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HIGHLIGHTED HANDOUT

ECO607

Mathematical Economics II

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Lesson 01

INTRODUCTION TO DYNAMIC ANALYSIS

TOPIC 001: NEED AND IMPORTANCE OF DYNAMIC ANALYSIS IN ECONOMICS

Pre-requisites: Basic Mathematical Economics

Types of Economic Analyses: Static, Comparative-Static & Dynamic

Static: ساکن, Not moving (over time)

Comparative-Static: مقابلتاً ساکن, Moving (due to exogenous variables)

Dynamic: متحرک, Moving (over time)

COMPARISON

#	Grounds	Static/Comparative-Static Analysis	Dynamic Analysis
1	Time dimension	Time dimension is absent as all economic variables refer to the same point in time – Static economy/Timeless economy.	Time element has pivotal role. Variables can occur in different points of time.
2	Equilibrium	Studies only a particular point of equilibrium.	Studies the process by which equilibrium is achieved. Therefore, it deals with both, equilibrium and disequilibrium.
3	Transition	Doesn't show the path of change – only the condition(s) of equilibrium, like a 'still picture' of the market.	Shows the path of changes – like a 'movie' of the market.
4	Nature of Model	'Static model' evaluates the endogenous variables that satisfy some specified equilibrium condition(s) e.g. $Q_d = Q_s$, $AD = AI$, $S = I$	'Dynamic model' delineates the time path of some variable, on the basis of the rate of change. <i>Saving(time)</i> and <i>Investment(time)</i> etc.
5	Real-life Situation	Doesn't fully represent real-life situations. Based on the unrealistic assumptions of perfect competition, perfect knowledge, etc. Important economic variables like fashions, population, models of production, etc. are assumed to be constant.	Closer to reality. Considers that all economic variables as changeable.
6	Complexity	Easy but oversimplified	Complex but realistic

Rationale for Dynamic Analysis: Time is 'unstoppable' and affect variables of all disciplines including those of economics.

Better to conduct analysis over time rather than one point in time only.

Examples of Dynamic Analysis in Economics:

1. Decline in global GDP due to Covid-19 during first half of 2020.
2. Rise in unemployment in Pakistan due to Covid-19 during first half of 2020.
3. Decline of fuel (Petrol) prices due to Covid-19 during first half of 2020.
4. Change of share value over time.
5. Daily activity in Pakistan Stock Exchange (PSX).

TOPIC 002: DYNAMIC ANALYSIS: EXAMPLE FROM POPULATION

Say, population size H changes over time at the rate: $\frac{dH}{dt} = t^{-\frac{1}{2}}$

What time path of population $H = H(t)$ can yield the rate of change $\frac{dH}{dt} = t^{-\frac{1}{2}}$?

Here $H(t)$ can be called time path as 'population' depends on 'time'.

If $H(t) = 2t^{\frac{1}{2}}$, then its derivative is $\frac{dH(t)}{dt} = 2 \left(\frac{1}{2}\right) t^{-\frac{1}{2}} \Rightarrow \frac{dH}{dt} = t^{-\frac{1}{2}}$.

Interpretation: If rate of change of population over time is $\frac{dH}{dt} = t^{-\frac{1}{2}}$, then the relationship (time path) between population and time is $H(t) = 2t^{\frac{1}{2}}$.

However, same $\frac{dH}{dt}$ can be had with a large number of similar $H(t)$:

If $H(t) = 2t^{\frac{1}{2}} \pm 15$	then $\frac{dH}{dt} = t^{-\frac{1}{2}}$
$H(t) = 2t^{\frac{1}{2}} \pm 99$	again $\frac{dH}{dt} = t^{-\frac{1}{2}}$
...	...
...	...
...	...
$H(t) = 2t^{\frac{1}{2}} \pm c$ or $H(t) = 2t^{\frac{1}{2}} + c^*$	again $\frac{dH}{dt} = t^{-\frac{1}{2}}$

* instead of mentioning $-c$, $-ve$ sign is merged into c .

c is used as general notation for all possible constant values.

It's better to call c as the 'arbitrary constant'.

As arbitrary is something which is based on 'random choice', rather than any reason. c is arbitrary as it can assume any value at random.

$H(t) = 2t^{\frac{1}{2}} \pm c$ is time path of population, however, without determination of c the time path will remain 'general' in nature and not 'definite'.

Therefore, we try to determine value of c , which is possible.

We need the starting point of the function for it – Initial condition.

Initial condition: Value of the dependent variable when independent variable is zero. Here dependent variable is H and independent variable t .

Therefore, initial condition for function $H(t)$ is $H(0)$ or H_0 .

Now for the initial condition, general time path $H(t)$ becomes:

$$H(t) = 2t^{\frac{1}{2}} + c \Rightarrow H(0) = 2(0)^{\frac{1}{2}} + c \Rightarrow H(0) = c \Rightarrow \boxed{c = H(0)}$$

Value of arbitrary constant c is $H(0)$.

Plugging it back in 'general' time path: $H(t) = 2t^{\frac{1}{2}} + c$

We get 'definite' time path: $\boxed{H(t) = 2t^{\frac{1}{2}} + H(0)}$

$$\underbrace{H(t)}_{\text{Definite Time Path}} = \underbrace{2t^{\frac{1}{2}}}_{\text{Term involving time}} + \underbrace{H(0)}_{\text{Initial Population}}$$

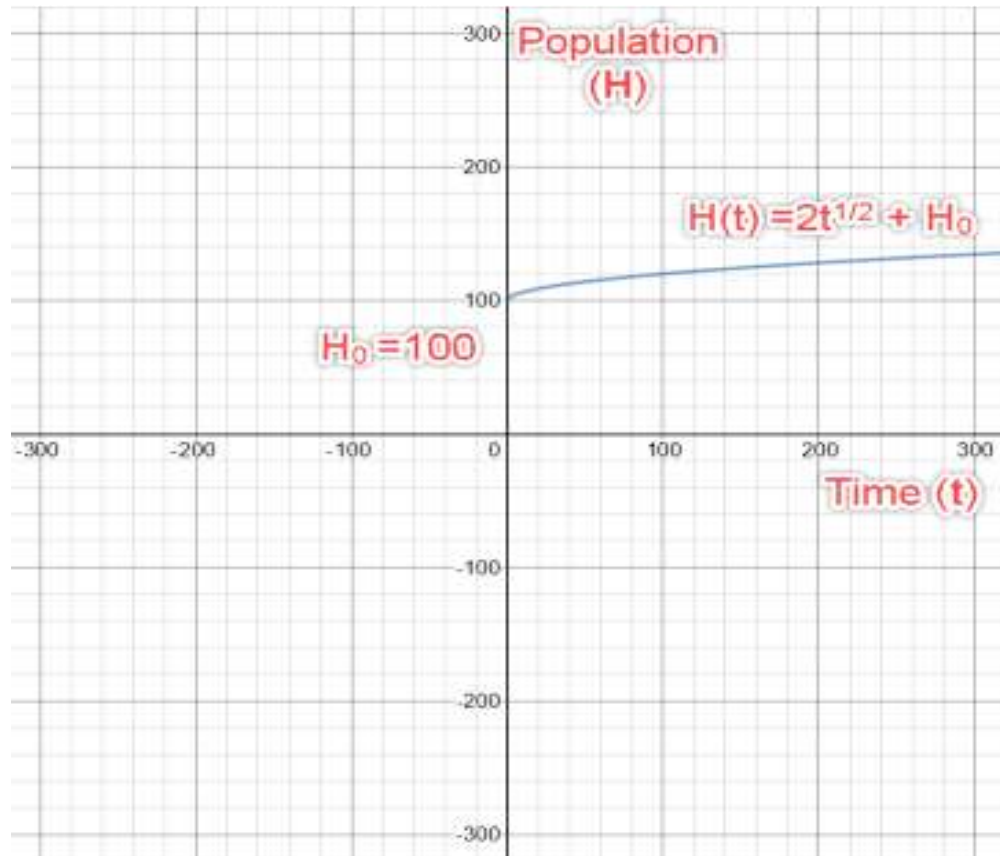
Arbitrary constant determined is not arbitrary any more $c = H(0)$.

Multiple arbitrary constants can exist for which initial conditions should be available.

Complete Map of the function – Solution of the dynamic model.

We can experiment some value of $H(0)$. e.g. $H(0) = 100$.

$$H(t) = 2t^{\frac{1}{2}} + 100$$



Lesson 02

INTEGRATION AND RULES

TOPIC 003: INTRODUCTION TO INTEGRAL CALCULUS: INDEFINITE INTEGRALS

RATIONALE FOR INTEGRATION:

History Perspective of Integration

Calculus is divided into differential and integral calculus. In this discussion, we are concerned with integral calculus. Economists often face the mathematical problem of finding a function from its derivative.

This process of reconstructing a function from its derivative can be regarded as the 'inverse' of differentiation. Mathematicians call this process *integration*.

There are simple formulas that have been known since ancient times for calculating the area of any triangle and polygon. Over 4000 years ago, however, the Babylonians were concerned with accurately measuring the area of plane surfaces, like circles, that are not bounded by straight lines. Solving this kind of area problem is related to integration. This topic will also include some important applications of integrals that economists are expected to know.

Mathematical Formulation

Let $F(x)$ be the 'primitive/original function' while $f(x)$ be the 'derivative', rather than $f'(x)$.

Under differentiation, process is from primitive function to derived function: $F(x) \Rightarrow f(x)$.

However, in Integration the shoe is on the other foot – From derived function to back to primitive function. $f(x) \Rightarrow F(x)$. Therefore, Integration is also called anti-derivative.

Elongated 'S' $\int f(x) dx$ represents the integral sign.

dx is written after the function here $f(x)$ showing that function is integrated w.r.t variable x .

Function to be integrated is known as '**integrand**'.

Differentiation and Integration are opposite of each other:

$$\frac{d}{dx}F(x) = f(x) \Rightarrow \int f(x) dx = F(x) + c$$

c is arbitrary constant of integration.

Multiple parentage of the integrand – multiple possibilities of c .

TOPIC 004: BASIC RULES OF INTEGRATION: CONSTANT FUNCTION RULE & CONSTANT MULTIPLE RULE

1. **Constant Function Rule:** It is known that a constant function can be written as:

$$y = f(x)$$

$$y = ax^0$$

Where a is a constant.

$$y = a$$

Integrating w.r.t. y and x on L.H.S and R.H.S, respectively:

$$\int y dy = \int a dx$$

$$\boxed{\int y dy = ax + c}$$

c is the constant of integration.

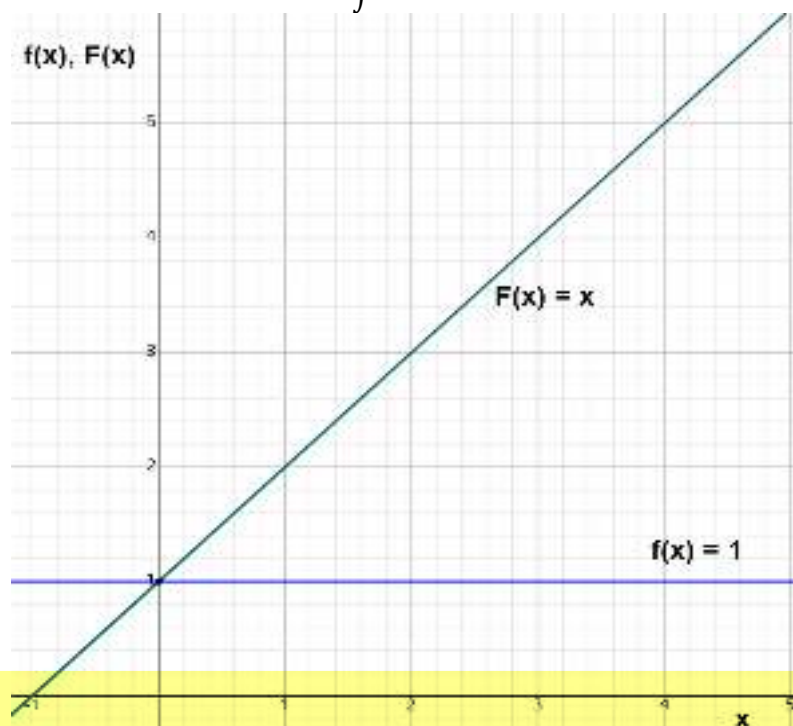
Numerical example: Let $y = f(x)$

$$f(x) = a$$

Let $a = 1$, then $y = 1$.

$$y = \int f(x) dx = \int (1) dx = (1)x + c$$

$$F(x) = \int (1) dx = x + c$$



2. Constant Multiple Rule: If a constant (a) is being multiplied with a function, its integration shall appear like this:

$$y = a \cdot f(x)$$

$$\int y dy = \int \{a \cdot f(x)\} dx$$

$$\int y dy = a \cdot \int f(x) dx$$

$$\boxed{\int y dy = a \cdot \{F(x)\} + c} \quad \left[\because \int f(x) dx = F(x) \right]$$

And $F(x)$ is the primitive.

Therefore, similar to differentiation, a constant that appears as a multiple of a function in an integral, comes out of the integral expression.

Numerical example: Let $a = 5$, then $y = 5$.

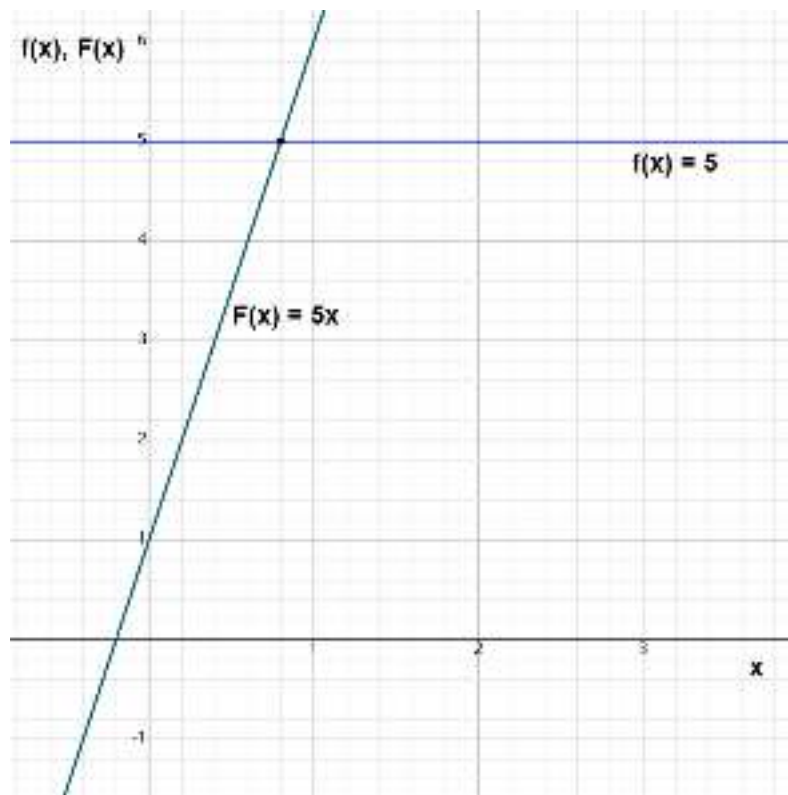
$$y = 5 \cdot f(x)$$

$$\int y dy = \int \{5 \cdot f(x)\} dx$$

$$\int y dy = 5 \cdot \int f(x) dx$$

$$\int y dy = 5 \cdot \{F(x)\} + c \quad \left[\because \int f(x) dx = F(x) \right]$$

Visualizing the integrand before after integration i.e. $f(x)$ and $F(x)$.



Before integration the function was a constant function with degree zero (0) and after it has become a linear function with degree one (1).

TOPIC 005: POWER RULE OF INTEGRATION

Power Rule: When one is to integrate a function that has power other than -1 , power rule is applicable:

Assuming that the power is symbolized by n :

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \text{ where } n \neq -1$$

Parametric Stipulation: Now it is clear why $n \neq -1$. Putting $n = -1$ in the formula:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$\int x^{(-1)} dx = \frac{x^{-1+1}}{-1+1} + c$$

$$\int x^{(-1)} dx = \frac{x^0}{0} + c$$

$$\int x^{(-1)} dx = \frac{1}{0} + c$$

$$\int x^{(-1)} dx = \infty + c$$

$$\int x^{(-1)} dx = \infty \text{ (undefined)}$$

Numerical Example: $y = f(x) = x^3$

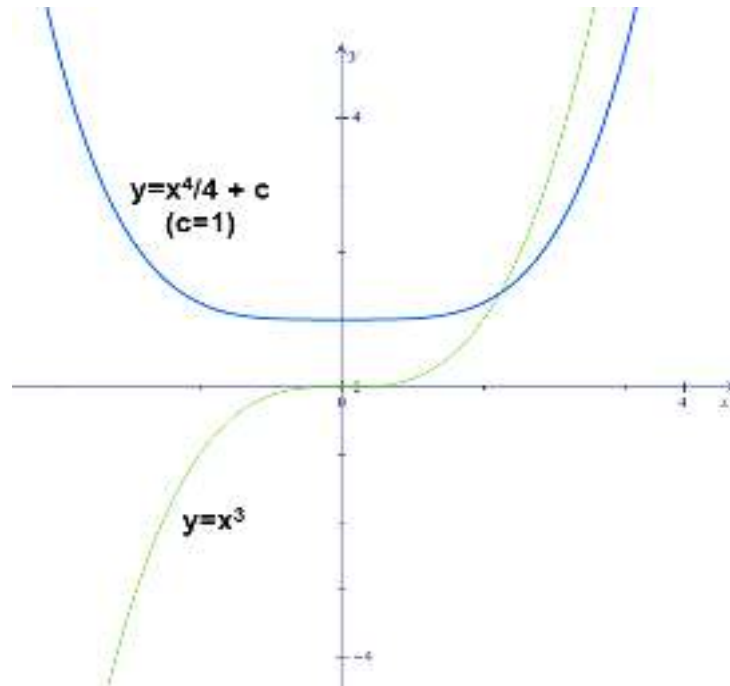
Integrating on both sides

$$\int y dy = \int x^3 dx$$

$$\int y dy = \frac{x^{3+1}}{3+1} + c$$

$$\int y dy = \boxed{\frac{x^4}{4} + c}$$

Visualizing the pre and post graphs of the integrand; i.e. $f(x) = x^3$ and $F(x) = \frac{x^4}{4} + c$ with ($c = 1$).



Power Rule and Constant Function Rule

One can use power rule to derive constant function rule:

Assuming the standard form of constant function:

$$y = a$$

Where $a = 1$ is a constant.

Rewriting the function:

$$y = (1)x^0$$

Integrating w.r.t x on R.H.S. $\int y dy = \int (x^0) dx$

Applying power rule: $= \frac{x^{0+1}}{0+1} + c$

$$= \frac{x^1}{1} + c$$

$$\boxed{\int y dy = x + c}$$

Which is the constant function rule with constant (a) = 1.

TOPIC 006: DERIVATIVE OVER FUNCTION RULE OF INTEGRATION

When the integrand happens to be in a ratio between derivative and its primitive, then the integration is done using Derivative over Function Rule.

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

The rule states that if integrand is a derivative-primitive ratio, then the integration is the natural logarithm of the primitive.

In other words, the primitive lies in denominator while its derivative is placed in the numerator.

Numerical Example(s): Assuming a function: $y = \frac{4x}{x^2+1}$

Integrating w.r.t. x on R.H.S.

$$\int y \, dy = \int \frac{4x}{x^2+1} dx$$

As per the rule the primitive lies in denominator, therefore, $f(x) = x^2 + 1$ The derivative of $f(x)$ is $f'(x) = 2x$.

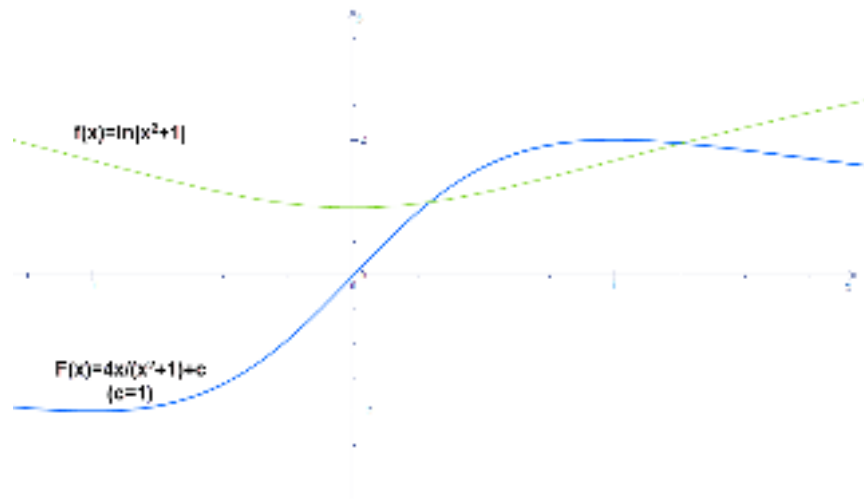
The numerator does not have exact term of $f'(x)$ i.e. $2x$. Therefore, we create it by re-writing the numerator.

$$\begin{aligned}\int y \, dy &= \int \frac{2(2x)}{x^2+1} dx \\ \int y \, dy &= 2 \int \frac{2x}{x^2+1} dx \\ \int y \, dy &= 2 \int \frac{\frac{d}{dx}(x^2+1)}{(x^2+1)} dx\end{aligned}$$

Recalling and applying the derivative over function rule:

$$\begin{aligned}\int \frac{f'(x)}{f(x)} dx &= \ln|f(x)| + c \\ \int y \, dy &= 2 \int \frac{2x}{x^2+1} dx \\ \boxed{\int y \, dy} &= \boxed{2(\ln|x^2+1|) + c}\end{aligned}$$

Visualizing the pre and post graphs of the integrand; $f(x) = \frac{4x}{x^2+1}$ and $F(x) = 2(\ln|x^2+1|) + c$ with $(c = 1)$.



Lesson 03

INTEGRATION AND RULES (CONTINUED)

TOPIC 007: EXPONENTIAL FUNCTION RULE OF INTEGRATION

Exponential Function Rule states that if integrand is an exponential function, then its integration is equal to:

$$\int a^{f(x)} dx = \frac{a^{f(x)}}{f'(x) \ln|a|} + c$$

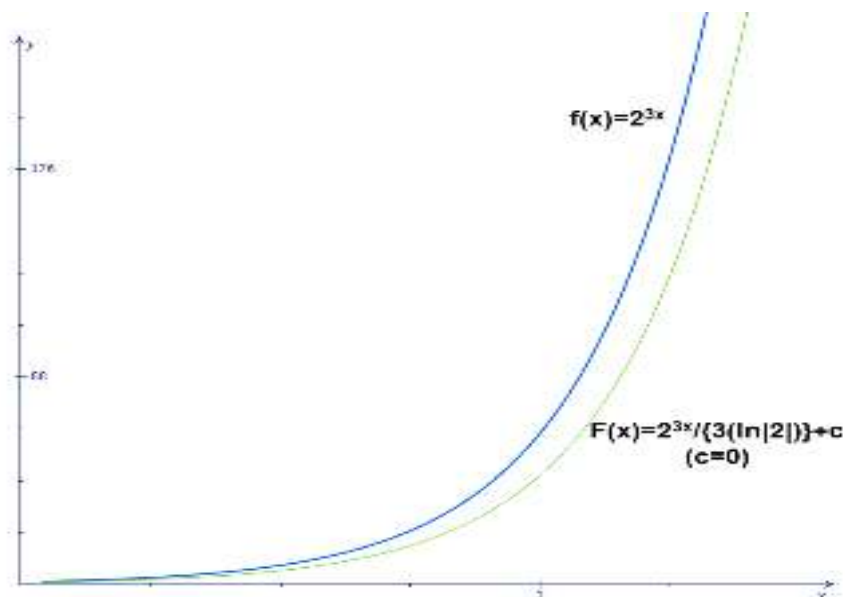
Numerical Example:

$$\int 2^{3x} dx = \frac{2^{3x}}{\frac{d}{dx}(3x) \cdot \ln|2|}$$

$$\int 2^{3x} dx = \frac{2^{3x}}{3 \cdot (0.693)}$$

$$\int 2^{3x} dx = \boxed{\frac{2^{3x}}{1.386} + c} \quad \text{or} \quad \boxed{\int 2^{3x} dx = \frac{2^{3x}}{3 \cdot \ln|2|} + c}$$

Visualizing the pre and post graphs of the integrand; i.e. $f(x) = 2^{3x}$ and $F(x) = \frac{2^{3x}}{3 \cdot \ln|2|}$ with ($c = 0$).



TOPIC 008: NATURAL EXPONENTIAL FUNCTION RULE OF INTEGRATION

Natural Exponential Function Rule: $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x) \ln|e|} + c = \frac{e^{f(x)}}{f'(x)} + c$

Natural Exponential Function Rule states that if integrand is a natural exponential function, then its integration is calculated similar to Exponential Function Rule:

$$\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x) \boxed{\ln|e|}} + c$$

$$\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x) \times \boxed{1}} + c$$

$$\boxed{\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + c}$$

Numerical Example: Assuming a numerical example: $y = 13e^{4x}$

Integrating w.r.t. x on R.H.S.

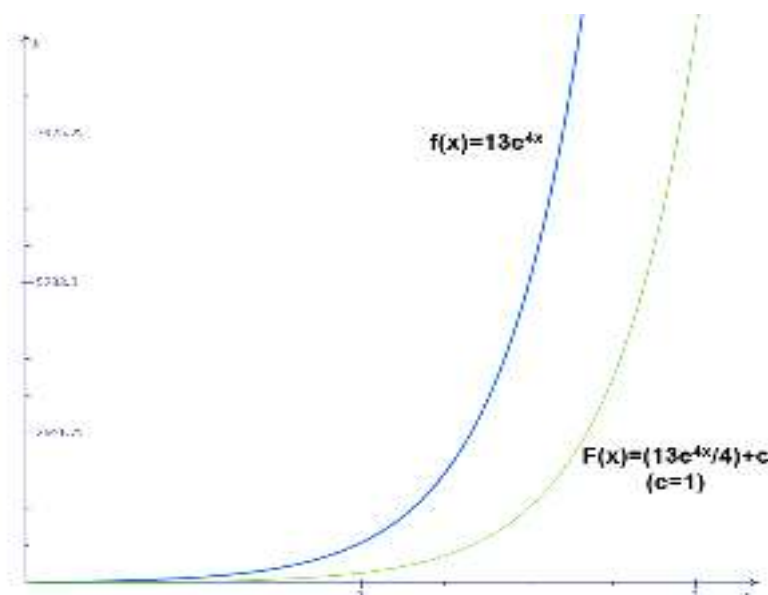
$$= \int 13e^{4x} dx$$

$$= 13 \left\{ \frac{e^{4x}}{\frac{d}{dx}(4x)} \right\}$$

$$= 13 \left(\frac{e^{4x}}{4} \right)$$

$$\boxed{\int 13e^{4x} dx = \frac{13}{4} e^{4x} + c}$$

Visualizing the pre and post graphs of the integrand, i.e. $f(x) = 13e^{4x}$ and $F(x) = \frac{13}{4}e^{4x} + c$ with ($c = 1$).



TOPIC 009: SUM-DIFFERENCE RULE OF INTEGRATION

Sum-Difference rule states that the integration of an integral which is either a sum/difference or two or more functions, is equal to the sum/difference of the integrals of those functions.

For convenience we assume only two functions here; i.e. $f(x) = g(x) \pm h(x)$

$$\int f(x) dx = \int \{g(x) \pm h(x)\} dx = \int g(x) dx \pm \int h(x) dx + c$$

Numerical Example(s): Assuming a function:

$$f(x) = (3x^3 - x + 1)$$

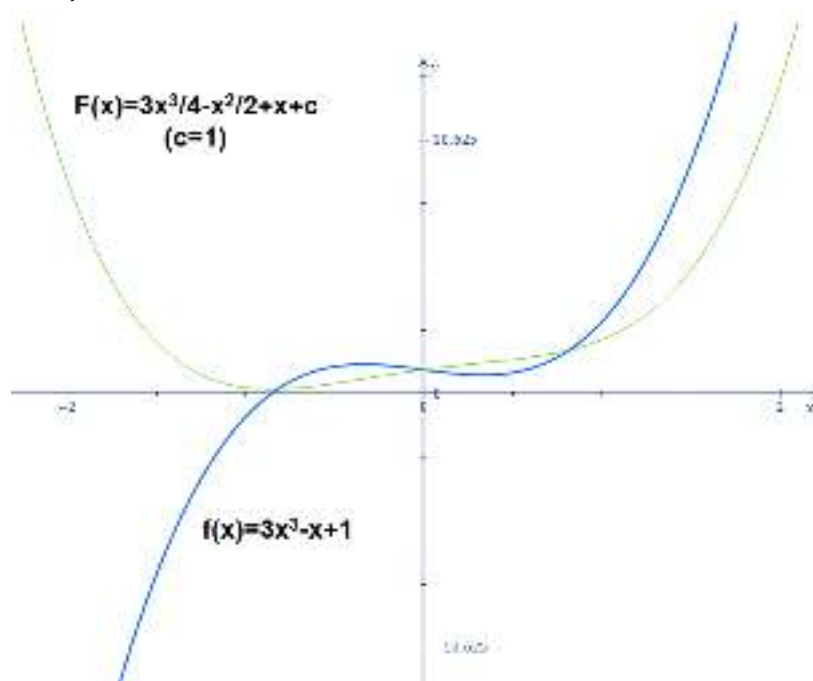
Integrating w.r.t. x .

$$\int f(x) dx = \int (3x^3 - x + 1) dx$$

Here, instead of two functions, there are three functions; i.e. $\int g(x)$, $h(x)$ and $i(x)$. Moreover, both of the arithmetic operations i.e. sum and difference, are used. Therefore, the formula shall be:

$$\begin{aligned}
 \int f(x) dx &= \int \{g(x) - h(x) + i(x)\} dx \\
 \int \{g(x) - h(x) + i(x)\} dx &= \int g(x) dx - \int h(x) dx + \int i(x) dx + c \\
 &= \int 3x^3 dx - \int x dx + \int 1 dx \\
 &= 3 \int x^3 dx - \int x dx + \int 1 dx \\
 &= 3 \left(\frac{x^{3+1}}{3+1} \right) - \frac{x^{1+1}}{1+1} + x + c \\
 \boxed{F(x) = \frac{3x^4}{4} - \frac{x^2}{2} + x + c}
 \end{aligned}$$

Visualizing the pre and post graphs of the integrand, i.e. $f(x) = (3x^3 - x + 1)$ and $F(x) = \frac{3x^4}{4} - \frac{x^2}{2} + x + c$ with $(c = 1)$.



TOPIC 010: MORE COMPLEX RULES: INTEGRATION BY SUBSTITUTION

Integration of a product or quotient of two differentiable functions can be done via 'Integration by Substitution'.

If integrand can be expressed as a constant-multiple n of another function u and its derivative $\frac{du}{dx}$.

Integration of a product or quotient of two differentiable functions of x , such as:

$$\int f(x) dx \Rightarrow \int n \cdot \{g(x) \cdot h(x)\} dx$$

Elaborating the expression:

$$\int f(x) dx \Rightarrow \int \underbrace{n}_{(\neq 0)} \cdot \left\{ \overbrace{\boxed{g(x)} \cdot h(x)}^{(\text{product of function and its derivative})} \right\} dx$$

$$\int f(x) dx \Rightarrow \int \underbrace{n}_{(\neq 0)} \cdot \left\{ \overbrace{\boxed{h'(x)} \cdot h(x)}^{(\text{product of function and its derivative})} \right\} dx$$

Method of substitution: Breakup $f(x)$ into a product of two function – $u = h(x)$ and its derivative $\frac{du}{dx} = h'(x)$.

$$\begin{aligned} &= \int \underline{f(x)} dx \\ &= \int u \left(\frac{du}{dx} \right) dx \\ &= \int u du \\ &= F(u) + c \end{aligned}$$

Hack: Look for the function and its derivative in the function.

Example 5: $\int 12x^2(x^3 + 2) dx$
(its derivative will be quadratic)

Intuition: $\int \underbrace{12x^2}_{(\text{quadratic function})} \underbrace{(x^3 + 2)}_{(\text{cubic function})} dx$

Derivative of $(x^3 + 2)$:

$$\frac{d}{dx}(x^3 + 2) = 3x^2$$

However, in the given example, we have $12x^2$ instead of $3x^2$.

So, we breakup $12x^2$ into $4 \times 3x^2$.

$$\begin{aligned} &= \int 12x^2(x^3 + 2) dx \\ &= \int \underbrace{4}_n \times \underbrace{3x^2}_{\left(\frac{du}{dx}\right)} \underbrace{(x^3 + 2)}_u dx \\ &= \int 4 \times \left(\frac{du}{dx} \right) u dx \end{aligned}$$

$$\begin{aligned}
 &= 4 \int (u) du \\
 &= 4 \left(\frac{u^2}{2} \right) + c \\
 &= 2u^2 + c \\
 &= 2 \underbrace{u^2}_{(x^3+2)^2} + c
 \end{aligned}$$

Restoring the expression in terms of x :

$$\int 12x^2(x^3 + 2) dx = 2(x^3 + 2)^2 + c$$

Verification (via differentiation):

$$\begin{aligned}
 &\frac{d}{dx} \{2(x^3 + 2)^2 + c\} \\
 \frac{d}{dx} \{2(x^3 + 2)^2 + c\} &= 2 \cdot \{2(x^3 + 2)\} \left\{ \frac{d}{dx} (x^3 + 2) \right\} + 0 \\
 \frac{d}{dx} \left\{ \int 12x^2(x^3 + 2) dx \right\} &= 4(x^3 + 2)(3x^2) \\
 \underbrace{12x^2(x^3 + 2)}_{\text{original function}} &= 4(3x^2)(x^3 + 2) \\
 12x^2(x^3 + 2) &= \underbrace{12x^2(x^3 + 2)}_{\substack{\text{original function} \\ \text{is restored}}}
 \end{aligned}$$

TOPIC 011: MORE COMPLEX RULES: INTEGRATION BY PARTS

Rationale for integration by parts:

Example 5: $\int 4x(x + 1)^3 dx$

(its derivative will not be linear)

Intuition: $\int \underbrace{4x}_{\substack{\text{linear} \\ \text{function}}} \underbrace{(x + 1)^3}_{\substack{\text{cubic} \\ \text{function}}} dx$

Derivative of $(x + 1)^3$:

$$\frac{d}{dx} (x + 1)^3 = 3(x + 1)^2$$

However, in the given example, we have $\underbrace{4x}_{\text{linear}}$ instead of $\underbrace{3(x + 1)^2}_{\text{quadratic}}$.

Coefficient n can be developed however, degree of function cannot be changed.

As Integration by Substitution fails here, we resort to a superior alternative 'Integration by Parts'.

Formula for Integration by Parts:

Used when integrand is a product or quotient of differentiable functions of x and Integration by substitution fails.

Derivation of the Rule: Reversing Product Theorem

$$\frac{d}{dx}\{g(x)h(x)\} = g(x)h'(x) + h(x)g'(x)$$

Taking Integral:

$$\int \frac{d}{dx}\{g(x)h(x)\} dx = \int \{g(x)h'(x)\} dx + \int \{h(x)g'(x)\} dx$$

$$\{g(x)h(x)\} = \int \{g(x)h'(x)\} dx + \int \{h(x)g'(x)\} dx$$

$$g(x)h(x) - \int \{g(x)h'(x)\} dx = \int \{h(x)g'(x)\} dx$$

$$\int \{h(x)g'(x)\} dx = g(x)h(x) - \int \{g(x)h'(x)\} dx$$

$$\boxed{\int \{h(x)g'(x)\} dx = g(x)h(x) - \int \{g(x)h'(x)\} dx}$$

Formula for 'Integration by Parts'

Example: $\int 4x(x+1)^3 dx$ **Hack:** Consider smaller function as $h(x)$, while more complicated one as $g'(x)$.

$$\int \underbrace{4x}_{\text{(simpler)}} \underbrace{(x+1)^3}_{\text{(more complicated)}} dx$$

$$\int \underbrace{4x}_{h(x)} \underbrace{(x+1)^3}_{g'(x)} dx$$

Formula for Integration by Substitution:

$$\int \{h(x)g'(x)\} dx = g(x)h(x) - \int \{g(x)h'(x)\} dx$$

$$\int \left\{ \overbrace{h(x)}^{\text{given}} \overbrace{g'(x)}^{\text{given}} \right\} dx = \underbrace{g(x)}_{\text{missing}} \overbrace{h(x)}^{\text{given}} - \int \left\{ \underbrace{g(x)}_{\text{missing}} \underbrace{h'(x)}_{\text{missing}} \right\} dx$$

Given extracted terms; $h(x) = 4x$, $g'(x) = (x+1)^3$.

Missing Terms	
$g(x)$	$h'(x)$
$g(x) = \int \{g'(x)\} dx$	$h'(x) = \frac{d}{dx}\{h(x)\}$
$g(x) = \int (x+1)^3 dx$	$= \frac{d}{dx}\{4x\}$
$\boxed{g(x) = \frac{(x+1)^4}{4} + c_1}$	$\boxed{h'(x) = 4}$

Plugging values in the formula:

$$\int \{ \overbrace{h(x)} \underbrace{g'(x)} \} dx = \underbrace{g(x)} \overbrace{h(x)} - \int \{ \underbrace{g(x)} \overbrace{h'(x)} \} dx$$

$$\int \underbrace{4x}_{\widetilde{4x}} \cdot \underbrace{(x+1)^3}_{\widetilde{(x+1)^3}} dx = \underbrace{\left\{ \frac{(x+1)^4}{4} + c_1 \right\}}_{\widetilde{\left\{ \frac{(x+1)^4}{4} + c_1 \right\}}} \underbrace{(4x)}_{\widetilde{(4x)}} - \int \left[\underbrace{\left\{ \frac{(x+1)^4}{4} + c_1 \right\}}_{\widetilde{\left\{ \frac{(x+1)^4}{4} + c_1 \right\}}} \underbrace{4}_{\widetilde{4}} \right] dx$$

$$\begin{aligned}
 \int 4x \cdot (x+1)^3 dx &= \left\{ 4x \frac{(x+1)^4}{4} + 4c_1 x \right\} - \int \left\{ 4 \frac{(x+1)^4}{4} + 4c_1 \right\} dx \\
 &= \{x(x+1)^4 + 4c_1 x\} - \int \{(x+1)^4\} dx - \int 4c_1 dx \\
 &= x(x+1)^4 + 4c_1 x - \left\{ \frac{(x+1)^5}{5} + c_2 \right\} - (4c_1 x + c_3) \\
 &= x(x+1)^4 + \cancel{4c_1 x} - \frac{(x+1)^5}{5} - c_2 - \cancel{4c_1 x} - c_3 \\
 &= x(x+1)^4 - \frac{(x+1)^5}{5} - \overbrace{(c_2 + c_3)}^{\text{yet another constant (c)}} \\
 \boxed{\int 4x \cdot (x+1)^3 dx = x(x+1)^4 - \frac{(x+1)^5}{5} - c}
 \end{aligned}$$

Verification (via differentiation):

$$\begin{aligned}
 &\frac{d}{dx} \left\{ x(x+1)^4 - \frac{(x+1)^5}{5} - c \right\} \\
 &\frac{d}{dx} \left\{ \overbrace{x(x+1)^4}^{\text{product rule}} - \overbrace{\frac{(x+1)^5}{5}}^{\text{power rule}} - c \right\} = \overbrace{x \cdot 4(x+1)^3 + 1 \cdot (x+1)^4}^{\text{product rule}} - \overbrace{\frac{5(x+1)^4}{5}}^{\text{power rule}} - 0 \\
 &\frac{d}{dx} \left\{ \int 4x(x+1)^3 dx \right\} = 4x \cdot (x+1)^3 + \cancel{(x+1)^4} - \cancel{(x+1)^4} \\
 &\quad \underbrace{4x(x+1)^3}_{\text{original function}} = \underbrace{4x \cdot (x+1)^3}_{\substack{\text{original function} \\ \text{is restored}}}
 \end{aligned}$$

Lesson 04

DEMAND AND SUPPLY ANALYSIS USING INTEGRATION

TOPIC 012: DEMAND ANALYSIS USING INTEGRATION BY SUBSTITUTION

The price p (dollars) of each unit of a particular commodity is estimated to be changing at the rate: $\frac{dp}{dx} = \frac{-135x}{\sqrt{9+x^2}}$. Here x is the Consumer demand in (hundred) units. If demand is ($x =$) 4 then

price would be ($p =$) **PKR 30**. Find demand function $p(x)$.

a. Find the demand function $p(x)$.

b. At what price will 300 units be demanded? At what price will no units be demanded?

c. How many units are demanded at a price of PKR 20 per unit?

Given the rate of change of inverse demand function; $\frac{dp}{dx} = \frac{-135x}{\sqrt{9+x^2}}$, integration can give inverse demand function:

$$\int \frac{dp}{dx} dx = \int \frac{-135x}{\sqrt{9+x^2}} dx \Rightarrow p = \int \frac{-135x}{\sqrt{9+x^2}} dx$$

$p(x) = \int \frac{-135x}{\sqrt{9+x^2}} dx$ using integration by substitution as basic rules of integration don't work here.

Let, the relatively higher degree polynomial be substituted:

$$u = 9 + x^2$$

Which implies that: $\frac{du}{dx} = 2x \Rightarrow \frac{du}{2} = x dx$

Substituting in the integrand, we get: $p(x) = \int \frac{-135x}{\sqrt{9+x^2}} dx = \int \frac{-135}{\sqrt{u}} (x dx)$

$$p(x) = \int \frac{-135}{\sqrt{u}} \left(\frac{du}{2}\right) = -\frac{135}{2} \int u^{-\frac{1}{2}} du = -\frac{135}{2} \left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + c$$

$$p(x) = -\frac{135}{2} \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = -\frac{135}{2} \times 2 u^{\frac{1}{2}} + c$$

$$p(x) = -135 u^{\frac{1}{2}} + c$$

Restoring the solution in terms of x instead of u .

$$p(x) = -135 (9 + x^2)^{\frac{1}{2}} + c$$

$p(x) = -135 \sqrt{9 + x^2} + c$ is a general solution as c is still arbitrary in nature.

As given, that $(p = 30)_{x=4}$, therefore, plugging values in $p(x) = -135 (9 + x^2)^{\frac{1}{2}} + c$, we get:

$$p(4) = 30 = -135 \{9 + (4)^2\}^{\frac{1}{2}} + c$$

$$30 = -135 \{9 + (4)^2\}^{\frac{1}{2}} + c \Rightarrow 30 = -135 \{25\}^{\frac{1}{2}} + c = -135 \times 5 + c$$

$$30 = -135 \times 5 + c \Rightarrow c = 705$$

Substituting the value of c in general solution we get the definite solution:

$$p(x) = -135 \sqrt{9 + x^2} + 705$$

$$\text{Graph of } p(x) = -135 \sqrt{9 + x^2} + 705$$



c. How many units are demanded at a price of PKR 20 per unit?

