dots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap \dots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$\stackrel{\text{sub set}}{\uparrow} = E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

$$m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m^*(E_1 - E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m^*(E_1) - \lim_{n \rightarrow \infty} m^*(E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m^*(E_1) - \lim_{n \rightarrow \infty} m^*(E_n)$$

$$\Rightarrow m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} m^*(E_n) \quad (\text{proved})$$

Topic # 61

Borel Set:

e.g. $G_\delta = \bigcap_{n=1}^{\infty} U_n$; $F_\sigma = \bigcup_{n=1}^{\infty} F_n$

Definition: A set 'E' is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g. $\{x\} = \bigcap_{n=1}^{\infty} (x - \frac{1}{n}, x + \frac{1}{n})$ Borel set $\rightarrow G_\delta$

MCQ

Topic # 62

Borel Measure Space:

MCQ'S

Restriction of Lebesgue measure "m" to the class of Borel set "B" is said to be Borel measure space (R, B, m)

IMP

Theorem: Every Borel set is measurable.

Proof: Every open or closed set is measurable $\Rightarrow$ countable union or intersection of closed or open set is also measurable.
 $\therefore$ Borel set is also measurable being union or intersection of closed or open sets.

Theorem: Countable set are Borel sets with zero measure.

proof: Case I: Singleton:

$$\forall a \in \mathbb{R}; \{a\} \in \mathcal{B}_1(a - \frac{1}{n}, a + \frac{1}{n}) \quad \left\{ \begin{array}{l} \text{Countable} \\ \downarrow \\ \text{finite} \rightarrow \text{infinite} \\ \mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{Q}, \dots \end{array} \right.$$

$\therefore \{ \{a - \frac{1}{n}, a + \frac{1}{n}\} \}$ is decreasing sequence

with $m(E_n) = \frac{2}{n}$

where $E_n = (a - \frac{1}{n}, a + \frac{1}{n})$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n \rightarrow \infty} m(E_n)$$

$$= \lim_{n \rightarrow \infty} \frac{2}{n}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) = \frac{2}{\infty}$$

$$= 0$$

Finite Set

MCQ'S

$$\{x_1, x_2, x_3, \dots, x_n\} = \bigcup_{i=1}^n \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^n \{x_i\}\right) = \sum_{i=1}^n m\{x_i\} = 0$$

Infinite set

IMP

$$\{x_1, x_2, x_3, \dots, x_n\} = \bigcup_{i=1}^{\infty} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\} = 0$$

Topic # 64

Sum Modulo 1:

Let $x, y \in A = [0, 1)$ and we define sum modulo '1' of x and y as

IMP

$$x \hat{+} y = \begin{cases} x+y & \text{if } x+y < 1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

MCQ'S

Translation Modulo 1:

Let $E \subset A = [0, 1)$ and $y \in A$, we define a translation modulo '1' of 'E' as

$$E \hat{+} y = \{x \hat{+} y, x \in E\}$$

Set of all "Sum Modulo 1"

Topic # 65

Theorem: Let $E \subset A = [0, 1)$ be a measurable set. Then for any each $y \in A$ the set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

proof: Suppose $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$

$\therefore E_1, [0, 1-y), [1-y, 1)$ are measurable

$\rightarrow E_1$ and E_2 are also measurable being the intersection of two measurable sets.

$$\begin{aligned} \text{Also } E \cap E_2 &= \{E \cap [1-y, 1)\} \cap \{E \cap [0, 1-y]\} \\ &= E \cap \{[0, 1-y) \cup [1-y, 1)\} \\ &= E \cap \phi = E \end{aligned}$$

$$\text{and } E_1 \cup E_2 = E \cup \phi = E$$

$$\Rightarrow m(E) = m(E_1) + m(E_2) \rightarrow \textcircled{1}$$

"m" is finitely additive

let $y \in A$, if $y \in E_1$

$$\Rightarrow y_1 = y \Rightarrow y_1 + y \in I \Rightarrow y, \hat{y} = y + y$$

$\downarrow$
 $E + y$

$$\Rightarrow E \hat{\cap} y = E_1 + y \rightarrow \textcircled{2}$$

$$\{ y_2 \in E_2 \Rightarrow y_2 \geq (y-1) \Rightarrow y + y_2 \geq 1$$

$$\Rightarrow y_2 \hat{\cap} y = y_2 + y_1$$

$$\Rightarrow E_2 \hat{\cap} y = E_2 + (y-1) \rightarrow \textcircled{3}$$

$\therefore$ "m" is translation invariant

$$\Rightarrow E_1 + y \text{ and } E_2 + (y-1) \text{ are measurable}$$

$$\Rightarrow E_1 \hat{\cap} y \text{ and } E_2 \hat{\cap} (y-1) \text{ are measurable}$$

$$\Rightarrow m(E_1 \hat{\cap} y) = m(E_1 + y) = m(E_1)$$

Similarly

$$m(E_2 \hat{\cap} y) = m(E_2 + (y-1)) = m(E_2)$$

$$\therefore E_1 \cup E_2 = E, \quad E_1 \cap E_2 = \phi$$

$$\Rightarrow E \hat{\cap} y = (E_1 \hat{\cap} y) \cup (E_2 \hat{\cap} y) \text{ and } (E_1 \hat{\cap} y) \cap (E_2 \hat{\cap} y) = \phi$$

" $E \hat{\cap} y$ " being union of two disjoint measurable sets its measurable

$$\Rightarrow m(E \hat{\cap} y) = m(E_1 \hat{\cap} y) + m(E_2 \hat{\cap} y)$$

$$= m(E_1 + y) + m(E_2 + y)$$

$$= m(E_1) + m(E_2) = m(E)$$

$\therefore$ "m" is also invariant w.r.t translation modulo 1.

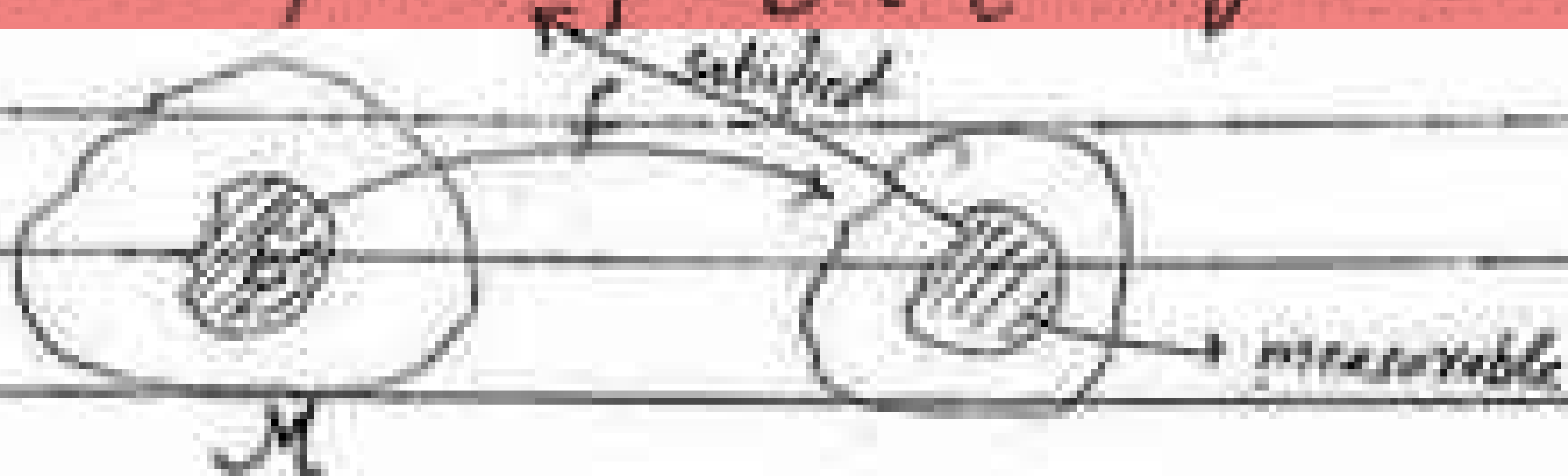
Topic # 66

Measurable Function:

Let M be the measurable space (σ -algebra of measurable sets). A function $f: E \rightarrow \mathbb{R}$ is Lebesgue measurable function on $E \in M$ if

$$\{x \in E: f(x) > a\} \in M \text{ for every } a \in \mathbb{R}$$

MCQ



For Example ①. Suppose f is real valued identically constant function $f(x) = p$. Then:

$$\{x: f(x) > a\} = \begin{cases} R & \text{if } p > a \\ \emptyset & \text{if } p \leq a \end{cases}$$

$$\therefore R, \emptyset \in M$$

$\Rightarrow f$ is measurable.