Given that $p = 20$, the definite solution of Inverse demand function becomes:

$$20 = -135\sqrt{9 + (x)^2} + 705 \Rightarrow \boxed{x = 4.09} \text{ demand in (hundred) units}$$

TOPIC 013: SUPPLY ANALYSIS USING INTEGRATION BY SUBSTITUTION

The owner of a fast-food chain determines that if x thousand units of a new meal item are supplied, then the marginal price at that level of supply is given by $p'(x) = \frac{x}{(x+3)^2}$. Where $p(x)$ is the price (in dollars) per unit at which all x meal units will be sold. Currently, 5,000 units are being supplied at a price of **PKR 2.20** per unit.

Find the supply (price) function $p(x)$.

$$p'(x) = \frac{x}{(x+3)^2}$$

Find the supply function $p(x)$.

Solution: Given the rate of change of (inverse) supply function;

$p'(x) = \frac{dp}{dx} = \frac{x}{(x+3)^2}$, integration can give (inverse) supply function:

$$\int \frac{dp}{dx} dx = \int \frac{x}{(x+3)^2} dx \Rightarrow p = \int \frac{x}{(x+3)^2} dx$$

$p(x) = \int \frac{x}{(x+3)^2} dx$ using integration by substitution as basic rules of integration don't work here.

Let, the denominator be substituted:

$$u = x + 3$$

Which implies that: $\frac{du}{dx} = 1 \Rightarrow du = dx$

Substituting in the integrand, we get:

$$p(x) = \int \frac{x}{(x+3)^2} dx = \int \frac{u-3}{(u)^2} (du) \quad \because [u = x + 3 \Rightarrow x = u - 3]$$

$$p(x) = \int \left(\frac{u-3}{u^2} \right) du = \int \left(\frac{1}{u} - \frac{3}{u^2} \right) du = \int \frac{1}{u} du - 3 \int \frac{1}{u^2} du$$

$$p(x) = \ln|u| + c_1 - \left\{ 3 \times \frac{(-1)}{u} + c_2 \right\} \Rightarrow \ln|u| + c_1 + \frac{3}{u} - c_2$$

$$p(x) = \ln|u| + \frac{3}{u} + c_1 - c_2 = \ln|u| + \frac{3}{u} + c \quad \because [c_1 - c_2 = c]$$

Restoring the solution in terms of x in place of u : $u = x + 3$

$$p(x) = \ln|u| + \frac{3}{u} + c$$

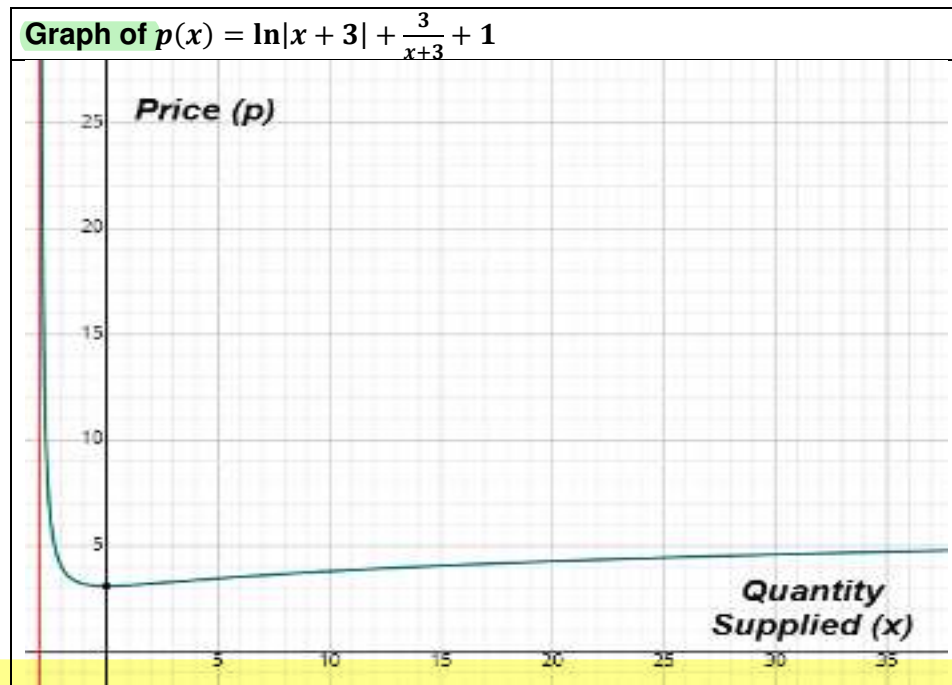
$$p(x) = \ln|x+3| + \frac{3}{x+3} + c$$

Due to the arbitrary constant ' c ', supply function $p(x)$ remains **general solution**.

Assuming value of $c (= 1)$ in **general solution**, we get **definite solution**:

$$p(x) = \ln|x+3| + \frac{3}{x+3} + 1$$

$$p(x) = \ln|x+3| + \frac{3}{x+3} + 1$$



b. If 10,000 meal units are supplied to restaurants in the chain, what unit price should be charged so that all the units will be sold?

$$p(10000) = \ln|10000 + 3| + \frac{3}{10000 + 3} + 1$$

$$p(10000) = \ln|10003| + \frac{3}{10003} + 1$$

$$\boxed{p(10000) = 10.21}$$

Interpretation: If 10000 meal units are supplied to restaurants in the chain, then unit price will be **PKR 10.21**.

TOPIC 014: REVENUE ANALYSIS USING INTEGRATION BY SUBSTITUTION

The marginal revenue from the sale of x units of a particular commodity is estimated to be:

$$R'(x) = 50 + 3.5xe^{-0.01x^2}$$

Dollars per unit where $R(x)$ is revenue in dollars.

a. Find $R(x)$, assuming that $R(0) = 0$.

b. What revenue should be expected from the sale of 1,000 units?

Given the rate of change of Revenue function;

$R'(x) = \frac{dR}{dx} = 50 + 3.5xe^{-0.01x^2}$, integration can give Revenue function:

$$\int \frac{dR}{dx} dx = \int (50 + 3.5xe^{-0.01x^2}) dx \Rightarrow R = \int 50 dx + \int 3.5xe^{-0.01x^2} dx$$

$$R = R(x) = 50 \int 1 dx + 3.5 \int xe^{-0.01x^2} dx$$

$$R = R(x) = 50x + c_1 + 3.5 \int xe^{-0.01x^2} dx$$

Applying integration by parts to solve underlined integrand:

$$u = -0.01x^2 \Rightarrow \frac{du}{dx} = -0.01(2x) \Rightarrow \frac{du}{dx} = -0.02x \Rightarrow \frac{du}{-0.02} = x dx$$

Rearranging the integrand to get $x dx$ term in:

$$R(x) = 50x + c_1 + 3.5 \int x e^{-0.01x^2} dx$$

$$R(x) = 50x + c_1 + 3.5 \int e^{-0.01x^2} \underline{x dx}$$

Introducing u instead of x .

$$= 50x + c_1 + 3.5 \int e^u \frac{du}{-0.02} \Rightarrow 50x + c_1 + \frac{3.5}{-0.02} \int e^u du$$

$$= 50x + c_1 + (-175) \int e^u du$$

Using Natural Exponential Function Rule: $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + c$

$$= 50x + c_1 + (-175)e^u + c_2 \Rightarrow 50x - 175e^u + c_1 + c_2$$

$$= 50x - 175e^u + c$$

Restoring the solution in terms of x in place of u : $u = -0.01x^2$

$$R(x) = 50x - 175e^{-0.01x^2} + c$$

Due to the arbitrary constant ' c ', revenue function $R(x)$ remains **general solution**.

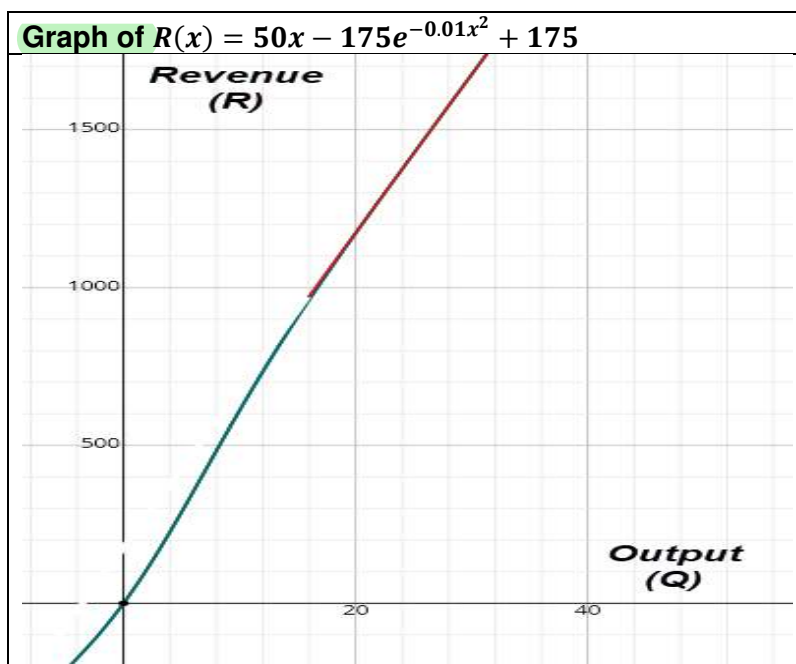
Given that $x = 0$, $R(0) = 0$, the general solution becomes:

$$R(0) = 50(0) - 175e^{-0.01(0)^2} + c \Rightarrow R(0) = 0 - 175e^0 + c$$

$$R(0) = 0 - 175 + c \Rightarrow 0 = -175 + c \Rightarrow c = 175$$

Substituting value of c in **general solution**, we get **definite solution**:

$$R(x) = 50x - 175e^{-0.01x^2} + 175$$



If 1,000 units are being produced then revenue shall be:

$$R(1000) = 50(1000) - 175e^{-0.01(1000)^2} + 175 = \text{PKR } 50175$$

Interpretation: If 1,000 units are sold, then the revenue will be **PKR 50175.**

TOPIC 015: INVESTMENT PORTFOLIO ANALYSIS USING INTEGRATION BY SUBSTITUTION

An investment portfolio changes value at the rate:

$$V'(t) = 12e^{-0.05t}(e^{0.3t} - 3)$$

dollars per unit where $V(t)$ is investment in dollars. Find $V(t)$.

Given the rate of change of investment portfolio function;

$V'(t) = \frac{dV}{dt} = 12e^{-0.05t}(e^{0.3t} - 3)$, integration can give investment portfolio function:

$$\int \frac{dV}{dt} dt = \int \{12e^{-0.05t}(e^{0.3t} - 3)\} dt \Rightarrow V(t) = 12 \int \{e^{-0.05t}(e^{0.3t} - 3)\} dt$$

Converting decimals into ratios for ease.

$$V(t) = 12 \int \left\{ e^{-\frac{1}{20}t} \left(e^{\frac{3}{10}t} - 3 \right) \right\} dt$$

Applying substitution method: $u = -\frac{1}{20}t \Rightarrow \frac{du}{dt} = -\frac{1}{20} \Rightarrow -20du = dt$

$$V(t) = 12 \int \left\{ e^{-\frac{1}{20}t} \left(e^{\frac{3}{10}t} - 3 \right) \right\} dt$$

$$-\frac{1}{20}t = u \Rightarrow t = -20u \Rightarrow \frac{3}{10}t = \frac{3}{10} \times (-20u) \Rightarrow \frac{3}{10}t = -6u$$

$$V(u) = 12 \int \left\{ e^u (e^{-6u} - 3) \right\} (-20du)$$

$$V(u) = 12 \times (-20) \int \{e^u(e^{-6u} - 3)\} du$$

$$V(u) = -240 \int \{e^u(e^{-6u} - 3)\} du$$

$$V(u) = -240 \int (e^{-5u} - 3e^u) du \Rightarrow 240 \int (3e^u - e^{-5u}) du$$

$$V(u) = 240 \left\{ \int (3e^u) du - \int (e^{-5u}) du \right\}$$

$$V(u) = 240 \left\{ 3 \int (e^u) du - \int (e^{-5u}) du \right\}$$

Using Natural Exponential Function Rule $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + c$

$$V(u) = 240 \left\{ 3e^u + c_1 - \left(\frac{e^{-5u}}{(-5)} + c_2 \right) \right\}$$

$$V(u) = 240 \left\{ 3e^u + \frac{e^{-5u}}{5} + c_1 - c_2 \right\}$$

$$V(u) = 240 \left\{ 3e^u + \frac{e^{-5u}}{5} + c_3 \right\}$$

$$V(u) = 720e^u + 48e^{-5u} + 240c_3$$

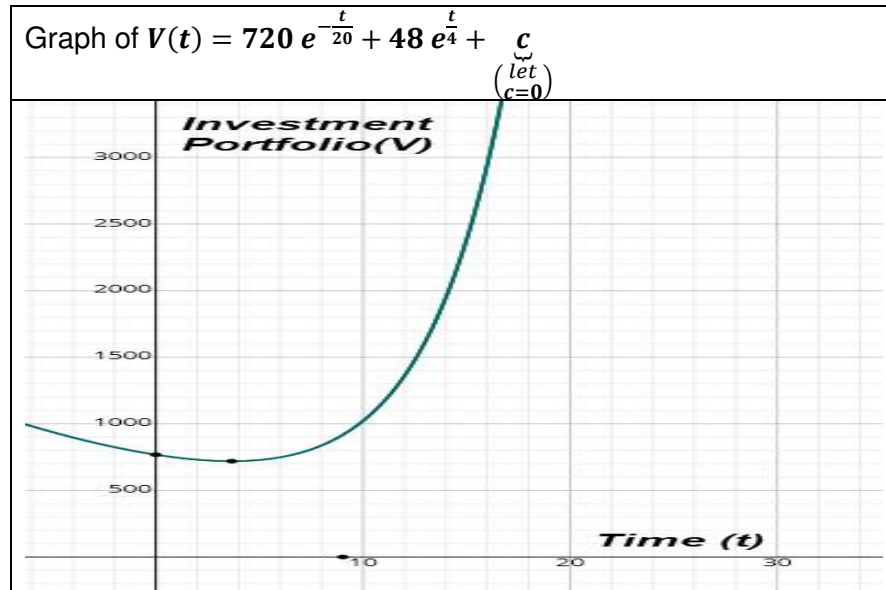
$$V(u) = 720e^u + 48e^{-5u} + c$$

Restoring the function in terms of t , instead of u , as $u = -\frac{1}{20}t$

$$V(t) = 720 e^{\left(-\frac{1}{20}t\right)} + 48 e^{-5\left(-\frac{1}{20}t\right)} + c$$

$$V(t) = 720 e^{-\frac{t}{20}} + 48 e^{\frac{t}{4}} + c$$

Due to the arbitrary constant ' c ', investment portfolio function $V(t)$ remains **general solution**.



Lesson 05

DEFINITE INTEGRALS

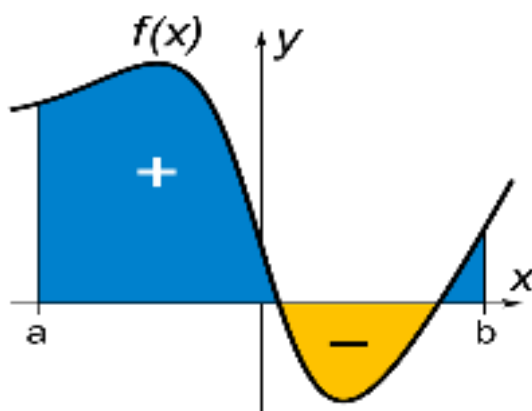
TOPIC 016: DEFINITE INTEGRALS

By the virtue of indefinite integral, we know:

$$\int f(x) dx = F(x) + c$$

However, if starting & ending values from domain (here, values of x) are chosen then function has starting & ending points.

Diagrammatically

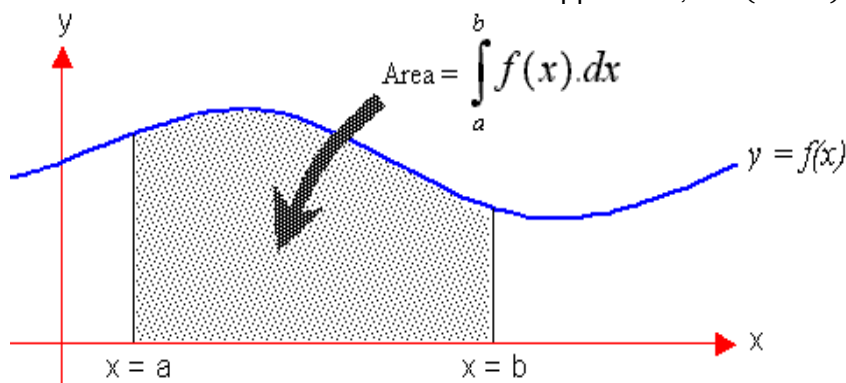


Here the between the curve and x -axis has opposite signs (+ve & -ve), therefore, it's a signed area. Properly defining, definite integral is the signed area between curve, x -axis and lines ($x = a$ and $x = b$).

Using calculus, the expression and diagram looks like this:

$$Area = \int_a^b f(x) dx$$

Where ' a ' is the lower limit and ' b ' is the upper limit, i.e. ($a < b$).



Solution will be

$$\begin{aligned} \int_a^b f(x) dx &= \{F(x) + c\}_a^b = [F(b) + c] - [F(a) + c] \\ &= F(b) + c - F(a) - c \Rightarrow \boxed{F(b) - F(a)} \end{aligned}$$

c – the arbitrary constant of integration, is cancelled out in definite integral, therefore, usually it is omitted.

Interpretation: Result of a definite integral will be numerical in nature.

Since, diagrammatically it is 'Area', therefore, it has a 'definite' numerical value.

Numerical Example # 1:

$$\int_1^5 3x^2 dx = (x^3 + c) \Big|_1^5 = [(5)^3 + c] - [(1)^3 + c]$$

$$= 125 + c - 1 - c \Rightarrow \boxed{124}$$

Numerical Example # 2:

$$\int_a^b k e^x dx = k(e^x) \Big|_a^b \text{ [constant of definite integrals gets omitted]}$$

$$= k(e^b - e^a)$$

Numerical Example # 3:

$$\int_0^4 \left(\frac{1}{1+x} + 2x \right) dx = \left(\underbrace{\ln|1+x|}_{\text{Derivative over Function Rule}} + \underbrace{x^2}_{\text{Power Rule}} \right) \Big|_0^4$$

$$= (\ln|1+4| - 4^2) + (\ln|1+0| - 0^2)$$

$$= (\ln|5| - 16) + (\ln|1| - 0) \Rightarrow (\ln|5| - 16) + (0 - 0)$$

$$= \boxed{\ln|5| - 16}$$

TOPIC 017: DEFINITE INTEGRAL: AREA UNDER CURVE

In addition to calculus-based (precise) method, Area under the curve can be 'approximated' graphically by dividing into vertical rectangles.

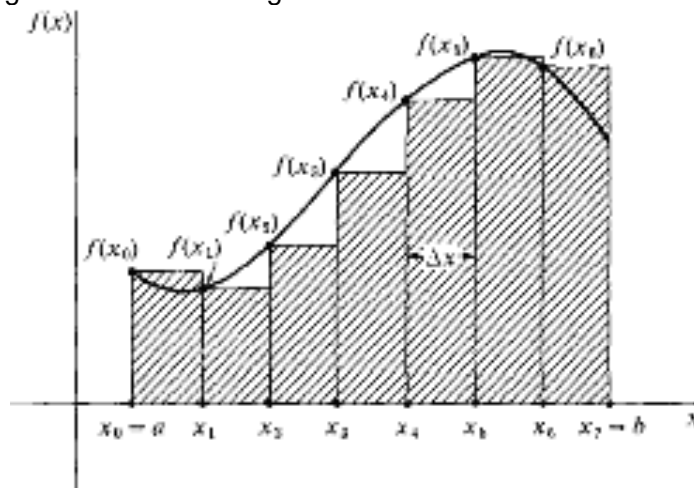


Diagram shows that the area under the curve between $[a, b]$, is divided into seven (7) vertical rectangles. The width of each of these rectangles is equal, however, height is not. Lengths are $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, $f(x_5)$ and $f(x_6)$, respectively.

Therefore, the respective areas of each rectangle will be $f(x_0) \times (x_1 - x_0)$, $f(x_1) \times (x_2 - x_1)$, $f(x_2) \times (x_3 - x_2)$, $f(x_3) \times (x_4 - x_3)$, $f(x_4) \times (x_5 - x_4)$, $f(x_5) \times (x_6 - x_5)$, and $f(x_6) \times (x_7 - x_6)$.

Adding the area (A) of the seven rectangles, we get overall 'shaded' area:

$$A = f(x_0) \times (x_1 - x_0) + f(x_1) \times (x_2 - x_1) + f(x_2) \times (x_3 - x_2) + f(x_3) \times (x_4 - x_3) + f(x_4) \times (x_5 - x_4) + f(x_5) \times (x_6 - x_5) + f(x_6) \times (x_7 - x_6)$$

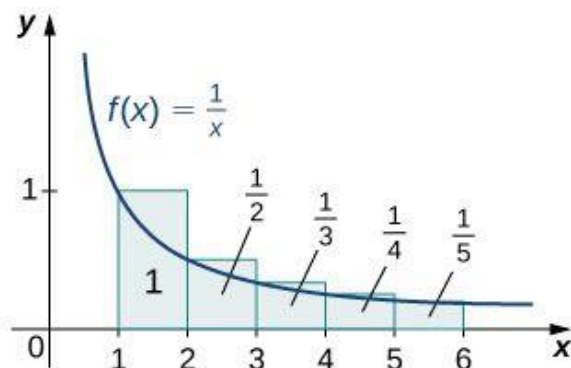
$$A = f(x_0) \times (\Delta x_0) + f(x_1) \times (\Delta x_1) + f(x_2) \times (\Delta x_2) + f(x_3) \times (\Delta x_3) + f(x_4) \times (\Delta x_4) + f(x_5) \times (\Delta x_5) + f(x_6) \times (\Delta x_6) + f(x_7) \times (\Delta x_7)$$

In summation form:

$$A = \sum_{i=0}^n \{f(x_i) \times (\Delta x_i)\}$$

The greater the value of n , the smaller will be the rectangles and more precise will be calculation. This method is usually known as the 'Riemann Sum'.

Numerical depiction:



$$A = f(x_1) \times (x_2 - x_1) + f(x_2) \times (x_3 - x_2) + f(x_3) \times (x_4 - x_3) + f(x_4) \times (x_5 - x_4) + f(x_5) \times (x_6 - x_5)$$

$$A = 1 \times (2 - 1) + \frac{1}{2} \times (3 - 2) + \frac{1}{3} \times (4 - 3) + \frac{1}{4} \times (5 - 4) + \frac{1}{5} \times (6 - 5)$$

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$A = 2.283$$

Interpretation: Approximation of the area under the curve between 1 and 6.

However, if calculus-based method is adopted: $\int_1^6 \left(\frac{1}{x}\right) dx = 1.792$

It is observable that Area using Riemann Sum (2.283) is over-approximated as compared to that using calculus-based method (1.792).

TOPIC 018: PROPERTIES OF DEFINITE INTEGRALS

Property – 1: Interchanging limits of integration changes the sign of the definite integral:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Proof: Using L.H.S: $\int_a^b f(x) dx = F(b) - F(a)$

Taking –1 common, we get: $= -\{-F(b) + F(a)\}$

Interchanging places of the two terms: $= -\{F(a) - F(b)\}$

In general form: $\int_{\text{lower limit}}^{\text{upper limit}} f(x) dx = \{F(\text{upper limit}) - F(\text{lower limit})\}$

Using the general form write the integral form: $-\int_b^a f(x) dx = \text{R.H.S}$

Property – 2: Definite integral has a value of zero when the two limits of integration are identical:

$$\int_a^a f(x) dx = 0$$

Proof: Using L.H.S: $\int_a^a f(x) dx = F(a) - F(a)$
 $0 = \text{R.H.S}$

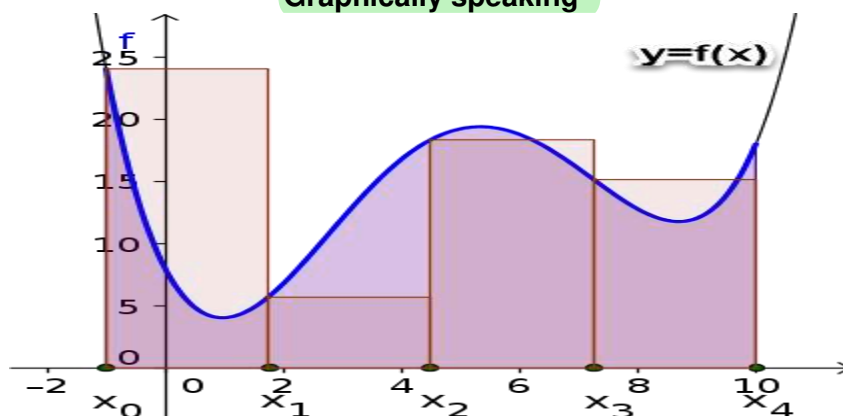
Property – 3: Definite integral can be expressed as a sum of a finite number of definite sub-integrals:

$$\int_a^d f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^d f(x) dx$$

Where $(a < b < c < d)$

It can be termed as Additive property of definite integrals.

Graphically speaking



Property – 4: A coefficient in definite integral appears a coefficient outside the integral:

$$\int_a^b k f(x) dx = k \int_a^b f(x) dx$$

For example:

$$\begin{aligned} \int_1^5 3cx^2 dx &= c \{(x^3 + c)\}_1^5 = c [\{(5)^3 + c\} - \{(1)^3 + c\}] \\ &= c (125 + c - 1 - c) \Rightarrow \boxed{c(124)} \end{aligned}$$

Property – 5: Definite integrals follow sum & difference rules of integration as followed by indefinite integrals:

$$\int_a^b \{g(x) \pm h(x)\} dx = \int_a^b \{g(x) dx\} \pm \int_a^b \{h(x)\} dx + c$$

Numerical Example(s):

$$\int_{-1}^2 (9 - x^2) dx = 9x - \frac{x^3}{3} \Big|_{-1}^2 \Rightarrow \left\{ 9(2) - \frac{(2)^3}{3} \right\} - \left\{ 9(-1) - \frac{(-1)^3}{3} \right\} \Rightarrow \left(\frac{46}{3} \right) - \left(-\frac{26}{3} \right) \Rightarrow \frac{72}{3} \Rightarrow \boxed{24}$$

TOPIC 019: IMPROPER INTEGRALS

Improper implies 'unfitting' – something not in accordance with accepted standards
Here, mathematically speaking, the element of 'undefined' (∞) will be present.

Can have two reasons for undefined answer:

1. Infinite limits of integration
2. Infinite integrand

1. INFINITE LIMITS OF INTEGRATION:

Consider the following two definite integrals:

$$\int_a^{\infty} f(x) dx$$

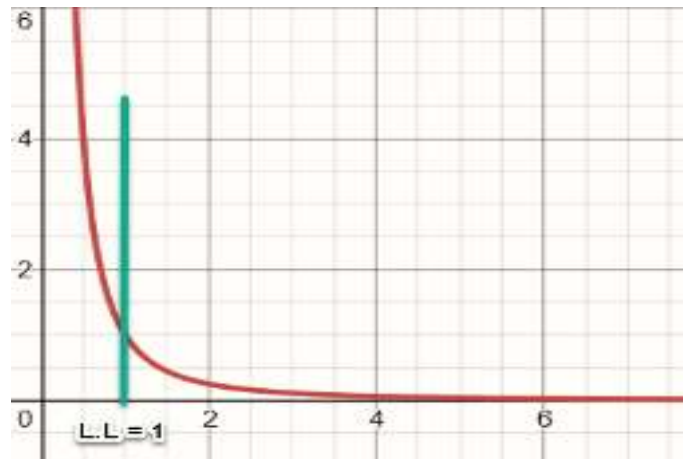
Or

$$\int_{-\infty}^b f(x) dx$$

Either of upper or lower limits are ∞ in these cases. Therefore, an improper integral is there.

Numerical Example # 1:

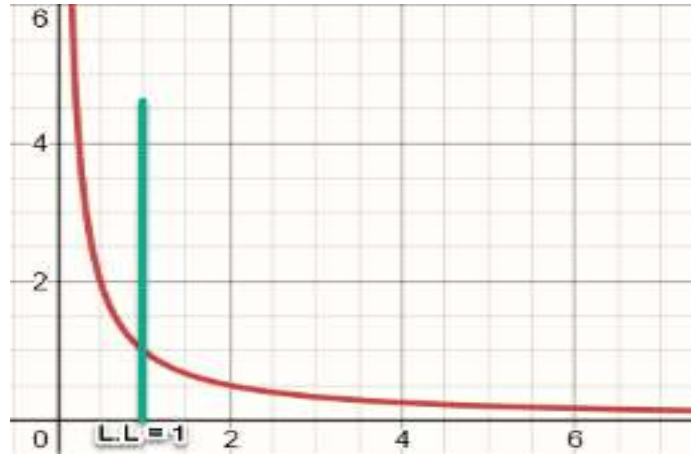
$$\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x} \Big|_1^{\infty} \Rightarrow -\left\{\left(\frac{1}{\infty}\right) - \left(\frac{1}{1}\right)\right\} \Rightarrow -\{(0) - (1)\} \Rightarrow \boxed{1}$$



Or

Numerical Example # 2:

$$\int_1^{\infty} \frac{dx}{x} = \ln|x| \Big|_1^{\infty} \Rightarrow \ln|\infty| - \ln|1| \Rightarrow \infty - 0 \Rightarrow \boxed{\infty}$$



2. INFINITE INTEGRAND

Even if, none of limits $[a, b]$ are infinite, integrand can become undefined – improper integral.

Numerical Example # 1:

$$\int_0^1 \frac{dx}{x} = \ln|x| \Big|_0^1 \Rightarrow \{\ln|1| - \ln|0|\} \Rightarrow \{(0) - (\infty)\} \Rightarrow \boxed{-\infty}$$

Numerical Example # 2:

$$\int_0^9 \frac{dx}{\sqrt{x}} = 2x^{\frac{1}{2}} \Big|_0^9 \Rightarrow 2 \left\{ (9)^{\frac{1}{2}} - (0)^{\frac{1}{2}} \right\} \Rightarrow \boxed{6}$$

Economic applications of improper integrals include a situation with interest from a perpetual bond or the revenue from an indestructible capital asset such as land.

Lesson 06

MARGINAL TO TOTAL ANALYSIS USING INTEGRALS

TOPIC 020: MARGINAL TO TOTAL COST USING INTEGRALS

Given that, $C'(Q) = 2e^{0.2Q}$, $C_F = 90$, $C(Q) = ?$

$$\int C'(Q) dQ = \int 2e^{0.2Q} dQ = 2 \int e^{0.2Q} dQ$$

$$\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)}$$

$$\int C'(Q) dQ = 2 \int e^{0.2Q} dQ \Rightarrow 2 \left(\frac{e^{0.2Q}}{0.2} \right) + c \Rightarrow 10(e^{0.2Q}) + c$$

Since derivative & integral are inverse operators: $C(Q) = 10e^{0.2Q} + c$

Indeterminate answer as (c) is still arbitrary.

Use $C_F = 90 \Rightarrow C(Q) = 10e^{0.2Q} + c$

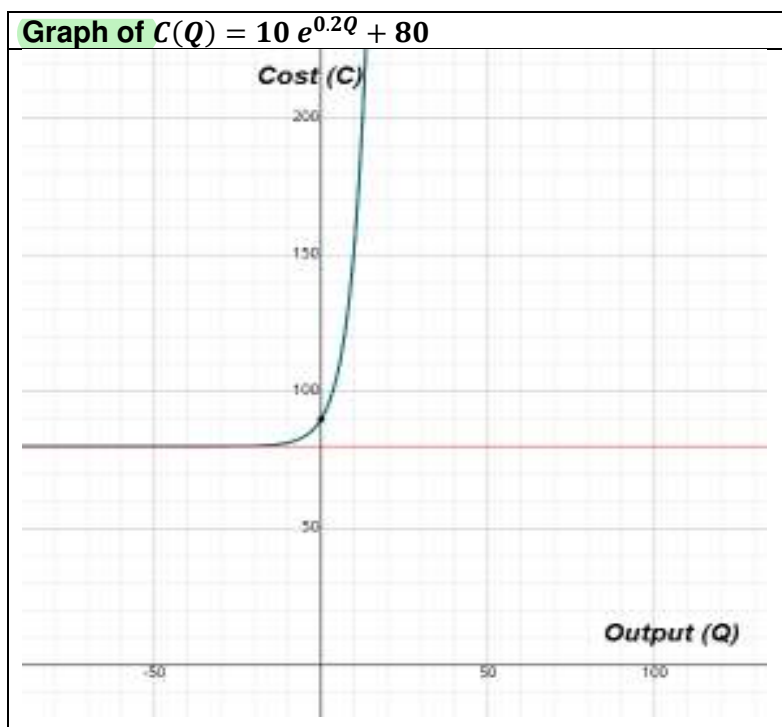
Initial Condition requires: $Q = 0$

$$C(0) = 10e^{0.2(0)} + c \Rightarrow C(0) = 10e^0 + c \Rightarrow C(0) = 10 + c$$

$$C(0) = 10 + c \text{ as } C(Q)_{Q=0} = C(0) = C_F$$

$$\Rightarrow C_F = 10 + c \Rightarrow c = C_F - 10 \Rightarrow c = 90 - 10 \Rightarrow c = 80$$

Substituting value of c : $C(Q) = 10e^{0.2Q} + 80$



TOPIC 021: MARGINAL TO TOTAL REVENUE USING INTEGRALS

Given that, $R'(Q) = 28Q - e^{0.3Q}$, $R(Q) = ?$

$$\begin{aligned} \int R'(Q) dQ &= \int (28Q - e^{0.3Q}) dQ = 28 \int Q dQ - \int e^{0.3Q} dQ \\ &= 14Q^2 + c_1 - \left(\frac{1}{0.3} \times 0.3 \right) \int e^{0.3Q} dQ = 14Q^2 - \frac{1}{0.3} \int 0.3 e^{0.3Q} dQ \end{aligned}$$

$$\begin{aligned} \text{Since: } \int f'(x) e^{f(x)} dx &= e^{f(x)} + c \\ &= 14Q^2 + c_1 - \left(\frac{1}{0.3} e^{0.3Q} + c_2 \right) = 14Q^2 - \frac{1}{0.3} e^{0.3Q} + c \quad [\because c_1 - c_2 = c] \\ &= 14Q^2 - \frac{10}{3} e^{0.3Q} + c \end{aligned}$$

Since derivative & integral are inverse operators: $\int R'(Q) dQ = R(Q)$

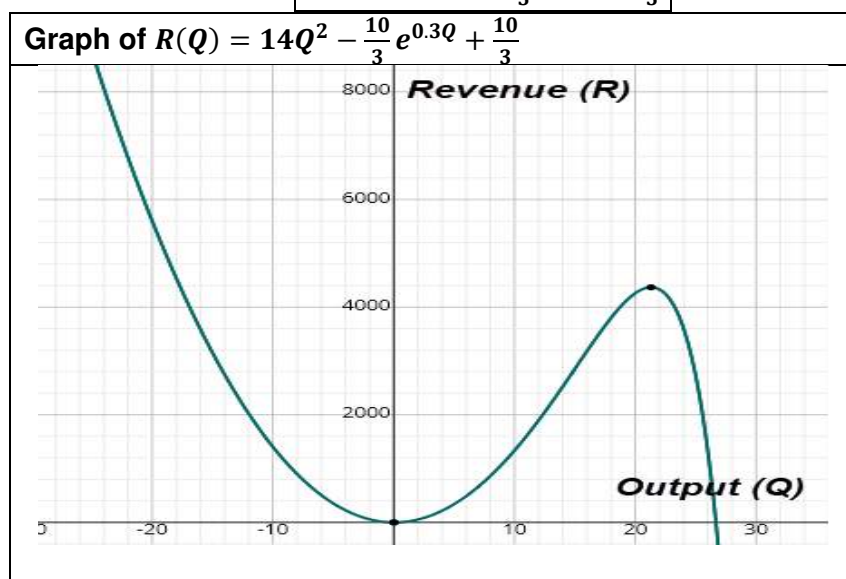
Finally, we have $R(Q) = 14Q^2 - \frac{10}{3} e^{0.3Q} + c$ **General Solution**

Indeterminate answer as (c) is still arbitrary.

Use initial condition: $Q = 0 \Rightarrow R(0) = 14(0)^2 - \frac{10}{3} e^{0.3(0)} + c \Rightarrow R(0) = -\frac{10}{3} + c \Rightarrow c = R(0) + \frac{10}{3}$ $[\because R(0) = 0]$: no revenue at no output.

$$\Rightarrow c = \frac{10}{3}$$

Substituting value of c : $R(Q) = 14Q^2 - \frac{10}{3} e^{0.3Q} + \frac{10}{3}$ **Definite Solution**



TOPIC 022: MARGINAL TO TOTAL SAVINGS USING INTEGRALS

Given that, $S'(Y) = 0.3 - 0.1Y^{-\frac{1}{2}}$, $(S)_{Y=81} = 0$, $S(Y) = ?$

Caveat: $(S)_{Y=81} = 0$ not $(Y)_{S=81} = 0$ as income must not be zero for a positive level of savings $\{(Y \neq 0)_{S>0}\}$.

Given condition is an initial condition of dependent variable in terms of independent variable.

$$\int S'(Y) dY = \int \left(0.3 - 0.1Y^{-\frac{1}{2}} \right) dY \Rightarrow \int 0.3 dY - \int 0.1Y^{-\frac{1}{2}} dY$$

$$S(Y) = 0.3Y - 0.1 \left(\frac{Y^{\frac{1}{2}}}{\frac{1}{2}} \right) + c \Rightarrow 0.3Y - 0.2 \left(Y^{\frac{1}{2}} \right) + c$$

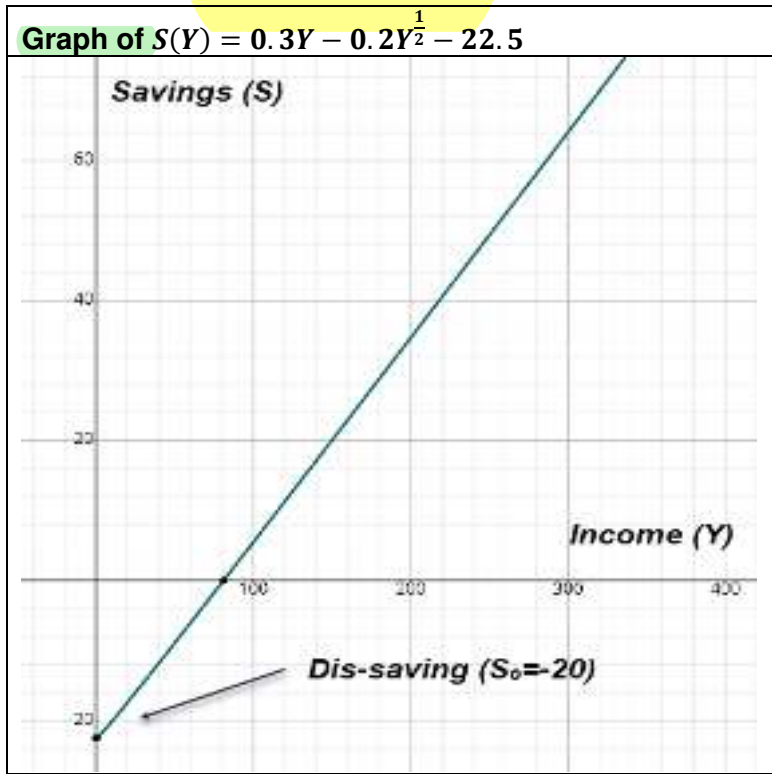
$$S(Y) = 0.3Y - 0.2 \left(Y^{\frac{1}{2}} \right) + c \quad \text{Indeterminate savings function}$$

Given $(S)_{Y=81} = 0$, therefore substituting $Y = 81$:

$$S(81) = 0.3(81) - 0.2 \left(81^{\frac{1}{2}} \right) + c \Rightarrow 0 = 22.5 + c \Rightarrow c = -22.5$$

Substituting value of c , we get determinate savings function:

$$S(Y) = 0.3Y - 0.2Y^{\frac{1}{2}} - 22.5$$



TOPIC 023: SHORT RUN PRODUCTION FUNCTION FROM MARGINAL PRODUCTS USING INTEGRALS

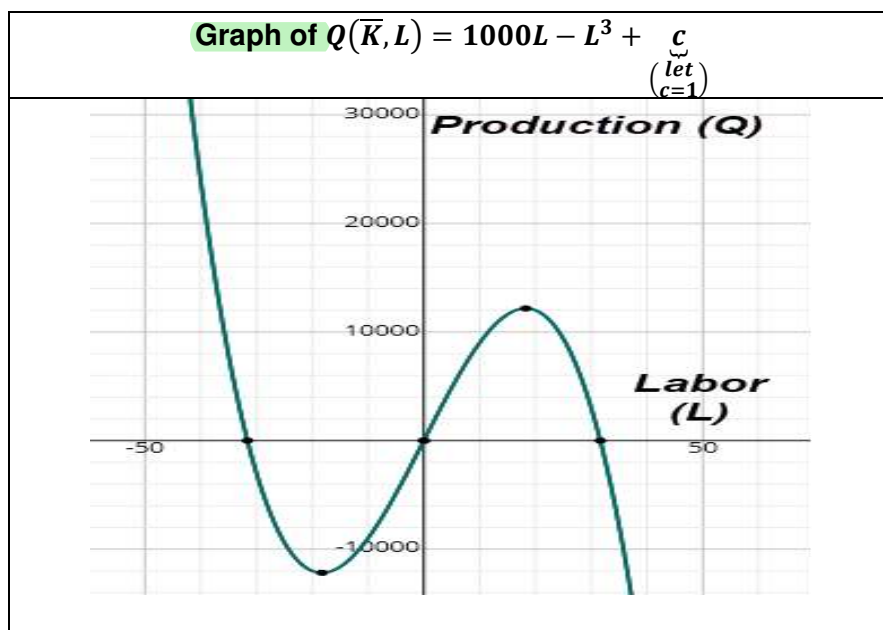
Short-run production functions can be found by using marginal product function.

Let $MP_L = \frac{d}{dL}\{Q(\bar{K}, L)\} = 1000 - 3L^2$.

Here $\bar{K}$ shows the constancy of capital in the short run.