For Example ②.

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

MCQ

$$\{x: f(x) > a\} = \begin{cases} \emptyset & \text{if } a \geq 1 \\ A & \text{if } 0 < a < 1 \\ X & \text{if } a < 0 \end{cases}$$

$\emptyset, A, X \in M \Rightarrow f$ is measurable.

Example (3): $(X, \mathcal{B}, m) \rightarrow (\mathbb{R} \cup \{\infty\}, \mathcal{B})$

$$f(x) = x$$

$$\Rightarrow \{x: f(x) > a\} = \{(a, +\infty) \in \mathcal{B}\}$$

i.e. always measurable

$\Rightarrow f$ is measurable.

Topic # 67

Theorem: Let $f: E \rightarrow \mathbb{R} \cup \{\infty\}$, $\emptyset \neq E \in \mathcal{M}$. Then

- (i) " f is measurable" $\Leftrightarrow$ (ii) $\{x \in E: f(x) > a\}$ is measurable $\forall$ all $a \in \mathbb{R} \cup \{\infty\}$ $\Leftrightarrow$ (iii) $\{x \in E: f(x) < a\}$ is measurable $\forall$ $a \in \mathbb{R} \cup \{\infty\}$ $\Leftrightarrow$ (iv) $\{x \in E: f(x) \leq a\}$ is measurable $\forall$ $a \in \mathbb{R}$.

Proof: (i) $\Rightarrow$ (ii) " f is measurable" $\Rightarrow \{x \in E: f(x) > a\} \in \mathcal{M}$
 $\{x \in E: f(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in E: f(x) > a - \frac{1}{n}\}$

Let $y \in \{x \in E: f(x) > a\} \Rightarrow f(y) > a, \forall a \in \mathbb{R}$

$$\Rightarrow f(y) > a - \frac{1}{n}, \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \{x \in E: f(x) > a - \frac{1}{n}\}, \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) > a - \frac{1}{n}\}$$

$$\{x \in E: f(x) > a\} \subset \bigcap_{n=1}^{\infty} \{x \in E: f(x) > a - \frac{1}{n}\}$$

Let conversely: $t \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) > a - \frac{1}{n}\}$ $\Rightarrow$ (i)

$$\Rightarrow f(t) > a - \frac{1}{n}, \forall n \in \mathbb{N}$$

$$\Rightarrow f(t) \geq a, \forall a \in \mathbb{R} \text{ s.t.}$$

$$\Rightarrow t \in \{x \in E : f(x) \geq a\}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\} \subseteq \{x : f(x) \geq a\}$$

$f \in \mathcal{M}^+$

$$\Rightarrow \{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\} \quad \text{⑥}$$

$\therefore \{x \in E : f(x) > a - \frac{1}{n}\}$ is measurable

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ is also measurable
 $\forall n \in \mathbb{N}$

$\Rightarrow \{x \in E : f(x) \geq a\}$ is measurable

Topic #68

$$\text{iii} \Rightarrow \text{iv} \quad \{x \in E : f(x) \geq a\} \in \mathcal{M} \Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M}$$

$$\{x \in E : f(x) < a\} = \{x \in E : f(x) \geq a\}^c \quad \forall a \in \mathbb{R}$$

$$\therefore \{x \in E : f(x) \geq a\} \in \mathcal{M} \Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M}$$

$$\Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M} \text{ is measurable}$$

Topic #69

$$\text{iii} \Rightarrow \text{iv} \quad \underbrace{\{x \in E : f(x) < a\}}_{\in \mathcal{M}} \Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M}$$

We show that $\{x \in E : f(x) < a\}$

$$= \bigcup_{n=1}^{\infty} \{x \in E : f(x) < a - \frac{1}{n}\}$$

Let $y \in \{x \in E : f(x) \leq a\} \rightarrow \textcircled{a}$

$$\Rightarrow f(y) \leq a, \forall a \in \mathbb{R}$$

$$\Rightarrow f(y) < a + \frac{1}{n}, \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \{x \in E : f(x) < a + \frac{1}{n}\}, \forall n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \rightarrow \textcircled{b}$$

From $\textcircled{a}$ and $\textcircled{b} \Rightarrow \{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

Let $z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

$$\Rightarrow z \in \{x \in E : f(x) < a + \frac{1}{n}\} \Rightarrow f(z) < a + \frac{1}{n}, \forall n \in \mathbb{N}$$

$$\Rightarrow f(z) \leq a, \forall a \in \mathbb{R} \Rightarrow z \in \{x \in E : f(x) \leq a\}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \subseteq \{x \in E : f(x) \leq a\}$$

$$\therefore \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} = \{x \in E : f(x) \leq a\}$$

Given that $\{x \in E : f(x) < a + \frac{1}{n}\} \in \mathcal{M}$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \in \mathcal{M}$$

$$\Rightarrow \{x \in E : f(x) \leq a\} \in \mathcal{M} \text{ is measurable}$$

Please NOTE that topics from 70-80 are so very much important for subjective part :)

Topic # 70

(W) $\Rightarrow$ (i) $\{x \in E : f(x) \leq a\} \in \mathcal{M}$ if f is measurable.

$f: E \rightarrow \mathbb{R} \cup \{\infty\}$, $\{x \in E : f(x) \leq a\} \in \mathcal{M}$

$\{x \in E : f(x) \leq a\} \in \mathcal{M}$

$\forall a \in \mathbb{R}$

$$= \{x \in E: f(x) \leq a\} \in \mathcal{M} \quad \{x \in E: f(x) > a\} \in \mathcal{M}$$

$\Rightarrow f$ is measurable

Topic # 71

Theorem: Let $f \in \mathcal{M}$, $f: E \rightarrow \mathbb{R} \cup \{\pm\infty\}$,
Then $\{x \in E: f(x) = a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

Proof: Case I: $a \in \mathbb{R}$

$$\Rightarrow \{x \in E: f(x) = a\} = \{x \in E: f(x) \geq a\} \cap \{x \in E: f(x) \leq a\}$$

$$\because \{x \in E: f(x) \geq a\} \in \mathcal{M}, \{x \in E: f(x) \leq a\} \in \mathcal{M}$$

$$\Rightarrow \{x \in E: f(x) \geq a\} \cap \{x \in E: f(x) \leq a\} \in \mathcal{M}$$

=

MCQ

$$\Rightarrow \{x \in E: f(x) = a\} \in \mathcal{M} \text{ (measurable)}$$

Case II: If $a = \infty$, then $\{x \in E: f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\}, \forall n \in \mathbb{N}$

Let $y \in \{x \in E: f(x) = \infty\} \Rightarrow y \in \{x \in E: f(x) > n\}$
 $\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\}, \forall n \in \mathbb{N}$

$$\Rightarrow \{x \in E: f(x) = \infty\} \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\}$$

QED

Now let $t \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\}, \forall n \in \mathbb{N}$

$$\Rightarrow t \in \{x \in E: f(x) > n\}, \forall n \in \mathbb{N}$$

$$\Rightarrow f(t) > n, \forall n \in \mathbb{N}$$

$$\Rightarrow t \in \{x \in E: f(x) = \infty\}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\} \subseteq \{x \in E: f(x) = \infty\} \rightarrow \textcircled{a}$$

From $\textcircled{a}$ and $\textcircled{b}$

$$\text{So, } \{x \in E: f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E: f(x) > n\}$$

Case II: If $a = -\infty$, then $\{x \in E: f(x) = -\infty\} =$

$$\bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\}$$

Let $x \in \{x \in E: f(x) = -\infty\} \Rightarrow x \in \{x \in E: f(x) < -n\}$

$$\Rightarrow x \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\}, \forall n \in \mathbb{N}$$

$$\Rightarrow \{x \in E: f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\}$$

$\rightarrow \textcircled{a}$

Now let $x \in \bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\}, \forall n \in \mathbb{N}$

$$\Rightarrow x \in \{x \in E: f(x) < -n\}, \forall n \in \mathbb{N}$$

$$\Rightarrow f(x) < -n, \forall n \in \mathbb{N}$$

$$\Rightarrow x \in \{x \in E: f(x) = -\infty\}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\} \subseteq \{x \in E: f(x) = -\infty\} \rightarrow \textcircled{b}$$

From (a) and (b)

$$\{x \in E: f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E: f(x) < -n\}$$

Topic # 72

MOST IMP

Positive and Negative parts of Extended Real Valued Function:

$$|f(x)| = \begin{cases} +f(x) & \text{if } f(x) \geq 0 \\ \text{minus } -f(x) & \text{if } f(x) < 0 \end{cases}$$

MCQ'S

Blue underline is same for all

$$f^+ = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

S.Q

$$\max\{f(x), 0\} = f^+$$

$$f^- = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

S.Q

$$\max\{-f(x), 0\} = -f^-$$

where $f: A \rightarrow \mathbb{R} \cup \{\infty\}$, $A \neq \emptyset$, $\forall x \in A$

Topic # 73

MOST IMP

Theorem: Show that

$$(i) f^+ = \frac{f + |f|}{2}, \quad (ii) f^- = \frac{|f| - f}{2}, \quad (iii) f = f^+ - f^-$$

$$(iv) |f| = f^+ + f^-, \quad (v) \max(f, g) = \frac{(f+g) + |f-g|}{2}$$

$$(vi) \min(f, g) = \frac{(f+g) - |f-g|}{2}$$

proof:

$$(i) \text{ if } f(x) \geq 0 \Rightarrow f^+(x) = f(x) \Rightarrow f^+ = f$$

From above formulas:

$$|f(x)| = |f(x)| = f(x)$$

$$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f(x)|}{2} = \frac{f(x) + f(x)}{2} = f(x) = f(x)$$

$$\text{If } f(x) < 0 \Rightarrow f'(x) = 0, \quad |f(x)| = |f(x)| = -f(x)$$

$$\therefore \frac{f + |f|}{2}(x) = \frac{f(x) + |f(x)|}{2} = \frac{f(x) - f(x)}{2} = 0 = f'(x)$$

$$\Rightarrow \frac{f + |f|}{2} = f' \quad (\text{proved})$$

$$\text{ii) If } f(x) > 0 \Rightarrow f''(x) = 0, \quad |f(x)| = |f(x)| = f(x)$$

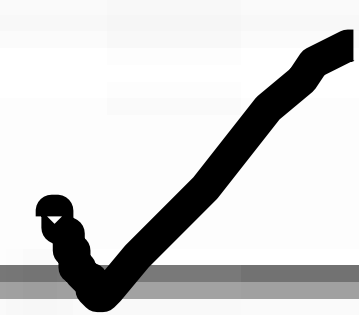
$$\therefore \frac{|f| - f}{2}(x) = \frac{|f(x)| - f(x)}{2} = \frac{f(x) - f(x)}{2} = 0 = f''(x)$$

$$\text{If } f(x) < 0 \Rightarrow f''(x) = -f(x), \quad |f(x)| = |f(x)| = -f(x)$$

$$\therefore \frac{|f| - f}{2}(x) = \frac{|f(x)| - f(x)}{2} = \frac{-f(x) - f(x)}{2} = -f(x) = f''(x)$$

$$\Rightarrow \frac{|f| - f}{2} = f'' \quad (\text{proved})$$

$$\text{iii) } f = f' - f''$$



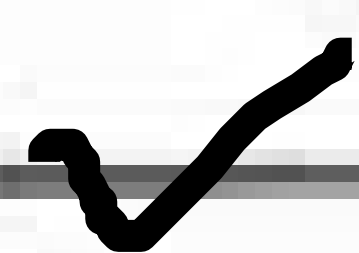
MOST IMP

$$\text{If } f(x) > 0 \Rightarrow f'(x) = f(x), \quad f''(x) = 0$$

$$\therefore (f' - f'')(x) = f'(x) - f''(x) = f(x) - 0 = f(x) = f$$

$$\Rightarrow f' - f'' = f \quad (\text{proved})$$

$$\text{iv) } |f| = f' + f''$$



$$\text{If } f(x) > 0 \Rightarrow f'(x) = f(x), \quad f''(x) = 0$$

$$\therefore (f' + f'')(x) = f'(x) + f''(x) = f(x) + 0 = f(x) = |f(x)|$$

$$\Rightarrow f' + f'' = |f| \quad (\text{proved})$$

(10)

$$\max(f, g) = \frac{(f+g) + |f-g|}{2}$$

$$|f-g| = \begin{cases} f-g & : f \geq g \\ -(f-g) & : f < g \end{cases}$$

$$|f-g|(x) = |f(x)-g(x)| = \begin{cases} f(x)-g(x) & ; f(x) \geq g(x) \\ g(x)-f(x) & ; f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x)-g(x) \geq 0$
 $\Rightarrow |f(x)-g(x)| = f(x)-g(x)$

$$\therefore \frac{(f+g)+|f-g|}{2}(x) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

$$= \frac{f(x)+g(x)+f(x)-g(x)}{2} = f(x)$$

Note: $f(x) \geq g(x)$ then $\max(f, g) = f(x)$

$$\Rightarrow \frac{(f+g)+|f-g|}{2} = \max(f, g)$$

Case II: If $f(x) < g(x) \Rightarrow f(x)-g(x) < 0$

$$\Rightarrow |f(x)-g(x)| = -(f(x)-g(x)) = g(x)-f(x)$$

and if $f(x) < g(x)$ then $\max(f, g) = g(x)$

$$\therefore \frac{(f+g)+|f-g|}{2}(x) = \frac{f(x)+g(x)+|f(x)-g(x)|}{2}$$

$$= \frac{f(x)+g(x)+g(x)-f(x)}{2} = g(x)$$

$$\Rightarrow \frac{(f+g)+|f-g|}{2} = \max(f, g) \quad (\text{proved})$$

$$(vi) \min(f, g) = \frac{(f+g) - |f-g|}{2}$$

$$|f-g|(x) = |f(x) - g(x)| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ g(x) - f(x) & \text{if } f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x) - g(x) \geq 0$