Integrating w.r.t. L on both sides: $\int \frac{d}{dL}\{Q(\bar{K}, L)\} dL = \int (1000 - 3L^2) dL$

$$\begin{aligned} Q(\bar{K}, L) &= \int (1000) dL - \int (3L^2) dL \Rightarrow 1000 \int (L^0) dL - 3 \int (L^2) dL \\ &= \{1000(L) + c_1\} - \left\{3 \left(\frac{L^3}{3}\right) + c_2\right\} \Rightarrow 1000L - L^3 + (c_1 - c_2) \\ Q(\bar{K}, L) &= 1000L - L^3 + c \quad \because c_1 - c_2 \end{aligned}$$



Lesson 07

MARGINAL TO TOTAL ANALYSIS USING INTEGRALS (CONTINUED)

TOPIC 024: CONSUMPTION FUNCTION FROM MPC USING INTEGRALS

Given that, $MPC = 20 + \frac{10}{Y^4}$ $(C)_{Y=16} = 420$, $C(Y) = ?$

Caveat: $(S)_{Y=81} = 0$ NOT $(Y)_{S=81} = 0$.
saving can happen with some income *saving can't happen with no income*
 $(Y \neq 0 \Rightarrow S > 0)$ $(Y = 0 \Rightarrow S \leq 0)$

$$\int (MPC) dY = \int C'(Y) dY = \int \left(20 + \frac{10}{Y^4} \right) dY \Rightarrow \int (20) dY + \int 10Y^{-\frac{3}{4}} dY$$

$$C(Y) = 20Y + 10 \left(\frac{Y^{\frac{1}{4}}}{\frac{1}{4}} \right) + c \Rightarrow 20Y + 40 \left(Y^{\frac{1}{4}} \right) + c$$

$$C(Y) = 20Y + 40 \left(Y^{\frac{1}{4}} \right) + c \quad \text{Indeterminate consumption function}$$

Given $(C)_{Y=16} = 420$, therefore substituting:

$$(C)_{Y=16} = C(16) = 20(16) + 40 \left(16^{\frac{1}{4}} \right) + c$$

$$420 = 320 + 80 + c \Rightarrow 420 = 400 + c$$

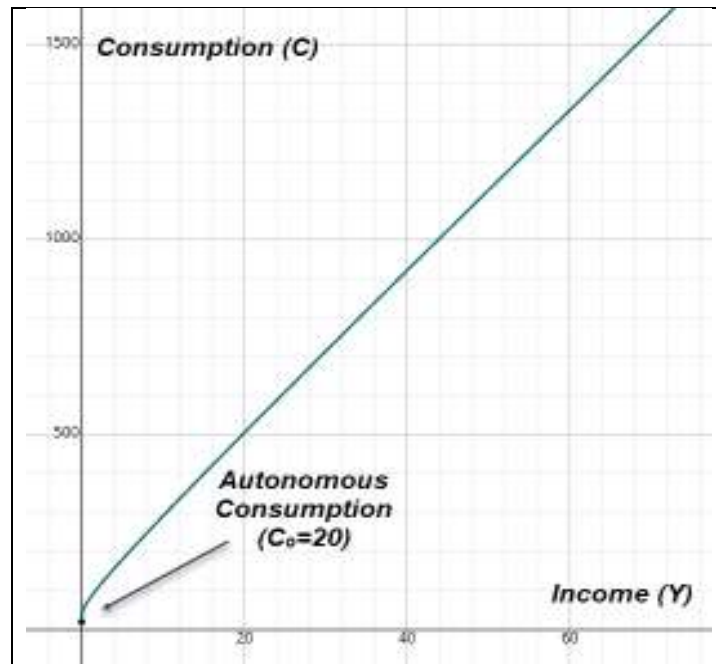
$$c = 420 - 400 \Rightarrow c = 20$$

Substituting value of c , we get determinate consumption function

$$C(Y) = 20Y + 40Y^{\frac{1}{4}} + 20$$

$$C(Y) = 20 \left(Y + 2Y^{\frac{1}{4}} + 1 \right)$$

$$\text{Graph of } C(Y) = 20Y + 40Y^{\frac{1}{4}} + 20$$



TOPIC 025: MARGINAL TO TOTAL IMPORTS USING INTEGRALS

Given that, $M'(Y) = 0.1$, $(M)_{Y=0} = 20$, $M(Y) = ?$

$$\int M'(Y) dY = \int 0.1 dY \Rightarrow 0.1 \int (1) dY \Rightarrow 0.1 Y$$

$$\int M'(Y) dY = 0.1 Y + c$$

Since derivative & integral are inverse operators: $M(Y) = 0.1 Y + c$

Indeterminate or General Solution as (c) is still arbitrary.

Initial Condition implies: $Y = 0$

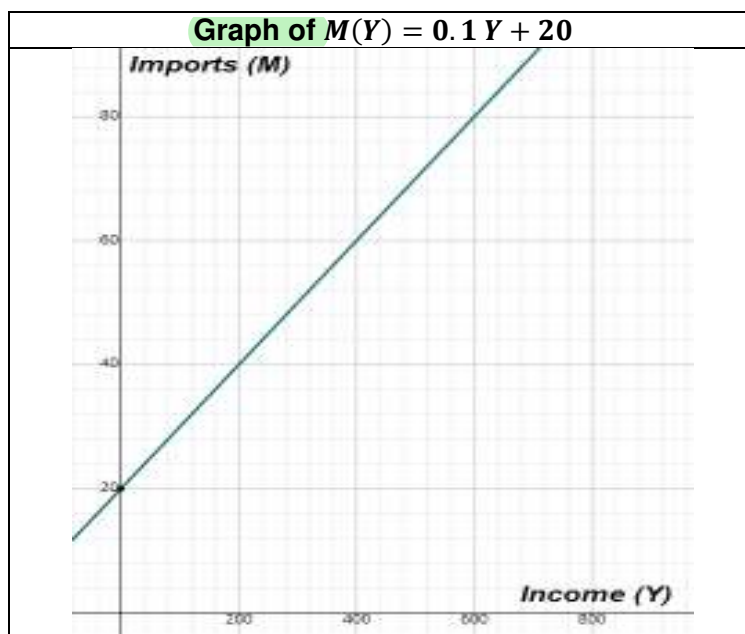
Use $(M)_{Y=0} = M(0) = 20 \Rightarrow M(0) = 0.1(0) + c$

$$20 = 0 + c \Rightarrow c = 20$$

Substituting value of c in:

$M(Y) = 0.1 Y + c$ becomes

$M(Y) = 0.1 Y + 20$ Definite Solution for Imports Function.



TOPIC 026: MARGINAL TO TOTAL PROFIT USING INTEGRALS

The marginal profit of a certain commodity is $\pi'(q) = 100 - 2q$ when q units are produced. When 10 units are produced, the profit is \$700.

$$\pi'(q) = 100 - 2q$$

Moreover, $\{\pi(q)\}_{q=10} = \$700$

$$\pi'(q) = \frac{d}{dq}\{\pi(q)\} = 100 - 2q.$$

Integrating w.r.t. q on both sides: $\int \frac{d}{dq}\{\pi(q)\} dq = \int (100 - 2q) dq$

$$\pi(q) = \int (100) dq - \int (2q) dq \Rightarrow 100 \int (1) dq - 2 \int (q) dq$$

$$\pi(q) = 100q - 2\left(\frac{q^2}{2}\right) + c \Rightarrow 100q - q^2 + c$$

$$\boxed{\pi(q) = 100q - q^2 + c} \text{ Indeterminate Profit function.}$$

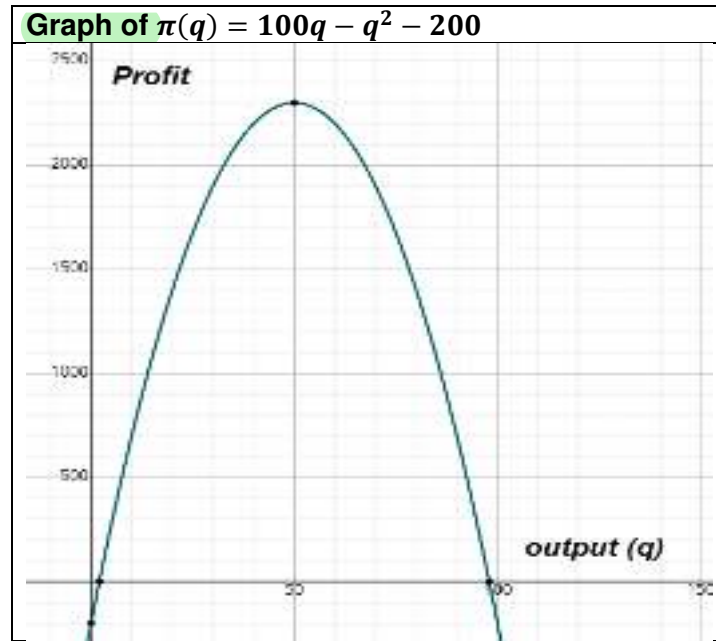
As given, $\{\pi(q)\}_{q=10} = \$700$, it can be substituted as $\pi(10) = \$700$

$$\pi(10) = 700 = 100(10) - (10)^2 + c \Rightarrow 700 = 1000 - 100 + c$$

$$700 = 900 + c$$

$$\boxed{c = -200}$$

$$\boxed{\pi(q) = 100q - q^2 - 200}$$



Determine Profit function.

If $q = 50$, then

$$\pi(50) = 100(50) - (50)^2 - 200 = \$2300$$
$$\pi(50) = \$2300$$

Lesson 08

VARIOUS ECONOMIC APPLICATIONS OF INTEGRALS

TOPIC 027: RATE OF GROWTH OF SHOPS USING INTEGRALS

The rate of growth in the number of shops, n , in a chain of local grocery stores after t months satisfies $\frac{dn}{dt} = \frac{3}{\sqrt{t}}$.

$$n(t) = \int \frac{3}{\sqrt{t}} dt \Rightarrow \int 3t^{-\frac{1}{2}} dt$$

$$n(t) = \int 3t^{-\frac{1}{2}} dt \Rightarrow 3 \left\{ \left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + c_1 \right\}$$

$$n(t) = 3 \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \right) + \underline{3c_1} \Rightarrow 6t^{\frac{1}{2}} + \underline{c}$$

$n(t) = 6t^{\frac{1}{2}} + c$ Indeterminate solution for shops' growth function due to c .
Introducing initial condition: $t = 0$, in the preceding solution.

$$n(0) = 6(0)^{\frac{1}{2}} + c \Rightarrow n(0) = c \Rightarrow \underline{c = n(0)}$$

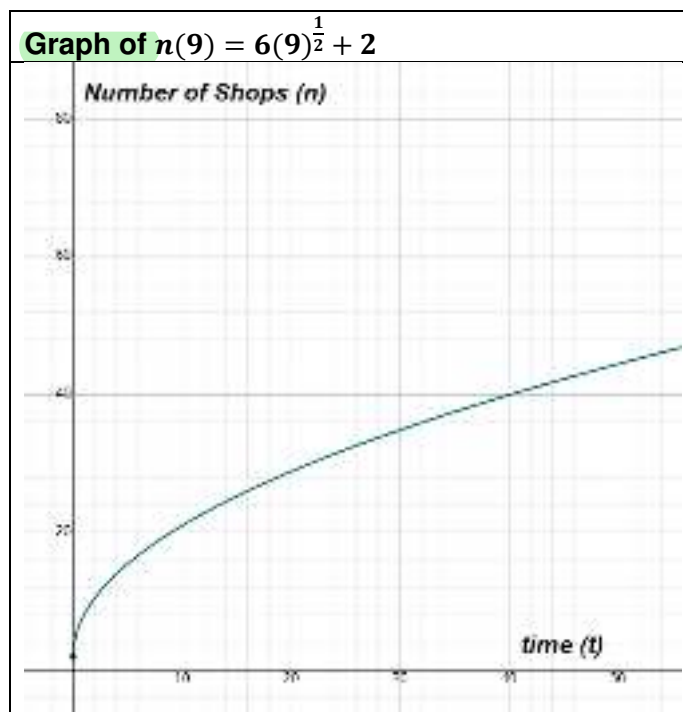
$$n(t) = 6t^{\frac{1}{2}} + n(0)$$

Scenario: If number of shops (n) are two in the very beginning ($n = 2$) _{$t=0$} = $n(0) = 2$, then the definite solution will become:

$$n(t) = 6t^{\frac{1}{2}} + 2$$

Furthermore, if 9 months pass ($t = 9$) after 'now' ($t = 0$) we use the value to $t = 9$ in the shops' growth function:

$$n(9) = 6(9)^{\frac{1}{2}} + 2$$



$n(9)$	$=$	$\frac{18}{\text{During 9 months}}$	$+$	$\frac{2}{\text{Pre-existing shops}}$
<i>Total Shops after 9 months</i>				

$n(9) = 20$

Interpretation: The number of shops after lapse of 9 months will be 20, where 2 shops were already there when the time of analysis started and 18 were established during the span of 9 months.

TOPIC 028: OIL EXTRACTION ANALYSIS USING INTEGRALS

A company begins its extraction of oil from a newly discovered oil field at $t = 0$. The rate of extraction, measured in thousands of barrels per year, is given by $\frac{dN}{dt} = 30t^2 - 4t^3$.

Given that, $\frac{dN}{dt} = 30t^2 - 4t^3$, $\{N(t)\}_{t=0} = \text{initial condition, } N(t) = ?$

$$\int \frac{dN}{dt} dt = \int (30t^2 - 4t^3) dt \Rightarrow \int 30t^2 dt - \int 4t^3 dt$$

$$N(t) = 30 \int t^2 dt - 4 \int t^3 dt \Rightarrow$$

$$N(t) = 30 \left(\frac{t^3}{3} + c_1 \right) - 4 \left(\frac{t^4}{4} + c_2 \right) \Rightarrow (10t^3 + 30c_1) - (t^4 + 4c_2)$$

$$N(t) = 10t^3 - t^4 + (30c_1 - 4c_2) \Rightarrow 10t^3 - t^4 + (c)$$

$$N(t) = 10t^3 - t^4 + c \Rightarrow 10t^3 - t^4 + c$$

$$N(t) = 10t^3 - t^4 + c$$

Indeterminate Oil Extraction function due to c .
 Introducing Initial Condition: $t = 0$ on both sides:

$$N(0) = 10(0)^3 - (0)^4 + c$$

$$N(0) = 10(0) - 0 + c$$

$c = N(0)$ arbitrary constant definitized in a symbolic manner rather than a numeric one.

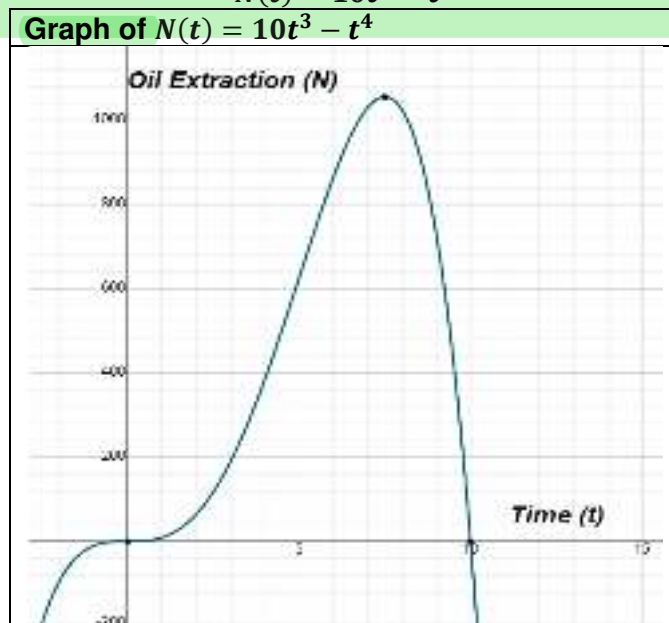
Substituting value of c , we get the definite solution for Oil Extraction function:

$$N(t) = 10t^3 - t^4 + N(0)$$

Scenario: If company starts extracting from very beginning ($t = 0$), no oil would be extracted till at that point of time.

$t = 0 \Rightarrow N(0) = 0$. Under this scenario, oil extraction function is:

$$N(t) = 10t^3 - t^4$$



If company extract oil for four years ($t = 4$), then the number of barrels (in thousands) would be:

$N(4) = 10(4)^3 - (4)^4 = 384$ thousand barrels **or 384,000** barrels of oil would be extracted after 4 years to oil extraction process.

TOPIC 029: CAPITAL FORMATION USING INTEGRALS

Given that, $I(t) = 3t^{\frac{1}{2}}$, $K(0) = 0$, $K(t) = ?$

$$K(t) = \int I(t) dt = \int 3t^{\frac{1}{2}} dt$$

$$K(t) = \int 3t^{\frac{1}{2}} dt \Rightarrow 2t^{\frac{3}{2}} + c$$

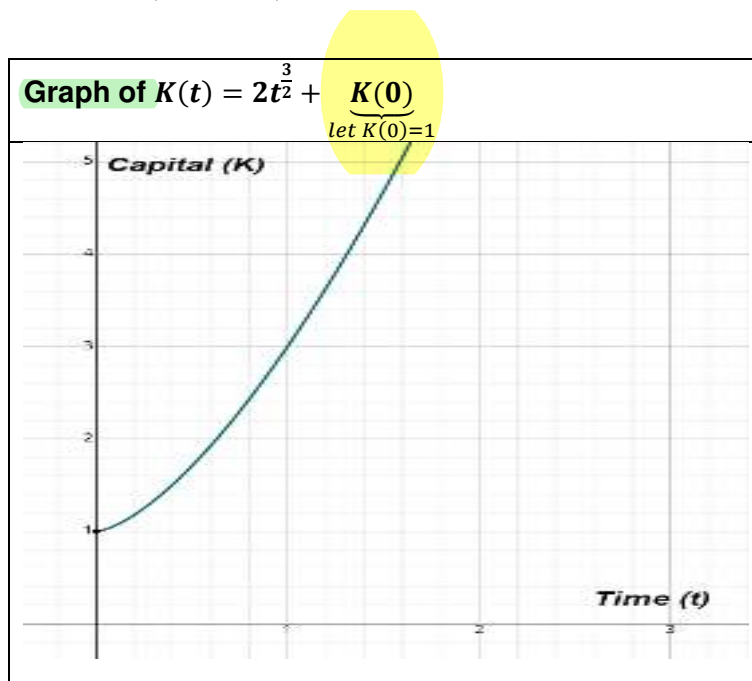
$K(t) = 2t^{\frac{3}{2}} + c$ Indeterminate Capital accumulation function due to c .

Introducing Initial Condition: $t = 0$ on both sides:

$$K(0) = 2(0)^{\frac{3}{2}} + c$$

$K(0) = c$ Substituting in Indeterminate Capital accumulation function

$K(t) = 2t^{\frac{3}{2}} + K(0)$ Determinate Capital accumulation function



TOPIC 030: RETAILER'S STORAGE COSTS USING INTEGRALS

A retailer receives a shipment of 10,000 kilograms of rice that will be used up over a 5-month period at the constant rate of 2,000 kilograms per month. Storage costs are 1 cent per kilogram per month. The storage costs function is $S(t) = 0.01(10000 - 2000t)$.

How much will the retailer pay in storage costs over the next 5 months?

$$\frac{dS}{dt} = 0.01(10000 - 2000t)$$

$$\begin{aligned} S(t) &= \int S(t) dt \Rightarrow \int \{0.01(10000 - 2000t)\} dt \Rightarrow 0.01 \int (10000 - 2000t) dt \\ &= 0.01 \int (10000) dt - 2000 \int (t) dt \\ &= 0.01 \left\{ (10000t) - 2000 \left(\frac{t^2}{2} \right) \right\} + c \Rightarrow 0.01(10000t - 1000t^2) + c \end{aligned}$$

$$\boxed{S(t) = 100t - 10t^2 + c} \text{ Indeterminate solution for Storage cost function due to } c.$$

Introducing initial condition: $t = 0$, in the preceding solution.

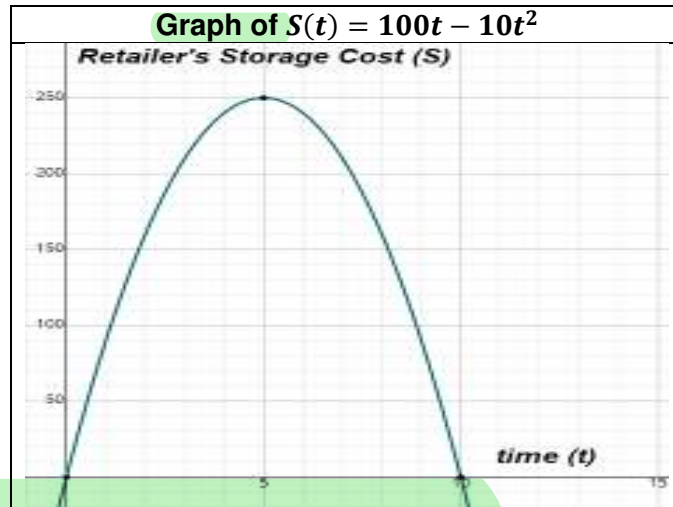
$$S(0) = 100(0) - 10(0)^2 + c \Rightarrow S(0) = c \Rightarrow \underline{c = S(0)}$$

$S(0)$ is the initial level of storage cost, when the ship arrives.

During this time of unloading, the storage is not amassed. Therefore, no storage is incurred.

$$S(0) = 0$$

$$\boxed{S(t) = 100t - 10t^2} \text{ Definite solution for Storage cost function.}$$



Scenario: Storage cost during five time-intervals ($t = 5$):
The definite solution will become:

$$S(t) = 100(5) - 10(5)^2$$

$$S(t) = \$250$$

Interpretation: The storage costs during the five months will be equal to \$ 250.

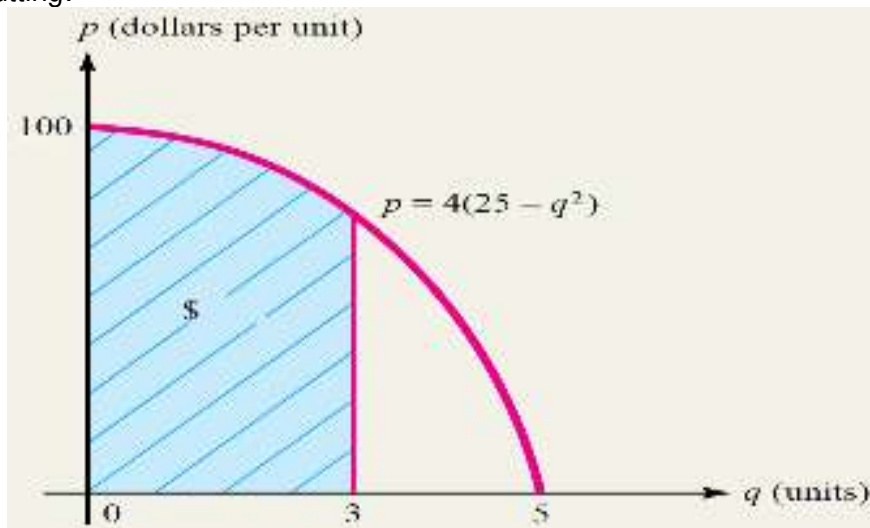
Lesson 09

CONSUMER AND PRODUCER SURPLUS ANALYSIS USING INTEGRALS

TOPIC 031: CONSUMER WILLINGNESS TO PAY USING INTEGRALS

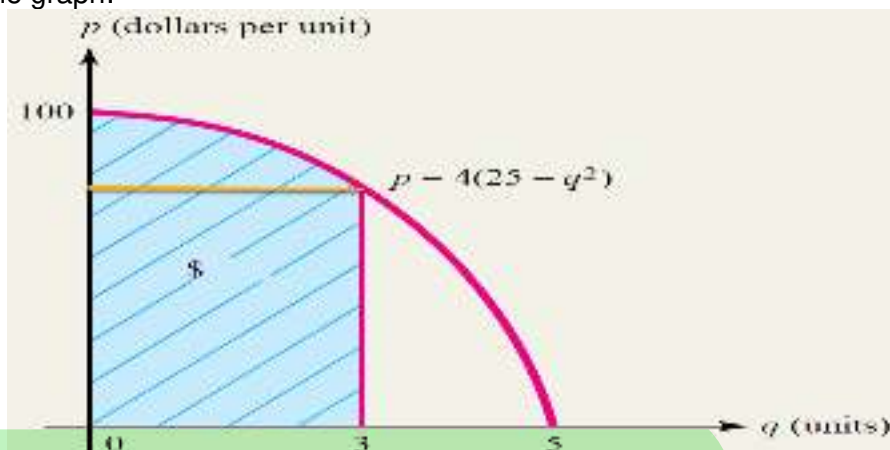
Suppose that the consumers' demand function for a certain commodity is $p(q) = 4(25 - q^2)$ dollars per unit. Find the consumer surplus, if consumer is willing to spend 3 units of the commodity.

Graphically putting:

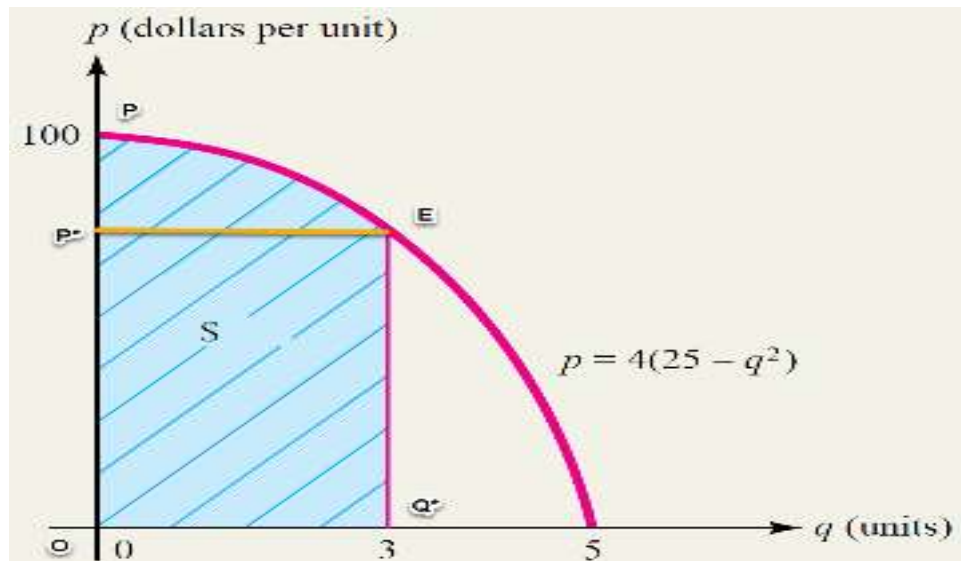


Given $q = 3$, therefore $p(3) = 4\{25 - (3)^2\} = 64$

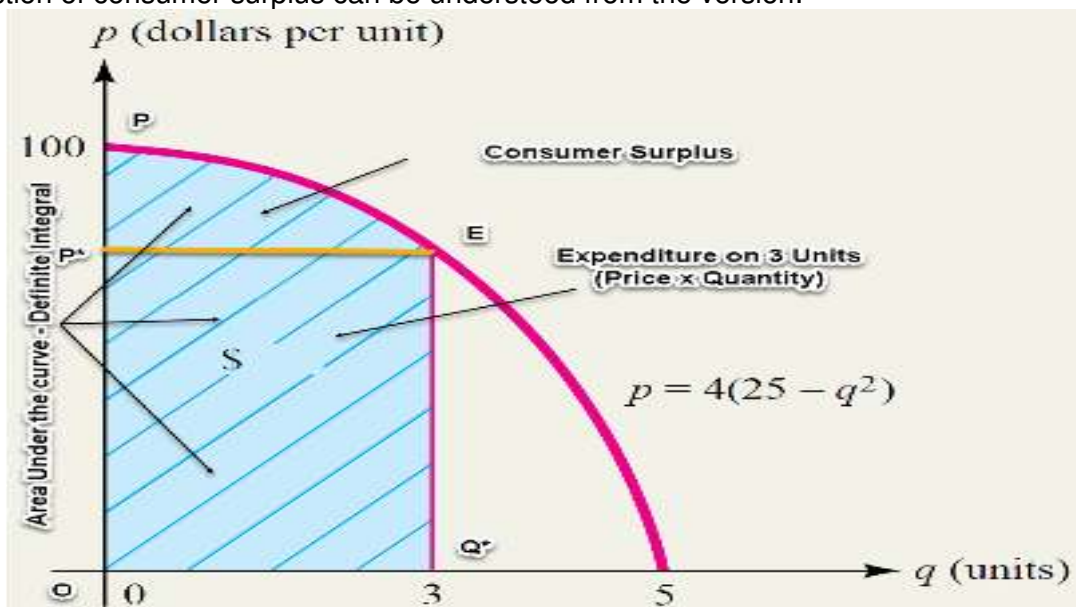
Visualizing the graph:



Yellow line demarcates an approximate triangle and a rectangle.
Allotting proper names to triangle $\triangle P^*EP^*$ and rectangle $\square OQ^*EP^*$:



Extraction of consumer surplus can be understood from the version:



Mathematically speaking:

Area under the curve – Definite Integral: $\int_0^3 p(q) dq = \int_0^3 \{4(25 - q^2)\} dq$

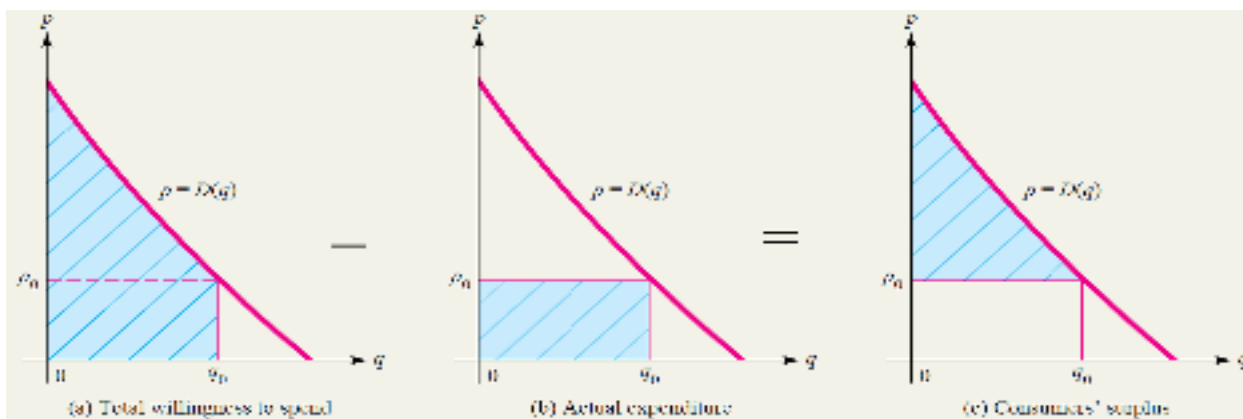
The shape of the whole shaded area is a Trapezoid OQ^*EP . (One pair of sides parallel). Economic sense of it is the 'consumers' willingness to pay' for 3 units.

Expenditure on 3 Units (Price $\times$ Quantity): $\square OQ^*EP^* = |\overline{OQ^*}| \times |\overline{OP^*}| = |\overline{P^*E}| \times |\overline{EQ^*}|$

This is a rectangular shape. Economic sense of it is 'Actual Expenditure'.

Triangular Area: $\triangle P^*EP$

This is a triangular shape. Economic sense of it is 'Consumer Surplus'.



It is evident from the diagram that the area under the curve is the sum of triangular area and the rectangular area:

$$\int_0^3 p(q) dq = \text{Area } OQ^*EP^* + \Delta P^*EP$$

Therefore, triangular area is the difference of area under the curve and expenditure on the 3 units:

$$\Delta P^*EP = \int_0^3 p(q) dq - \text{Area } OQ^*EP^*$$

$$\Delta P^*EP = \int_0^3 \{4(25 - q^2)\} dq - \{|\overline{OQ^*}| \times |\overline{OP^*}|\}$$

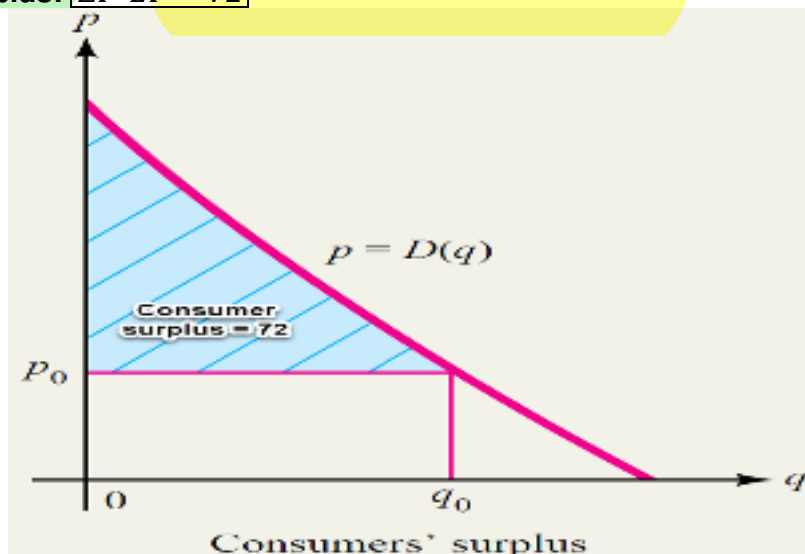
Putting the formula numerically:

$$\Delta P^*EP = 4 \left(\int_0^3 25 dq - \int_0^3 q^2 dq \right) - (3 \times 64) \Rightarrow 4 \left\{ 25(q)_0^3 - \frac{q^3}{3} \right\}_0^3 - (192)$$

$$\Delta P^*EP = 4 \left[25(3 - 0) - \left\{ \frac{(3)^3 - (0)^3}{3} \right\} \right] - (192) \Rightarrow 4[25(3) - (9)] - 192$$

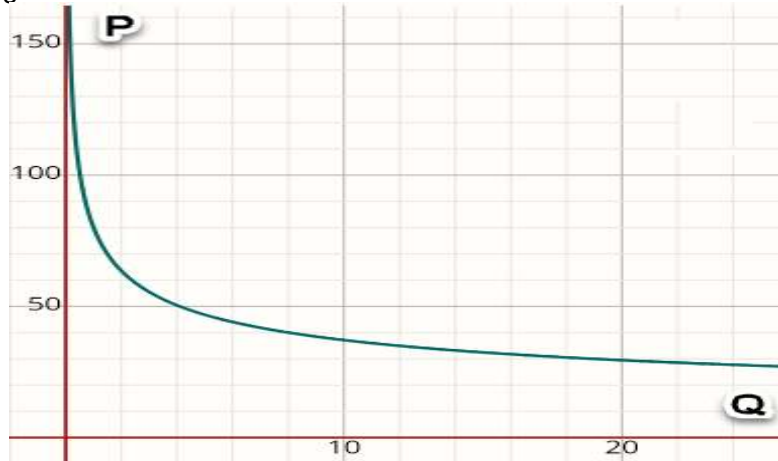
$$\Delta P^*EP = 4\{25(3) - (9)\} - 192 = 72$$

Consumer Surplus: $\Delta P^*EP = 72$



TOPIC 032: CONSUMER SURPLUS USING INTEGRALS

The demand function of a good is given by $P = \frac{80}{\sqrt[3]{Q}}$ dollars per unit. Shade the area which gives the consumer's surplus at $Q = 64$ and use integration to calculate its value. Graphically putting:

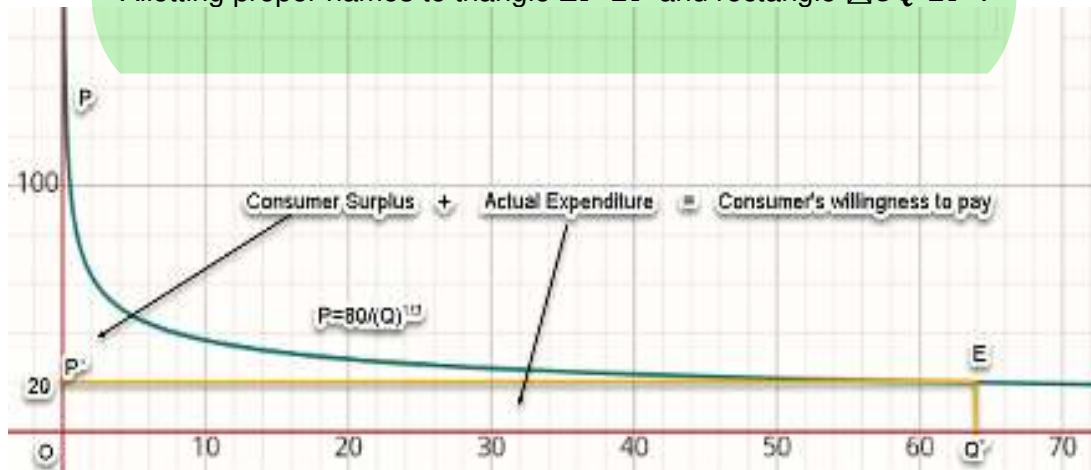


Given $Q = 64$, therefore $P(64) = \frac{80}{\sqrt[3]{64}} = \frac{80}{4} = 20$

Visualizing the graph for Consumer Surplus:

We are looking for consumer surplus.

Allotting proper names to triangle $\triangle P^*EP$ and rectangle $\square OQ^*EP^*$:

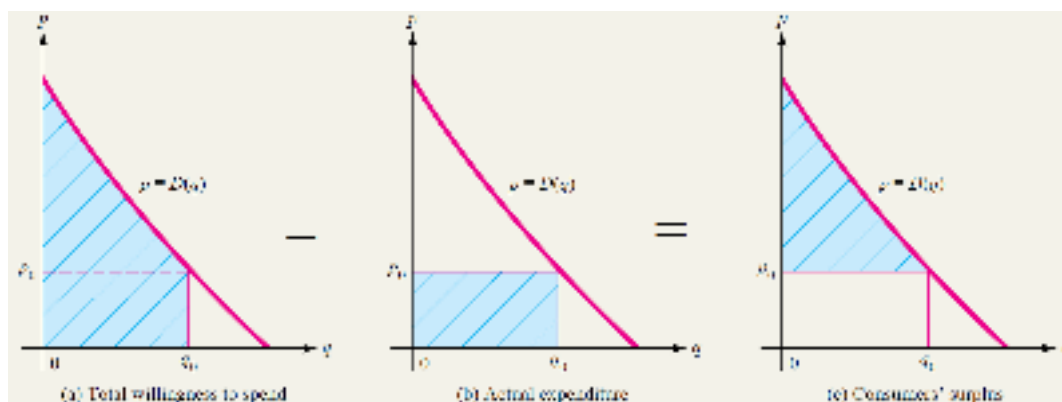


Mathematically speaking:

Consumer's willingness to pay (Area under the curve): $\int_0^{64} P(Q) dQ = \int_0^{64} \left(\frac{80}{\sqrt[3]{Q}} \right) dQ$

Actual Expenditure (Equilibrium Price $\times$ Equilibrium Quantity): $\square OQ^*EP^* := |\overline{OQ^*}| \times |\overline{OP^*}|$

Triangular Area: $\triangle P^*EP$ is the 'Consumer Surplus'.



It is evident from the diagram that the area under the curve is the sum of triangular area and the rectangular area:

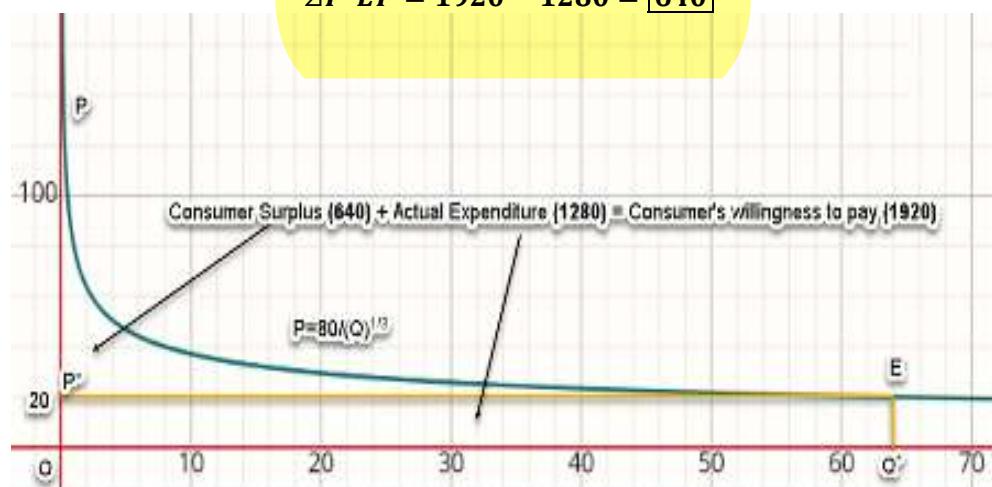
$$\int_0^{64} \left(\frac{80}{\sqrt[3]{Q}} \right) dQ = (|\overline{OQ^*}| \times |\overline{OP^*}|) + \Delta P^*EP$$

Triangular area is the difference of area under the curve and actual expenditure on the 64 units:

$$\Delta P^*EP = \int_0^{64} \left(\frac{80}{\sqrt[3]{Q}} \right) dQ - (|\overline{OQ^*}| \times |\overline{OP^*}|)$$

Putting the formula numerically:

$$\begin{aligned} \Delta P^*EP &= 80 \left\{ \int_0^{64} Q^{-\frac{1}{3}} dQ \right\} - (64 \times 20) \Rightarrow 80 \left(\frac{Q^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \right) \Bigg|_0^{64} - 1280 \\ &= 80 \left(\frac{Q^{\frac{2}{3}}}{\frac{2}{3}} \right) \Bigg|_0^{64} - 1280 \Rightarrow 80 \times \frac{3}{2} \left(Q^{\frac{2}{3}} \right) \Bigg|_0^{64} \Rightarrow 120 \left(64^{\frac{2}{3}} - 0^{\frac{2}{3}} \right) - 1280 \\ \Delta P^*EP &= 1920 - 1280 = \boxed{640} \end{aligned}$$

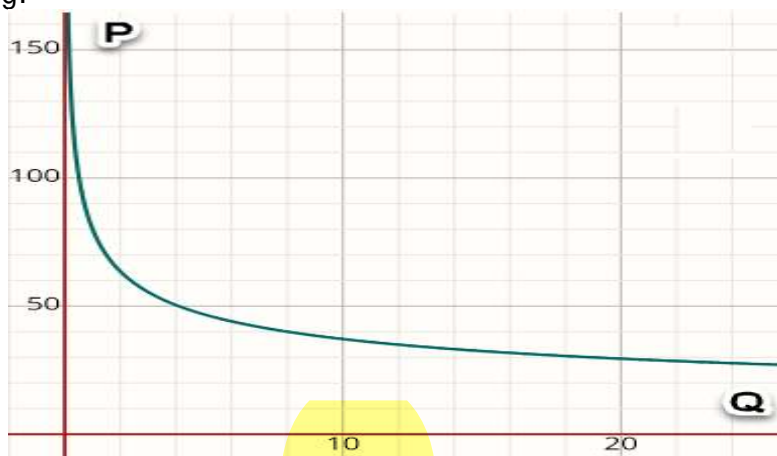


TOPIC 033: DIFFERENTIAL OF CONSUMER SURPLUS USING INTEGRALS

A price-discriminating monopolist faces the weekly market demand curve $P = 60 - 2Q^2$, where q is the quantity he sells. The monopolist charges two customers their reservation prices.

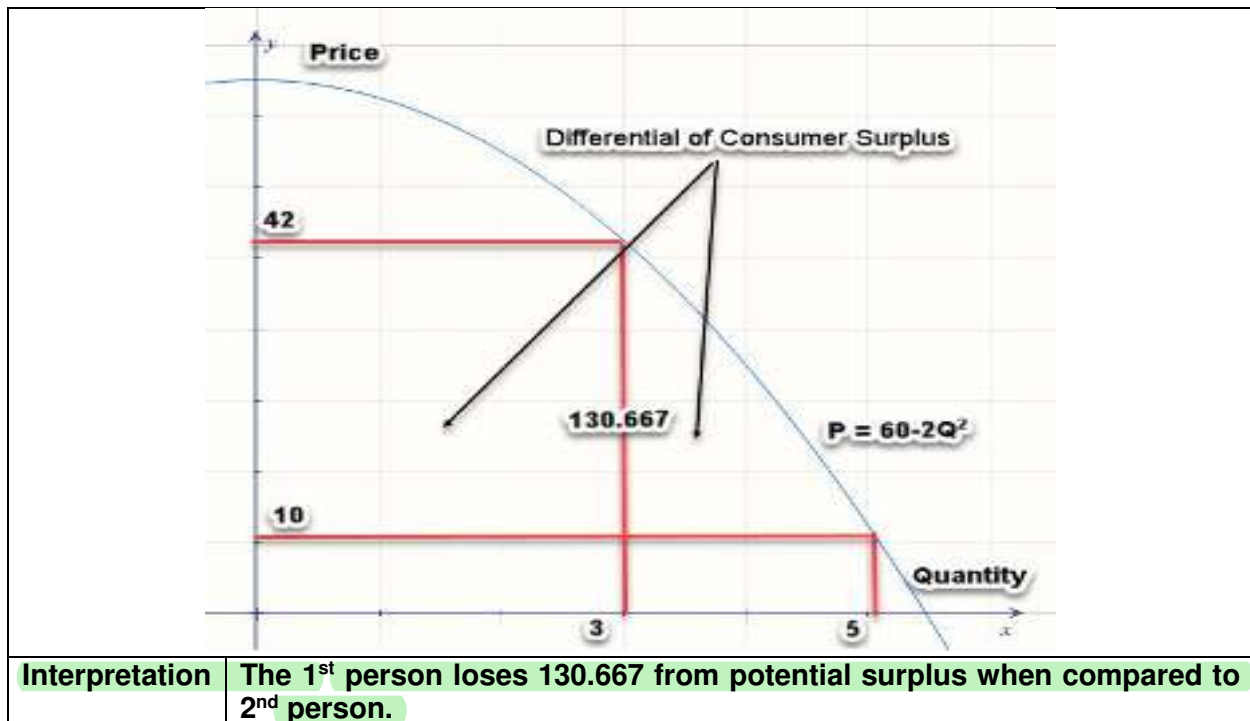
reservation price of the first customer is \$42. The second customer is charged \$10. How much surplus does the first customer lose from not being charged the lower price?

Graphically putting:



Given $P = 60 - 2Q^2$, and $P_1^R = 42$ and $P_2^R = 10$

	First Customer Analysis	Second Customer Analysis
Demand	$P_1^R = 60 - 2(Q_1)^2$	$P_2^R = 60 - 2(Q_2)^2$
Q_d	$42 = 60 - 2(Q_1)^2$ $(Q_1)^2 = \frac{60 - 42}{2} \Rightarrow (Q_1)^2 = 9$ $Q_1 = 3$ Quantity demanded by 1 st person at reservation price.	$10 = 60 - 2(Q_2)^2$ $(Q_2)^2 = \frac{60 - 10}{2}$ $(Q_2)^2 = 25$ $Q_2 = 5$ Quantity demanded by 2 nd person at reservation price.
Actual expenditure	$ \overline{OQ_1} \times \overline{OP_1^R} $ $3 \times 42 = 126$	$ \overline{OQ_2} \times \overline{OP_2^R} $ $5 \times 10 = 50$
Consumer Willingness to pay	$= \int_0^{Q_1} (60 - 2Q^2) dQ$ $= \int_0^3 (60 - 2Q^2) dQ$ $= \left(60Q \Big _0^3 - 2 \left(\frac{Q^3}{3} \right) \Big _0^3 \right)$ $= 60(3 - 0) - 2 \left(\frac{3^3}{3} - \frac{0^3}{3} \right)$ $= 60(3) - 2(9) \Rightarrow 180 - 18$ $= 162$	$= \int_0^{Q_2} (60 - 2Q^2) dQ$ $= \int_0^5 (60 - 2Q^2) dQ$ $= \left(60Q \Big _0^5 - 2 \left(\frac{Q^3}{3} \right) \Big _0^5 \right)$ $= 60(5 - 0) - 2 \left(\frac{5^3}{3} - \frac{0^3}{3} \right)$ $= 60(5) - 2 \left(\frac{125}{3} \right) \Rightarrow 300 - 83.33$ $= 216.667$
Consumers Surplus	$CS_1 = 162 - 126$ $CS_1 = 36$	$CS_2 = 216.667 - 50$ $CS_2 = 166.667$
Differential of Consumer Surpluses	$\Delta CS = CS_2 - CS_1 = 166.667 - 36$ $\Delta CS = 130.667$	



TOPIC 034: PRODUCER & SOCIAL SURPLUS USING INTEGRALS

Demand and Supply functions of a good (shirts) are $P = 60 - 0.6Q$ and $P = 20 + 0.2Q$. Find total (social) surplus.

For equilibrium output, let the prices in demand and supply functions be equal. $60 - 0.6Q = 20 + 0.2Q$

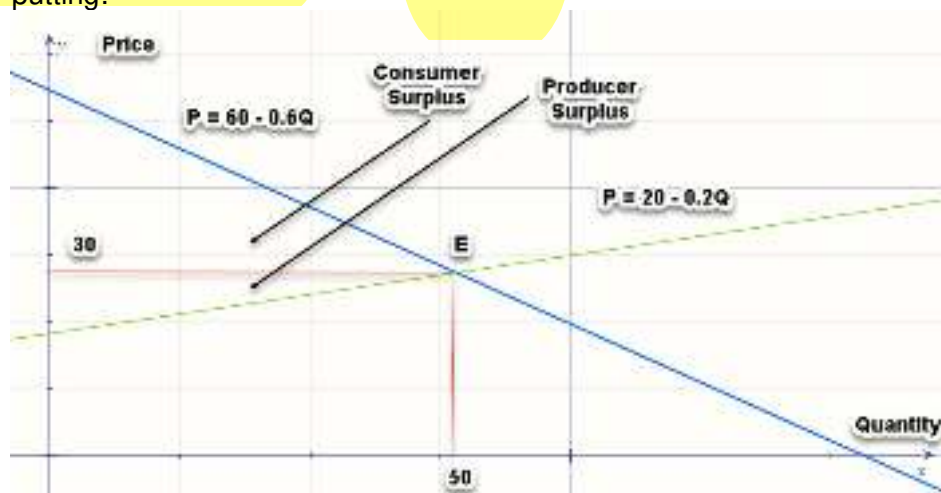
$$60 - 20 = 0.6Q + 0.2Q \Rightarrow 40 = 0.8Q$$

$$Q^* = 50$$

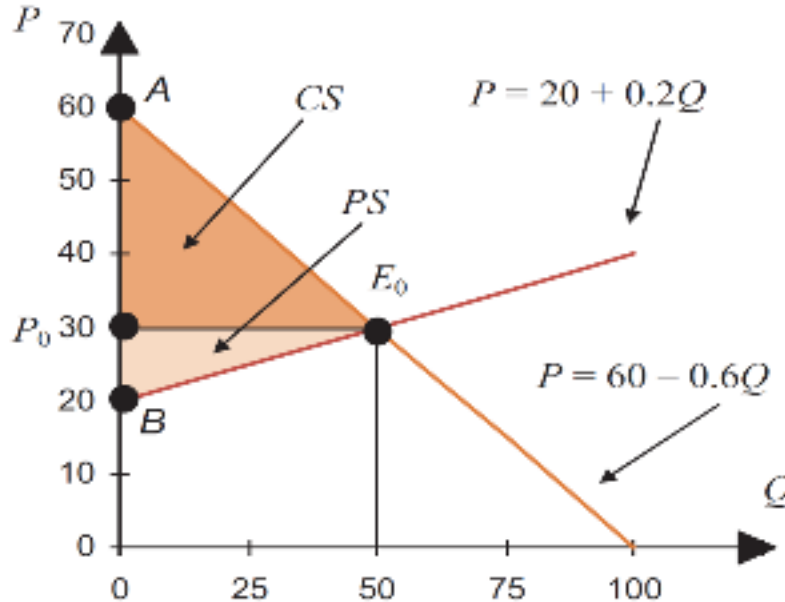
For equilibrium price, substitute Q^* in demand or supply function (we use supply function): $P^* = 20 + 0.2Q^* \Rightarrow P^* = 20 + 0.2(50)$

$$P^* = 30$$

Graphically putting:



Since both consumer surplus and producer surplus are right triangles, we can use area of right triangle formula to find both:



Consumer Surplus: $\triangle AP_0E_0$

Producer Surplus: $\triangle BP_0E_0$

Area of a $\triangle = \frac{1}{2} \times |\text{Base}| \times |\text{Altitude}|$

$$\begin{aligned}\text{Consumer Surplus: Area of } \triangle AP_0E_0 &= \frac{1}{2} \times |\overline{P_0E_0}| \times |\overline{AP_0}| \\ &= \frac{1}{2} \times |50| \times |30|\end{aligned}$$

Consumer Surplus: Area of $\triangle AP_0E_0 = 750$

$$\begin{aligned}\text{Producer Surplus: Area of } \triangle BP_0E_0 &= \frac{1}{2} \times |\overline{P_0E_0}| \times |\overline{BP_0}| \\ &= \frac{1}{2} \times |50| \times |10|\end{aligned}$$

Producer Surplus: Area of $\triangle BP_0E_0 = 250$

Finally, we calculate the Social (total) Surplus = Consumer Surplus + Producer Surplus

Geometrically; Social (total) Surplus: Area of $\triangle ABE_0 = \triangle AP_0E_0 + \triangle BP_0E_0$

$$\triangle ABE_0 = 750 + 250$$

$$\triangle ABE_0 = 1000$$

Hence the Social (total) Surplus is 1000 units.

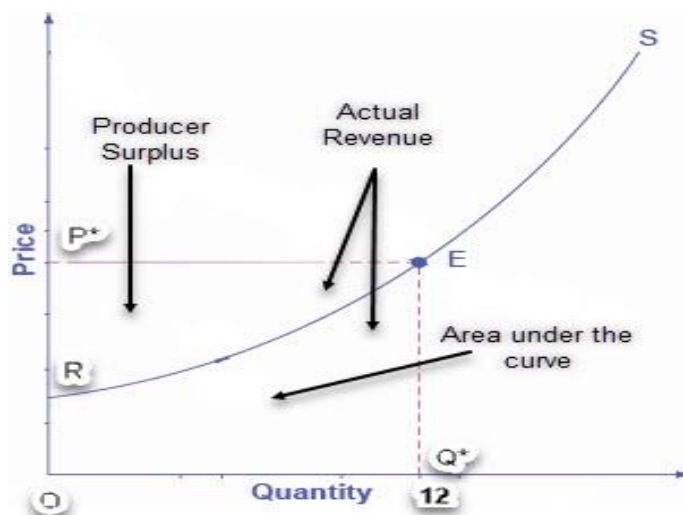
TOPIC 035: PRODUCER SURPLUS WITH TAX USING INTEGRALS

A firm producing computers has the following cost function $C(Q) = \frac{Q^3}{3} - 9Q^2 + 160Q + 20$. The producer has planned to produce 12 computers to meet market demand. What is his total producer surplus if the market price is 270? What is his loss in producer surplus when, due to taxation, he reduces production from 12 to 9 machines?

Producer's surplus at $Q = 12$ is to be calculated and market price is 270.

$$\begin{aligned}MC(Q) &= \frac{d}{dQ} C(Q) \Rightarrow MC(Q) = \frac{d}{dQ} \left(\frac{Q^3}{3} - 9Q^2 + 160Q + 20 \right) \\ MC(Q) &= Q^2 - 18Q + 160\end{aligned}$$

As marginal cost is considered equivalent to supply curve, therefore, we treat $MC(Q)$ as the supply curve.



Producer surplus from the diagram is the difference of Actual Revenue and Area under the curve:

$$\Delta RP^*E = \square OP^*EQ^* - \text{Trapezoid}(OREQ^*)$$

$$\Delta RP^*E = (|\overline{OQ^*}| \times |\overline{OP^*}|) - \int_0^{12} MC(Q) dQ$$

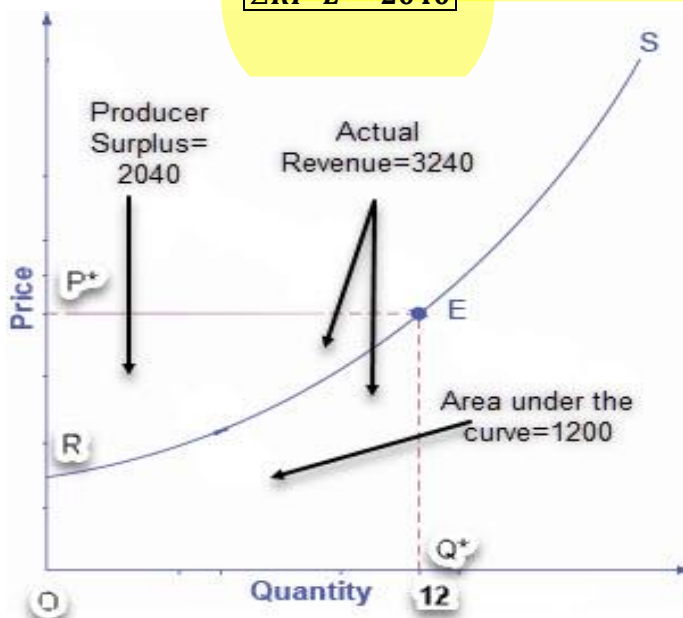
$$\Delta RP^*E = (|12| \times |270|) - \int_0^{12} (Q^2 - 18Q + 160) dQ$$

$$\Delta RP^*E = 3240 - \left(\frac{Q^3}{3} - 9Q^2 + 160Q \right) \Bigg|_0^{12}$$

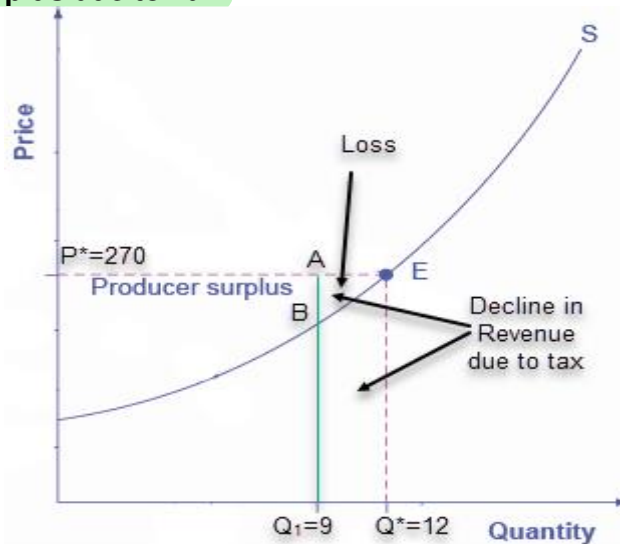
$$= 3240 - \left\{ \left(\frac{12^3}{3} - \frac{0^3}{3} \right) - 9(12^2 - 0^2) + 160(12 - 0) \right\}$$

$$= 3240 - (576 - 1296 + 1920) \Rightarrow 3240 - (1200)$$

$$\boxed{\Delta RP^*E = 2040}$$



Interpretation: 2040 is the producer surplus of the company if it produces 12 units with market price being 270.

Loss in Consumer Surplus due to Tax

Post-tax diagram shows that output is discouraged from Q^* to Q_1 . However, the price is still the same $P^* = 270$.

This causes a decline in the revenue: $\square AQ_1Q^*E = (|\overline{AQ_1}| \times |\overline{Q_1Q^*}|)$

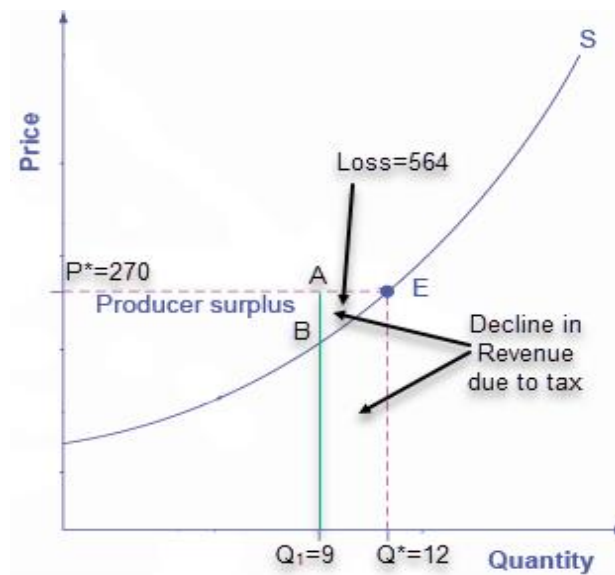
Within this rectangle, the triangle $\triangle ABE$ shows the loss in producer surplus.

Whereas, the area **Trapezoid**(BQ_1Q^*E) is the unwanted area. Mathematically speaking it is the area under the curve with lower and upper limits as 9 and 12, respectively: $\int_9^{12} MC(Q) dQ \Rightarrow \int_9^{12} (Q^2 - 18Q + 160) dQ$.

From the diagram, the triangle is the difference between decline in revenue and the area under the curve: $\triangle ABE = \square AQ_1Q^*E - \text{Trapezoid}(BQ_1Q^*E)$

$$\begin{aligned}\triangle ABE &= (|\overline{AQ_1}| \times |\overline{Q_1Q^*}|) - \int_9^{12} MC(Q) dQ \\ \triangle ABE &= 270 \times (12 - 9) - \int_9^{12} (Q^2 - 18Q + 160) dQ \\ \triangle ABE &= 810 - \left(\frac{Q^3}{3} - 9Q^2 + 160Q \right) \Big|_9^{12} \\ &= 810 - \left\{ \left(\frac{12^3}{3} - \frac{9^3}{3} \right) - 9(12^2 - 9^2) + 160(12 - 9) \right\} \\ &= 810 - (246) \\ \triangle ABE &= 564\end{aligned}$$

Interpretation: 564 units is the loss in producer surplus due to the imposition of the tax.



Lesson 10

FIRM'S ANALYSIS USING INTEGRALS

TOPIC 036: PRESENT VALUE OF CASH FLOW USING INTEGRALS

Present value of cash flow of a firm is represented by: $\pi = \int_0^t R(t) e^{-rt} dt$. Where $R(t)$ is the function defining the cash flows over time. t is time, which is usually measured in years. r is the discount rate. e^{-rt} is the exponential-decay expression that depicts the discounting phenomenon. What is the present value of a continuous revenue flow lasting for y years at the constant rate of D dollars per year and discounted at the rate of r per year?

As already known, that: $\pi = \int_0^t R(t) e^{-rt} dt$

$$\begin{aligned} \text{Here } t = y, R(t) = D. \text{ Therefore; } \pi &= \int_0^y D e^{-rt} dt \Rightarrow D \left\{ \frac{e^{-rt}}{\frac{d}{dt}(-rt)} \right\} \Big|_0^y \\ &= D \left\{ \frac{e^{-rt}}{(-r) \frac{d}{dt}(t)} \right\} \Big|_0^y \Rightarrow D \left\{ \frac{e^{-rt}}{(-r)} \right\} \Big|_0^y \\ &= -\frac{D}{r} (e^{-rt}) \Big|_0^y \\ &= -\frac{D}{r} (e^{-r(y)} - e^{-r(0)}) \Rightarrow -\frac{D}{r} (e^{-ry} - e^0) \\ \pi &= -\frac{D}{r} (e^{-ry} - 1) \Rightarrow \frac{D}{r} (1 - e^{-ry}) \end{aligned}$$

General formula for present value of revenue flows over y periods of time. Formula shows that the present value (π) depends on D , r and y .

Numerical: $D = 3000$, $r = 0.06$, and $y = 2$.

Substituting the values in general formula $\pi = \frac{D}{r} (1 - e^{-ry})$

$$\begin{aligned} \pi &= \frac{3000}{0.06} \{1 - e^{-(0.06)(2)}\} \\ &= \frac{3000}{0.06} (1 - e^{-0.12}) \Rightarrow 50000(1 - 0.8869) \\ \pi &= 50000(0.1131) = \$5655 \text{ approx.} \end{aligned}$$

Interpretation: Present value of revenue in constant dollar flows of \$3000 over 2 year, with discount rate of 6% is approximately equal to \$5655.

TOPIC 037: PRESENT VALUE OF PERPETUAL FLOW USING INTEGRALS

If a cash flow were to persist forever – a situation exemplified by the interest from a perpetual bond or the revenue from an indestructible capital asset such as land – the present value of the flow would be: $\pi = \int_0^\infty R(t) e^{-rt} dt$.

It's an example of improper integral.

Present value of a perpetual income stream flowing at constant rate of D dollars per year and discounted at the rate of r per year?

As already known, that: $\pi_{\text{perptual}} = \int_0^\infty R(t) e^{-rt} dt$

Here $R(t) = D$. Therefore; $\pi_{\text{perptual}} = \int_0^\infty D e^{-r(t)} dt \Rightarrow \lim_{y \rightarrow \infty} \{ \int_0^y D e^{-r(t)} dt \}$

$$\begin{aligned}
&= \lim_{y \rightarrow \infty} \left[D \left\{ \frac{e^{-rt}}{(-r) \frac{d}{dt}(t)} \right\} \right]_0^y \Rightarrow \lim_{y \rightarrow \infty} \left[D \left\{ \frac{e^{-rt}}{(-r)} \right\} \right]_0^y \\
&= \lim_{y \rightarrow \infty} \left\{ -\frac{D}{r} (e^{-rt}) \right\}_0^y \Rightarrow \lim_{y \rightarrow \infty} \left[-\frac{D}{r} \{e^{-r(y)} - e^{-r(0)}\} \right] \\
&= \lim_{y \rightarrow \infty} \left\{ -\frac{D}{r} (e^{-ry} - e^0) \right\} \\
\pi_{perptual} &= \lim_{y \rightarrow \infty} \left\{ -\frac{D}{r} (e^{-ry} - 1) \right\} \Rightarrow \lim_{y \rightarrow \infty} \left\{ \frac{D}{r} (1 - e^{-ry}) \right\} \\
&= \frac{D}{r} \{1 - e^{-r(\infty)}\} \Rightarrow \frac{D}{r} \{1 - e^{-\infty}\} \\
&= \frac{D}{r} \{1 - 0\} \quad [\because e^{-\infty} = 0] \\
\boxed{\pi_{perptual} = \frac{D}{r}}
\end{aligned}$$

General formula for present value of perpetual income stream flowing over time. Formula shows that this present value ($\pi_{perptual}$) depends on D and r .

Numerical: $D = \text{PKR } 1450$ and $r = 5\%$.

Substituting the values in general formula $\pi_{perptual} = \frac{D}{r}$

$$\begin{aligned}
\pi_{perptual} &= \frac{1450}{5\%} \Rightarrow \frac{1450}{0.05} \\
\boxed{\pi_{perptual} = \text{PKR } 29000}
\end{aligned}$$

Interpretation: Present value of the perpetual income stream flowing at constant rate, of 1450 dollars per year and discounted at the rate of 5% per year, is PKR 29,000.

TOPIC 038: WINE STORAGE PROBLEM WITH STORAGE COSTS USING INTEGRALS

Wine-storage problem of Sec. 10.6, we assumed zero storage cost. Permit the storage cost. Let the purchase cost of the case of wine be an amount C , incurred at the present time. Its (future) sale value, which varies with time, may be generally denoted as $V(t)$ – its present value being $V(t)e^{rt}$. Whereas the sale value represents a single future value (there can be only one sale transaction on this case of wine), the storage cost is a stream. Assuming this cost to be a constant stream at the rate of s dollars per year, the total present value of the storage cost incurred in a total of T years can be calculated as follows:

$$\text{Net Present value} = \text{Present value of Sale} - \text{Present value of Costs}$$

Instead of Present Value we use **Net** Present Value, as now there is 'net/difference' of Sales and Costs.

$$\begin{aligned}
\text{Net Present Value} &= (NPV)_{\text{wine}} \\
&= \text{Present value of sale} \\
&\quad - \text{Present value of storage cost} - \text{Purchase cost}
\end{aligned}$$

$$\text{Present value of sale} = \underbrace{V(t)}_{\text{Sale value}} \times \underbrace{(e^{rt})}_{\text{Discounting expression}} = V(t)e^{rt}$$

$$\text{Present value of storage cost} = \int_0^t \left\{ \underbrace{s}_{\text{Constant cost stream}} \times \underbrace{(e^{-rt})}_{\text{Discounting expression}} \right\} dt$$

Caveat: Both 'Present value of sale' and 'Present value of storage cost' are discounted as they occur in future. While 'Present value of sale' is not integrated as it occurs once when the wine will be sold. Whereas, 'Present value of storage cost' is integrated as it occurs repeatedly on yearly basis.

Now, simplifying the expression for 'present value of storage cost':

$$\begin{aligned} &= \int_0^t s e^{-rt} dt \Rightarrow s \int_0^t e^{-rt} dt \\ &= s \left[\frac{e^{-rt}}{\frac{d}{dt}(-rt)} \right]_0^t \\ &= s \left[\frac{e^{-rt}}{(-r)} \right]_0^t \Rightarrow s \left[\frac{e^{-rt}}{(-r)} \right]_0^t \\ &= -\frac{s}{r} (e^{-rt}) \Big|_0^t \\ &= -\frac{s}{r} (e^{-r(t)} - e^{-r(0)}) \Rightarrow -\frac{s}{r} (e^{-ry} - \underline{e^0}) \Rightarrow -\frac{s}{r} (e^{-ry} - \underline{1}) \\ &\text{Present value of storage cost} = \underline{\frac{s}{r} (1 - e^{-rt})} \end{aligned}$$

Substituting value in the formula for Net Present Value of Wine.

$$(NPV)_{\text{wine}} = \text{Present value of sale} - \text{Present value of storage cost} - \text{Purchase cost}$$

$$\begin{aligned} (NPV)_{\text{wine}} &= V(t)e^{rt} - \frac{s}{r}(1 - e^{-rt}) - C \\ &= V(t)e^{rt} - \frac{s}{r} + \frac{s}{r}e^{-rt} - C \Rightarrow V(t)e^{rt} + \frac{s}{r}e^{-rt} - \frac{s}{r} - C \\ \frac{(NPV)_{\text{wine}}}{NPV \text{ is function of } (t)} &= N(t) = \left\{ V(t) + \frac{s}{r} \right\} e^{-rt} - \frac{s}{r} - C \end{aligned}$$

Since the objective is to maximize the NPV of the wine, there is need to optimize:
Differentiating w.r.t. t :

$$N'(t) = \frac{d}{dt} \left[\left\{ V(t) + \frac{s}{r} \right\} e^{-rt} - \frac{s}{r} - C \right] \Rightarrow \frac{d}{dt} \left[\left\{ V(t) + \frac{s}{r} \right\} e^{-rt} \right] - \frac{d}{dt} \left(\frac{s}{r} \right) - \frac{d}{dt} (C)$$

Product Rule requires: $\frac{d}{dt} \{f(x)\} = \frac{d}{dt} \{g(x) \times h(x)\} = g'(x) \times h(x) + h'(x) \times g(x)$

Here $g(x) = \left\{ V(t) + \frac{s}{r} \right\}$ **and** $h(x) = e^{-rt}$

$$\begin{aligned} N'(t) &= \frac{d}{dt} \left[\underbrace{\left\{ V(t) + \frac{s}{r} \right\}}_{g(x)} \times \underbrace{(e^{-rt})}_{h(x)} \right] - \frac{d}{dt} \left(\frac{s}{r} \right) - \frac{d}{dt} (C) \\ N'(t) &= \frac{d}{dt} \left\{ V(t) + \frac{s}{r} \right\} \times e^{-rt} + \frac{d}{dt} (e^{-rt}) \times \left\{ V(t) + \frac{s}{r} \right\} - 0 - 0 \\ N'(t) &= \{V'(t) + 0\} \times e^{-rt} + \left[\left\{ \frac{d}{dt} (-rt) \right\} (e^{-rt}) \right] \times \left\{ V(t) + \frac{s}{r} \right\} \end{aligned}$$

$$N'(t) = \{V'(t)\} \times e^{-rt} + \{(-r)(e^{-rt})\} \times \left\{V(t) + \frac{s}{r}\right\}$$

$$N'(t) = V'(t)e^{-rt} - re^{-rt} \times \left\{V(t) + \frac{s}{r}\right\}$$

$$N'(t) = V'(t)e^{-rt} - V(t)re^{-rt} - \frac{s}{r}e^{-rt}$$

$$N'(t) = \{V'(t) - rV(t) - s\}e^{-rt}$$

First order condition requires $f'(x) = 0$:

$$N'(t) = \{V'(t) - rV(t) - s\}e^{-rt} = 0$$

$$\{V'(t) - rV(t) - s\} = 0 \Rightarrow \boxed{V'(t) = rV(t) + s}$$

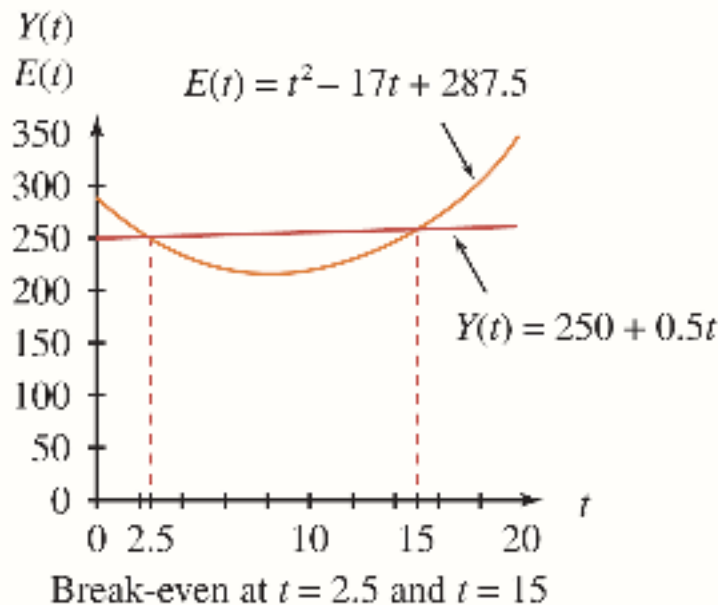
Interpretation: This is the condition for maximizing the net present value of the wine-sale (NPV_{wine}). The critical value of variable time (t^*) will lead to maximization of net present value of wine-sale. Moreover, the condition is replica of the usual condition for profit maximization $MC = MR$.

$$\text{i.e. } \underbrace{V'(t)}_{MR} = \underbrace{rV(t) + s}_{MC}$$

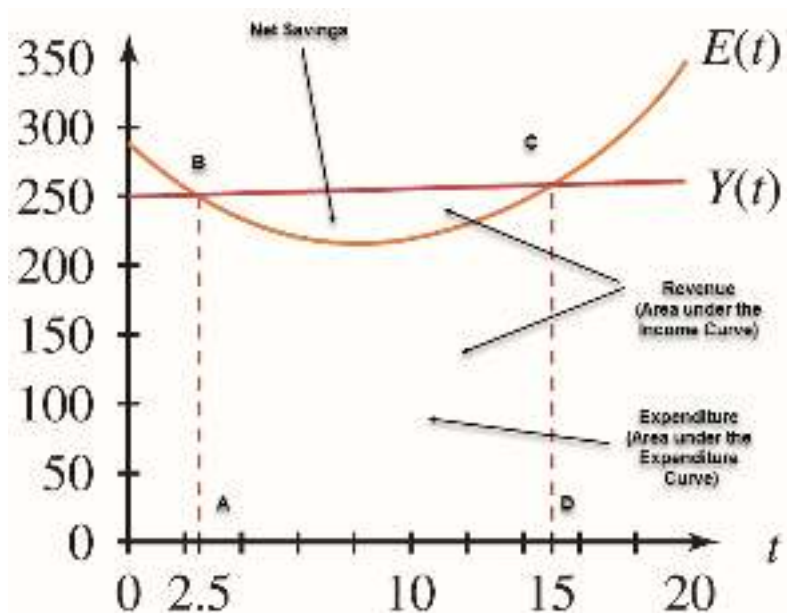
TOPIC 039: BREAKEVEN ANALYSIS OF A FIRM USING INTEGRALS

A firm expects its annual income, (Y), and expenditure (E) to change according to the equations $Y(t) = 250 + 0.5t$ and $E(t) = t^2 - 17t + 287.5$ (t is time in years).

- (a) Sketch the graphs of income and expenditure on the same diagram over the interval $0 \leq t \leq 20$. Confirm the break-even points algebraically.



- b) Use integration to calculate the net savings (total income – total expenditure) over the interval $2.5 \leq t \leq 15$.



From diagram, it clear that:

Net Savings = Area under the Revenue Curve – Area under the Expenditure Curve

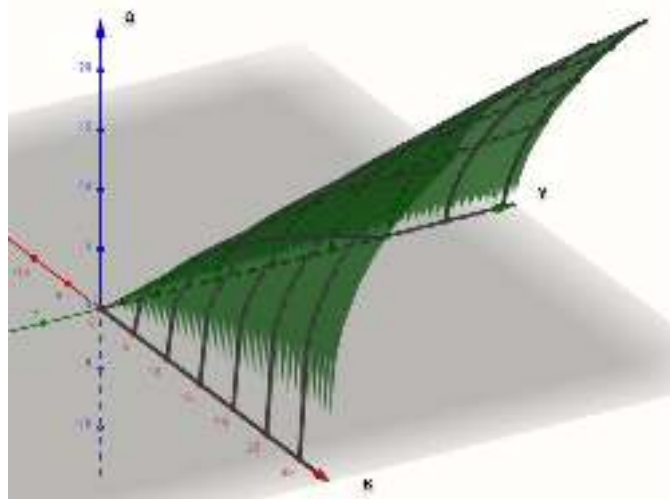
$$\begin{aligned}
 \text{Net Savings} &= \int_{2.5}^{15} (250 + 0.5t) dt - \int_{2.5}^{15} (t^2 - 17t + 287.5) dt \\
 &= \left(250t + 0.5 \times \frac{t^2}{2} \right) \Big|_{2.5}^{15} - \left(\frac{t^3}{3} - \frac{17t^2}{2} + 287.5t \right) \Big|_{2.5}^{15} \\
 &= (250t + 0.25t^2) \Big|_{2.5}^{15} - \left(\frac{t^3}{3} - 8.5t^2 + 287.5t \right) \Big|_{2.5}^{15} \\
 &= \{ [250(15) - 250(2.5)] + 0.25\{(15)^2 - (2.5)^2\} \} \\
 &\quad - \left\{ \frac{(15)^3}{3} - \frac{(2.5)^3}{3} - 8.5\{(15)^2 - (2.5)^2\} + 287.5(15 - 2.5) \right\} \\
 &\quad \boxed{\text{Net Savings} = 325.5}
 \end{aligned}$$

Interpretation: The annual net savings of the firm, if time is considered from 2.5 years till 15 years, will be 325.5 units:

TOPIC 040: SCALE OF PRODUCTION ANALYSIS USING INTEGRALS

Assume that the total quantity of a good produced by a firm using capital (K) and labor (L) is given by the Cobb–Douglas production function $f(K, L) = K^{0.7}L^{0.3}$. What will be the total increase in output if the firm increases the quantities of capital and labor to 10 units each from the present levels of 5 units each?

Visualizing the production function, we get a 3D graph:



If implies that we have (3D) volume under the curve instead of (2D) area under the curve. Therefore, developing an integral expression w.r.t. both capital and labor.

$$Q(K, L) = \int_5^{10} \left\{ \int_5^{10} (K^{0.7} L^{0.3}) dK \right\} dL$$

It is called a double-definite integral as integrand is integrated w.r.t K and L , respectively. Moreover, it has lower and upper limits of 5 and 10, respectively, as K and L are doubled from 5 to 10.

Solution: Integrating w.r.t to K .

$$Q(K, \bar{L}) = \int_5^{10} \left[\int_5^{10} \{K^{0.7} (\bar{L})^{0.3}\} dK \right] dL$$

$$Q(K, \bar{L}) = \int_5^{10} \left[\left\{ \frac{K^{0.7+1}}{0.7+1} \times (\bar{L})^{0.3} \right\} \right]_5^{10} dL$$

$$Q(K, \bar{L}) = \int_5^{10} \left[(\bar{L})^{0.3} \left\{ \frac{K^{1.7}}{1.7} \right\} \right]_5^{10} dL$$

$$Q(\bar{L}) = \int_5^{10} \left\{ \frac{(\bar{L})^{0.3}}{1.7} (34.69) \right\} dL$$

$$Q(\bar{L}) = \int_5^{10} \{(\bar{L})^{0.3} 20.41\} dL$$

$$Q(\bar{L}) = \int_5^{10} \{20.41(\bar{L})^{0.3}\} dL$$

Integrating w.r.t L :

$$Q(L) = \int_5^{10} \{20.41(L)^{0.3}\} dL$$

$$Q(L) = 20.41 \left(\frac{L^{0.3+1}}{0.3+1} \right) \Bigg|_5^{10}$$

$$Q(L) = 20.41 \left(\frac{L^{1.3}}{1.3} \right) \Bigg|_5^{10}$$

$$Q(L) = 15.7 \left\{ (L^{1.3}) \Big|_5^{10} \right\}$$

$$Q = 15.7 \{ (10)^{1.3} - (5)^{1.3} \}$$

$$Q = 186.03 \text{ units}$$

Interpretation: Therefore, the increase in output when the firm increases the quantities of capital and labor from 5 units each to 10 units each will be 186.03 units.

Lesson 11

FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 1)

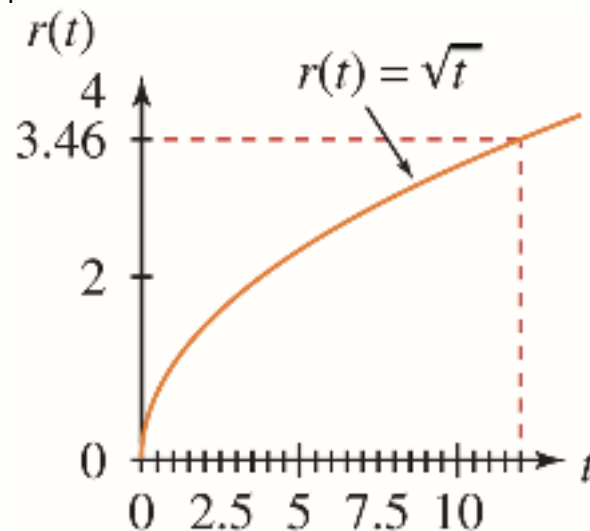
TOPIC 041: POLLUTION IN CATTLE FATTENING ANALYSIS USING INTEGRALS

Cattle Fattening involves environmental costs.

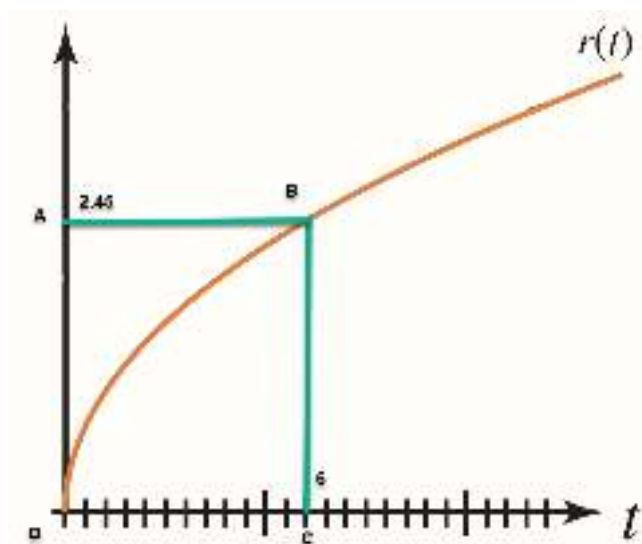


Here an intensive cattle-fattening unit is adding pollutant (in hundreds of gallons) to a river at the rate of $r(t) = \sqrt{t}$, where t is the number of months that the unit has been in operation.

(a) Graph the rate of pollution for $t = 0$ to 12 months. Comment.



(b) Calculate the volume of pollutant in the river during the first six months. Here $t = 6$, therefore diagram becomes:



$$\Delta OBC = \square OABC - \text{Area under the curve}$$

$$\Delta OBC = (|\overline{OA}| \times |\overline{OC}|) - \int_0^6 r(t) dt$$

$$\Delta OBC = (|2.45| \times |6|) - \int_0^6 \sqrt{t} dt$$

$$\Delta OBC = 14.7 - \left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) \bigg|_0^6$$

$$\Delta OBC = 14.7 - \frac{2}{3} \left\{ (6)^{\frac{3}{2}} - (0)^{\frac{3}{2}} \right\}$$

$$= 14.7 - \frac{2}{3} (14.7)$$

$$= 14.7 - 9.8$$

$$\Delta OBC = 4.9$$

As the units of pollutant is in hundreds of gallons, the pollutant in first 6 months will be:

$$\Delta OBC = 4.9 \times 100$$

$$\Delta OBC = 490 \text{ gallons}$$

TOPIC 042: CROP FARMING ANALYSIS USING INTEGRALS

It is estimated that t days from now a farmer's crop will be increasing at the rate of $0.3t^2 + 0.6t + 1$ bushels per day. By how much will the value of the crop increase during the next 5 days if the market price remains fixed at \$3 per bushel?

$$\frac{dC(t)}{dt} = 0.3t^2 + 0.6t + 1$$

Moreover, it is given that $P = \$3$.