$\Rightarrow |f(x) - g(x)| = f(x) - g(x)$ and

If $f(x) \geq g(x) \Rightarrow \min(f, g) = g(x)$

$$\begin{aligned} \frac{(f+g) - |f-g|}{2}(x) &= \frac{f(x) + g(x) - |f(x) - g(x)|}{2} \\ &= \frac{f(x) + g(x) - (f(x) - g(x))}{2} \\ &= \frac{f(x) + g(x) - f(x) + g(x)}{2} = g(x) \end{aligned}$$

$$\Rightarrow \frac{(f+g) - |f-g|}{2} = \min(f, g)$$

Case II: If $f(x) < g(x) \Rightarrow f(x) - g(x) < 0$

$\Rightarrow |f(x) - g(x)| = g(x) - f(x)$ and

If $f(x) < g(x) \Rightarrow \min(f, g) = f(x)$

$$\begin{aligned} \frac{(f+g) - |f-g|}{2}(x) &= \frac{f(x) + g(x) - |f(x) - g(x)|}{2} \\ &= \frac{f(x) + g(x) - (g(x) - f(x))}{2} = f(x) \end{aligned}$$

$$\Rightarrow \frac{(f+g) - |f-g|}{2} = \min(f, g) \quad (\text{proved})$$

Topic # 74

Characteristic Function:

Let $\phi \neq x$ and $A \subseteq X$, then characteristic function $\chi_A: X \rightarrow \mathbb{R}$ is defined and given by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

MCQ'S

If $A = \phi \Rightarrow \chi_A(x) = 0 = 0 \forall x \in X$
 $\Rightarrow \chi_A = 0$
and if $A = X \Rightarrow \chi_A(x) = 1 = 1 \forall x \in X$
 $\Rightarrow \chi_A = 1$

Topic # 75

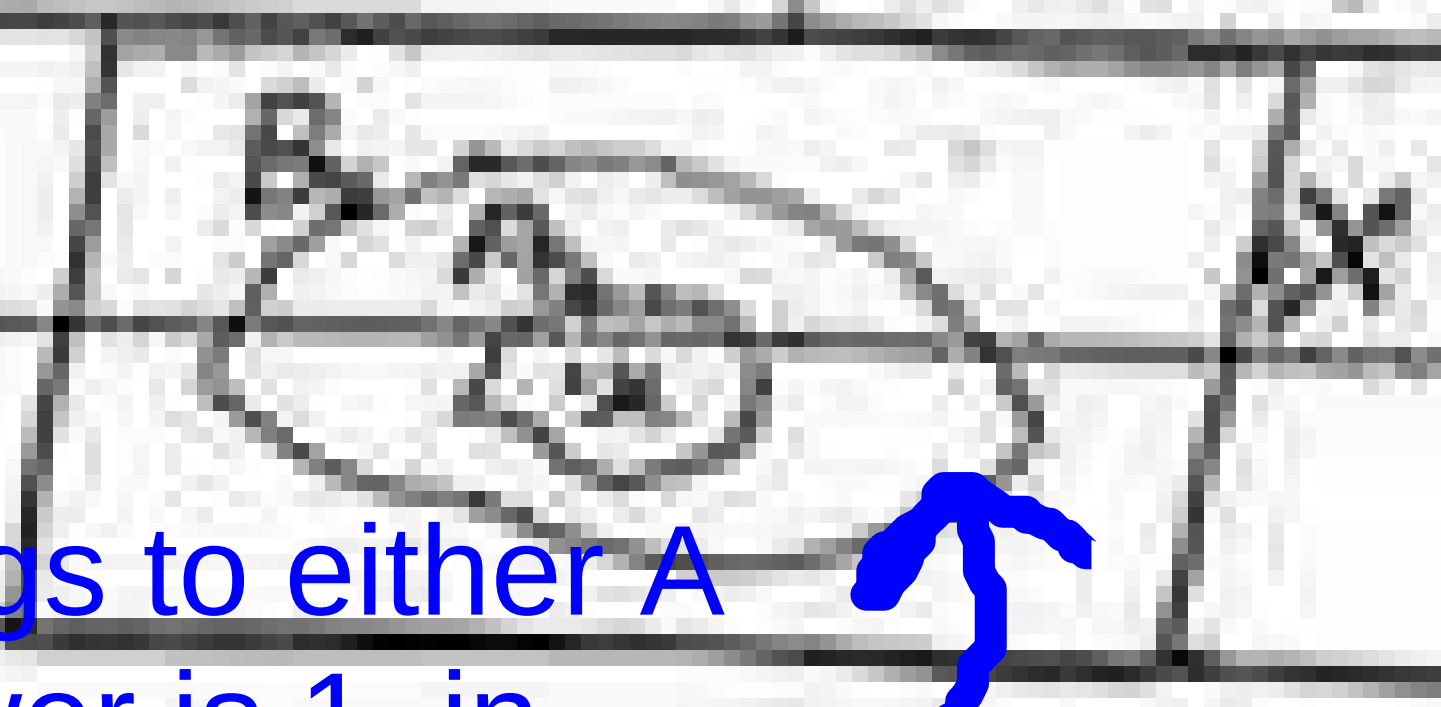
MOST IMP

Theorem: Prove that if $A \subseteq B$, then $\chi_A \subseteq \chi_B$

Proof: Case I: If $x \in A$ and

$A \subseteq B \Rightarrow x \in B$

$$\Rightarrow \chi_A = 1 = \chi_B \Rightarrow \chi_A \subseteq \chi_B$$



If x belongs to either A or B answer is 1, in this case x is belonging to both A.B so 1

Case II: If $x \in B$ but $x \notin A$

$$\Rightarrow \chi_A = 0 \text{ and } \chi_B = 1$$

$$\Rightarrow \chi_A \subseteq \chi_B$$

Here x belongs to B not A that's why answer of A is 0 and B is one



Case III: If $x \notin A$ nor $x \notin B$

$$\Rightarrow \chi_A = 0 = \chi_B \Rightarrow \chi_A \subseteq \chi_B \text{ (proved)}$$



apply same conditions as mentioned above

MOST IMP

Topic # 76

Theorem: Show that $\chi_{A \cap B} = \chi_A \cdot \chi_B$

Proof: If $x \in A \cap B \Rightarrow x \in A$ and $x \in B$ CASE I

$$\Rightarrow \chi_{A \cap B} = 1 = \chi_A = \chi_B = 1 \cdot 1 = \chi_A \cdot \chi_B$$

CASE II

$$\text{If } x \notin A \cap B \Rightarrow \begin{cases} x \in A \text{ but } x \notin B \Rightarrow \chi_A = 1, \chi_B = 0 \\ x \in B \text{ but } x \notin A \Rightarrow \chi_B = 1, \chi_A = 0 \\ x \notin A \text{ and } x \notin B \Rightarrow \chi_A = \chi_B = 0 \end{cases}$$

$$x \notin A \cap B \Rightarrow \chi_{A \cap B} = 0 \quad \left\{ \because \chi_A \cdot \chi_B = 0 = 0(x) \right.$$

$$\text{For all } x, x \notin A \cap B: \chi_{A \cap B} = 0 = 0(x) = \chi_A \cdot \chi_B$$

$$\chi_{A \cap B} = \chi_A \cdot \chi_B \quad (\text{proved})$$

Topic # 77

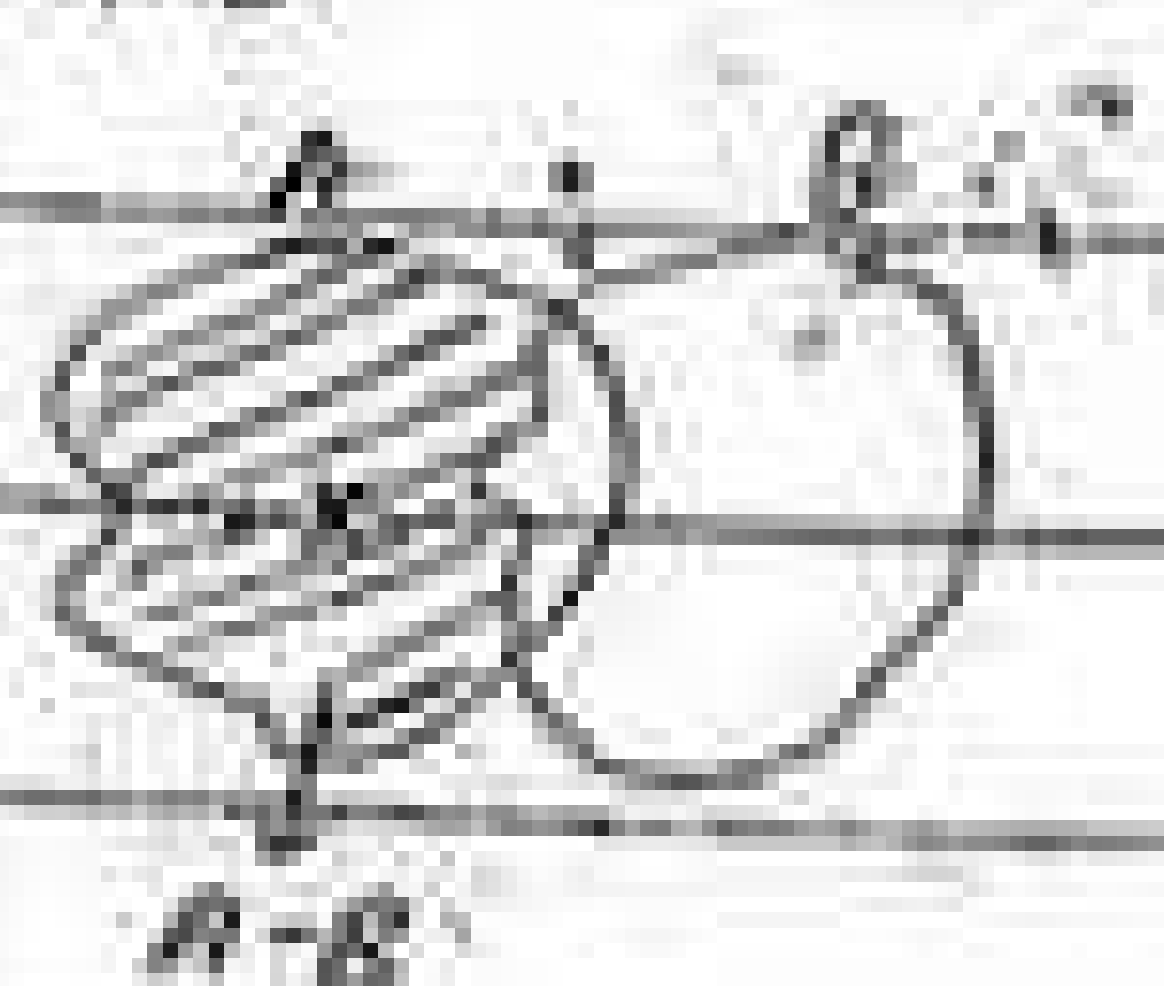
Theorem: Show that $\chi_{A-B} = \chi_A - \chi_{A \cap B}$

Proof: Case I: If $x \in A-B$

$$x \in A-B \Rightarrow x \in A \text{ but } x \notin B, x \notin A \cap B$$

$$\Rightarrow \chi_{A \cap B} = 0$$

$$\chi_A - \chi_{A \cap B} = 1 - 0 = 1 = \chi_{A-B}$$



$$\text{Case II If } x \notin A-B \Rightarrow \begin{cases} x \notin A, x \notin B, x \notin A \cap B \rightarrow \textcircled{a} \\ x \notin A, x \in B, x \notin A \cap B \rightarrow \textcircled{b} \\ x \in A, x \in B, x \in A \cap B \rightarrow \textcircled{c} \end{cases}$$

$$\Rightarrow \chi_{A-B} = 0 - 0(x)$$

$$\chi_A - \chi_{A \cap B} = \begin{cases} \textcircled{a} \Rightarrow 0 - 0 = 0 \\ \textcircled{b} \Rightarrow 0 - 0 = 0 \\ \textcircled{c} \Rightarrow 1 - 1 = 0 \end{cases}$$

JIRA NEEHSON

∴ For all cases: $\chi_{A \cap B} = \chi_A \cdot \chi_B$ (proved)

Same as previous(76) topic just change is we're using multiplication sign rather than intersection

Topic #78

MOST IMP

Theorem: Show that $\chi_{A \cap B} = \chi_A \cdot \chi_B$

Proof: $A \times B = \{ (x, y) : x \in A, y \in B \}$

Case 1:

$(x, y) \in (A \times B) \Rightarrow \{ x \in A \text{ and } y \in B \}$

$$\Rightarrow \chi_A = 1 = \chi_B$$

$$\Rightarrow \chi_{A \cap B}(x, y) = 1$$

$$\therefore \chi_{A \cap B} = \chi_A \cdot \chi_B$$

Case 2:

$$\checkmark \quad \text{If } (x, y) \notin A \times B \Rightarrow \begin{cases} x \in A \text{ but } y \notin B \rightarrow \textcircled{A} \\ x \notin A \text{ but } y \in B \rightarrow \textcircled{B} \\ x \notin A \text{ and } y \notin B \rightarrow \textcircled{C} \end{cases}$$

$$\Rightarrow \chi_{A \cap B}(x, y) = 0$$

$$\text{Also } \chi_A \cdot \chi_B = \begin{cases} \textcircled{A} \Rightarrow 1 \cdot 0 = 0 \\ \textcircled{B} \Rightarrow 0 \cdot 1 = 0 \\ \textcircled{C} \Rightarrow 0 \cdot 0 = 0 \end{cases}$$

In all cases: $\chi_{A \cap B} = \chi_A \cdot \chi_B$ (proved)

Topic #79

Theorem: If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X then show that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Proof: If $x \in A \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ exactly for one

$x \notin \bigcup_{n=1}^{\infty} A_n = \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1, \chi_{A_n}(x) = 0, \forall n \neq k$

$x \in A \Rightarrow \chi_A(x) = 1$



$$\therefore \chi_A(x) = 1 = 0 + 0 + \dots + 1 + 0 + 0 + \dots$$

$$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x) \Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$$

Case II: $x \notin A = \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin A_n, \forall n \in \mathbb{N}$

$\Rightarrow \chi_A(x) = 0 \quad \left\{ \Rightarrow \chi_{A_n}(x) = 0, \forall n \in \mathbb{N} \right.$

$$= 0 + 0 + \dots + 0 + \dots = \sum_{n=1}^{\infty} \chi_{A_n}(x)$$

$$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n} \quad (\text{proved})$$

MCQ

Topic # 80

Theorem: Let $\phi: X \rightarrow \mathbb{R}$ and ϕ characteristic

function $\chi_A: X \rightarrow \mathbb{R}$. χ_A is measurable $\Leftrightarrow A$ is measurable.

Proof: Suppose χ_A is measurable (function)

$\Rightarrow \forall a \in \mathbb{R}$ is set; $\{x: \chi_A(x) \geq a\}$ is measurable

In particular for $a=0$; $\{x: \chi_A(x) \geq 0\} = A$

by definition of χ_A

$\therefore A \in \mathcal{M}$ (measurable)

Conversely, let $A \in \mathcal{M}$ (measurable) and we prove χ_A is measurable.