Let $C(t)$ represent the rate of increase in crop bushels per day.

$$\int_0^t \frac{dC(t)}{dt} dt = \int_0^t (0.3t^2 + 0.6t + 1) dt$$

$$\int_0^5 \frac{dC(t)}{dt} dt = \int_0^5 (0.3t^2 + 0.6t + 1) dt$$

$$C(t) = \left(0.3 \times \frac{t^3}{3} + 0.6 \frac{t^2}{2} + t \right) \Big|_0^5$$

$$C(t) = (0.1t^3 + 0.3t^2 + t) \Big|_0^5$$

$$C(t) = [0.1\{(5)^3 - (0)^3\} + 0.3\{(5)^2 - (0)^2\} + (5 - 0)]$$

$$\boxed{C(t) = 25}$$

The crop after 5 months will be 25 bushels.

It is the crop function over time, however, if the increase in value of crop is required, **Value of**

crop = crop × price

$$V_C(t) = C(t) \times P$$

$$V_C(t) = (0.1t^3 + 0.3t^2 + t) \Big|_0^5 \times \$3$$

$$V_C(t) = (0.3t^3 + 0.9t^2 + 3t) \Big|_0^5$$

Applying the lower and upper limits in $V_C(t)$, we get:

$$V_C(t) = [0.3\{(5)^3 - (0)^3\} + 0.9\{(5)^2 - (0)^2\} + 3(5 - 0)]$$

$$\boxed{V_C(t) = \$75}$$

Interpretation: The value of crop will be \$75, after 5 months, if the price is \$3 per bushel.

TOPIC 043: MOBILE PRODUCTION ANALYSIS USING INTEGRALS

Bejax Corporation has set up a production line to manufacture a new type of mobiles. The rate of production of the mobiles is $\frac{dP}{dt} = 1500 \left(2 - \frac{t}{2t+5} \right)$ units/month. How many telephones are produced during the third month?

Given the rate of change of production over time:

$$\frac{dP}{dt} = 1500 \left(2 - \frac{t}{2t+5} \right)$$

Integrating w.r.t t to get the production as a function of time (t).

$$\int \frac{dP}{dt} dt = \int \left\{ 1500 \left(2 - \frac{t}{2t+5} \right) \right\} dt$$

$$P(t) = 1500 \left(\int 2 dt - \int \frac{t}{2t+5} dt \right)$$

$$P(t) = 1500 \left(2t - \int \frac{t}{2t+5} dt \right)$$

$\int \frac{t}{2t+5} dt$ requires integration by substitution:

$$\text{Let } \underline{u = 2t + 5} \Rightarrow \underline{\frac{u-5}{2} = t}, \underline{\frac{du}{dt} = 2} \Rightarrow \underline{\frac{du}{2} = dt}$$

Substituting, we get:

$$\int \frac{\left\{ \frac{(u-5)}{2} \right\}}{u} \frac{du}{2} \Rightarrow \int \frac{(u-5)}{4u} du$$

$$= \frac{1}{4} \int \left(1 - \frac{5}{u} \right) du$$

$$\begin{aligned}
&= \frac{1}{4} \left(\int 1 \, du - 5 \int \frac{1}{u} \, du \right) \\
&= \frac{1}{4} (u - 5 \ln|u|) + C_u \\
&= \frac{1}{4} \{(2t + 5) - 5 \ln|2t + 5|\} + C_{(2t+5)} \\
&= \frac{1}{4} (2t + 5 - 5 \ln|2t + 5|) + c \quad [\because C_{(2t+5)} = c'] \\
&\int \frac{t}{2t+5} \, dt = \frac{1}{4} \{(2t + 5) - 5 \ln|2t + 5|\} + c' \\
&P(t) = 1500 \left(2t - \int \frac{t}{2t+5} \, dt \right) \\
&P(t) = 1500 \left\{ 2t - \left[\frac{1}{4} \{(2t + 5) - 5 \ln|2t + 5|\} \right] \right\} \\
&P(t) = 1500 \left\{ 2t - \left(\frac{2t+5}{4} - \frac{5}{4} \ln|2t+5| \right) \right\} \\
&P(t) = 1500 \left(2t - \frac{2t+5}{4} + \frac{5}{4} \ln|2t+5| \right)
\end{aligned}$$

Recalling the given information regarding time: “during the 3rd month”. Mathematically; $\int_2^3 P(t) \, dt$ which implies $t_{lower} = 2$ and $t_{upper} = 3$.

$$\begin{aligned}
&= 1500 \left[2(3) - 2(2) - \left\{ \frac{2(3)+5}{4} - \frac{2(2)+5}{4} \right\} + \frac{5}{4} \{ \ln|2(3)+5| - \ln|2(2)+5| \} \right] \\
&= 1500 \left[2 - \left(\frac{11}{4} - \frac{9}{4} \right) + \frac{5}{4} (\ln|11| - \ln|9|) \right] \\
&= 1500 \left\{ 2 - \frac{1}{2} + \frac{5}{4} \left(\ln \left| \frac{11}{9} \right| \right) \right\} \\
&= 1500 \left\{ \frac{3}{2} + \frac{5}{4} \left(\ln \left| \frac{11}{9} \right| \right) \right\} \\
&\boxed{P(t) = 2626.2575}
\end{aligned}$$

Interpretation: Approximately, 2626 telephones will be produced during the third month by Bejax Corporation.

TOPIC 044: FIRM'S AVERAGE PROFIT ANALYSIS USING INTEGRALS

Firm's profit function is $\pi = f(Q)$ and the level of output (Q), varies between a and b units, then the firm's average profit is $\pi = \frac{1}{b-a} \int_a^b f(Q) \, dQ$. Calculate a firm's average profit when $TR = 100(1 - e^{-0.1Q})$, $TC = 0.1Q^2 + 2Q + 1$ and output varies from 3 to 8 units.

Solution: $\pi = f(Q) = TR(Q) - TC(Q)$

$$\begin{aligned}
\pi &= f(Q) = \{100(1 - e^{-0.1Q})\} - \{0.1Q^2 + 2Q + 1\} \\
\pi &= f(Q) = 100 - 100e^{-0.1Q} - 0.1Q^2 - 2Q - 1 \\
\pi &= f(Q) = 99 - 100e^{-0.1Q} - 2Q - 0.1Q^2
\end{aligned}$$

Introducing integral with upper and lower limits on general formula of average profit:

$$\begin{aligned}
\pi_{avg} &= \frac{1}{b-a} \int_a^b f(Q) \, dQ \Rightarrow \frac{1}{8-3} \int_3^8 (99 - 100e^{-0.1Q} - 2Q - 0.1Q^2) \, dQ \\
\pi_{avg} &= \frac{1}{5} \int_3^8 (99 - 100e^{-0.1Q} - 2Q - 0.1Q^2) \, dQ
\end{aligned}$$

$$\begin{aligned}
 \pi_{avg}(Q) &= \frac{1}{5} \left\{ 99Q - 100 \left(\frac{e^{-0.1Q}}{-0.1} \right) - Q^2 - 0.03Q^3 \right\} \Bigg|_3^8 \\
 \pi_{avg}(Q) &= \frac{1}{5} \{ 99Q + 1000(e^{-0.1Q}) - Q^2 - 0.03Q^3 \} \Bigg|_3^8 \\
 \pi_{avg}(Q) &= \frac{1}{5} [\{ 99(8 - 3) \} + 1000 \{ e^{-0.1(8)} - e^{-0.1(3)} \} - (8^2 - 3^2) - 0.03(8^3 - 3^3)] \\
 \pi_{avg}(Q) &= \frac{1}{5} \{ 99(5) + 1000(e^{-0.8} - e^{-0.3}) - (55) - 0.03(485) \} \\
 \pi_{avg}(Q) &= \frac{1}{5} \{ 495 + 1000(0.449 - 0.741) - 55 - 14.55 \} \\
 \pi_{avg}(Q) &= 26.46
 \end{aligned}$$

Interpretation: Between 3 and 8 units of output, the firm makes 26.46 units of average profit.

TOPIC 045: FIRM'S AVERAGE COST ANALYSIS USING INTEGRALS

The cost of producing x units of a new product is $C(Q) = 3Q^2 - 2$ hundred dollars. What is the average cost of producing 3 to 7 units?

Solution: Given the total cost function, $C(Q) = 3Q^2 - 2$, we can find the average cost between the lower and upper limits $[3, 7]$ by general formula for average of a function using definite integrals:

$$\begin{aligned}
 f_{avg}(x) &= \frac{1}{b-a} \int_a^b f(x) dx \\
 C_{avg}(Q) &= \frac{1}{7-3} \int_3^7 (3Q^2 - 2) dQ \\
 C_{avg}(Q) &= AC(Q) = \frac{1}{4} \int_3^7 (3Q^2 - 2) dQ \\
 AC(Q) &= \frac{1}{4} \left\{ 3 \int_3^7 (Q^2) dQ - 2 \int_3^7 (1) dQ \right\} \\
 AC(Q) &= \frac{1}{4} \left\{ 3 \left(\frac{Q^3}{3} \right) \Bigg|_3^7 - 2(Q) \Bigg|_3^7 \right\} \\
 AC(Q) &= \frac{1}{4} \{ (Q^3) \Big|_3^7 - 2(Q) \Big|_3^7 \} \\
 AC(Q) &= \frac{1}{4} \{ (7^3 - 3^3) - 2(7 - 3) \} \\
 AC(Q) &= \frac{1}{4} \{ (316) - 2(4) \} \\
 AC(Q) &= 77
 \end{aligned}$$

Interpretation: The firm will incur 77 units as average cost if we consider production of 3 to 7 units.

Lesson 12

FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 2)

TOPIC 046: AVERAGE FOOD PRICE ANALYSIS USING INTEGRALS

Records indicate that t months after the beginning of the year, the price of ground beef in local supermarkets was $P(t) = 0.09t^2 - 0.2t + 4$ dollars per pound. What was the average price of ground beef during the first 3 months of the year?

Solution: Given the price function, $P(t) = 0.09t^2 - 0.2t + 4$, we can find the average price of food between the lower and upper limits $[0, 3]$ by general formula for average of a function using definite integrals:

$$f_{avg}(x) = \frac{1}{b-a} \int_a^b f(x) dx$$

Here averaging of price is over time (t) instead of output.

$$\begin{aligned} P_{avg}(t) &= \frac{1}{3-0} \int_0^3 (0.09t^2 - 0.2t + 4) dt \\ P_{avg}(t) &= \frac{1}{3} \times \left(0.09 \frac{t^3}{3} - 0.2 \frac{t^2}{2} + 4t \right) \Bigg|_0^3 \\ P_{avg}(t) &= \frac{1}{3} \times (0.03Q^3 - 0.1t^2 + 4t) \Bigg|_0^3 \\ P_{avg}(t) &= \frac{1}{3} \times [0.03\{(3)^3 - (0)^3\} - 0.1\{(3)^2 - (0)^2\} + 4(3 - 0)] \\ P_{avg}(t) &= \frac{1}{3} \times [0.03(27) - 0.1(9) + 4(3)] \\ P_{avg}(t) &= \frac{1}{3} \times [0.03(27) - 0.1(9) + 4(3)] \\ P_{avg}(t) &= \frac{1}{3} \times (11.91) \\ \boxed{P_{avg}(t) = 3.97} \end{aligned}$$

Interpretation: The average price of beef during first 3 months in the local supermarket will be 3.97 dollar per pound.

TOPIC 047: AVERAGE PRODUCTION ANALYSIS USING INTEGRALS

A company determines that if L worker-hours of labor are employed, then Q units of a particular commodity will be produced, where $Q(L) = 500L^{\frac{2}{3}}$. What is the average production as labor varies from 1,000 to 2,000 worker-hours?

Solution: Given the production function, $Q(L) = 500L^{\frac{2}{3}}$, we can find the average production between the lower and upper limits $[1000, 2000]$ by general formula for average of a function using definite integrals:

$$f_{avg}(x) = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\begin{aligned}
 Q_{avg}(L) &= \frac{1}{2000 - 1000} \int_{1000}^{2000} (500L^{\frac{2}{3}}) dL \\
 Q_{avg}(L) = AP_L &= \frac{1}{1000} \times 500 \left(\frac{L^{\frac{2}{3}+1}}{\frac{2}{3}+1} \right) \Bigg|_{1000}^{2000} \\
 AP_L &= \frac{1}{1000} \times 500 \left(\frac{L^{\frac{5}{3}}}{\frac{5}{3}} \right) \Bigg|_{1000}^{2000} \\
 AP_L &= \frac{1}{1000} \times 500 \times \frac{3}{5} (L^{\frac{5}{3}}) \Bigg|_{1000}^{2000} \\
 AP_L &= \frac{3}{10} (L^{\frac{5}{3}}) \Bigg|_{1000}^{2000} \\
 AP_L &= \frac{3}{10} \left\{ (2000)^{\frac{5}{3}} - (1000)^{\frac{5}{3}} \right\} \\
 AP_L &= \frac{3}{10} \{217480.21\} \\
 \boxed{AP_L} &= \boxed{65244.06}
 \end{aligned}$$

Interpretation: The average product of labor between 1000 and 2000, in the company will be 65244.06.

TOPIC 048: AVERAGE INVENTORY ANALYSIS USING INTEGRALS

An inventory of 60,000 kilograms of a certain commodity is used at a constant rate and is exhausted after 1 year. If $y(t) = -60,000(t - 1)$, then what is the average inventory for the year?

Solution: Given the inventory function, $y(t) = -60,000(t - 1)$, we can find the average inventory function between the lower and upper limits $[0, 1]$ by general formula for average of a function using definite integrals:

$$\begin{aligned}
 f_{avg}(x) &= \frac{1}{b-a} \int_a^b f(x) dx \\
 I_{avg}(t) &= \frac{1}{1-0} \int_0^1 \{-60000(t-1)\} dt \\
 I_{avg}(t) = AI_t &= \int_0^1 \{-60000(t-1)\} dt \\
 AI_t &= -60000 \int_0^1 (t-1) dt \\
 AI_t &= -60000 \left(\int_0^1 (t) dt - \int_0^1 (1) dt \right) \\
 AI_t &= -60000 \left(\left[\frac{t^2}{2} \right]_0^1 - t \Big|_0^1 \right)
 \end{aligned}$$

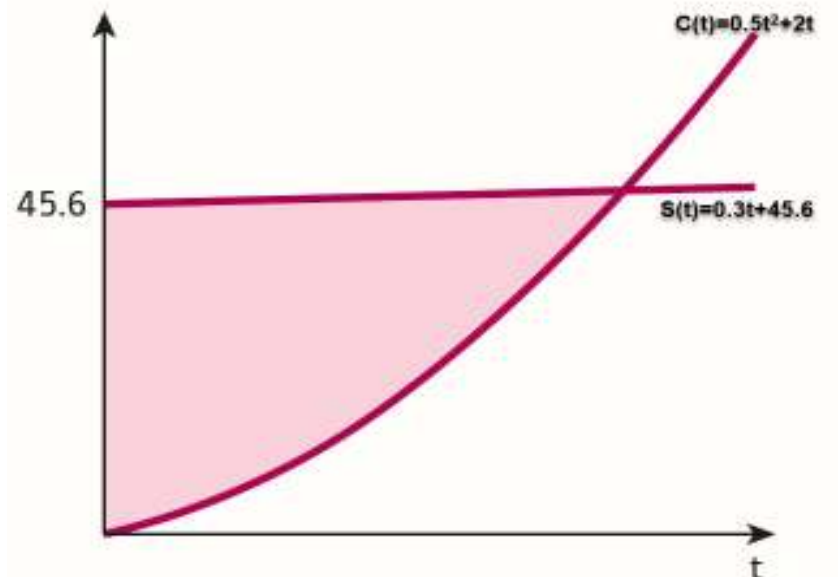
$$\begin{aligned}
 AI_t &= -60000 \left\{ \left(\frac{1^2 - 0^2}{2} \right) - (1 - 0) \right\} \\
 AI_t &= -60000 \left\{ \frac{1}{2} - 1 \right\} \\
 AI_t &= -60000 \left\{ -\frac{1}{2} \right\} \\
 \boxed{AI_t} &= \boxed{30000}
 \end{aligned}$$

Interpretation: The average inventory of the commodity during the year will be 30000 kilograms.

TOPIC 049: NET SAVINGS ANALYSIS USING INTEGRALS

A hospital trust decides to install a new rooftop solar power system which will save money on its energy bills. It estimates that t years after installation the savings flow (in \$1000s per year) will steadily increase according to $S(t) = 0.3t + 45.6$. The flow of money needed to repair and maintain the system in subsequent years is modelled by $C(t) = 0.5t^2 + 2t$:

(a) Draw a sketch graph of the savings and cost functions on the same diagram.



(b) Solve the equation $S(t) = C(t)$ to determine the period during which the savings exceed the maintenance costs, and shade the area which determines the net savings during this time.

Solution: Visualizing two functions i.e. Savings $S(t)$ and Cost $C(t)$.

The intersection point of the two functions $S(t)$ and $C(t)$ shall exist at a certain t . It can be found by equating the two functions, i.e. $S(t) = C(t)$.

$$0.5t^2 + 2t = 0.3t + 45.6$$

$$0.5t^2 + 2t - 0.3t - 45.6 = 0$$

$$0.5t^2 + 1.7t - 45.6 = 0 \Rightarrow t^2 + 3.4t - 91.2 = 0$$

$$t^2 + 3.4t - 91.2 = 0 \text{ is comparable with } ax^2 + bx + c = 0$$

This implies: $x = t$, $a = 1$, $b = 3.4$ and $c = -91.2$. and using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$t = \frac{-(3.4) \pm \sqrt{(3.4)^2 - 4(1)(-91.2)}}{2(1)}$$

$$t = \frac{-3.4 \pm \sqrt{11.56 + 364.8}}{2}$$

$$t = \frac{-3.4 \pm \sqrt{376.36}}{2}$$

$$t = \frac{-3.4 \pm 19.4}{2} \Rightarrow \frac{-3.4 + 19.4}{2}, \frac{-3.4 - 19.4}{2}$$

$$t = 8, -11.4$$

$$\boxed{t^* = 8} \quad [\because t \geq 0]$$

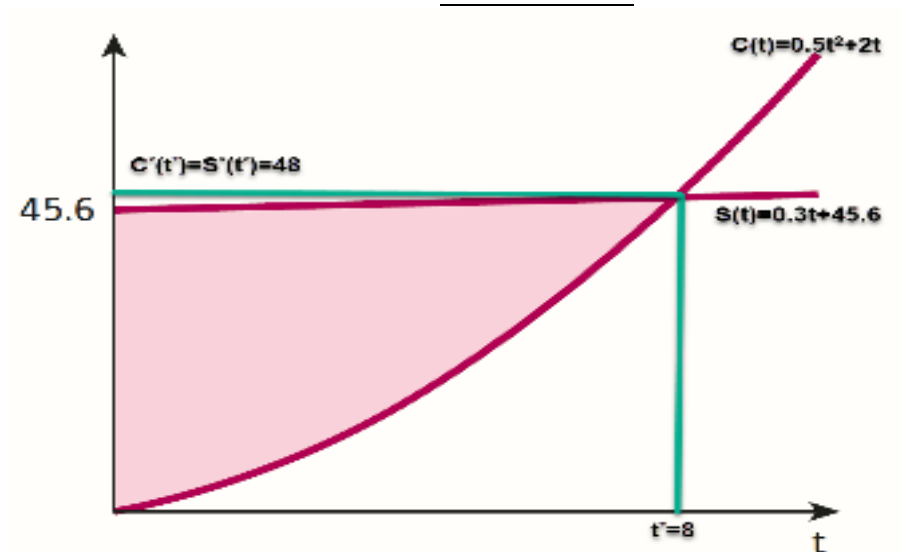
Putting $t^* = 8$ in $S(t)$ or $C(t)$, we get their common value $C^*(t^*) = S^*(t^*)$:

$$0.5(t^*)^2 + 2t^* = 0.3t^* + 45.6$$

$$0.5(8)^2 + 2(8) = 0.3(8) + 45.6$$

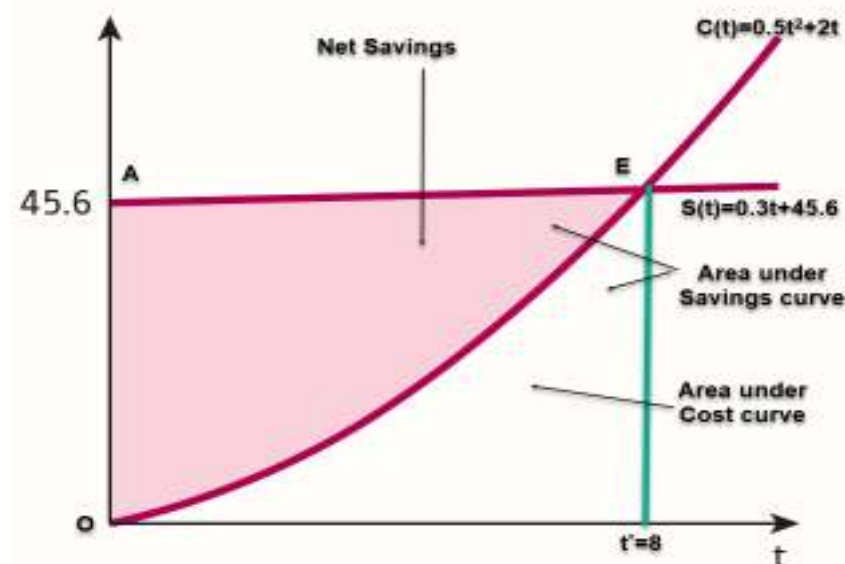
$$32 + 16 = 2.4 + 45.6$$

$$48 = 48 = C^*(t^*) = S^*(t^*)$$



(c) Use integration to find the exact value of net savings

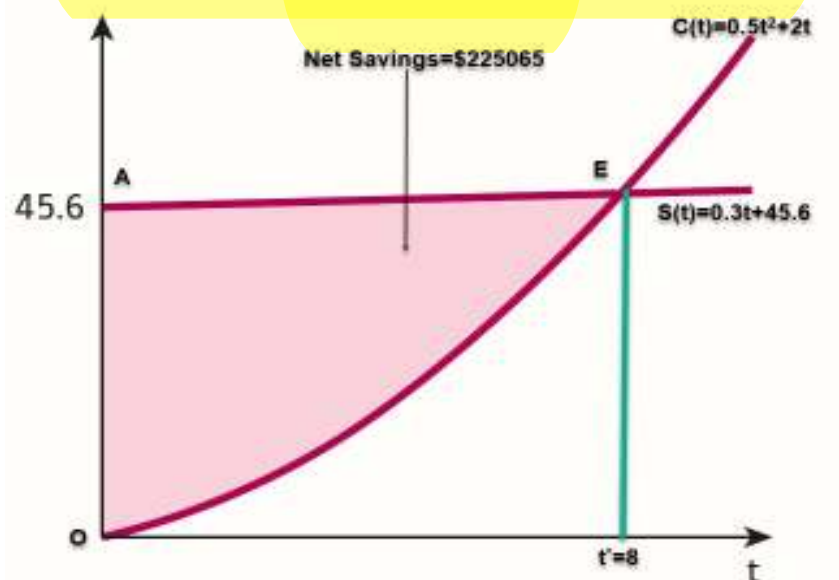
Net Savings: Net savings is the shaded area, which is the difference of the two integrals:



i.e. **Net Savings = Area under Savings curve – Area under Savings curve**

Mathematically:

$$\begin{aligned}\Delta OEA &= \int_0^8 S(t) dt - \int_0^8 C(t) dt \\ \Delta OEA &= \int_0^8 (0.3t + 45.6) dt - \int_0^8 (0.5t^2 + 2t) dt \\ \Delta OEA &= 0.3 \int_0^8 (t) dt + 45.6 \int_0^8 (1) dt - \left\{ 0.5 \int_0^8 (t^2) dt + 2 \int_0^8 (t) dt \right\} \\ \Delta OEA &= 0.3 \int_0^8 (t) dt + 45.6 \int_0^8 (1) dt - \left\{ 0.5 \int_0^8 (t^2) dt + 2 \int_0^8 (t) dt \right\} \\ \Delta OEA &= 0.3 \left(\frac{t^2}{2} \right)_0^8 + 45.6(t)_0^8 - \left\{ 0.5 \left(\frac{t^3}{3} \right)_0^8 + 2 \left(\frac{t^2}{2} \right)_0^8 \right\} \\ \Delta OEA &= 0.3 \left\{ \frac{(8)^2}{2} - \frac{(0)^2}{2} \right\} + 45.6(8 - 0) - \left[0.5 \left\{ \frac{(8)^3}{3} - \frac{(0)^3}{3} \right\} + 2 \left\{ \frac{(8)^2}{2} - \frac{(0)^2}{2} \right\} \right] \\ \Delta OEA &= 0.3(32) + 45.6(8) - \{ 0.5(170.67) + 2(32) \} \\ \Delta OEA &= 225.065\end{aligned}$$



Since given data is in \$1000s per year, then answer shall be $225.065 \times 1000 = \$225065$.

Interpretation: Net savings in installing a new rooftop solar power system, during 8 years, shall be \$225065.

TOPIC 050: NET EXCESS PROFIT ANALYSIS USING INTEGRALS

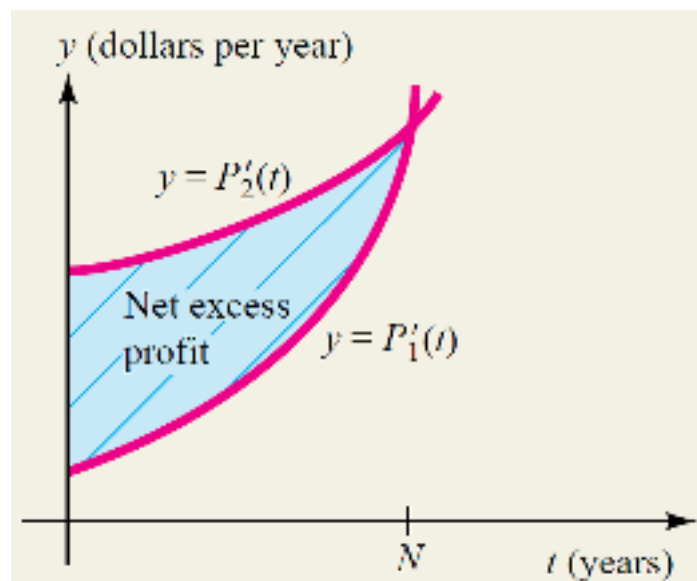
The area between curves can sometimes be used as a way of measuring the amount of a quantity that has been accumulated during a particular procedure. For instance, suppose that t years from now, two investment plans will be generating profit $P_1(t)$ and $P_2(t)$, respectively, and that their respective rates of profitability, $P'_1(t)$ and $P'_2(t)$, are expected to satisfy $P'_2(t) \geq P'_1(t)$ for the next N years; that is, over the time interval $0 \leq t \leq N$. Then $E(t) = P_2(t) - P_1(t)$ represents the **excess profit** of plan 2 over plan 1 at time t , and the **net excess profit** $NE = E(N) - E(0)$ over the time interval $0 \leq t \leq N$ is given by the definite integral:

$$NE = NE = E(N) - E(0) = \int_0^N E'(t) dt$$

$$E(t) = P_2(t) - P_1(t) \Rightarrow E'(t) = P_2'(t) - P_1'(t)$$

$$NE = \int_0^N \{P_2'(t) - P_1'(t)\} dt$$

Graphically:



Numerical Example: Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year.

a. For how many years does the rate of profitability of the second investment exceed that of the first?

From the diagram, before the intersection point one of the rates of profit is greater than other. Therefore, we need to find the point of intersection.

Mathematically speaking; $P_1'(t) = P_2'(t)$

$$50 + t^2 = 200 + 5t \Rightarrow t^2 + 50 - 200 - 5t = 0$$

$$t^2 - 5t - 150 = 0$$

Factorization:

$$t^2 - 15t + 10t - 150 = 0$$

$$t(t - 15) + 10(t - 15) = 0$$

$$(t + 10)(t - 15) = 0$$

$$(t + 10) = 0 \text{ OR } (t - 15) = 0$$

$$t = -10 \text{ OR } t = 15$$

Negative value of time ($t < 0$) is meaningless in current analysis.

b. Compute the net excess profit for the time period determined in part (a). Interpret the net excess profit as an area.

Given the formula of Net Excess Profit:

$$NE = \int_0^N \{P'_2(t) - P'_1(t)\} dt$$

Substituting values of $P'_2(t)$, $P'_1(t)$ and N , we get:

$$NE = \int_0^{15} \{(200 + 5t) - (50 + t^2)\} dt$$

$$NE = \left\{ 200(t) + \frac{5(t)^2}{2} \right\} \Big|_0^{15} - \left\{ 50(t) + \frac{(t)^3}{3} \right\} \Big|_0^{15}$$

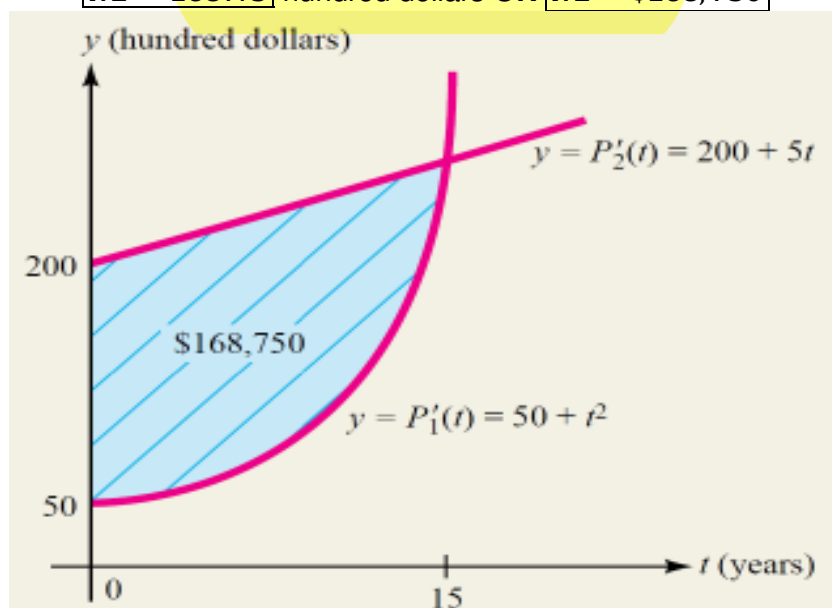
$$NE = \left\{ 200(15 - 0) + \frac{5(15 - 0)^2}{2} \right\} - \left\{ 50(15 - 0) + \frac{(15 - 0)^3}{3} \right\}$$

$$NE = \left\{ 200(15) + \frac{5(225)}{2} \right\} - \left\{ 50(15) + \frac{3375}{3} \right\}$$

$$NE = \{3000 + 562.5\} - \{750 + 1125\}$$

$$NE = 3000 + 562.5 - 750 - 1125$$

$$NE = 1687.5 \text{ hundred dollars OR } NE = \$168,750$$



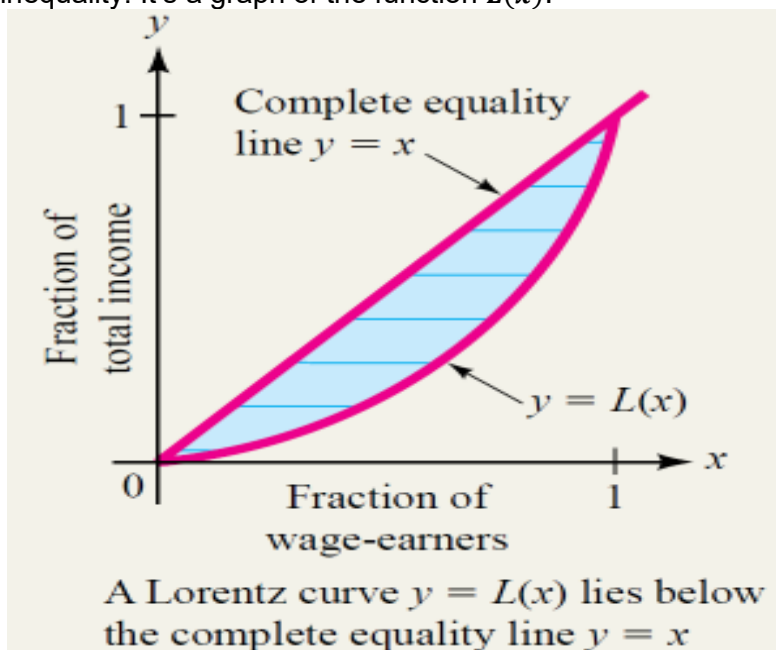
Interpretation: Thus, the net excess profit (EN) is **\$168,750**. The graphs of the rate of profitability functions $P'_1(t)$ and $P'_2(t)$ are shown in Figure. The net excess profit can be interpreted as the area of the (shaded) region between the rate of profitability curves over the interval $0 \leq t \leq N$.

Lesson 13

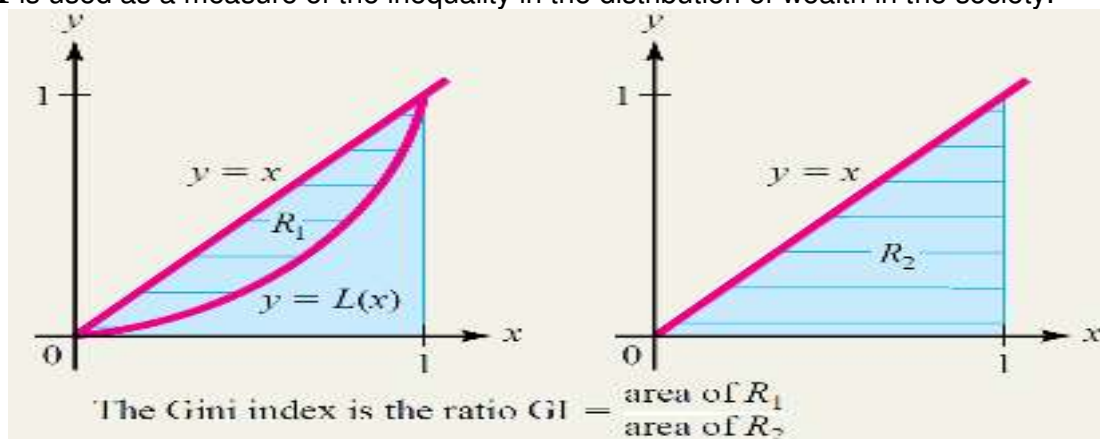
FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 3)

TOPIC 051: GINI INDEX USING INTEGRALS

Area also plays an important role in the study of Lorentz curves which is used by economists to measure income inequality. It's a graph of the function $L(x)$:



The ratio of area R_1 to the area of the region R_2 under the complete equality line $y = x$ over $0 \leq x \leq 1$ is used as a measure of the inequality in the distribution of wealth in the society.



This ratio, called the Gini index, denoted GI (also called the index of income inequality).

More precisely, $GI = \frac{\text{area b/w } y=L(x) \text{ \& } y=x}{\text{area under } y=x \text{ over } 0 \leq x \leq 1}$

$$GI = \frac{\int_0^1 [x - L(x)] dx}{\int_0^1 x dx}$$

$$GI = \frac{\int_0^1 [x - L(x)] dx}{\left(\frac{x^2}{2}\right)\bigg|_0^1}$$

$$GI = \frac{\int_0^1 [x - L(x)] dx}{\left\{\frac{(1)^2 - (0)^2}{2}\right\}}$$

$$GI = \frac{\int_0^1 [x - L(x)] dx}{\frac{1}{2}}$$

$$GI = 2 \int_0^1 \{x - L(x)\} dx$$

Numerical Example: A governmental agency determines that the Lorenz curves for the distribution of income for dentists and contractors in a certain state are given by the functions $L_1(x) = x^{1.7}$ and $L_2(x) = 0.8x^2 + 0.2x$ respectively. For which profession is the distribution of income more fairly distributed?

Solution: Gini Index, for both dentists and contractors, needs to be calculated and compared.

$$GI_{dentists} \leq GI_{contractors}$$

Here $L_1(x)$: Lorenz curve for the distribution of income for dentists.

and $L_2(x)$: Lorenz curve for the distribution of income for contractors.

Firstly, calculating Gini Index for dentists:

$$GI_{dentists} = 2 \int_0^1 \{x - L_1(x)\} dx$$

$$GI_{dentists} = 2 \int_0^1 \{x - (x)^{1.7}\} dx$$

$$GI_{dentists} = 2 \left\{ \left(\frac{x^2}{2}\right)\bigg|_0^1 - \left(\frac{x^{2.7}}{2.7}\right)\bigg|_0^1 \right\}$$

$$GI_{dentists} = 2 \left[\left\{ \frac{(1)^2 - (0)^2}{2} \right\} - \left\{ \frac{(1)^{2.7} - (0)^{2.7}}{2.7} \right\} \right]$$

$$GI_{dentists} = 2 \left[\left\{ \frac{1}{2} \right\} - \left\{ \frac{1}{2.7} \right\} \right]$$

$$GI_{dentists} = 0.2593$$

Secondly, calculating Gini Index for contractors:

$$GI_{contractors} = 2 \int_0^1 \{x - (0.8x^2 + 0.2x)\} dx$$

$$GI_{contractors} = 2 \int_0^1 \{x - 0.8x^2 - 0.2x\} dx$$

$$GI_{contractors} = 2 \int_0^1 \{0.8x - 0.8x^2\} dx$$

$$GI_{contractors} = 2 \times 0.8 \int_0^1 (x - x^2) dx$$

$$GI_{contractors} = 1.6 \left\{ \int_0^1 (x) dx - \int_0^1 (x^2) dx \right\}$$

$$\begin{aligned}
 GI_{contractors} &= 1.6 \left\{ \left(\frac{x^2}{2} \right) \Big|_0^1 - \left(\frac{x^3}{3} \right) \Big|_0^1 \right\} \\
 GI_{contractors} &= 1.6 \left[\left\{ \frac{(1)^2 - (0)^2}{2} \right\} - \left\{ \frac{(1)^3 - (0)^3}{3} \right\} \right] \\
 GI_{contractors} &= 1.6 \left[\left\{ \frac{1}{2} \right\} - \left\{ \frac{1}{3} \right\} \right] \\
 GI_{contractors} &= 2.667
 \end{aligned}$$

Interpretation: As $GI_{dentists} = 0.2593$ and $GI_{contractors} = 2.667$, and it is evident that $GI_{dentists} < GI_{contractors}$. It is straightforward to suggest that income inequality is higher among contractors as measured through Gini Index.

TOPIC 052: SALES ANALYSIS USING INTEGRALS

A manufacturer determines that t months after introducing a new product, the company's sales will be $S(t)$ thousand dollars, where

$$S(t) = \frac{750t}{\sqrt{4t^2 + 25}}$$

What are the average monthly sales of the company over the first 6 months after the introduction of the new product?

Solution:

Given sales function:

$$S(t) = \frac{750t}{\sqrt{4t^2 + 25}}$$

Time span is first 6 month: $[0, 6]$.

Average monthly sales are to be calculated, so using the integral for the average function:

$$f_{avg}(x) = \frac{1}{b-a} \int_a^b f(x) dx$$

Substituting the values in the general formula above.

$$S_{avg}(t) = \frac{1}{6-0} \int_0^6 S(t) dt$$

$$S_{avg}(t) = \frac{1}{6} \int_0^6 \left(\frac{750t}{\sqrt{4t^2 + 25}} \right) dt$$

$$S_{avg}(t) = \frac{1}{6} \int_0^6 \left(\frac{750t}{\sqrt{4t^2 + 25}} \right) dt$$

Let $X = 750 \int_0^6 \left(\frac{t dt}{\sqrt{4t^2 + 25}} \right)$ then $S_{avg}(t) = \frac{1}{6}(X)$

$X = 750 \int_0^6 \left(\frac{t dt}{\sqrt{4t^2 + 25}} \right)$ requires integration by substitution:

Derivative	Limits Conversion
Let $u = 4t^2 + 25$	If Lower Limit: $t = 0 \Rightarrow u = 4(0)^2 + 25$ Then $u = 25$

$\frac{du}{dt} = 8t \Rightarrow \frac{du}{8} = t dt$	If Upper Limit: $t = 6 \Rightarrow u = 4(6)^2 + 25$ Then $u = 169$
--	---

Substituting the tabulated values in $X = 750 \int_0^6 \left(\frac{t dt}{\sqrt{4t^2 + 25}} \right)$, we get:

$$\begin{aligned}
 X &= 750 \int_{25}^{169} \left(\frac{\frac{du}{8}}{\sqrt{u}} \right) \Rightarrow \frac{750}{8} \int_{25}^{169} \left(\frac{du}{u^{\frac{1}{2}}} \right) \\
 X &= 93.75 \int_{25}^{169} \left(u^{-\frac{1}{2}} \right) du \Rightarrow 93.75 \left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) \Bigg|_{25}^{169} \\
 X &= 93.75 \left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right) \Bigg|_{25}^{169} \Rightarrow 93.75 \times 2 \left(u^{\frac{1}{2}} \right) \Bigg|_{25}^{169} \\
 X &= 187.5 \left\{ (u^{\frac{1}{2}}) \right\} \Bigg|_{25}^{169} \\
 &= 187.5 \left\{ (169)^{\frac{1}{2}} - (25)^{\frac{1}{2}} \right\} \\
 X &= 1500
 \end{aligned}$$

Putting value of X back in $S_{avg}(t) = \frac{1}{6}(X)$

$$S_{avg}(t) = \frac{1}{6}(1500)$$

$$S_{avg}(t) = 250 \text{ thousand dollars}$$

Interpretation: Thus, for the 6-month period immediately after the introduction of the new product, the company's sales average \$250,000 per month.

TOPIC 053: EFFICIENCY ANALYSIS USING INTEGRALS

After t months on the job, a postal clerk can sort $Q(t) = 700 - 400e^{-0.5t}$ letters per hour. What is the average rate at which the clerk sorts mail during the first 3 months on the job?

Solution: Given the sorting-efficiency function, $Q(t) = 700 - 400e^{-0.5t}$, we can find the average sorting-efficiency function between the lower and upper limits $[0, 3]$ by general formula for average of a function using definite integrals:

$$\begin{aligned}
 f_{avg}(x) &= \frac{1}{b-a} \int_a^b f(x) dx \\
 Q_{avg}(t) &= \frac{1}{3-0} \int_0^3 (700 - 400e^{-0.5t}) dt \\
 Q_{avg}(t) &= \frac{1}{3} \left\{ 700 \int_0^3 (1) dt - 400 \int_0^3 (e^{-0.5t}) dt \right\} \\
 Q_{avg}(t) &= \frac{1}{3} \left\{ 700(t) \Big|_0^3 - 400 \left(\frac{e^{-0.5t}}{-0.5} \right) \Big|_0^3 \right\}
 \end{aligned}$$

$$\begin{aligned}
 Q_{avg}(t) &= \frac{1}{3} \{ 700(t|_0^3) + 800(e^{-0.5t})|_0^3 \} \\
 Q_{avg}(t) &= \frac{1}{3} [700(3 - 0) + 800\{e^{-0.5(3)} - e^{-0.5(0)}\}] \\
 Q_{avg}(t) &= \frac{1}{3} \{ 700(3) + 800(e^{-1.5} - e^0) \} \\
 Q_{avg}(t) &= \frac{1}{3} \{ 700(3) + 800(0.223 - 1) \} \\
 Q_{avg}(t) &= \frac{1}{3} \{ 700(3) + 800(-0.777) \} \\
 Q_{avg}(t) &= \frac{1}{3} (2100 - 621.6) \\
 Q_{avg}(t) &= \frac{1}{3} (1478.4) \\
 Q_{avg}(t) &= 492.83
 \end{aligned}$$

Interpretation: The average efficiency of the clerk in sorting mail during the first 3 months is 492.83 letter per hour.