In other words we want to prove the $\{x: \chi_A(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

$$\{x: \chi_A(x) > a\} = \begin{cases} \emptyset & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ \mathbb{R} & \text{if } a < 0 \end{cases}$$

(i) $a \geq 1, \chi_A(x) > 1 \Rightarrow$ there doesn't exist $x \in \mathbb{R}$ such that $\chi_A(x) > 1$
 $\Rightarrow \emptyset, \chi_A(x) > 0.7 \Rightarrow \emptyset$

(ii) $0 \leq a < 1, \chi_A(x) > a \Rightarrow x \in A$

In all cases: if $A \in \mathcal{M}$ (measurable)

$\Rightarrow A^c \in \mathcal{M}$ (measurable)

$\Rightarrow \mu(A^c) \in \mathcal{M}$ (measurable)

$\phi \in \mathcal{M}$ (measurable)

$\Rightarrow \{x: \chi_A(x) > a\} \in \mathcal{M}$ (measurable)

So " χ_A " is measurable. (Proved)

MCQ'S

Topic # 81

Algebra of Measurable Functions:

If " f " and " g " are two extended real valued and measurable functions defined on same domain

$f, g \in \mathcal{M}$ then (i) $f+k$, (ii) kf , (iii) $f+g$, (iv) fg

are measurable, (v) $|f|$, (vi) $f \vee g$, (vii) $f \wedge g$ are measurable functions.

max min

MCQ'S

NOTE

In coming lecture from Topic 82_87 all topics are proof of above indices in which we will repeat the first line as it is For all proofs.

MOST IMP

Topic # 82 & 83

JIRA NEEHSON

Proof: is $f \circ k$ is measurable, $k \in R$

$\therefore$ "f" is measurable $\Rightarrow \{x: f(x) > a\} \in M, \forall a \in R$

$\Rightarrow \{x: f(x) > a - k\} \in M$ (measurable)

$\Rightarrow \{x: f \circ k(x) > a\} \in M, \forall a \in R$

$\Rightarrow$ "fok" is measurable.

83 (a) kf is measurable, $\forall k \in R$

f-kf
Subs. capital

Proof: "f" is measurable

$\Rightarrow \{x: f(x) > a\} \in M, \forall a \in R$

① If $k > 0$: $f(x) > a, \forall a \in R$

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow k f(x) > a, \forall a \in R$

$\therefore \{x: k f(x) > a\} \in M, \forall a \in R$

$\Rightarrow$ "kf" is measurable

② If $k < 0$: $f(x) > a, \forall a \in R$ (given)

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow k f(x) < a$ ($\because k < 0$)

$\Rightarrow \{x: k f(x) < a\} \in M$

$\Rightarrow$ "kf" is measurable

③ If $k = 0 \Rightarrow \{x: k f(x) > a\}, \forall a \in R$

$\Rightarrow \{x: 0 f(x) > a\}, \forall a \in R$

$\Rightarrow \{x: 0 > a\}, \forall a \in R$

$\Rightarrow \{x: 0 > a\} = \emptyset, \forall a > 0$

$\{x: 0 > a\} = \begin{cases} \emptyset, & a > 0 \\ D, & a \leq 0 \end{cases}$

"D" is measurable (given) and $\emptyset$ is measurable

i. $\{x: K f(x) > a\}$ is measurable.

$\Rightarrow$ " Kf " is measurable.

Topic # 84

(iii) " $f \pm g$ " is measurable.

MCQ

if we have to prove " $f \pm g$ " is measurable.

$\Rightarrow \{x: (f \pm g)(x) > a\} \in \mathcal{M} \text{ (or } \mathcal{M}_f \text{)}$

$\{x: (f \pm g)(x) > a\} = \{x: f(x) \pm g(x) > a\}$

$= \{x: f(x) > a - g(x)\}$, $\forall a \in \mathbb{R}$

$\therefore \{g(x), -g(x)\} \subset \mathcal{M} \text{ and } \exists a \in \mathbb{R}$ such that

$f(x) > p > a - g(x)$

because " $\mathcal{M}$ " is a σ -algebra in $\mathbb{R}$: $\overline{\mathcal{M}} = \mathcal{M}$

$\{x: f(x) > a - g(x)\} = \bigcap_{p \in \mathbb{Q}} \{x: f(x) > p > a - g(x)\}$

ie no claim

Let $y \in \{x: (f \pm g)(x) > a\} = y \in \{x: f(x) \pm g(x) > a\}$

$\Rightarrow f(y) \pm g(y) > a \Rightarrow f(y) > a - g(y)$

$\Rightarrow$ there exists $p \in \mathbb{Q}$ such that $f(y) > p > a - g(y)$

$\mathcal{M} \subset \mathcal{M}$

$\Rightarrow f(y) > p$ and $a - g(y) < p$

$\Rightarrow y \in \{x: f(x) > p\}$ and $y \in \{x: a - g(x) < p\}$

$\Rightarrow y \in \{x: f(x) > p\} \cap \{x: a - g(x) < p\}$

$\Rightarrow y \in \bigcap_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}]$

$$\Rightarrow \{x: f(x) > a\} \in \bigcup_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}]$$

Now $\{x: f(x) > a\} \in \bigcup_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}]$ (1)

$$\Rightarrow t \in \{x: f(x) > p\} \cap \{x: a - g(x) < p\} \text{ for some } p \in \mathbb{Q}$$

$$\Rightarrow t \in \{x: f(x) > p\} \text{ and } t \in \{x: a - g(x) < p\}$$

$$\Rightarrow f(t) > p \text{ and } a - g(t) < p \Rightarrow p > a - g(t)$$

$$\Rightarrow f(t) > p > a - g(t) \Rightarrow f(t) > a - g(t)$$

$$\Rightarrow f(t) - g(t) > a \Rightarrow (f - g)(t) > a$$

$$\Rightarrow t \in \{x: (f - g)(x) > a\}$$

$$\Rightarrow \bigcup_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}] \subseteq \{x: (f - g)(x) > a\}$$

From (1) and (2) $\Rightarrow \{x: (f - g)(x) > a\} =$

$$= \bigcup_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}]$$

.. $\{f\}$ is measurable $\Rightarrow \{x: f(x) > p\} \in \mathcal{M}$

$\{g\}$ is measurable $\Rightarrow \{x: a - g(x) < p\} \in \mathcal{M}$

$$\Rightarrow \{x: f(x) > p\} \cap \{x: a - g(x) < p\} \in \mathcal{M} \quad \text{for } p \in \mathbb{Q}$$

$$\Rightarrow \bigcup_{p \in \mathbb{Q}} [\{x: f(x) > p\} \cap \{x: a - g(x) < p\}] \in \mathcal{M}$$

$$\Rightarrow \{x: (f - g)(x) > a\} \in \mathcal{M} \text{ (measurable)}$$

$$\Rightarrow f - g \text{ is measurable function.}$$

Now we have to prove " $f \cdot g$ " is measurable
 $\Rightarrow$ g is measurable $\rightarrow \text{csg} \cdot g$ is also Me.
 $\Rightarrow f + \text{csg} \cdot g = f \cdot g$ is also measurable.
 (proved)

Topic#85

iv) " $f \circ g$ " is measurable.

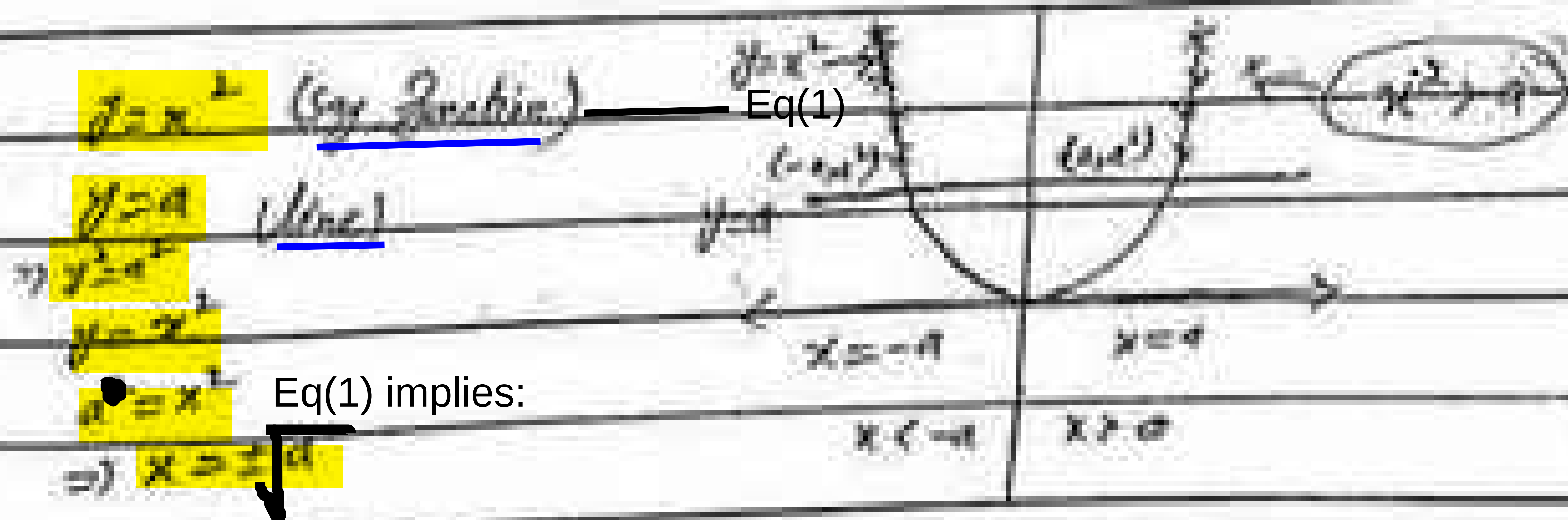
~~proof: $f \cdot g = \frac{1}{4} [(f+g)^2 - (f-g)^2]$~~

FORMULA

As we know that $F+G$ and $F-G$ are measurable but here square terms are used so we have to prove for squares of these terms are also measurable for that we are considering the below highlighted

Considering the below highlighted

$\rightarrow$ we need to prove $\{2, f^2(u) \in \mathcal{A}_1\} \subseteq \mathcal{A}_2$



Both measurable

$$\int f^2(x) dx \Rightarrow f(x) > a, f(x) < a$$
$$\{x, f(x), x^2\} \begin{cases} \{x, f(x), x^2\} \cup \{x, f(x), x^2\} \text{ is a } \mathcal{O} \\ \text{all elements in } (D) \text{ is a } \mathcal{O} \end{cases}$$

$\therefore \mathcal{I} \text{ is measurable} \Rightarrow \{x: f(x) \geq t_0\} \in \mathcal{M}$

and 92-1095-56911

This is just how we can show that both +ve and -ve square roots are measurable so, $f^2(x)$ will also be measurable

~~$\Rightarrow \{x: f(x) > 0\} \cup \{x: f(x) = 0\} \in \mathcal{M}$~~
 and "Domain" is then to be measurable i.e. $\in \mathcal{M}$
 ~~$\{x: f(x) > 0\} \in \mathcal{M} \Rightarrow f$ is measurable~~
 $\therefore f, g$ is measurable $\Rightarrow (f+g)$ is also measurable
 $\Rightarrow (f+g)$ is measurable i.e.
 $\Rightarrow \frac{1}{4} [(f+g)^2 - (f-g)^2] = fg$ is also measurable.
 $(\because kf$ is also measurable)
 $\forall k \in \mathbb{R}$

Topic # 86

Q. prove that if f is measurable function, then
 $\frac{1}{f}$ is also measurable.

Proof: f is measurable $\Rightarrow \{x: \underline{f(x)} > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

To prove $\frac{1}{f}$ is measurable, we show that

$\{x: \frac{1}{f(x)} > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

$\uparrow$
 $\frac{1}{f(x)}$

$$\frac{1}{f(x)} = \frac{1}{f(x)} > a$$

$$\{x: \frac{1}{f(x)} > a\} = \begin{cases} \{x: f(x) < \frac{1}{a}\} \cap \{x: f(x) > 0\}; a > 0 \\ \{x \in D: f(x) > 0\}; a = 0 \end{cases}$$

$$\{x: f(x) < \frac{1}{a}\} \cup \{x: f(x) > 0\}$$

$$\{x: f(x) < \frac{1}{a}\} \cup \{x: f(x) > 0\}$$

All the sets on R.H.S are measurable.

$\Rightarrow \{x: \frac{1}{f(x)} > a\}$ is also measurable $\forall a \in \mathbb{R}$

$\Rightarrow f$ is measurable.

(ii) $f, g \geq 0$ is measurable.

proof: f is also measurable because $f+g$ is measurable being product of two measurable functions.

Topic # 87

(iii) $|f|$ is measurable.

proof: f is measurable, $\{x: f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

To prove $|f|$ is measurable, we need to show

$$\{x: |f(x)| > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$$

$$f(x)$$

$$\{x: |f(x)| > a\} = \{x: f(x) > a\} \cup \{x: f(x) < -a\}$$

$$= \{x: f(x) > a\} \cup \{x: f(x) < -a\}, \forall a > 0$$

$$\{x: |f(x)| > a\} = \{x: f(x) > a\} \cup \{x: f(x) < -a\}$$

In all cases on RHS, we get a measurable

$$\{x: |f(x)| > a\} \in \mathcal{M} \text{ (measurable)}, \forall a > 0$$

$\Rightarrow |f|$ is also measurable.

(vii) $F \vee g = \max(f, g)$ is measurable.

MOST IMP

Proof: $\therefore f \vee g = \max(f, g) = \frac{1}{2}[(f+g) + |f-g|]$

So $f \vee g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and kf is measurable $\forall k \in \mathbb{R}$.

(viii) $f \wedge g = \min(f, g)$ is measurable.

Proof: $\therefore f \wedge g = \min(f, g) = \frac{1}{2}[(f+g) - |f-g|]$

So $f \wedge g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and kf is measurable $\forall k \in \mathbb{R}$.

Topic # 88

Restriction of a function

MCQ

Let $f: X \rightarrow Y$ be a function and $A \subset X$. Then $g: A \rightarrow Y$ defined as $g(x) = f(x)$, $x \in A$ is called the restriction of f to $A \subset X$ and given by $f_A = g$.

Theorem: Let f be an extended real valued measurable function on D . Let $A \subset D$. Then restriction of f to A is also measurable and $A \in \mathcal{M}$.

Proof: $\therefore f$ is measurable $\Rightarrow \{x \in D: f(x) \leq a\} \in \mathcal{M}$ for each $a \in \mathbb{R}$.

For any $a \in \mathbb{R}$, to prove f^{-1} is measurable
we have $\{x \in A : f_n(x) > a\} \in \mathcal{M}$
Note: $\{x \in A : f_n(x) > a\} = \{x \in A : f_n(x) > a\} \cap A$

On R.H.S both sets are measurable.
 $\Rightarrow \{x \in A : f_n(x) > a\} \in \mathcal{M}$
 $\therefore$ Intersection of two measurable sets is measurable.
 $\Rightarrow f_n^{-1}$ is measurable.

Topic # 89

MCQ

Theorem: An extended real valued function defined on a set of measure zero is measurable.

Proof: Let $f: E \subseteq \mathbb{R}$, such that $m^*(E) = 0$
and $f: E \rightarrow \mathbb{R} \cup \{\pm\infty\}$. To prove f is measurable, we need to show that

$\{x \in E : f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.

$\therefore \{x \in E : f(x) > a\} \subseteq E$

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq m^*(E)$

$\therefore$ By monotonicity of m^*

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq 0 \quad \dots (3)$

But by nonnegativity of m^*

$\Rightarrow m^*\{x \in E : f(x) > a\} \geq 0$

$\Rightarrow m^*\{x \in E : f(x) > a\} = 0 \quad \forall a \in \mathbb{R}$

$\Rightarrow \{x \in E: f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$
 So f is measurable.