TOPIC 054: ADVERTISEMENT ANALYSIS USING INTEGRALS

An advertising agency begins a campaign to promote a new product and determines that t days later, the number of people $N(t)$ who have heard about the product is changing at a rate given by $N'(t) = 5t^2 - \frac{0.04t}{t^2+3}$ people per day. How many people learn about the product during the first week? During the second week?

Solution: Given the rate at which people hear about the new product: $N'(t) = 5t^2 - \frac{0.04t}{t^2+3}$.

Analysis for first week implies that lower limit is 0 and upper limit is 7; i.e. $[0, 7]$

Using this information, we can express the situation mathematically:

$$\begin{aligned}
 \int_0^7 \frac{d}{dt} \{N'(t)\} &= \int_0^7 \left(5t^2 - \frac{0.04t}{t^2+3} \right) dt \\
 N(t) &= 5 \int_0^7 (t^2) dt - 0.04 \int_0^7 \left(\frac{t}{t^2+3} \right) dt
 \end{aligned}$$

In attempt of applying: $\int_a^b \left\{ \frac{f'(x)}{f(x)} \right\} dx = \ln|f(x)|$, introducing $\left(\frac{2}{2} \right)$

$$\begin{aligned}
 N(t) &= 5 \int_0^7 (t^2) dt - 0.04 \times \left(\frac{2}{2} \right) \int_0^7 \left(\frac{t}{t^2+3} \right) dt \\
 N(t) &= 5 \int_0^7 (t^2) dt - 0.02 \int_0^7 \left(\frac{2t}{t^2+3} \right) dt \\
 N(t) &= 5 \left(\frac{t^3}{3} \right) \Big|_0^7 - 0.02 \left(\ln|t^2+3| \right) \Big|_0^7 \\
 N(t) &= \frac{5}{3} \{ (7)^3 - (0)^3 \} - 0.02 \{ \ln|(7)^2+3| - \ln|(0)^2+3| \} \\
 N(t) &= \frac{5}{3} (343) - 0.02 \{ 2.85 \}
 \end{aligned}$$

$$\boxed{N(t) = 572} \text{ approximately}$$

Interpretation: During the first week (seven days), approximately 572 people will learn about the new product.

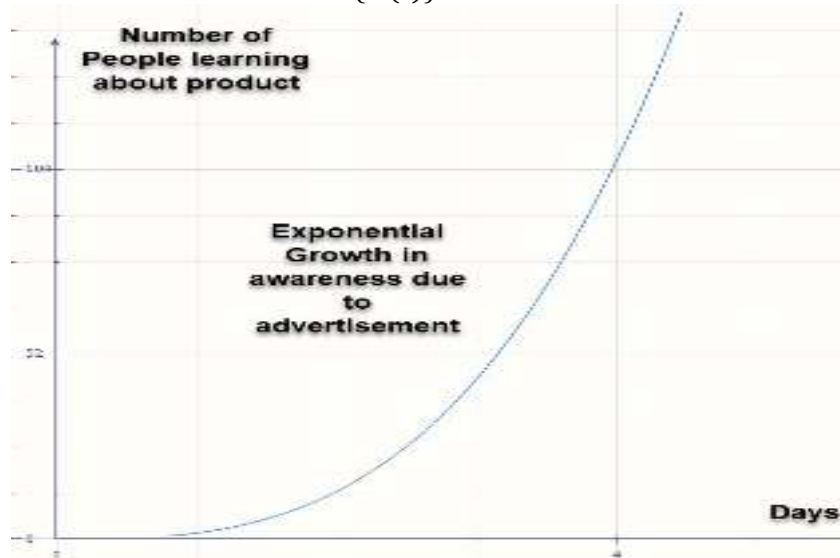
If we want to know about the same number during next week [7, 14], the mathematical formation will be:

$$\int_7^{14} \frac{d}{dt} \{N'(t)\} = \int_7^{14} \left(5t^2 - \frac{0.04t}{t^2 + 3} \right) dt$$

$$\boxed{\{N(t)\}_{2nd \text{ week}} = 4002} \text{ approximately}$$

Percentage increase in the number of people getting aware during first and second week:

$$\begin{aligned} \% \Delta \{N(t)\} &= \frac{[\{N(t)\}_{2nd \text{ week}} - \{N(t)\}_{1st \text{ week}}]}{\{N(t)\}_{1st \text{ week}}} \times 100 \\ &= \frac{4002 - 572}{572} \times 100 \\ &= \frac{3430}{572} \times 100 \\ \% \Delta \{N(t)\} &= 600\% \end{aligned}$$



Interpretation: It is noticeable that there is a rapid increase in the number of people getting aware of the product during 2nd week. The advertisement campaign appears a success.

TOPIC 055: PROFIT OVER USEFUL LIFE OF MACHINE USING INTEGRALS

Suppose that t years after being put into use, a machine has generated total revenue $R(t)$ and that the total cost of operating and servicing the machine up to this time is $C(t)$. Then the total profit generated by the machine up to time t is $P(t) = R(t) - C(t)$. Profit declines when costs accumulate at a higher rate than revenue; that is, when $C'(t) > R'(t)$. Thus, a manager may consider disposing of the machine at the time T when $C'(T) = R'(T)$, and for this reason, the time period $0 \leq t \leq T$ is called the useful life of the machine. The net profit over the useful life of the machine provides the manager with a measure of its value.

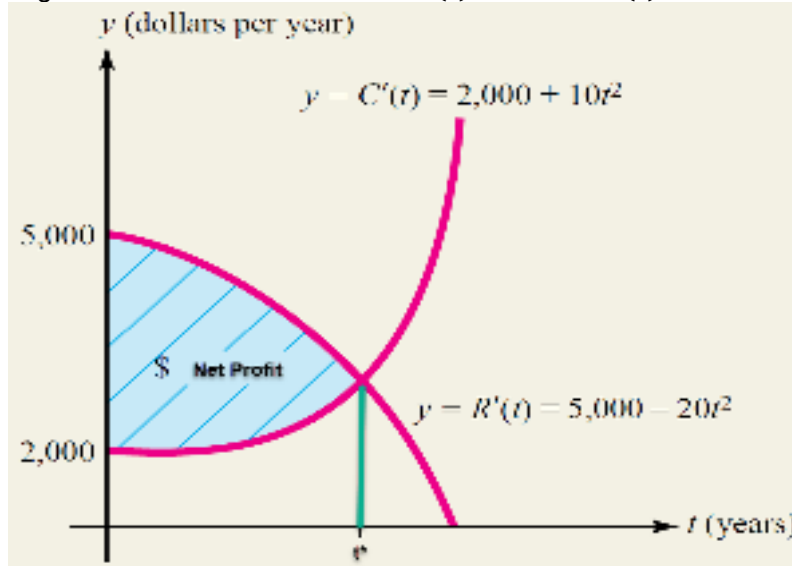
Numerical Instance: Suppose that when it is t years old, a particular industrial machine is generating revenue at the rate $R'(t) = 5,000 - 20t^2$ dollars per year and that operating and

servicing costs related to the machine are accumulating at the rate $C'(t) = 2,000 + 10t^2$ dollars per year.

a) What is the useful life of this machine?

b) Compute the net profit generated by the machine over its period of useful life.

Solution: Visualizing two functions i.e. Revenue $R(t)$ and Cost $C(t)$.



The intersection point of the two functions $R'(t)$ and $C'(t)$ shall exist at a certain point on the x-axis (t^*). It can be found by equating the two functions, i.e. $R'(t) = C'(t)$.

$$\begin{aligned}
 5000 - 20t^2 &= 2000 + 10t^2 \\
 20t^2 + 10t^2 - 5000 + 2000 &= 0 \\
 30t^2 - 3000 &= 0 \\
 t^2 - 100 &= 0 \\
 t^2 &= 100 \\
 \boxed{t = 10} & \quad [\because t \geq 0]
 \end{aligned}$$

Interpretation: Useful life of machine is till 10 years. This information gives us lower and upper limit; i.e. $[0, 10]$

To find the profit over the same interval, we evaluate the profit function $\{\pi(t) = R(t) - C(t)\}$ over $[0, 10]$.

$$\begin{aligned}
 \int_0^{10} \pi(t) dt &= \int_0^{10} \{R(t) - C(t)\} dt \\
 \int_0^{10} \pi(t) dt &= \int_0^{10} \{(5000 - 20t^2) - (2000 + 10t^2)\} dt \\
 \int_0^{10} \pi(t) dt &= \int_0^{10} (5000 - 20t^2 - 2000 - 10t^2) dt \\
 \int_0^{10} \pi(t) dt &= \int_0^{10} (3000 - 30t^2) dt \\
 \int_0^{10} \pi(t) dt &= 3000 \int_0^{10} (1) dt - 30 \int_0^{10} (t^2) dt
 \end{aligned}$$

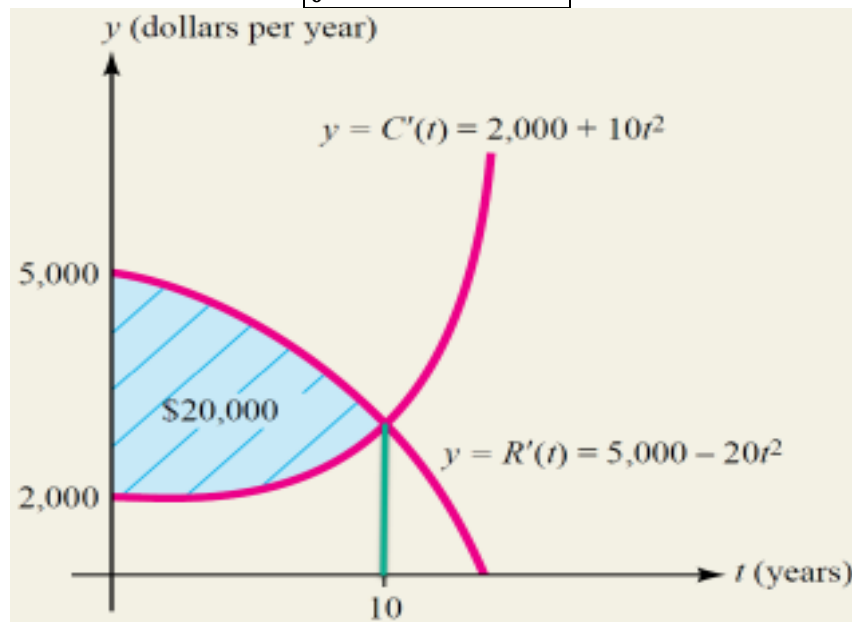
$$\int_0^{10} \pi(t) dt = 3000 \{ (t) \}_0^{10} - 30 \left(\frac{t^3}{3} \right) \Big|_0^{10}$$

$$\int_0^{10} \pi(t) dt = 3000 \{ (t) \}_0^{10} - 10(t^3) \Big|_0^{10}$$

$$\int_0^{10} \pi(t) dt = 3000(10 - 0) - 10\{(10)^3 - (0)^3\}$$

$$\int_0^{10} \pi(t) dt = 30000 - 10000$$

$$\int_0^{10} \pi(t) dt = 20000$$



Interpretation: Profit generated by the machine over its useful life (10 years) is **\$20,000**.

Lesson 14

FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 4)

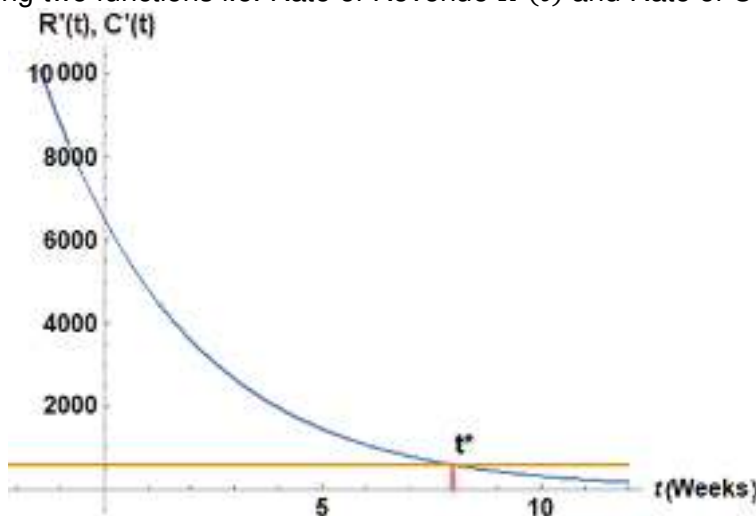
TOPIC 056: FUND-RAISING NET EARNINGS ANALYSIS USING INTEGRALS

It is estimated that t weeks from now, contributions in response to a fundraising campaign will be coming in at the rate of $R(t) = 6,537e^{-0.3t}$ dollars per week, while campaign expenses are expected to accumulate at the constant rate of \$593 per week.

For how many weeks does the rate of revenue exceed the rate of cost?

What net earnings will be generated by the campaign during the period of time determined?

Solution: Visualizing two functions i.e. Rate of Revenue $R'(t)$ and Rate of Cost $C'(t)$.

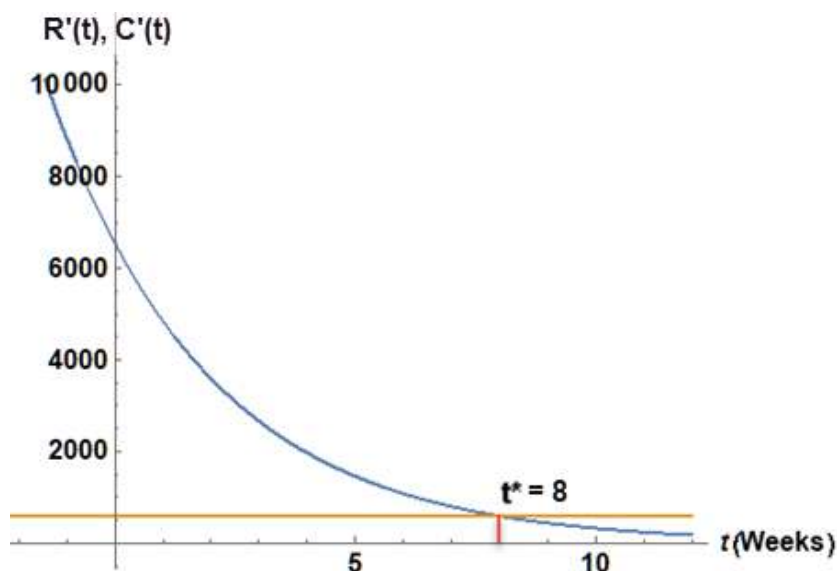


To find t^* mathematically, equate $R'(t)$ and $C'(t)$.

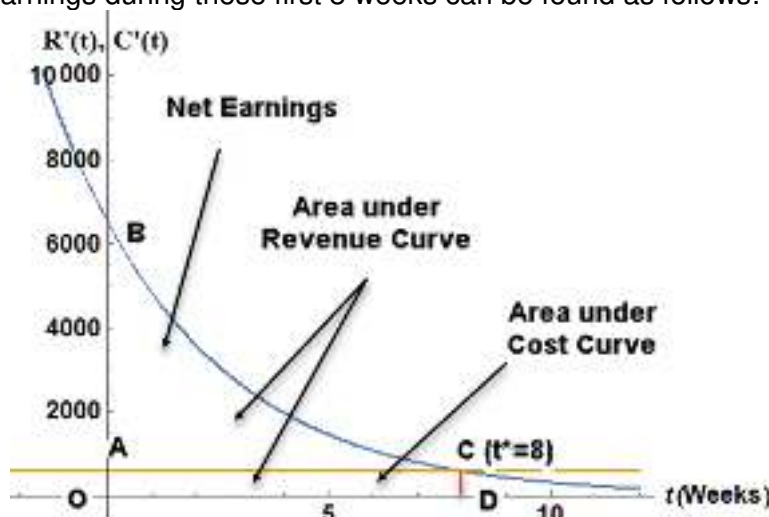
$$\begin{aligned} R'(t) &= C'(t) \\ 6537e^{-0.3t} &= 593 \\ e^{-0.3t} &= \frac{593}{6537} \end{aligned}$$

Introducing natural logarithm to eliminate exponential term (e).

$$\begin{aligned} \ln e^{-0.3t} &= \ln|0.0907| \\ -0.3t &= -2.4 \\ t &= \frac{2.4}{0.3} \\ t^* &= 8 \end{aligned}$$



Interpretation: For 8 weeks, the rate of revenue exceeds the rate of cost. While the difference between the net earnings during these first 8 weeks can be found as follows:



Net Earnings = Area under Revenue Curve – Area under Revenue Curve

Mathematically speaking,

$\{NE(t) = R(t) - C(t)\}$ over $[0, 8]$.

$$\Delta ABC = \int_0^8 \{R(t) - C(t)\} dt$$

$$\Delta ABC = \int_0^8 \{(6537e^{-0.3t}) - (593)\} dt$$

$$\Delta ABC = 6537 \int_0^8 (e^{-0.3t}) dt - \int_0^8 (593) dt$$

$$\Delta ABC = 6537 \left(\frac{e^{-0.3t}}{-0.3} \right)_0^8 - 593t \Big|_0^8 \left[\because \int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} \right]$$

$$\Delta ABC = -\frac{6537}{0.3} \{e^{-0.3(8)} - e^{-0.3(0)}\} - 593(8 - 0)$$

$$\Delta ABC = -\frac{6537}{0.3} \{e^{-2.4} - e^0\} - 593(8)$$

$$\Delta ABC = -21790\{0.0907 - 1\} - 4744$$

$$\Delta ABC = 15069.647$$

Interpretation: Net earnings in fund raising during first 8 weeks shall be \$15069.647.

TOPIC 057: RETIREMENT ANNUITY ANALYSIS USING INTEGRALS

Suppose money is being transferred continuously into an account over a time period $0 \leq t \leq T$ at a rate given by the function $f(t)$ and that the account earns interest at an annual rate r compounded continuously. Then the future value FV of the income stream over the term T is given by the definite integral

$$FV = \int_0^T f(t) e^{r(T-t)} dt$$

$$FV = e^{rT} \int_0^T f(t) e^{-rt} dt$$

$$FV = e^{rT} \int_0^T f(t) e^{-rt} dt$$

Economic Application: At age 25, Tom starts making annual deposits of \$2,500 into an Individual Retirement Account (IRA) that pays interest at an annual rate of 5% compounded continuously. Assuming that his payments are made as a continuous income flow, how much money will be in his account if he retires at age 60? At age 65?

Tom is undergoing a process of continuous compounding.

Annual rate of compounding = 5% = 0.05

Annual deposits in the account = \$2,500

Upper limit of time (T) is $(60 - 25) = 35$

Lower limit of time is = 0

Resorting to formula of continuous compounding:

$$FV = e^{rT} \int_0^T f(t) e^{-rt} dt$$

$$FV = e^{(0.05)(35)} \int_0^{35} (2500) e^{-(0.05)t} dt$$

$$FV = e^{1.75} \times 2500 \int_0^{35} (e^{-0.05t}) dt$$

$$FV = e^{1.75} \times 2500 \left\{ \left(\frac{e^{-0.05t}}{-0.05} \right) \right\}_0^{35}$$

$$FV = 5.7546 \times \frac{2500}{-0.05} (e^{-0.05t})_0^{35}$$

$$FV = -287730 \{e^{-0.05(35)} - e^{-0.05(0)}\}$$

$$FV = -287730(e^{-1.75} - e^0)$$

$$FV = 237730.02$$

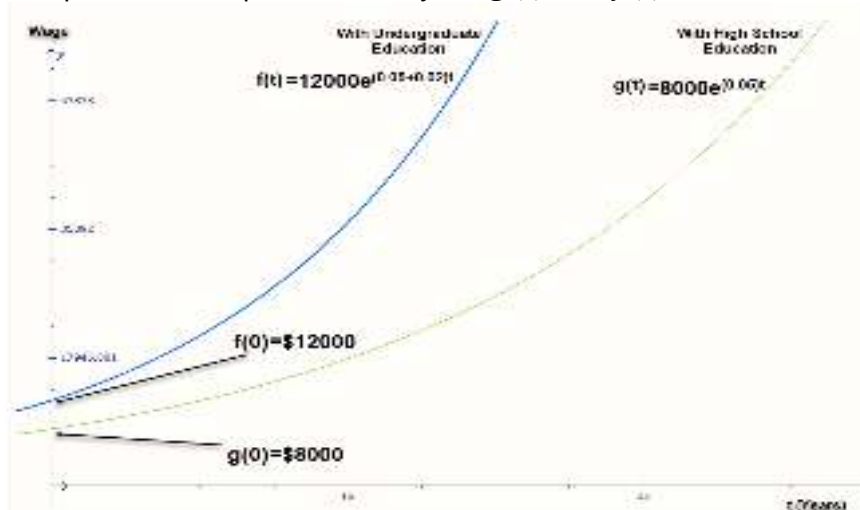
Interpretation: Tom, at age 25, starts making annual deposits of \$2,500 into an Individual Retirement Account (IRA) that pays interest at an annual rate of 5% compounded continuously. At the age 60, when he retires, \$237730.02 will be in his account.

TOPIC 058: WAGE DIFFERENTIAL USING INTEGRALS

Assume that an individual has a choice of either going for work with only high school education or going for work after obtaining an undergraduate degree in economics, business, or finance. If the individual decides to go for work with only high school education, the salary (in dollars) over the years will be given by the function $g(t) = 8000e^{0.05t}$; and if the individual decides to go for work after obtaining a degree, the salary over the years will be given by the function $f(t) = 12000e^{(0.05+0.02)t}$. If the individual works for 25 years, what will be the difference in total salary earned over the entire period of work?

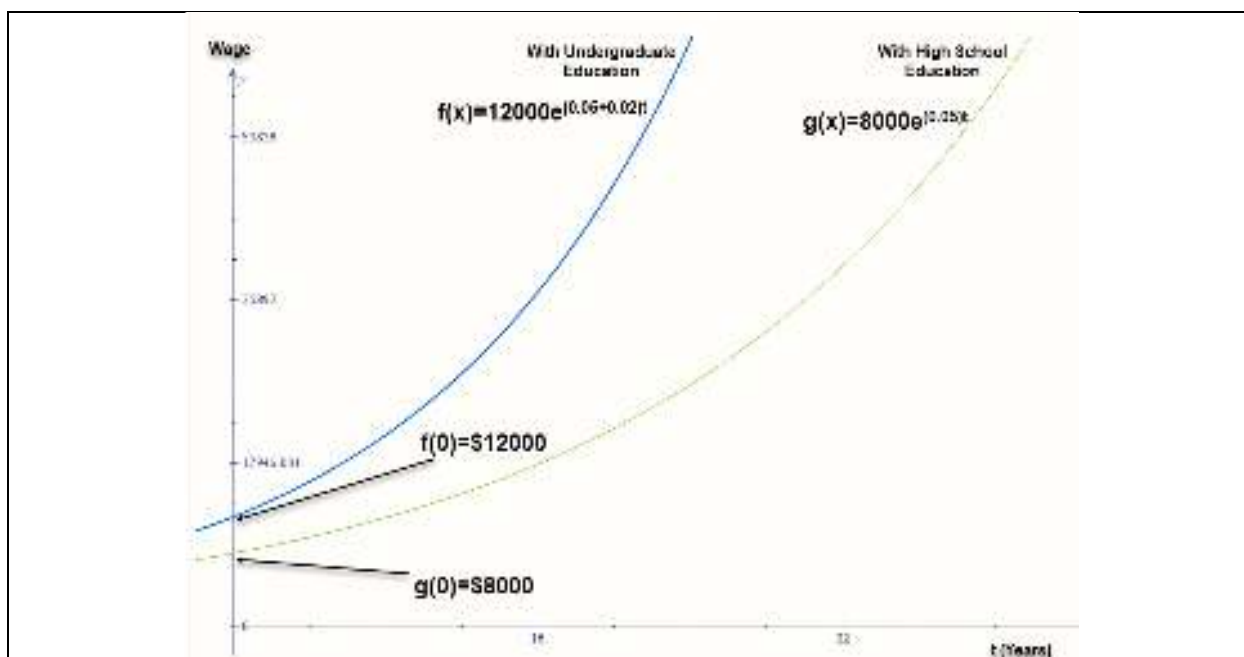
Solution:

Visualizing the two possible time paths of salary i.e. $g(t)$ and $f(t)$.



Finding the initial condition of the time paths, we introduce $t = 0$ in $g(x)$ and $f(x)$, respectively.

Initial condition with High School Education	Initial condition with Undergraduate Education
$g(x) = 8000e^{(0.05)t}$ $g(0) = 8000e^{(0.05)(0)}$ $g(0) = 8000(e^0)$ $g(0) = 8000(1)$ $g(0) = 8000$	$f(x) = 12000e^{(0.05+0.02)t}$ $f(0) = 12000e^{(0.05+0.02)(0)}$ $f(0) = 12000(e^0)$ $f(0) = 12000(1)$ $f(0) = 12000$



Interpretation

Starting job with High School Education will get the individual lower starting-wage i.e. \$8,000

Starting job with Undergraduate Education will get the individual higher starting-wage i.e. \$12,000

Determination of Time Path

Since the job is done for 25 years, the lower and upper limits will be 0 and 25.
Applying the limits on both of the functions

$$\begin{aligned}
 \int_0^{25} g(t) dt &= \int_0^{25} 8000e^{(0.05)t} dt \\
 &= 8000 \int_0^{25} e^{(0.05)t} dt \\
 &= 8000 \left\{ \frac{e^{(0.05)t}}{0.05} \right\}_0^{25} \\
 &= \frac{8000}{0.05} \{ e^{(0.05)t} \}_0^{25} \\
 &= 160000 \{ e^{(0.05)(25)} - e^{(0.05)(0)} \} \\
 &= 160000 (e^{1.25} - e^0) \\
 &= 160000 (3.49 - 1) \\
 &= 160000 (2.49)
 \end{aligned}$$

$$\int_0^{25} g(t) dt = 398400$$

$$\begin{aligned}
 \int_0^{25} f(t) dt &= \int_0^{25} 12000e^{(0.05+0.02)t} dt \\
 &= 12000 \int_0^{25} e^{(0.05+0.02)t} dt \\
 &= 12000 \left\{ \frac{e^{(0.07)t}}{0.07} \right\}_0^{25} \\
 &= \frac{12000}{0.07} \{ e^{(0.07)t} \}_0^{25} \\
 &= 171428.57 \{ e^{(0.07)(25)} - e^{(0.07)(0)} \} \\
 &= 171428.57 \{ e^{1.75} - e^0 \} \\
 &= 171428.57 \{ 5.75 - 1 \} \\
 &= 171428.57 \{ 4.75 \}
 \end{aligned}$$

$$\int_0^{25} f(t) dt = 814285.71$$

Wage Differential = Wage Time Path with Undergraduate Education - Wage Time Path with High School Education

$$WD_{GH} = \int_0^{25} f(t) dt - \int_0^{25} g(t) dt$$

$$WD_{GH} = 814285.71 - 398400$$

$$WD_{GH} = 415885.71$$

Interpretation: Obtaining Undergraduate Education before starting job, instead of starting after High School Education, earns the individual additional 415885.71 dollars.

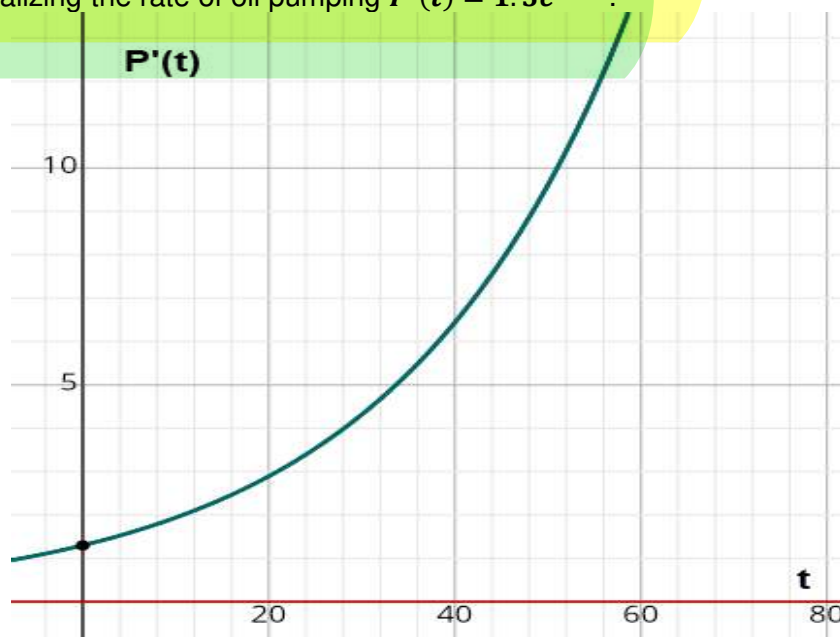
TOPIC 059: DEPLETION OF ENERGY RESOURCES ANALYSIS USING INTEGRALS

Oil is being pumped from an oil field t years after its opening at the rate of $P'(t) = 1.3e^{0.04t}$ billion barrels per year. The field has a reserve of 20 billion barrels, and the price of oil holds steady at \$112 per barrel.

a) Find $P(t)$, the amount of oil pumped from the field at time t . How much oil is pumped from the field during the first 3 years of operation?

b) For how many years T does the field operate before it runs dry?

Solution: Visualizing the rate of oil pumping $P'(t) = 1.3e^{0.04t}$:



To find how much oil is pumped during first 3 years, we apply lower and upper limits $[0, 3]$.

$$\int_0^3 P'(t) dt = \int_0^3 (1.3e^{0.04t}) dt$$

$$\int_0^3 P'(t) dt = 1.3 \int_0^3 (e^{0.04t}) dt$$

$$= 1.3 \left\{ \left(\frac{e^{0.04t}}{0.04} \right) \right\}_0^3$$

$$= \frac{1.3}{0.04} \{ e^{0.04(t)} \}_0^3$$

$$= 32.5 \{ e^{0.04(3)} - e^{0.04(0)} \}$$

$$\begin{aligned}
 &= 32.5(e^{0.12} - e^0) \\
 &= 32.5(1.127 - 1) \\
 &= 32.5(0.127) \\
 &\int_0^3 P'(t) dt = 4.13
 \end{aligned}$$

Interpretation: The amount extracted from the oil field during the first year is 4.13 billion barrels. To find the time when the field is used up all of its reservoir (20 billion barrels) it holds, we need $P(t)$ function.

$$\begin{aligned}
 P(t) &= \int P'(t) dt = \int (1.3e^{0.04t}) dt \\
 P(t) &= 1.3 \int (e^{0.04t}) dt \\
 P(t) &= 1.3 \left\{ \frac{e^{0.04t}}{0.04} + c' \right\} \\
 P(t) &= \frac{1.3}{0.04} \{e^{0.04t} + 1.3c'\} \\
 P(t) &= 32.5e^{0.04t} + 32.5c' \quad [\because 1.3c' = c] \\
 P(t) &= 32.5e^{0.04t} + c \quad [\because 32.5c' = c]
 \end{aligned}$$

Substituting the initial condition ($t = 0$), for getting rid of arbitrary constant c .

$$P(0) = 32.5e^{0.04(0)} + c$$

Since at the beginning of pumping process, no oil is pumped out, $P(0) = 0$

$$0 = 32.5e^0 + c$$

$$0 = 32.5 + c$$

$$c = -32.5$$

Substituting the value of c in general time path $P(t) = 32.5e^{0.04t} + c$:

$$P(t) = 32.5e^{0.04t} + (-32.5)$$

$$P(t) = 32.5e^{0.04t} - 32.5$$

Now substituting the 20 billion barrels in place of oil pumping function $P(t)$.

$$20 = 32.5e^{0.04t^*} - 32.5$$

$$20 + 32.5 = 32.5e^{0.04t^*}$$

$$32.5e^{0.04t^*} = 52.5$$

$$32.5e^{0.04t^*} = \frac{52.5}{32.5}$$

$$e^{0.04t^*} = 1.615$$

$$\ln(e^{0.04t^*}) = \ln(1.615)$$

$$0.04t^* = 0.479$$

$$t^* = \frac{0.479}{0.04}$$

$$t^* = 11.975 \cong 12 \text{ years}$$

Interpretation: It will take approximately 12 years to pump out the oil field till no more oil is left.

TOPIC 060: PRESENT VALUE OF GOLD MINE USING INTEGRALS

Given the function of income earned from a gold mine is $P(t) = 600e^{-0.5t}$. If money is discounted at 10% a year within a period of 20 years, what is the present value of the gold mine?

As already known, that: $\pi = \int_0^t R(t) e^{-rt} dt$

Here $t = 20$, $R(t) = P(t)$, $r = 10\% = 0.1$, $\pi = PV_{GM}$.

Therefore, substitution of values gives:

$$PV_{GM} = \int_0^{20} (600e^{-0.5t}) e^{-0.1t} dt$$

$$PV_{GM} = 600 \int_0^{20} e^{-0.5t} e^{-0.1t} dt$$

$$PV_{GM} = 600 \int_0^{20} e^{-0.6t} dt$$

$$PV_{GM} = 600 \left\{ \left(\frac{e^{-0.6t}}{-0.6} \right) \right\}_0^{20}$$

$$PV_{GM} = \frac{600}{-0.6} \{ (e^{-0.6t}) \}_0^{20}$$

$$PV_{GM} = -1000(e^{-0.6(20)} - e^{-0.6(0)})$$

$$PV_{GM} = -1000(e^{-12} - e^0)$$

$$PV_{GM} = -1000\{(6.14 \times 10^{-6}) - 1\}$$

$$PV_{GM} = -1000(-0.99999386)$$

$$PV_{GM} = 999.99 \cong 1000$$

Interpretation: The present value of the gold mine that mined for 20 years, is \$1000 thousands of dollars i.e. (\$10,00,000).

Lesson 15

FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 5)

TOPIC 061: LOTTERY PAYOUT ANALYSIS USING INTEGRALS

The winner of a state lottery is offered a choice of either receiving \$10 million now as a lump sum or of receiving A dollars a year for the next 6 years as a continuous income stream. If the prevailing annual interest rate is 5% compounded continuously and the two payouts are worth the same, what is A ?

Solution: Winner of the lottery has two options:

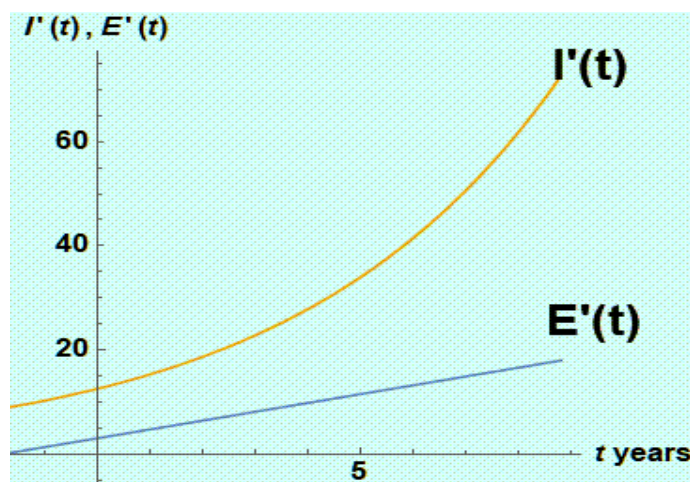
Option – I Receiving \$10 million now as a lump sum.	Option – II 'A' dollars a year for the next 6 years as a continuous income stream (annual interest rate is 5%)
The unknown in the two options is 'A': i.e. how much should be given to the individual for next 6 years so that he remains <u>indifferent</u> between two options. Mathematically, we can use equation to represent the indifference of the individual between two options. i.e. $\begin{aligned} \$10 \text{ million now} &= 'A' \text{ annually for 6 years} \\ 10 &= \int_0^6 A e^{-0.05t} dt \\ 10 &= A \int_0^6 e^{-0.05t} dt \\ 10 &= A \left(\frac{e^{-0.05t}}{-0.05} \right) \Big _0^6 \\ 10 &= \frac{A}{-0.05} (e^{-0.05(6)} - e^{-0.05(0)}) \\ -0.05 \times 10 &= A \{e^{-0.3} - 1\} \\ -0.5 &= A(e^{-0.3} - 1) \\ -0.5 &= A(0.741 - 1) \\ -0.5 &= A(-0.259) \\ A &= \frac{-0.5}{-0.259} \\ \boxed{A = \$1.931 \text{ million}} \end{aligned}$ Interpretation: Present value analysis shows the lottery winner can be kept indifferent between Option – I and Option – II, he can be given \$1.931 for next 6 years.	

TOPIC 062: BALANCE OF TRADE ANALYSIS USING INTEGRALS

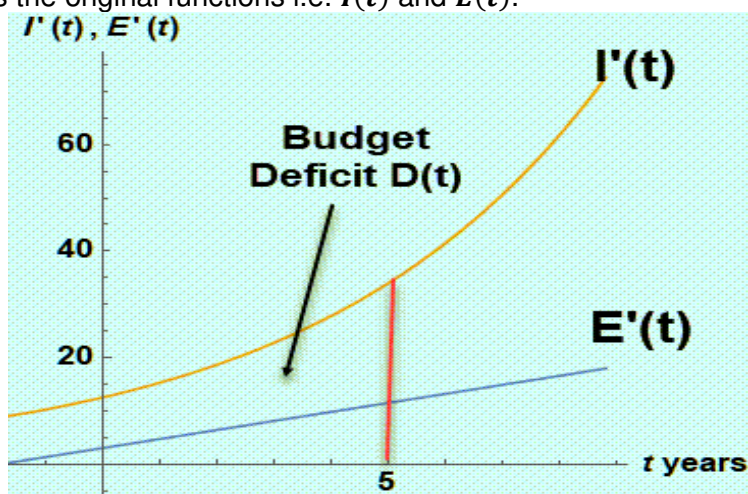
The government of a certain country estimates that t years from now, imports will be increasing at the rate I_t and exports at the rate E_t , both in billions of dollars per year, where $I'(t) = 12.5e^{0.2t}$ and $E'(t) = 1.7t + 3$.

The trade deficit is $D(t) = I(t) - E(t)$. By how much will the trade deficit for this country change over the next 5 years? Will it increase or decrease during this time period?

Solution: Visualizing the given functions:



We are given the rate of import and export functions i.e. $I'(t)$ and $E'(t)$. While in the trade deficit $D(t)$ formula needs the original functions i.e. $I(t)$ and $E(t)$.



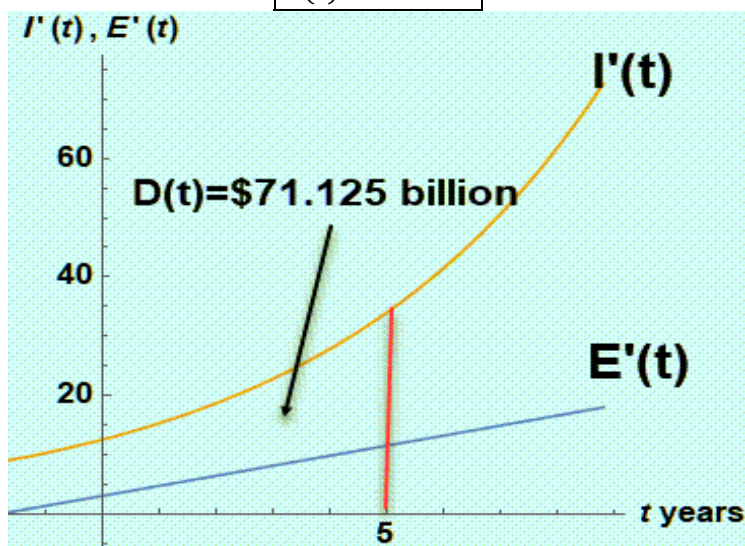
Therefore, we integrate the two functions, respectively, w.r.t time (t).

Moreover, we need to apply the lower and upper limits $[0, 5]$. As deficit is to be quantified for five years.

$I'(t) = 12.5e^{0.2t}$	$E'(t) = 1.7t + 3$
$\int_0^5 I'(t) dt = \int_0^5 12.5e^{0.2t} dt$	$\int_0^5 E'(t) dt = \int_0^5 (1.7t + 3) dt$
$\int_0^5 I'(t) dt = 12.5 \int_0^5 e^{0.2t} dt$	$\int_0^5 E'(t) dt = 1.7 \int_0^5 (t) dt + 3 \int_0^5 (1) dt$
$\int_0^5 I'(t) dt = 12.5 \left\{ \left(\frac{e^{0.2t}}{0.2} \right) \right\}_0^5$	$\int_0^5 E'(t) dt = 1.7 \left\{ \left(\frac{t^2}{2} \right) \right\}_0^5 + 3(t)_0^5$
$\int_0^5 I'(t) dt = 12.5 \left\{ \frac{e^{0.2(5)} - e^{0.2(0)}}{0.2} \right\}$	$\int_0^5 E'(t) dt = 1.7 \left(\frac{(5)^2 - (0)^2}{2} \right) + 3(5 - 0)$
$\int_0^5 I'(t) dt = 62.5(e^1 - e^0)$	$\int_0^5 E'(t) dt = 1.7 \left\{ \frac{25}{2} \right\} + 3(5)$
$\int_0^5 I'(t) dt = 62.5(2.718 - 1)$	$\int_0^5 E'(t) dt = 1.7(12.5) + 15$
$\int_0^5 I'(t) dt = 62.5(1.718)$	$\int_0^5 E'(t) dt = 21.25 + 15$
$I(t) = \int_0^5 I'(t) dt = 107.375$	$E(t) = \int_0^5 E'(t) dt = 36.25$

With both $I(t)$ and $E(t)$ available, we can find the value of deficit $D(t)$:

$$\begin{aligned} D(t) &= I(t) - E(t) \\ D(t) &= \int_0^5 I'(t) dt - \int_0^5 E'(t) dt \\ &= 107.375 - 36.25 \\ \boxed{D(t) &= 71.125} \end{aligned}$$



Interpretation: During the next 5 years, the trade deficit $D(t)$ shall be 71.125 billion dollars.

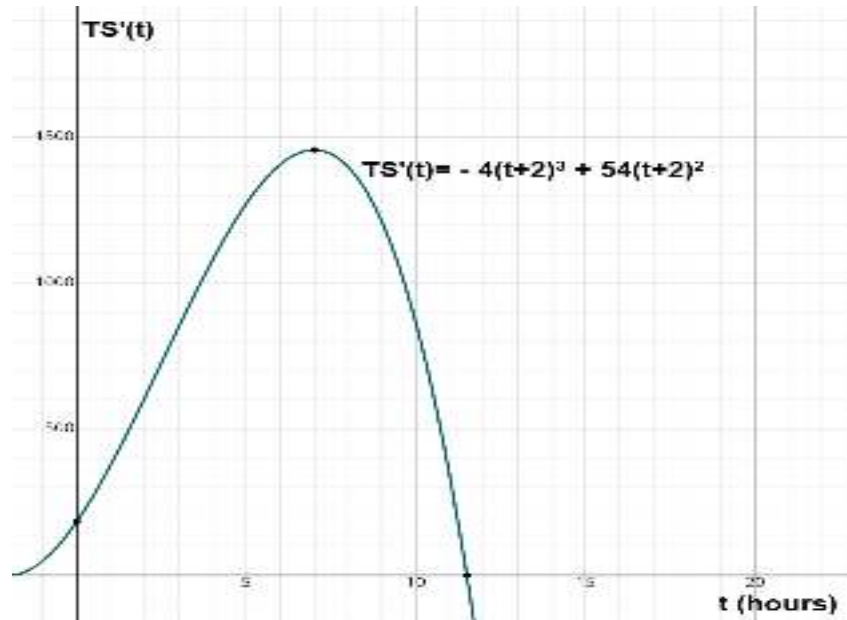
TOPIC 063: TICKET SALES ANALYSIS USING INTEGRALS

The promoters of a county fair estimate that t hours after the gates open at **9:00 A.M.** visitors will be entering the fair at the rate of $-4(t+2)^3 + 54(t+2)^2$ people per hour. How many people will enter the fair between **10:00 A.M.** and noon?

Solution: Primarily one can assume that 'number of tickets sold = visitors entering'. Therefore, Rate of Ticket Sales $TS'(t)$ function is same as that of Rate of visitors entering $VE'(t)$ function, i.e.

$$VE'(t) = TS'(t) = -4(t+2)^3 + 54(t+2)^2$$

Visualizing the function:



As mentioned, the Ticket Sales are to be found between **10:00 A.M.** till noon.

Since county fair starts from **9:00 A.M.**, therefore, at **9:00 A.M.**, $t = 0$

Accordingly, **10:00 A.M.**, therefore, at **10:00 A.M.**, $t = 1$

At noon, **12:00 P.M.**, therefore, at **12:00 A.M.**, $t = 3$

Summary Table:

9:00 A.M. $\Rightarrow t = 0$
10:00 A.M. $\Rightarrow t = 1$
11:00 A.M. $\Rightarrow t = 2$
12:00 A.M. $\Rightarrow t = 3$

Therefore, the lower and upper limits of the function are $[1, 3]$.

$$\begin{aligned}
 \int_1^3 TS'(t) dt &= \int_1^3 \{-4(t+2)^3 + 54(t+2)^2\} dt \\
 \int_1^3 TS'(t) dt &= \int_1^3 \{-4(t+2)^3\} dt + \int_1^3 \{54(t+2)^2\} dt \\
 \int_1^3 TS'(t) dt &= -4 \int_1^3 (t+2)^3 dt + 54 \int_1^3 (t+2)^2 dt \\
 \int_1^3 TS'(t) dt &= -4 \left\{ \frac{(t+2)^4}{4} \right\}_1^3 + 54 \left\{ \frac{(t+2)^3}{3} \right\}_1^3 \\
 \int_1^3 TS'(t) dt &= -\frac{4}{4} \{(t+2)^4\}_1^3 + \frac{54}{3} \{(t+2)^3\}_1^3 \\
 \int_1^3 TS'(t) dt &= -\{(3+2)^4 - (1+2)^4\} + 18\{(3+2)^3 - (1+2)^3\} \\
 \int_1^3 TS'(t) dt &= -\{(5)^4 - (3)^4\} + 18\{(5)^3 - (3)^3\} \\
 \int_1^3 TS'(t) dt &= -(625 - 81) + 18(125 - 27) \\
 \int_1^3 TS'(t) dt &= -(544) + 18(98) \\
 \int_1^3 TS'(t) dt &= -544 + 1764 \\
 \int_1^3 TS'(t) dt &= 1220
 \end{aligned}$$

Interpretation: Hence, between 10:00 A.M. till noon (12:00 A.M.), the Ticket Sales will be 1220 in number. In other words, 1220 people will be entering the country fair.

TOPIC 064: DEMAND ANALYSIS OF OIL USING INTEGRALS

It is estimated that t years from the beginning of the year 2005, the demand for oil in a certain country will be changing at the rate of $D'(t) = (1 + 2t)^{-1}$ billion barrels per year. Will more oil be consumed (demanded) during 2006 or during 2009? How much more?

Solution: Rate of change of demand in certain country for Oil is $D'(t) = (1 + 2t)^{-1}$.

Analysis of demand for oil starts from year 2005, therefore, during 2005 A.M., $t = 0$ to $t = 1$.

Demand for oil during year 2006, will be; $t = 1$ to $t = 2$.

Summary Table:

Demand for oil during 2005 $\Rightarrow t = 0$ to $t = 1$
Demand for oil during 2006 $\Rightarrow t = 1$ to $t = 2$
Demand for oil during 2007 $\Rightarrow t = 2$ to $t = 3$
Demand for oil during 2008 $\Rightarrow t = 3$ to $t = 4$
Demand for oil during 2009 $\Rightarrow t = 4$ to $t = 5$

Therefore, the lower and upper limits of the function, for year 2006 are $[1, 2]$.

$$\int_1^2 D'(t) dt = \int_1^2 \{(1+2t)^{-1}\} dt$$

$$\int_1^2 D'(t) dt = \int_1^2 \left(\frac{1}{1+2t}\right) dt$$

Setting stage for application of: $\int \left\{\frac{f'(x)}{f(x)}\right\} dx = \ln|f(x)|$.

As derivative of $(1+2t)$ is 2, we can multiply and divide by $\boxed{2}$.

$$\int_1^2 D'(t) dt = \boxed{\frac{2}{2}} \int_1^2 \left(\frac{1}{1+2t}\right) dt$$

$$\int_1^2 D'(t) dt = \boxed{\frac{1}{2}} \int_1^2 \left(\frac{\boxed{2}}{1+2t}\right) dt$$

Applying the rule: $\int \left\{\frac{f'(x)}{f(x)}\right\} dx = \ln|f(x)|$

$$\int_1^2 D'(t) dt = \boxed{\frac{1}{2}} \{ \ln|1+2t| \}_1^2$$

$$\int_1^2 D'(t) dt = \boxed{\frac{1}{2}} (\ln|1+2(2)| - \ln|1+2(1)|)$$

$$\int_1^2 D'(t) dt = \boxed{\frac{1}{2}} (\ln|5| - \ln|3|)$$

$$D(2006) = \int_1^2 D'(t) dt = \mathbf{0.2554} \text{ billion barrels approximately.}$$

Interpretation: The Demand for oil in a certain country during 2006 shall be $0.2554 \times 1,000,000,000 = \boxed{2,55,400,000}$ barrels.

Similarly, for the year **2009** same analysis can be done. However, the lower and upper limits will be $[4, 5]$:

$$\int_4^5 D'(t) dt = \int_4^5 \{(1+2t)^{-1}\} dt$$

$$\int_4^5 D'(t) dt = \int_4^5 \left(\frac{1}{1+2t}\right) dt$$

Setting stage for application of: $\int \left\{\frac{f'(x)}{f(x)}\right\} dx = \ln|f(x)|$.

Borrowing the solution of integral from above, as the function is the same.

$$\int_4^5 D'(t) dt = \boxed{\frac{1}{2}} \int_4^5 \left(\frac{\boxed{2}}{1+2t}\right) dt$$

Applying the rule: $\int \left\{\frac{f'(x)}{f(x)}\right\} dx = \ln|f(x)|$

$$\int_4^5 D'(t) dt = \boxed{\frac{1}{2}} \{ \ln|1+2t| \}_4^5$$

$$\int_4^5 D'(t) dt = \boxed{\frac{1}{2}} (\ln|1+2(5)| - \ln|1+2(4)|)$$

$$\int_4^5 D'(t) dt = \boxed{\frac{1}{2}} (\ln|11| - \ln|9|)$$

$$D(2009) = \int_4^5 D'(t) dt = \mathbf{0.1003} \text{ billion barrels approximately.}$$

Interpretation: The Demand for oil in a certain country during 2006 shall be $0.1003 \times 1,000,000,000 = \boxed{1,00,300,000}$ barrels.

Comparative Analysis:

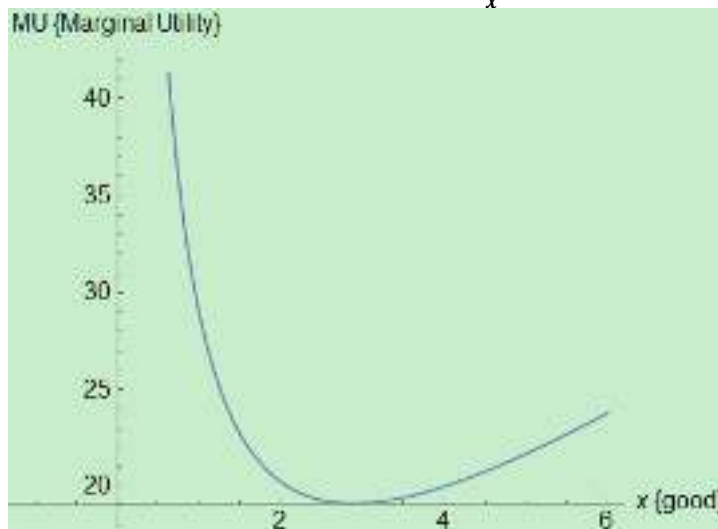
$$D(2006) = 2,55,400,000 > D(2009) = 1,00,300,000$$

Demand for Oil in the country under consideration is higher during **2006** as compared to that in **2009**.

TOPIC 065: MARGINAL TO TOTAL UTILITY USING INTEGRALS

If the marginal utility of a consumer is $MU_x = \frac{3x^2+6\sqrt{x}+20}{x}$, where x is the number of units consumed of a good, find the increase in his total utility when x goes up from **16** to **100** units.

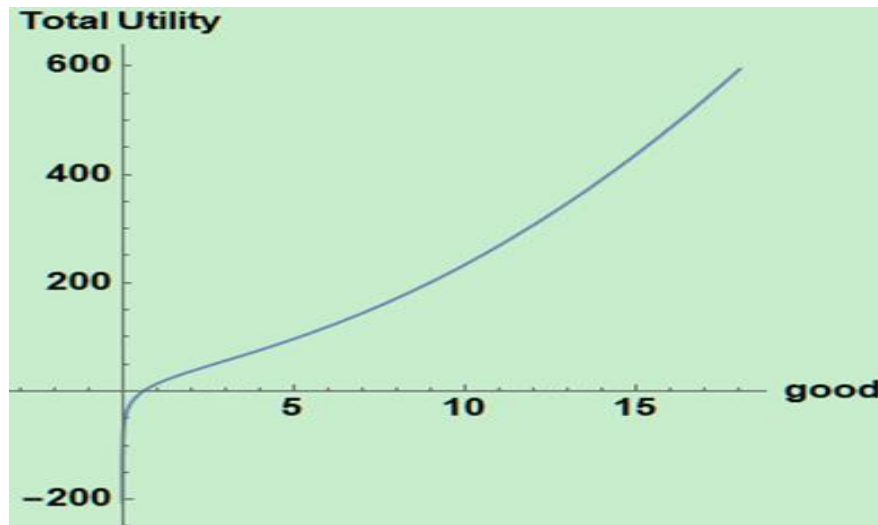
Solution: Given the marginal utility function: $MU_x = \frac{3x^2+6\sqrt{x}+20}{x}$.



Additional total utility from **16th** to **100th** unit is to be found.

Mathematically, it will be: $\int_{16}^{100} MU(x) dx = \int_{16}^{100} \left(\frac{3x^2+6\sqrt{x}+20}{x} \right) dx$

$$\begin{aligned} \int_{16}^{100} MU(x) dx &= \int_{16}^{100} \left(\frac{3x^2}{x} \right) dx + \int_{16}^{100} \left(\frac{6\sqrt{x}}{x} \right) dx + \int_{16}^{100} \left(\frac{20}{x} \right) dx \\ \int_{16}^{100} MU(x) dx &= 3 \int_{16}^{100} (x) dx + 6 \int_{16}^{100} \left(\frac{(x)^{\frac{1}{2}}}{x} \right) dx + 20 \int_{16}^{100} \left(\frac{1}{x} \right) dx \\ \int_{16}^{100} MU(x) dx &= 3 \int_{16}^{100} (x) dx + 6 \int_{16}^{100} \left(x^{-\frac{1}{2}} \right) dx + 20 \int_{16}^{100} \left(\frac{1}{x} \right) dx \end{aligned}$$



$$\begin{aligned}
 \int_{16}^{100} TU(x) dx &= 3 \left\{ \left(\frac{x^2}{2} \right) \right\}_{16}^{100} + 6 \left\{ \left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right) \right\}_{16}^{100} + 20 \{ (\ln|x|) \}_{16}^{100} \\
 &= 3 \left\{ \frac{(100)^2}{2} - \frac{(16)^2}{2} \right\} + 12 \left\{ (100)^{\frac{1}{2}} - (16)^{\frac{1}{2}} \right\} + 20 (\ln|100| - \ln|16|) \\
 &= 3(5000 - 128) + 12(10 - 4) + 20(4.6052 - 2.7726) \\
 &= 3(4872) + 12(6) + 20(1.8326) \\
 \int_{16}^{100} TU(x) dx &= 14616 + 72 + 36.652 \\
 \boxed{\int_{16}^{100} TU(x) dx = 14724.652}
 \end{aligned}$$

Interpretation: The total utility of the good from 16th to 100th unit is equal to 14724.652.

Lesson 16

FIRM'S ANALYSIS USING INTEGRALS (CONTINUED 6)

TOPIC 066: REAL ESTATE ANALYSIS USING INTEGRALS

In a certain community the fraction of the homes placed on the market that remain unsold for at least t weeks is approximately $f(t) = e^{-0.2t}$. If 200 homes are currently on the market the function for existing unsold homes will be $g(t) = 200e^{-0.2t}$. Now, if additional homes are placed on the market at the rate of 8 per week, approximately, the function for unsold new houses will be $h(t) = 8e^{-0.2(10-t)}$. How many homes will be on the market 10 weeks from now?

Solution: The total number of unsold homes will be sum of **existing** and **new** ones, i.e.:

$$\text{Total Unsold} = \text{preexisting unsold} + \text{New unsold}$$

After 10 years, the **existing** unsold houses: $g(10) = 200e^{-0.2(10)}$

Caveat: $g(t)$ function didn't use limits as homes are pre-existing and no integration over time is needed.

After 10 years, the **new** unsold houses: $\int_0^{10} h(t) dt = \int_0^{10} \{8e^{-0.2(10-t)}\} dt$

Caveat: In case of $h(t)$, integration over time is needed as the new homes are being added on weekly basis (as a stream).

Resuming the function:

$$\text{Total Unsold} = \text{preexisting unsold} + \text{New unsold}$$

$$\begin{aligned} U(t) &= g(10) + \int_0^{10} h(t) dt \\ U(t) &= 200e^{-0.2(10)} + \int_0^{10} \{8e^{-0.2(10-t)}\} dt \\ U(t) &= 200e^{-2} + 8 \int_0^{10} (e^{-2+0.2t}) dt \\ U(t) &= 200e^{-2} + 8 \left(\frac{e^{-2+0.2t}}{0.2} \right) \Big|_0^{10} \\ U(t) &= 200e^{-2} + \frac{8}{0.2} (e^{-2+0.2t}) \Big|_0^{10} \\ U(t) &= 200e^{-2} + 40 \{e^{-2+0.2(10)} - e^{-2+0.2(0)}\} \\ U(t) &= 200e^{-2} + 40(e^0 - e^{-2}) \\ U(t) &= 200(0.135) + 40(1 - 0.135) \\ U(t) &= 27 + 40(0.865) \\ \underbrace{U(t)}_{\text{Total unsold}} &= \underbrace{27}_{\text{preexisting unsold}} + \underbrace{34.6}_{\text{New unsold}} \\ U(t) &= 61.6 \end{aligned}$$

$U(t) = 62$ **unsold homes** approximately.

Interpretation: If 200 homes are currently on the market the function for existing unsold homes, 27 will be left unsold after 10 weeks. If additional homes are placed on the market at the rate of 8 per week, approximately 34.6 homes will be left unsold after 10 weeks. Consequently, $(27 + 34.6 = 62)$ will be left unsold on the whole at the end of 10 weeks.

TOPIC 067: DYNAMICS OF LABOR COST USING INTEGRALS

The marginal labor cost function is given by the equation $MLC = 3 + 4L$.

(a) Find the equation of the total labor cost function if total labor cost is zero when $L = 0$.

(b) Calculate the cost of employing successive labor units from $L = 1$ to $L = 7$.

Solution: Initial condition of Marginal Labor Cost (MLC) can be found at $L = 0$.

$$MLC(L) = 3 + 4L$$

$$MLC(0) = 3 + 4(0)$$

$$MLC(0) = 3$$

Successive labor cost of employing 1 to 7 units of labor is found using the definite integral with lower and upper limits $[1, 7]$.

$$\int_1^7 MLC(L) dL = \int_1^7 (3 + 4L) dL$$

$$\int_1^7 MLC(L) dL = 3 \int_1^7 (1) dL + 4 \int_1^7 (L) dL$$

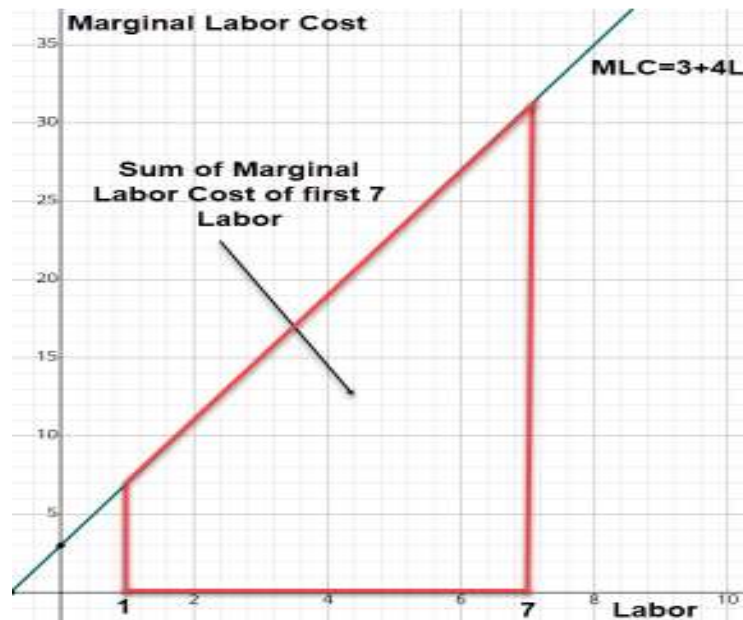
$$\int_1^7 MLC(L) dL = 3(L) \Big|_1^7 + 4 \left(\frac{L^2}{2} \right) \Big|_1^7$$

$$\int_1^7 MLC(L) dL = 3(L) \Big|_1^7 + 2(L^2) \Big|_1^7$$

$$\int_1^7 MLC(L) dL = 3(7 - 1) + 2\{(7)^2 - (1)^2\}$$

$$\int_1^7 MLC(L) dL = 18 + 96$$

$$\boxed{\int_1^7 MLC(L) dL = 114}$$



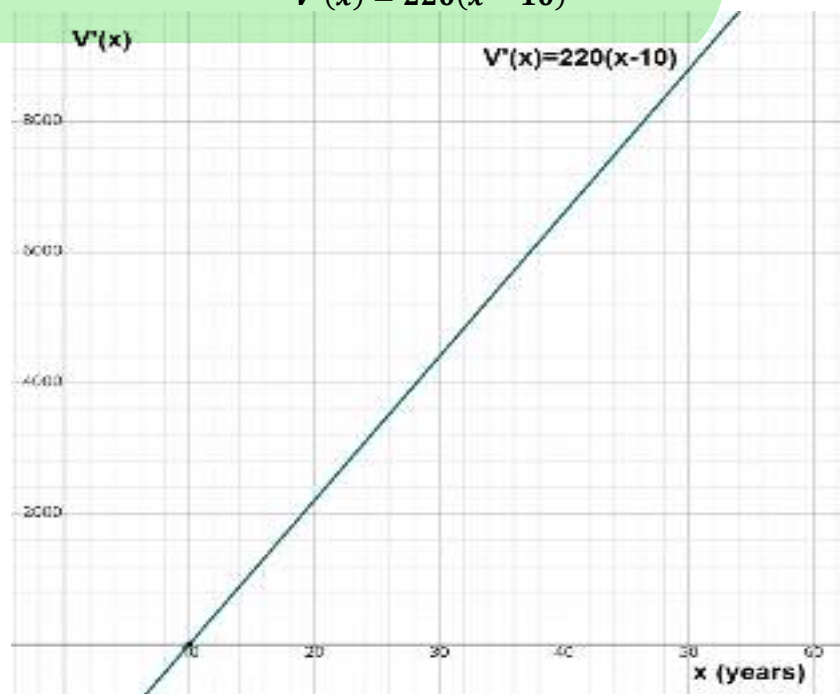
Interpretation: The Marginal labor cost of higher first 7 labor sums to 114.

TOPIC 068: DEPRECIATION ANALYSIS USING INTEGRALS

The resale value of a certain industrial machine decreases over a 10-year period at a rate that changes with time. When the machine is x years old, the rate at which its value is changing is $220(x - 10)$ dollars per year. By how much does the machine depreciate during the second year?

Solution: Let $V'(x)$ be the rate of change of value of machine, then:

$$V'(x) = 220(x - 10)$$



$$\begin{aligned}
 V(2) - V(1) &= \int_1^2 V'(x) dx = \int_1^2 \{220(x - 10)\} dx \\
 \int_1^2 V'(x) dx &= 220 \left\{ \int_1^2 x dx - 10 \int_1^2 (1) dx \right\} \\
 \int_1^2 V'(x) dx &= 220 \left[\left\{ \left(\frac{x^2}{2} \right) \right\}_1^2 - 10 \{ (x) \}_1^2 \right] \\
 \int_1^2 V'(x) dx &= 220 \left[\left\{ \frac{(2)^2}{2} - \frac{(1)^2}{2} \right\} - 10(2 - 1) \right] \\
 \int_1^2 V'(x) dx &= 220 \left(\frac{3}{2} - 10 \right) \\
 \int_1^2 V'(x) dx &= 220(-8.5) \\
 \boxed{\int_1^2 V'(x) dx} &= \boxed{-1870}
 \end{aligned}$$

Interpretation: Change in value of machine is -1870 , which means a decline of \$1870 in the value of machine. In other words, the machine depreciates by \$1870 during the 2nd year.

TOPIC 069: COST EFFECTIVENESS DECISION USING DIFFERENTIAL EQUATIONS

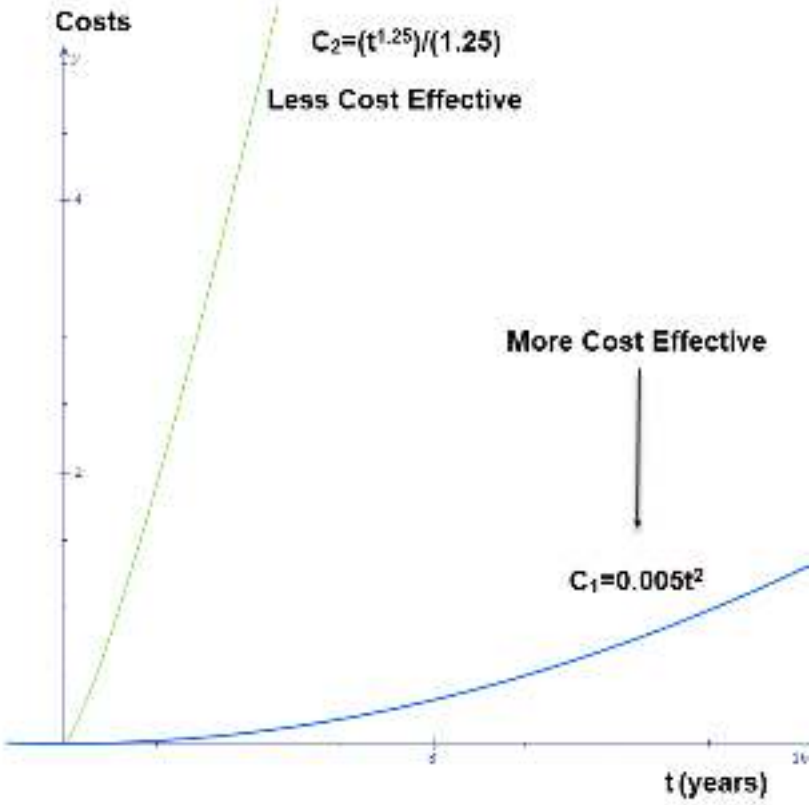
A hospital is considering the installation of two alternative heating systems. The rate of increase in costs for each system is given by the equations

$$\frac{dc_1}{dt} = 0.01t \text{ and } \frac{dc_2}{dt} = t^{0.25}$$

(a) Derive expressions for the total cost of each system at any time t given that cost = £50000 at $t = 0$.

(b) Which system is more cost effective in the next 10 years.

Solution: Rate of increase in costs for each heating system are $\frac{dc_1}{dt} = 0.01t$ and $\frac{dc_2}{dt} = t^{0.25}$, respectively. Lower and upper limit can be $[0, 10]$

$\frac{dC_1}{dt} = 0.01t$	$\frac{dC_2}{dt} = t^{0.25}$
Definite Integral implies:	
$\int_0^{10} \left(\frac{dC_1}{dt}\right) dt = \int_0^{10} (0.01t) dt$ $C_1(t) = 0.01 \int_0^{10} (t) dt$ $C_1(t) = 0.01 \left\{ \left(\frac{t^2}{2}\right) \right\}_0^{10}$ $C_1(t) = \frac{0.01}{2} \{(t^2)\}_0^{10}$ $C_1(t) = 0.005\{(10)^2 - (0)^2\}$ $C_1(t) = 0.005(100)$ $C_1(t) = 0.5$	$\int_0^{10} \left(\frac{dC_2}{dt}\right) dt = \int_0^{10} (t^{0.25}) dt$ $C_2(t) = \left\{ \left(\frac{t^{0.25+1}}{0.25+1}\right) \right\}_0^{10}$ $C_2(t) = \left\{ \left(\frac{t^{1.25}}{1.25}\right) \right\}_0^{10}$ $C_2(t) = \frac{(10)^{1.25} - (0)^{1.25}}{1.25}$ $C_2(t) = \frac{17.7828 - 0}{1.25}$ $C_2(t) = 14.226$
Clearly $C_1(t) < C_2(t)$ therefore, 1 st heating system is more cost effective. Same is visible in the graph.	
 The graph illustrates the cumulative costs for two heating systems over a 10-year period. The x-axis represents time in years (t), and the y-axis represents costs. The blue curve, labeled $C_1 = 0.005t^2$, represents the first heating system. The green curve, labeled $C_2 = (t^{1.25})/(1.25)$, represents the second heating system. The graph shows that C_1 is consistently lower than C_2 for the same time period, indicating that the first system is more cost-effective. The area under the C_2 curve is shaded yellow, and the area under the C_1 curve is shaded blue.	

Lesson 17

DOMAR GROWTH MODEL

TOPIC 070: DOMAR GROWTH MODEL: FRAMEWORK

PREMISES:

Stimulus: A change in the rate of investment will affect both aggregate demand and the productive capability of an economy: $(\Delta \kappa \leftarrow \Delta I \rightarrow \Delta AD)$

Here κ is the Productive Capacity and I & AD are investment and aggregate demand as usual.

Purpose: A Time path of investment $I(t)$ is needed along which an economy could grow while maintaining full utilization of its productive capacity (κ).

(ΔI) Change in Investment causes dual change:	
ΔY (Change in Income/Aggregate Demand)	$\Delta \kappa$ (Change in Productive Capacity)
Keynes Investment Multiplier Theory suggests that: $\frac{dY}{dI} = K = \frac{1}{s}$ OR $\frac{dY}{dI} = \left(\frac{1}{s}\right)$ Rearranging for dY: $\underbrace{dY}_{\text{Change in Income}} = \underbrace{\left(\frac{1}{s}\right)}_{\text{Multiplier}} \times \underbrace{dI}_{\text{Change in Investment}}$ For instance: $\underbrace{500}_{\text{Change in Income}} = \underbrace{5}_{\text{Multiplier}} \times \underbrace{100}_{\text{Change in Investment}}$ Dynamize the changes in national income (dY) and investment (dI): $(dY \ \& \ dI) \Rightarrow \left(\frac{dY}{dt} \ \& \ \frac{dI}{dt}\right)$ Differentiating w.r.t t on both sides the equation, $dY = \frac{1}{s} dI$, we get: $\frac{dY}{dt} = \left(\frac{1}{s}\right) \frac{dI}{dt}$ Here s (Marginal Propensity to Save) is assumed to be constant.	'Productive capacity per unit of capital' is: $\frac{\kappa}{K} \equiv \rho$. Here κ is the Productive Capacity and K is capital. Numerically speaking Productive Capacity-Capital ratio: $\rho = \frac{\kappa}{K} = 5 = \frac{5}{1}$ Each unit of capital is capable of producing 5 units of output. ρ is assumed to be constant as technology is assumed to be constant. Rewriting Productive Capacity-Capital ratio from, $\rho = \frac{\kappa}{K}$: $\kappa = \rho K$ For dynamic analysis, we need to dynamize the equation. Differentiating $\kappa = \rho K$ w.r.t. t: $\frac{d\kappa}{dt} = \rho \frac{dK}{dt}$ Since rate of change of capital is investment; $\frac{dK}{dt} = I$. We get: $\frac{d\kappa}{dt} = \rho I$
Static sense of Equilibrium Condition: At equilibrium, productive capacity is fully utilized: $\underbrace{Y}_{\text{Current Income (production)}} = \underbrace{\kappa}_{\text{potential production}}$	
Dynamic sense of Equilibrium Condition: If economy starts from this (initial & equilibrium) condition, then respective changes should be balanced, i.e. $\frac{dY}{dt} = \frac{d\kappa}{dt}$	

$\frac{dY}{dt} = \frac{d\kappa}{dt}$ Rate of Change of Income over time Rate of Change of Productive Capacity over time
Above mentioned is the condition for dynamic equilibrium in the economy under Domar Growth model.

TOPIC 071: DOMAR GROWTH MODEL: SOLUTION

From the framework of Domar Growth Model, productive capacity remains fully utilized over time:

$$\frac{dY}{dt} = \frac{d\kappa}{dt}$$

Rate of Change of Income over time Rate of Change of Productive Capacity over time

In the framework, we found:

$$\frac{dY}{dt} = \frac{1}{s} \frac{dI}{dt}$$

$$\frac{d\kappa}{dt} = \rho I$$

Substituting the values in $\frac{dY}{dt} = \frac{d\kappa}{dt}$, we get:

$$\frac{1}{s} \frac{dI}{dt} = \rho I$$

Rearranging for I in terms of t :

$$\frac{dI}{I} = \rho s dt$$

Introducing integral on both sides:

$$\int \frac{dI}{I} = \int \rho s dt$$

$$\ln|I| + c_I = \rho s t + c_t$$

$$\ln|I| = \rho s t + c_t - c_I$$

$$\ln|I| = \rho s t + c \quad [\because c_t - c_I = c \text{ (yet another constant)}]$$

We need solution as a function of I and not of its logarithm ($\ln|I|$).

Therefore, raising both side to natural exponent (e).

$$e^{\ln|I|} = e^{\rho s t + c}$$

$$I = e^{\rho s t} e^c$$

$$I = e^{\rho s t} A \quad [\because e^c = A \text{ (another constant)}]$$

Rearranging R.H.S and writing I as function is in terms of t :

$$I(t) = A e^{\rho s t}$$

Indeterminate/General Time Path of investment due to A .

Introducing initial condition to definitize arbitrary constant A , i.e. $t = 0$

$$I(0) = A e^{\rho s (0)}$$

$$I(0) = A$$

Substituting value of arbitrary constant A in general solution, $I(t) = A e^{\rho s t}$

$$I(t) = I(0) e^{\rho s t}$$

Here $I(0)$ is the initial investment. Moreover, marginal propensity to save (s) and productive capacity-capital ratio (ρ) are already assumed to be constants.

Therefore, the definite solution of the time path of investment is:

$$I(t) = I(0)e^{\rho st}$$

TOPIC 072: DOMAR GROWTH MODEL: NUMERICAL

From the solution of Domar Growth Model, we got the definite solution of time path of investment: $I(t) = I(0)e^{\rho st}$

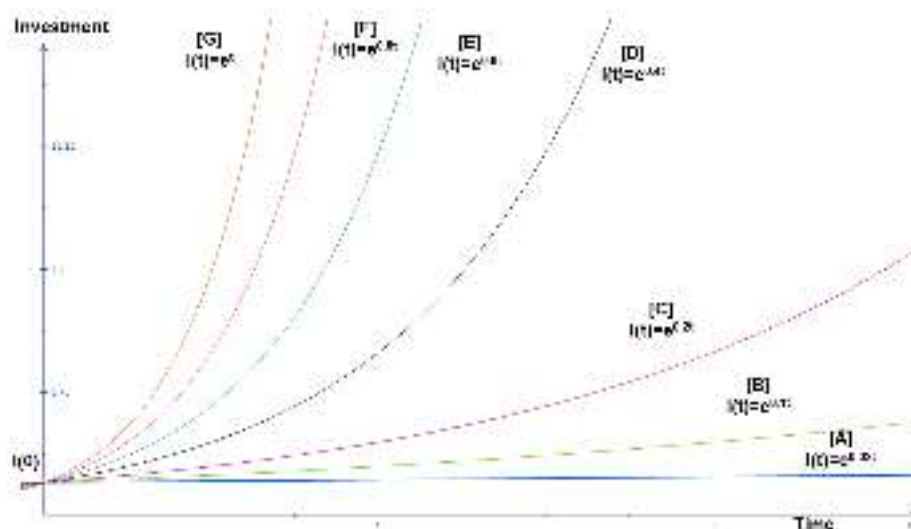
It's an exponential function and when compared with standard form e^{rt} , we get $r = \rho s$

So, the growth of investment depends on the expression ρs .

Assuming values of ρ and s : $\rho = 2$ and $s = 0.01$ to 0.5 .

Time Path	s	ρs	$I(t) = I(0)e^{\rho st}$	$I(t) = e^{\rho st} [\because I(0) = 1]$
A	0.01	0.02	$I(0)e^{0.02t}$	$I(t) = e^{0.02t}$
B	0.05	0.1	$I(0)e^{0.1t}$	$I(t) = e^{0.1t}$
C	0.1	0.2	$I(0)e^{0.2t}$	$I(t) = e^{0.2t}$
D	0.2	0.4	$I(0)e^{0.4t}$	$I(t) = e^{0.4t}$
E	0.3	0.6	$I(0)e^{0.6t}$	$I(t) = e^{0.6t}$
F	0.4	0.8	$I(0)e^{0.8t}$	$I(t) = e^{0.8t}$
G	0.5	1	$I(0)e^t$	$I(t) = e^t$

Plotting the numerical form of definite solution of time paths; A, B, C, D, E, F, G:



Observations: The composite graph shows 7-time paths. These time paths are generated due to alternative values of Marginal Propensity to Save (s). However, all time paths have a common initial condition $I(0)$.

It is observable that increase in value of s uplifts the time path. Specifically, increase in marginal propensity to save increases the rate at which the investment takes place. As increased savings lead to increased investment, the rate of growth of investment also increases.

Lesson 18

FIRST ORDER DIFFERENTIAL EQUATIONS

TOPIC 073: FIRST-ORDER DIFFERENTIAL EQUATIONS: INTRODUCTION

A differential equation is an equation which expresses an explicit or implicit relationship between a function $y = f(t)$ and one or more of its derivatives or differentials. For example:

$$\frac{dy}{dx} = 10x$$

It is a First-Order differential equation, since the highest derivative is a first-order derivative.

$y' = 12y$ is also a First-Order differential equation as it contains y' .

$y'' - 2y' + 19 = 0$ is a Second-Order differential equation as it contains second order derivative y'' .

In addition to order of differential equation, degree can also be noted. It is the power of the highest order of the derivative in the equation.

Differential Equation	Order	Degree
$\frac{dy}{dt} = 2x + 6$ OR $\left(\frac{dy}{dt}\right)^1 = 2x + 6$	1 st Order	1 st Degree
$\left(\frac{dy}{dt}\right)^4 - 5t^5 = 0$	1 st Order	4 th Degree
$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + x^2 = 0$ OR $\left(\frac{d^2y}{dt^2}\right)^1 + \left(\frac{dy}{dt}\right)^3 + x^2 = 0$	2 nd Order	1 st Degree
$\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 = 75y$ OR $\left(\frac{d^2y}{dt^2}\right)^7 + \left(\frac{d^3y}{dt^3}\right)^5 = 75y$	3 rd Order	5 th Degree

Solution: Solution of such an equation will be $y = f(x)$, where no derivative term would be present.

A standard form of First-Order Differential Equation:

$$\frac{dy}{dt} + u(t) \cdot y = w(t)$$

$\frac{dy}{dt}$: is the derivative of dependent variable y w.r.t independent variable t .

$u(t)$: Coefficient of y . A function in terms of t .

$w(t)$: Term. A function in terms of t .

Caveat: Here we assume $u(t)$ and $w(t)$ are constant. Therefore, using constant symbols for $u(t)$ and $w(t)$:

$$u(t) = a$$

$$w(t) = b$$

Therefore, the standard form $\frac{dy}{dt} + \underline{u(t)} \cdot y = \underline{w(t)}$ becomes:

$$\frac{dy}{dt} + \underline{a} \cdot y = \underline{b}$$

$\frac{dy}{dt} + \underbrace{a}_{\substack{\text{constant} \\ \text{coefficient}}} \underbrace{y}_{\substack{\text{dependent} \\ \text{variable}}} = \underbrace{b}_{\substack{\text{constant} \\ \text{term}}}$
--

TOPIC 074: FIRST-ORDER DIFFERENTIAL EQUATIONS: HOMOGENEOUS CASE

Standard Form of First-Order Differential Equation

$$\frac{dy}{dt} + a \cdot y = b$$

Depending on the values of a and b , we get 3 possible cases:

Case – I: $a \neq 0, b = 0$ [Homogenous Case]

Case – II: $a \neq 0, b \neq 0$ [Non-Homogenous Case – I]

Case – III: $a = 0, b \neq 0$ [Non-Homogenous Case – II]

Case – I: Homogenous Case: $a \neq 0, b = 0$

Standard form of First-Order Differential Equation $\frac{dy}{dt} + ay = b$, becomes:

$$\frac{dy}{dt} + ay = \underbrace{0}_{\substack{\text{homogeneous} \\ \text{case}}}$$

$$\frac{dy}{dt} = -ay$$

$$\frac{1}{y} \frac{dy}{dt} = -a$$

$$\frac{\left(\frac{dy}{dt}\right)}{y} = -a$$

Integrating w.r.t. t

$$\int \frac{\left(\frac{dy}{dt}\right)}{y} dt = - \int a dt$$

$$\int \frac{(dy)}{y} = - \int a dt$$

$$\int \left(\frac{1}{y}\right) dy = - \int a dt$$

$$\ln|y| + c_1 = -at + c_2$$

$$\ln|y| = -at + \underline{c_2 - c_1} \quad [\because c_2 - c_1 = c]$$

$$\ln|y| = -at + \underline{c}$$

To get rid of natural logarithm, raise exponent on both sides:

$$e^{\ln|y|} = e^{-at+c}$$

$$y = e^{-at} \underline{e^c}$$

$$y(t) = e^{-at} \underline{A} [\because e^c = A] \text{ yet another constant}$$

$$y(t) = Ae^{-at} \text{ (General Solution)}$$

Introducing the initial condition to definitize the arbitrary constant; $(t = 0)$: $y(t) = Ae^{-at}$ becomes

$$y(0) = Ae^{-a(0)}$$

$$y(0) = Ae^0$$

$$y(0) = A$$

$$A = y(0)$$

$$y(t) = y(0)e^{-at}$$

It is Definite Solution as the arbitrary constant A is determined.

TOPIC 075: FIRST-ORDER DIFFERENTIAL EQUATIONS: NONHOMOGENEOUS CASE – I

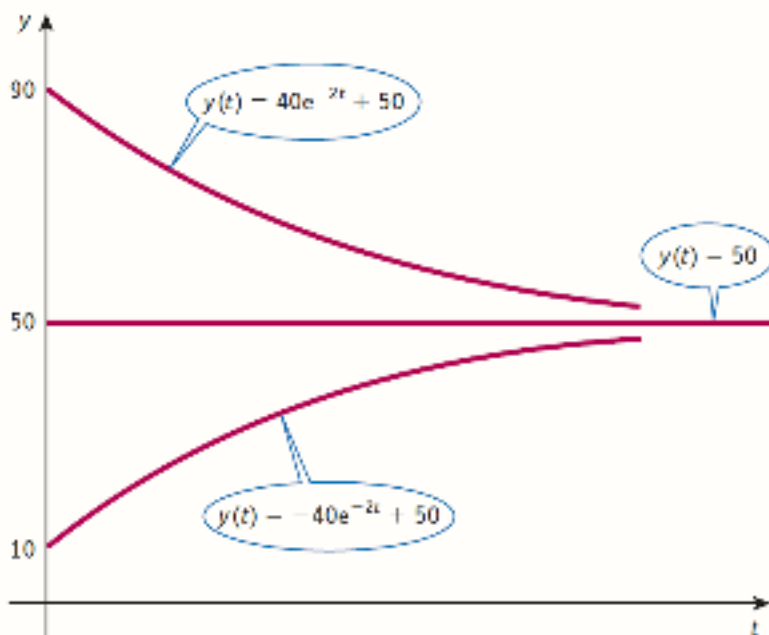
Case – II: Non-Homogenous Case – I: $a \neq 0, b \neq 0$. ($NHC_{a \neq 0}$)

Standard form of First-Order Differential Equation $\frac{dy}{dt} + ay = b$, remains:

$$\frac{dy}{dt} + ay = \underbrace{b}_{\text{Non-Homogenous case}}$$

As no term is equal to zero ($b \neq 0$) in this case, Non-Homogenous case is also termed as Complete Equation.

In the Complete Equation of time path, there is equilibrium value and the deviation from it. As in graph of time path, 50 is equilibrium value and $\pm 40e^{-2t}$ is the deviation from equilibrium. Time path convergences towards equilibrium (50) in either case. i.e. when deviation is positive $+40e^{-2t}$ or negative $-40e^{-2t}$.