Topic # 90

Theorem: Let $\{f_n\}$ be a sequence of extended real valued measurable function on S , then

(i) $\sup_{n \in \mathbb{N}} f_n, \inf_{n \in \mathbb{N}} f_n$ are measurable

MCQ'S

(ii) $\max_{1 \leq i \leq n} (f_i), \min_{1 \leq i \leq n} (f_i)$ are measurable.

(iii) $\limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$ are measurable.

Suprimum with line in superscript

Minimum with line in subscript

(iv) If $\lim_{n \rightarrow \infty} f_n = f$, Then f is measurable.

Proof: Let $f = \lim_{n \rightarrow \infty} f_n$ and To prove f is measurable.

Then we have to show that $\{x: f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

We claim: $\{x: f(x) > a\} = \bigcup_{n=1}^{\infty} \{x: f_n(x) > a\}$

Let $y \in \{x: f(x) > a\} \Rightarrow f(y) > a$

$\therefore f = \lim_{n \rightarrow \infty} f_n \Rightarrow f(y) = \lim_{n \rightarrow \infty} f_n(y), \forall n \in \mathbb{N}$

$\therefore f(y) > a \Rightarrow a < f(y) = \lim_{n \rightarrow \infty} f_n(y), \forall n \in \mathbb{N}$

$\Rightarrow f_n(y) > a, \forall n \in \mathbb{N} \Rightarrow y \in \{x: f_n(x) > a\}, \forall n \in \mathbb{N}$

$\Rightarrow y \in \bigcup_{n=1}^{\infty} \{x: f_n(x) > a\}$

$\Rightarrow \{x: f(x) > a\} \subseteq \bigcup_{n=1}^{\infty} \{x: f_n(x) > a\} \rightarrow \textcircled{1}$

Now let $t \in \bigcap_{n \in \mathbb{N}} \{x: f_n(x) > a\}$

$$\Rightarrow t \in \{x: f_n(x) > a\} \quad \forall a \in \mathbb{R}, \forall n \in \mathbb{N}$$

$$\Rightarrow f_n(t) > a, \forall n \in \mathbb{N}$$

$$\Rightarrow a < f_n(t), \forall n \in \mathbb{N}$$

$$\Rightarrow 'a' \text{ is lower bound } \{f_1(t), f_2(t), \dots\}$$

But $f = \sup_{n \in \mathbb{N}} f_n$ is also the lower bound

$$\Rightarrow a < \inf_{n \in \mathbb{N}} f_n, \forall n \in \mathbb{N}$$

$$\Rightarrow \inf_{n \in \mathbb{N}} f_n > a \Rightarrow f(t) > a$$

$$f(t) > a \Rightarrow t \in \{x: f(x) > a\} \quad \text{--- (1)}$$

$$\Rightarrow \bigcap_{n \in \mathbb{N}} \{x: f_n(x) > a\} \subset \{x: f(x) > a\} \quad \text{--- (2)}$$

$$\text{From (1) and (2) } \{x: f(x) > a\} = \bigcap_{n \in \mathbb{N}} \{x: f_n(x) > a\}$$

So $f = \sup_{n \in \mathbb{N}} f_n$ is measurable.

Now let $h = \sup_{n \in \mathbb{N}} f_n$

The set $\{x: h(x) > a\} = \bigcap_{n \in \mathbb{N}} \{x: f_n(x) > a\}$ is

measurable. So $h = \sup_{n \in \mathbb{N}} f_n$ is measurable.

Topic 9/

$\max_{1 \leq i \leq n} f_i$ & $\min_{1 \leq i \leq n} f_i$ are measurable.

$\{f_i\}_{i=1}^{\infty}$ is seq of measurable functs.

Proof: let $f = \max_{1 \leq i \leq n} f_i$, To prove f is measurable we need to show that $\{x: f(x) > a\} \in \mathcal{M}$ and we claim

$$\{x: f(x) > a\} = \bigcup_{i=1}^n \{x: f_i(x) > a\} \text{ if } a \in \mathbb{R}$$

$$\text{let } y \in \{x: f(x) > a\} \Rightarrow f(y) > a$$

$$\therefore f(x) = \max_{1 \leq i \leq n} f_i \text{ for then exist } i = \{i_2, \dots, i_n\}$$

$$\Rightarrow f(x) = f_{i_2}(x) \Rightarrow f(y) = f_{i_2}(y) > a$$

$$\Rightarrow y \in \{x: f_{i_2}(x) > a\} \text{ for some } i_2 = \{i_2, \dots, i_n\}$$

$$\text{Hence } y \in \bigcup_{i=1}^n \{x: f_i(x) > a\}$$

$$\text{Now let } y \in \bigcup_{i=1}^n \{x: f_i(x) > a\}$$

$$\text{Then } y \in \{x: f_{i_2}(x) > a\} \text{ for some } i_2 = \{i_2, \dots, i_n\}$$

$$\Rightarrow f_{i_2}(y) > a, \text{ By definition } f_{i_2}(y) \leq \max_{1 \leq i \leq n} f_i(y) = f(y)$$

$$\forall i_2 = 1, \dots, n$$

$$\text{and } x f_{i_2}(y) \leq f(y) \Rightarrow f(y) > a$$

$$\Rightarrow y \in \{x: f(x) > a\}$$

The countable union of measurable sets is measurable. So $f = \max_{1 \leq i \leq n} f_i$ is measurable.

$\lim_{n \rightarrow \infty} f_n \rightarrow$ Limit superior
 $\lim_{n \rightarrow \infty} f_n \rightarrow$ Limit inferior

JIRA NEEHSON

let $g = \min_{1 \leq n} f_n$. To prove g is measurable
 we need to show that $\{x: g(x) \leq a\}$

and we claim $\{x: g(x) \leq a\} = \bigcup_{n=1}^{\infty} \{x: f_n(x) \geq a\}$

The countable intersection of measurable sets is
 measurable. So $g = \min_{1 \leq n} f_n$ is measurable.

Topic #92

IMP

Show that $\lim_{n \rightarrow \infty} f_n$ and $\lim_{n \rightarrow \infty} f_n$ are measurable

Proof:

$$\lim_{n \rightarrow \infty} f_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} f_k \right) \quad (*)$$

$$\sup_{k \geq n} f_k = \sup \{f_k, f_{k+1}, f_{k+2}, \dots\}, \quad k=1, 2, 3, \dots$$

$\sup \{f_1, f_2, f_3, \dots\} = f_1$ is measurable being "sup" of measurable functions

$$\sup \{f_2, f_3, f_4, \dots\} = f_2, \dots$$

$\Rightarrow \{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions

$\Rightarrow \inf_{n \in \mathbb{N}} \{f_n\}_{n=1}^{\infty}$ is also measurable

$\Rightarrow \lim_{n \rightarrow \infty} f_n = \inf_{n \in \mathbb{N}} \left(\sup_{k \geq n} f_k \right)$ is also measurable

FIRA NEEHSON

Similarly

$\liminf_{n \rightarrow \infty} f_n - \sup_{n \in \mathbb{N}} (f_n)$ is measurable

Topic # 9.3

IMP

If $\liminf_{n \rightarrow \infty} f_n$ is measurable, then " f " is measurable

Proof: $\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} f_n = \limsup_{n \rightarrow \infty} f_n = f$

{
if f_n is measurable
it will be measurable

But $\liminf_{n \rightarrow \infty} f_n$ and $\limsup_{n \rightarrow \infty} f_n$ are measurable

$\Rightarrow$ " f " is also measurable

Wishing you all Success

and Excellence in your finals!

Highlighted with care By Fira"

MTH644 - Measure Theory Handouts