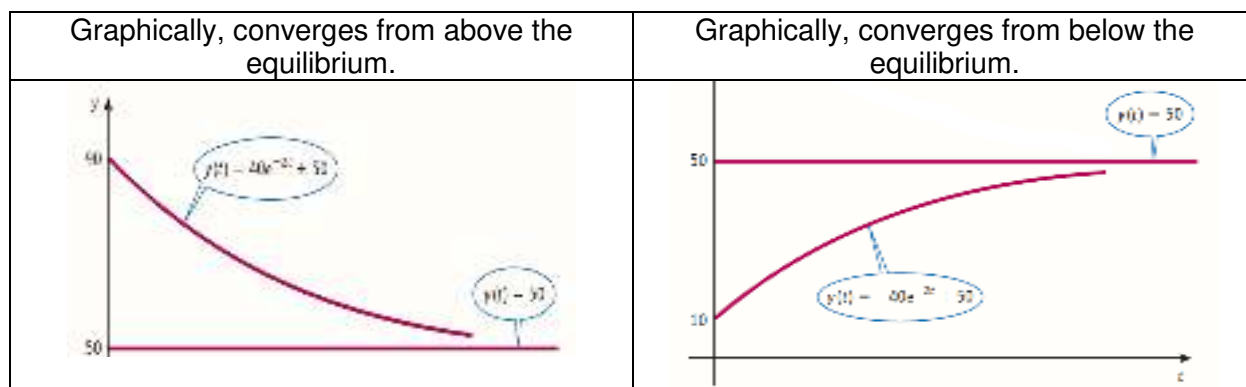


Therefore, solution has two components i.e. Complementary Function ($y_c = \pm 40e^{-2t}$) and Particular Solution ($y_p = 50$).

Complementary Function (y_c) is **complementary** (تكمیلی - پورا کرنے والا) to the Equilibrium value (y_p) of the time path.

Equilibrium value (y_p) is called Particular Solution as it is one Particular (مخصوص) value of y out of infinite values on whole of the time path.

Case of Positive y_c ($y_c = 40e^{-2t}$)	Case of Negative y_c ($y_c = -40e^{-2t}$)
$y(t) = + \underbrace{40e^{-2t}}_{y_c[t \rightarrow \infty, y_c \rightarrow 0, y(t) \rightarrow y_p]} + \underbrace{50}_{y_p}$	$y(t) = - \underbrace{40e^{-2t}}_{y_c[t \rightarrow \infty, y_c \rightarrow 0, y(t) \rightarrow y_p]} + \underbrace{50}_{y_p}$
Mathematically, addition decreases	Mathematically, deduction decreases



Complementary Function (y_c): is equivalent to the general solution of homogenous case.

$$(y_c)_{NHC} = \{y(t)\}_{HC}$$

$$(y_c)_{NHC} = Ae^{-at}$$

$$y(t) = Ae^{-at} + y_p$$

Particular Solution (y_p): of the Non-Homogenous Case is as follows:

Two possibilities of particular values of y : constant and non-constant.

1. Can be a constant (say k) when ($a \neq 0$):

Let,

$$y = k$$

$$\frac{dy}{dt} = 0$$

Substituting value of $\frac{dy}{dt}$ in $\left(\frac{dy}{dt} + ay = b\right)$, we get:

$$0 + ay = b$$

$$y = \frac{b}{a}$$

In proper notion as per the definition is: $y_p = \frac{b}{a}$

Substituting the value of y_p in $y(t)$:

$$y(t) = Ae^{-at} + \frac{b}{a} \text{ (General Solution of Non-homogenous Case with } a \neq 0 \text{)}$$

For Definite Solution: Put $t = 0$ in $y(t) = Ae^{-at} + \frac{b}{a}$

$$y(0) = Ae^{-a(0)} + \frac{b}{a}$$

$$y(0) = Ae^0 + \frac{b}{a}$$

$$y(0) = A + \frac{b}{a}$$

$$A = \left\{y(0) - \frac{b}{a}\right\}$$

Putting value of A in $y(t) = Ae^{-at} + \frac{b}{a}$

$$y(t) = \left\{y(0) - \frac{b}{a}\right\}e^{-at} + \frac{b}{a}$$

(Definite Solution of Non-homogenous Case with $a \neq 0$).

What if, $a = 0$?

TOPIC 076: FIRST-ORDER DIFFERENTIAL EQUATIONS: NONHOMOGENEOUS CASE – II**Case – III: Non-Homogenous Case – II: $a = 0, b \neq 0$. ($NHC_{a=0}$)**

Standard form of First-Order Differential Equation $\frac{dy}{dt} + ay = b$, at $a = 0$ becomes:

$$\frac{dy}{dt} + (0)y = b$$

$$\frac{dy}{dt} = b$$

Solution $y(t)$ has two components:

$$y(t) = y_c + y_p$$

Complementary Function (y_c): is equivalent to the general solution of homogenous case.

$$(y_c)_{NHC} = \{y(t)\}_{HC}$$

$$(y_c)_{NHC} = Ae^{-at}$$

However here $a = 0$ therefore y_c reduces to:

$$y_c = Ae^{-(0)t}$$

$$y_c = Ae^0$$

$$y_c = A$$

$$y(t) = A + y_p$$

Particular Solution (y_p): If y is considered as a constant k as in case of $NHC_{a \neq 0}$, $y = k$

$$\frac{dy}{dt} = 0$$

Substituting value of $\frac{dy}{dt}$ in $\left(\frac{dy}{dt} = b\right)$, we get:

$$0 = b$$

$$0 \neq b$$

So, equality is not established and some other value of y should be experimented.

Therefore, we experiment a non-constant ($y = kt$) can be experimented.

$$y = kt$$

Taking derivative w.r.t t :

$$\frac{dy}{dt} = k \left(\frac{dt}{dt} \right)$$

$$\frac{dy}{dt} = \underline{k}$$

We already have the $NHC_{a=0}$ differential equation $\frac{dy}{dt} = \underline{b}$

$$\underline{k} = \underline{b}$$

Therefore, $y = \underline{kt}$ can be written as $y = \underline{bt}$.

More properly, it is the proper solution: $y_p = \underline{bt}$.

Substituting the value of y_p in $y(t) = A + y_p$:

$$y(t) = A + \underline{bt}$$

$$\boxed{y(t) = A + bt} \text{ is the General Solution of } NHC_{a=0}.$$

Definite Solution: Put $t = 0$ in

$$y(t) = A + bt$$

$$y(0) = A + b(0)$$

$$y(0) = A + 0$$

$$y(0) = A$$

$$A = y(0)$$

Putting value of A in **General Solution** of $NHC_{a=0}$.

$$y(t) = y(0) + bt$$

Definite Solution of Non-homogenous Case with $a = 0$.

TOPIC 077: VERIFICATION OF SOLUTION OF FIRST-ORDER DIFFERENTIAL EQUATIONS

The validity of all solutions of differential equations can always be checked by differentiation.

Consider the definite solution of First Order Differential Equation where $a \neq 0$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Differentiating w.r.t t :

$$\begin{aligned} \frac{d}{dt}\{y(t)\} &= \frac{d}{dt} \left[\left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a} \right] \\ \frac{dy}{dt} &= \frac{d}{dt} \left[\left\{ y(0) - \frac{b}{a} \right\} e^{-at} \right] + \frac{d}{dt} \left(\frac{b}{a} \right) \\ \frac{dy}{dt} &= \left\{ y(0) - \frac{b}{a} \right\} \left\{ \frac{d}{dt} (-at) \right\} (e^{-at}) + 0 \\ \frac{dy}{dt} &= \left\{ y(0) - \frac{b}{a} \right\} \frac{-a(e^{-at})}{1} \\ \frac{dy}{dt} &= -a \left\{ y(0) - \frac{b}{a} \right\} e^{-at} \end{aligned}$$

Recalling standard form of First Order Differential Equation $\frac{dy}{dt} + a[y] = b$ and substituting boxed values:

$$\begin{aligned} -a \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + a \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a} &= b \\ -a \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + a \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + a \times \frac{b}{a} &= b \\ \underline{b} &= b \\ \text{L.H.S} &= \text{R.H.S} \end{aligned}$$

Numerical Example: Verify if $\{y(t) = 4(1 - e^{-t})\}$ the definite solution of $\left(\frac{dy}{dt} + y = 4\right)$ is true:

Definite Solution: $y(t) = 4(1 - e^{-t})$

Differentiating w.r.t t .

$$\begin{aligned} \frac{d}{dt}\{y(t)\} &= \frac{d}{dt}\{4(1 - e^{-t})\} \\ \frac{dy}{dt} &= 4 \frac{d}{dt}(1 - e^{-t}) \\ \frac{dy}{dt} &= 4 \left[0 - \left\{ \frac{d}{dt} (-t) \right\} e^{-t} \right] \\ \frac{dy}{dt} &= 4 \{(-1)e^{-t}\} \\ \frac{dy}{dt} &= -4e^{-t} \end{aligned}$$

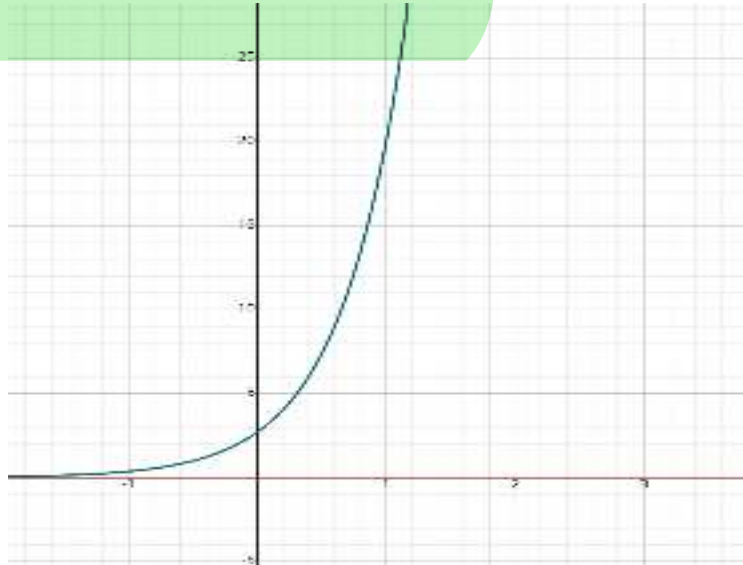
Substituting values in the First Order Differential Equation, $\frac{dy}{dt} + [y] = 4$;

$$-4e^{-t} + 4(1 - e^{-t}) = 4$$

$$\begin{aligned}
 -4e^{-t} + 4(1 - e^{-t}) &= 4 \\
 -4e^{-t} + 4 - 4e^{-t} &= 4 \\
 \underbrace{4}_{L.H.S=R.H.S} &= 4
 \end{aligned}$$

TOPIC 078: FIRST-ORDER DIFFERENTIAL EQUATIONS: SUMMARY & NUMERICALS

1. GRAPH of $\frac{dy}{dt} - 2y = 0$: A HOMOGENOUS CASE:



Solution: Comparing the given differential equation with the standard form

$$\begin{aligned}
 \frac{dy}{dt} - 2y &= 0 \\
 \frac{dy}{dt} + ay &= b
 \end{aligned}$$

$a = -2$, $b = 0$ (a Homogenous case)

Formula for General Solution of Homogenous case:

$$y(t) = Ae^{-at}$$

Substituting value of a :

$$\begin{aligned}
 y(t) &= Ae^{-(-2)t} \\
 \boxed{y(t) = Ae^{2t}} &\text{ (General Solution)} \\
 \boxed{y(t) = \underbrace{Ae^{2t}}_{y_c} + \underbrace{0}_{y_p}} &\text{ (General Solution)}
 \end{aligned}$$

Interpretation: The graph verifies, that there is an exponential growth term in the solution (e^{2t}). While A remains the arbitrary element, which can be determined by introducing the initial condition.

Definite Solution: requires the introduction of initial condition; $t = 0$:

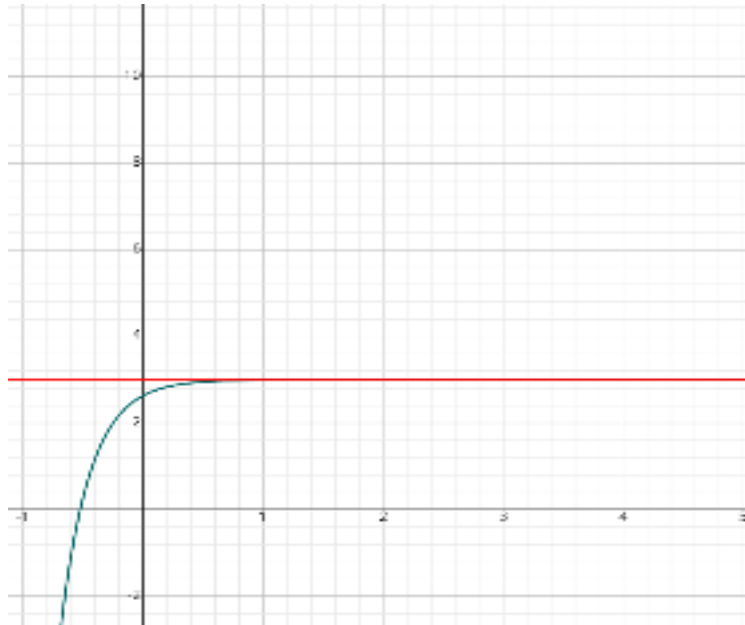
$$\begin{aligned}
 y(0) &= Ae^{2(0)} \\
 y(0) &= Ae^0 \\
 y(0) &= A(1) \\
 \underline{A} &= \underline{y(0)}
 \end{aligned}$$

Substituting the values of A in General Solution: $y(t) = \underline{A}e^{2t}$

$$y(t) = \underline{y(0)}e^{2t}$$

$y(t) = y(0)e^{2t}$ is the definite solution of the differential equation $\frac{dy}{dt} - 2y = 0$.

2. GRAPH of $\frac{dy}{dt} + 4y = 12$: A NON-HOMOGENOUS CASE: with $(a \neq 0, b \neq 0)$



Solution: Comparing the given differential equation with the standard form

$$\frac{dy}{dt} + 4y = 12$$

$$\frac{dy}{dt} + ay = b$$

$a = 4, b = 12$ (a Non – Homogenous case with $a \neq 0$)

Formula for General Solution of Non-Homogenous case:

$$y(t) = Ae^{-at} + \frac{b}{a}$$

Substituting value of a :

$$y(t) = Ae^{-(4)t} + \frac{12}{4}$$

$$y(t) = \underbrace{Ae^{-4t}}_{y_c} + \underbrace{3}_{y_p} \text{ (General Solution)}$$

Interpretation: In the graph it is visible with a red horizontal line that value of particular integral is 3. However, the complementary function has an exponential decay that leads to its convergence towards equilibrium (y_p).

Definite Solution: For its sake, we introduce; $t = 0$:

$$y(t) = Ae^{-4t} + 3$$

$$y(0) = Ae^{-4(0)} + 3$$

$$y(0) = Ae^0 + 3$$

$$y(0) = A(1) + 3$$

$$y(0) = A + 3$$

$$y(0) - 3 = A$$

$$y(0) - 3 = A$$

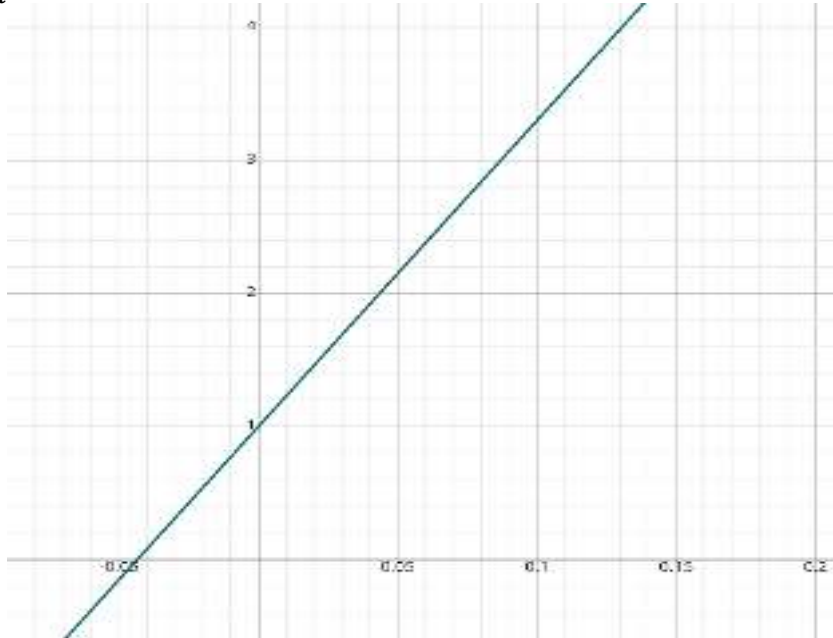
$$A = \{y(0) - 3\}$$

Substituting the value of arbitrary constant in general solution: $y(t) = \underline{A}e^{-4t} + 3$

$$y(t) = \{y(0) - 3\}e^{-4t} + 3$$

$y(t) = \{y(0) - 3\}e^{-4t} + 3$ is the definite solution of the differential equation $\frac{dy}{dt} + 4y = 12$.

3. GRAPH of $\frac{dy}{dt} = 23$: A NON-HOMOGENOUS CASE: with $(a = 0, b \neq 0)$



Solution: Comparing the given differential equation with the standard form

$$\frac{dy}{dt} = 23$$

Rewriting for comparability

$$\frac{dy}{dt} + (0)y = 23$$

$$\frac{dy}{dt} + ay = b$$

$a = 0, b = 23$ (a Non – Homogenous case with $a = 0$)

Formula for General Solution of Non-Homogenous case:

$$y(t) = A + bt$$

Substituting value of b :

$$y(t) = A + 23t$$

$$y(t) = \underbrace{A}_{y_c} + \underbrace{23t}_{y_p} \text{ (General Solution)}$$

Interpretation: In the graph it is visible with a red horizontal line that value of particular integral is 3. However, the complementary function has an exponential decay that leads to its convergence towards equilibrium (y_p).

Definite Solution: For its sake, we introduce; $t = 0$:

$$y(t) = A + 23t$$

$$y(0) = A + 23(0)$$

$$y(0) = A + 0$$

$$y(0) = A$$

$$A = y(0)$$

Substituting the value of arbitrary constant in general solution: $y(t) = \underline{A} + 23t$

$$y(t) = \underline{y(0)} + 23t$$

$$\boxed{y(t) = y(0) + 23t}$$

$y(t) = y(0) + 23t$ is the definite solution of the differential equation $\frac{dy}{dt} = 23$.

Summary of Homogenous and Non-Homogenous Cases of First Order Differential

Equations:

$\frac{dy}{dt} + ay = b$						
	Constant Coefficient	Constant Term	Complementary Function	Particular Solution	General Solution	Definite Solution
	$\bar{a}$	$\bar{b}$	y_c	y_p	$y(t)_{t \neq 0}$	$y(t)_{t=0}$
HC	$a \neq 0$	$b = 0$	Ae^{-at}	N.A.	Ae^{-at}	$y(0)e^{-at}$
NHC	$a \neq 0$	$b \neq 0$	Ae^{-at}	$\frac{b}{a}$	$Ae^{-at} + \frac{b}{a}$	$\left\{y(0) - \frac{b}{a}\right\}e^{-at} + \frac{b}{a}$
	$a = 0$	$b \neq 0$	A	bt	$A + bt$	$y(0) + bt$

Lesson 19

DYNAMICS OF MARKET PRICE

TOPIC 079: DYNAMICS OF MARKET PRICE: FRAMEWORK

Famous Microeconomic Model: Demand and Supply equations are given.

$$Q_d = \alpha - \beta P \quad [\alpha, \beta > 0]$$

$$Q_s = -\gamma + \delta P \quad [\gamma, \delta > 0]$$

Static Sense of Equilibrium: In Static Sense the equilibrium shall be:

$$Q_d = Q_s$$

Substituting values of Q_d and Q_s in equilibrium condition:

$$\begin{aligned} Q_d &= Q_s \\ \alpha - \beta P &= -\gamma + \delta P \\ \alpha + \gamma &= \beta P + \delta P \\ \alpha + \gamma &= P(\beta + \delta) \\ \frac{\alpha + \gamma}{\beta + \delta} &= P \\ \boxed{P^* = \frac{\alpha + \gamma}{\beta + \delta}} \end{aligned}$$

Parametric Interpretation: Since α, β, γ and δ are positive constants by given parametric stipulations, the answer of their addition and division shall give rise to yet another positive constant. i.e. $P^* = \text{constant} (> 0)$.

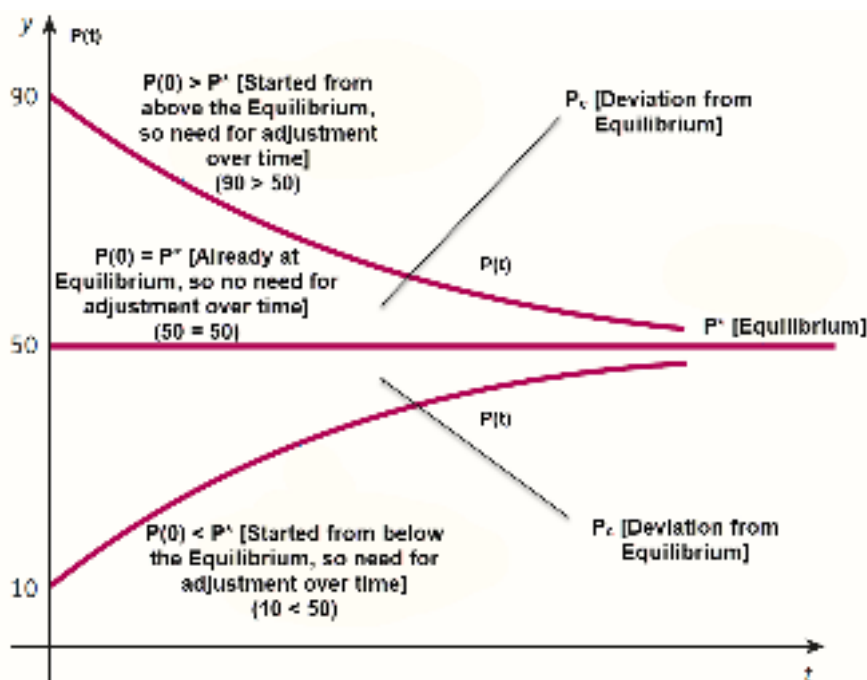
Dynamic Sense of Equilibrium: In dynamic sense, the price shall follow a time-path $P(t)$, rather being limited to a single equilibrium (P^*). So, we write the expression of definite time-path for price.

$$P(t) = P_c + P_p$$

Here, particular solution of price P_p is the static equilibrium value of price (P^*). $P(t) = P_c + P^*$

$$P(t) = P_c + P^*$$

Possibilities of Initial Conditions: Time path of price can start $\{P(0)\}$ either from above, at or below the equilibrium value of price (P^*):



Putting all 3 cases summarily: $[P(0) \leq P^*]$

Case – I: $[P(0) = P^*]$ requires no adjustments over time. However, $[P(0) \leq P^*]$ do.

Case – II: $[P(0) > P^*]$ requires adjustments over time.

Case – III: $[P(0) < P^*]$ requires adjustments over time.

TOPIC 080: DYNAMICS OF MARKET PRICE: THE TIME PATH

Price Adjustment over time: Price adjustments over time can be measured by the rate of change of price over time $\left(\frac{dP}{dt}\right)$.

Mathematizing the shortage/surplus in static sense:

Shortage	Surplus
$Q_d > Q_s$	$Q_d < Q_s$
$(Q_d - Q_s) > 0$ {positive}	$(Q_d - Q_s) < 0$ {negative}
Excess Demand	Negative Excess Demand

Greater the shortage/surplus in the market, more responsive will be change in price and vice versa.

(Price Adjustments) are directly proportional to **(Excess demand)**

Mathematically speaking:

$$\left(\frac{dP}{dt}\right) \propto (Q_d - Q_s)$$

Developing an equation and bringing the constant of proportionality (j).

$$\frac{dP}{dt} = j(Q_d - Q_s)$$

Here $j > 0$, which means that the relationship between $\frac{dP}{dt}$ and $(Q_d - Q_s)$ shall be positive.

j is, suitably, called the 'adjustment' coefficient.

$$\underbrace{\frac{dP}{dt}}_{\text{Intertemporal Sense}} = \underbrace{j}_{\text{Adjustment Coefficient}} \times \underbrace{(Q_d - Q_s)}_{\text{Market Clearing Sense}}$$

At equilibrium, no need for adjustment in price over time in equilibrium. i.e. $\left(\frac{dP}{dt} = 0\right)$

At equilibrium, there is no excess demand in the market. i.e. $(Q_d - Q_s = 0)$

Putting the two values in $\frac{dP}{dt} = j \times (Q_d - Q_s)$:

$$0 = j \times (0)$$

$$0 = 0$$

Therefore, the adjustment equation makes sense for market equilibrium both in dynamic sense as well as static sense.

Now substituting the values of Q_d and Q_s in Adjustment equation:

$$\frac{dP}{dt} = j \times (Q_d - Q_s)$$

$$\frac{dP}{dt} = j \{ (\alpha - \beta P) - (-\gamma + \delta P) \}$$

$$\frac{dP}{dt} = j(\alpha - \beta P + \gamma - \delta P)$$

Gathering terms containing P .

$$\frac{dP}{dt} = j(\alpha + \gamma - \beta P - \delta P)$$

$$\frac{dP}{dt} = j\{(\alpha + \gamma) - (\beta + \delta)P\}$$

$$\frac{dP}{dt} = j(\alpha + \gamma) - j(\beta + \delta)P$$

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

Comparing with the standard form of First Order Differential Equation:

$$\frac{dP}{dt} + \underbrace{j(\beta + \delta)}_a P = \underbrace{j(\alpha + \gamma)}_b$$

Here $\frac{dy}{dt} = \frac{dP}{dt}$, $y = P$, $a = j(\beta + \delta)$, $b = j(\alpha + \gamma)$

Choice of Case of the Differential Equation:

$b = j(\alpha + \gamma)$; as j, α and γ are positive, the value of b shall be some positive constant. i.e.

$b \neq 0$ – **Non-Homogenous Case.**

$a = j(\beta + \delta)$; as j, β and δ are positive, the value of a shall be some positive constant. i.e.

$a \neq 0$ – **Non-Homogenous Case with $a \neq 0$.**

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{P}(t) = \left\{ \underline{P}(0) - \frac{j(\alpha + \gamma)}{j(\beta + \delta)} \right\} e^{-j(\beta + \delta)t} + \frac{j(\alpha + \gamma)}{j(\beta + \delta)}$$

$$= \left\{ P(0) - \frac{\alpha + \gamma}{\beta + \delta} \right\} e^{-j(\beta + \delta)t} + \frac{\alpha + \gamma}{\beta + \delta}$$

$$P(t) = \{P(0) - P^*\} e^{-j(\beta + \delta)t} + P^* \quad \text{Let } \{j(\beta + \delta)\} = k$$

$$P(t) = \{P(0) - P^*\} e^{-kt} + P^*$$

$$P(t) = \underbrace{\{P(0) - P^*\}}_{P_c} e^{-kt} + \underbrace{P^*}_{P_p}$$

Definite Time-path for the Market Equilibrium in Dynamic Sense.

TOPIC 081: DYNAMICS OF MARKET PRICE: DYNAMIC STABILITY

For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (P_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (P_c). Therefore, for dynamic stability; $P_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Market Price in Dynamic Sense.

$$P(t) = \{P(0) - P^*\}e^{-kt} + P^*$$

$$P(t) = \underbrace{\{P(0) - P^*\}}_{P_c} e^{-kt} + \underbrace{P^*}_{P_p}$$

Need for time path of price $\{P(t)\}$ to converge to equilibrium price (P^*) i.e. $P(t) \rightarrow P^*$ as $P_c \rightarrow 0$ when $t \rightarrow \infty$.

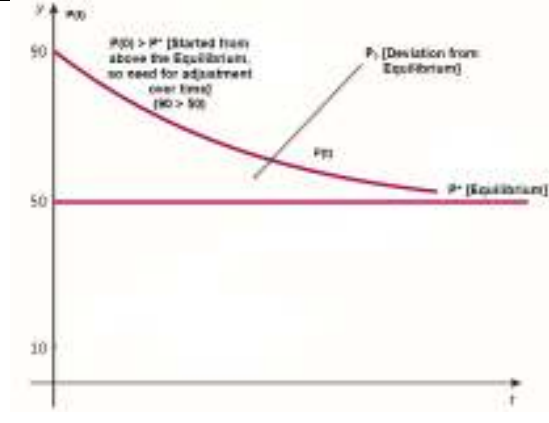
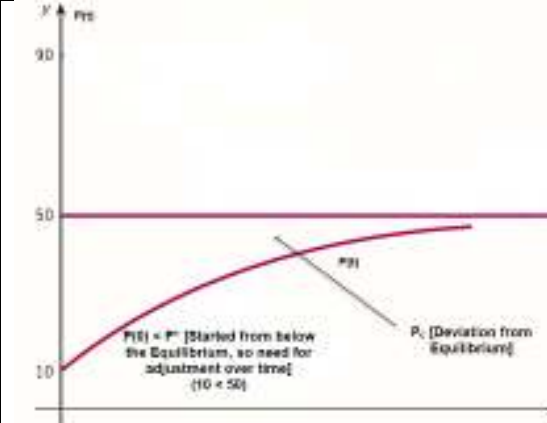
$$\begin{aligned} &P_c \rightarrow 0 \\ &\{P(0) - P^*\}e^{-kt} \rightarrow 0 \\ &\left[\underbrace{\left\{ \begin{array}{l} \frac{\alpha+\gamma}{\beta+\delta} > 0 \\ P(0) - P^* \\ P(0) \geq P^* \end{array} \right\}}_{\text{Deviation (+ve or -ve)}} \underbrace{e^{-\frac{k}{j(\beta+\delta)}t}}_{\substack{\text{Decays} \\ \text{over time}}} \right] \rightarrow 0 \end{aligned}$$

Therefore,

$$P(t) = \underbrace{\{P(0) - P^*\}}_{\rightarrow 0} e^{-kt} + P^*$$

$$\boxed{P(t) \rightarrow P^*}$$

Interpretation: It can be said that provided $\alpha, \beta, \delta, \gamma$ and j are positive, regardless of initial price $P(0)$ is greater or smaller than equilibrium price (P^*), the time-path of market price shall **converge towards equilibrium price** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.

$P(t) = \pm 40e^{-4t} + 50$				
$P(t) = 40e^{-4t} + 50$		$P(t) = -40e^{-4t} + 50$		
	Complementary Function (Deviation)	Time-Path	Complementary Function (Deviation)	Time-Path
t	$P_c = 40e^{-4t}$	$P(t) = 40e^{-4t} + 50$	$P_c = -40e^{-4t}$	$P(t) = -40e^{-4t} + 50$
0	40	90	-40	10
1	0.7326255555493	50.7326255555494	-0.7326255555493	49.2673744444506
2	0.0134185051161	50.0134185051161	-0.0134185051161	49.9865814948839
3	0.0002457684941	50.0002457684941	-0.0002457684941	49.9997542315059
4	0.0000045014069	50.0000045014070	-0.0000045014069	49.9999954985930
5	0.0000000824461	50.0000000824461	-0.0000000824461	49.9999999175539
6	0.0000000015100	50.0000000015101	-0.0000000015100	49.9999999984899
7	0.0000000000027	50.0000000000027	-0.0000000000027	49.9999999999723
8	0.0000000000000	50.0000000000000	-0.0000000000000	49.9999999999995
...	...	...	...	...
Remarks				
	Positive deviation decays over time	Time-path slides and converges to equilibrium ($P^* = 50$)	Negative deviation decays over time	Time-path rises and converges to equilibrium ($P^* = 50$)
				

TOPIC 082: DYNAMICS OF MARKET PRICE: NUMERICAL

Consider the Market Model:

$$Q_s = 3P - 4$$

$$Q_d = -5P + 20$$

$$\frac{dP}{dt} = 0.2(Q_d - Q_s)$$

Find expressions for $P(t)$, $Q_s(t)$ and $Q_d(t)$ when $P(0) = 2$. Is this system stable or unstable?

Solution: Given the adjustment equation $\frac{dP}{dt} = 0.2(Q_d - Q_s)$, values of Q_d and Q_s can be substituted.

$$\begin{aligned}\frac{dP}{dt} &= 0.2(Q_d - Q_s) \\ \frac{dP}{dt} &= 0.2\{(-5P + 20) - (3P - 4)\} \\ \frac{dP}{dt} &= 0.2\{(20 + 4) - (5P + 3P)\} \\ \frac{dP}{dt} &= 0.2\{(24) - (8P)\} \\ \frac{dP}{dt} &= 0.2(24 - 8P) \\ \frac{dP}{dt} &= 4.8 - 1.6P \\ \frac{dP}{dt} + 1.6P &= 4.8\end{aligned}$$

Comparing with the standard form:

$$\frac{dy}{dt} + \underbrace{1.6}_a P = \underbrace{4.8}_b$$

Here $\frac{dy}{dt} = \frac{dP}{dt}$, $y = P$, $a = 1.6$, $b = 4.8$

Choice of Case of the Differential Equation:

$b \neq 0$: Non-Homogenous Case.

$a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{P}(t) = \left(\underline{P(0)} - \frac{\underline{4.8}}{\underline{1.6}} \right) e^{-\underline{1.6}t} + \frac{\underline{4.8}}{\underline{1.6}}$$

$$P(t) = (2 - 3)e^{-1.6t} + 3$$

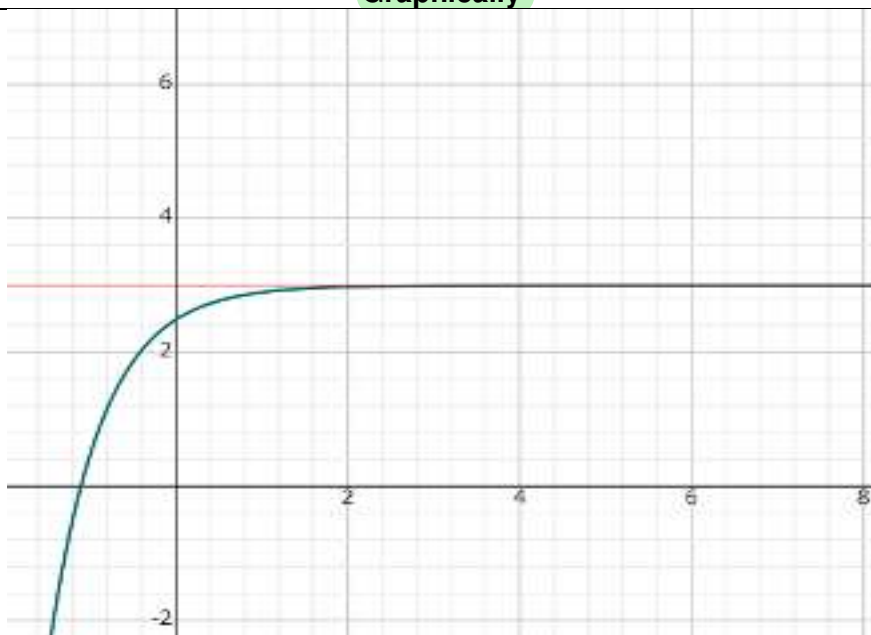
$$P(t) = (-1)e^{-1.6t} + 3$$

$$\boxed{P(t) = -e^{-1.6t} + 3}$$

It is the definite Time-Path of market price.

Dynamic Stability: The expression; $P(t) = -e^{-1.6t} + 3$ has an exponential decay term. i.e.

	Complementary Function (Deviation)	Time-Path
t	$P_c = -e^{-1.6t}$	$P(t) = -e^{-1.6t} + 3$
0	-1.0000000	2.00000000

1	-0.2018965	2.79810348
2	-0.0407622	2.95923779
3	-0.0082297	2.99177025
4	-0.0016616	2.99833844
5	-0.0003354	2.99966453
6	-0.0000677	2.99993227
7	-0.0000137	2.99998632
8	-0.0000027	2.99999723
9	-0.0000005	2.99999944
10	-0.0000001	2.99999988
...	...	...
Remarks		
Negative deviation decays over time	Time-path rises and converges to equilibrium ($P^* = 3$)	
As $t \rightarrow \infty$, $P_c \rightarrow 0$	And when $P_c \rightarrow 0$, $(P_t) \rightarrow (P^*)$	
Graphically		
		
Red line shows the equilibrium value of market price (P^*), while green line shows the deviation from equilibrium (P_c), which decreases over time and leads the time-path to convergence.		

Lesson 20

PRICE DYNAMICS USING DIFFERENTIAL EQUATIONS

TOPIC 083: CONSUMER'S RESERVATION PRICE USING DIFFERENTIAL EQUATIONS

Given the direct demand function $Q(P)$, if you know that the own price elasticity of demand is given by $\epsilon_d = \frac{-(3P+2P^2)}{Q}$, where $Q = 50$ and $P = 10$. What is the highest reservation price consumers are willing to pay?

Solution: The general formula of elasticity of demand: $\epsilon_d = \frac{dQ}{dP} \times \frac{P}{Q}$, we can rewrite the given

expression of elasticity of demand $\epsilon_d = \frac{(-3P+2P^2)}{Q}$.

$$\frac{dQ}{dP} \times \frac{P}{Q} = \frac{-(3P+2P^2)}{Q}$$

$$\frac{dQ}{dP} \times \frac{P}{Q} = \frac{-(3+2P)P}{Q}$$

$$\frac{dQ}{dP} = -3 - 2P$$

$$\frac{dQ}{dP} + 2P = -3$$

Standard form requires dependent variable Q . Instead, we have the independent variable Q . So, we cannot use the usual formula for first order differential equation. An alternative approach to solving the differential equation is to integrate it, so the derivative gets cancelled with integral and value of dependent variable (Q) comes out as a result.

Keeping derivative term $\frac{dQ}{dP}$ on L.H.S.

$$\frac{dQ}{dP} + 2P = -3$$

$$\frac{dQ}{dP} + 2P = -3$$

$$\frac{dQ}{dP} = -2P - 3$$

Integrating the result w.r.t. P :

$$\int \frac{dQ}{dP} dP = \int (-2P - 3) dP$$

$$\int \frac{dQ}{dP} dP = \int (-2P) dP - \int (3) dP$$

$$Q = -2 \int (P) dP - 3 \int (P^0) dP$$

$$Q = -2 \left(\frac{P^2}{2} \right) - 3P + c$$

$$Q(P) = -P^2 - 3P + c$$

General Solution as c is still indefinite.

Using the given information that at equilibrium $Q = 50$ if $P = 10$.

Putting the given information in General Solution.

$$50 = -(10)^2 - 3(10) + c$$

$$50 = -100 - 30 + c$$

$$50 + 100 + 30 = c$$

$$c = 180$$

Substituting the value of $c = 180$ back into the General Solution; $Q(P) = -P^2 - 3P + c$.

$$Q(P) = -P^2 - 3P + 180$$

We get the Definite Solution.

Reservation Price: Price at which no demand for good/service doesn't start.

Mathematically speaking, reservation price is: $P_R = (P)_{Q=0}$. Therefore, putting $Q = 0$ in the Definite Solution; $Q = -P^2 - 3P + 180$:

$$0 = -P^2 - 3P + 180$$

$$-P^2 - 3P + 180 = 0$$

$$P^2 + 3P - 180 = 0$$

Quadratic Formula is applicable: $x = P$, $a = 1$, $b = 3$, $c = -180$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$P = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-180)}}{2(1)}$$

$$P = \frac{-3 \pm \sqrt{729}}{2(1)}$$

$$P = \frac{-3 \pm 27}{2(1)}$$

Either

$$P = \frac{-3+27}{2} \text{ OR } P = \frac{-3-27}{2}$$

$$P = \frac{24}{2} \text{ OR } P = \frac{-30}{2}$$

$$P = 12 \text{ OR } P = -15$$

Dispelling the 'unrealistic' negative value of P , we get the Reservation Price (P_R).

$$P_R = 12$$

Verification: Since Reservation Price is one at which demand doesn't starts ($Q = 0$). Therefore, putting Reservation Price should make demand equal to zero:

$$Q(P_R) = 0$$

$$Q(P_R) = -P_R^2 - 3P_R + 180$$

$$Q(12) = -(12)^2 - 3(12) + 180$$

$$Q(12) = -144 - 36 + 180$$

$$Q(12) = -180 + 180$$

$$Q(12) = 0$$

Interpretation: Hence its evident that when price is 12, demand for the good doesn't exist.

TOPIC 084: DYNAMICS OF PRICE & MONEY MARKET USING DIFFERENTIAL EQUATIONS

Money supply M_s is fixed at a level determined by the government.

Transaction demand for money L is positively related to prices $P(t)$ and real output Y :

$$L = f\{P(t) \times Y\}$$

As given that the relationship is positive – direct proportionality ($\propto$) exist.

$$L \propto \{P(t) \times Y\}$$

Introducing the constant of proportionality (α) to develop equation.

$$L = \alpha PY \quad [\alpha > 0]$$

Constant of Proportionality is positive which will keep the relationship between variables positive.

If inflation is the increase in the overall price level with time $\frac{dP}{dt}$, it depends on excess money supply. Excess Money Supply is the difference between Money-Supply and Transaction Demand for Money. The model then becomes:

$$\frac{dP}{dt} = j(M_s - L) \quad [j > 0]$$

$j > 0$ implies that price level rises with excess money-supply.

Time-path of the price level in the economy can be found by substituting values in the given information.

$$\begin{aligned} \frac{dP}{dt} &= j(M_s - L) \\ \frac{dP}{dt} &= j(M_s - \alpha PY) \\ \frac{dP}{dt} &= j(M_s - \alpha PY) \\ \frac{dP}{dt} &= jM_s - \alpha jYP \\ \frac{dP}{dt} + \alpha jYP &= jM_s \end{aligned}$$

Comparing with the standard form of the differential equation; $\frac{dy}{dt} + ay = b$

$$\frac{dP}{dt} + \underbrace{\alpha jY}_{a} \underbrace{P}_{y} = \underbrace{jM_s}_{b}$$

$$\text{Here } \frac{dy}{dt} = \frac{dP}{dt}, y = P, a = \alpha jY, b = jM_s$$

Choice of Case of the Differential Equation:

$b = jM_s$; as j is positive constant, and money-supply (M_s) is fixed (some positive number) by the government, the product will be non-zero positive constant. i.e. $b \neq 0$ – **Non-Homogenous Case.**

$a = \alpha jY$; as α and j are positive constants, and Y is national income (some positive number), the value of a shall be some positive (non-zero) constant. i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$.**

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\begin{aligned} \underline{P}(t) &= \left\{ \underline{P}(0) - \frac{\underline{jM_s}}{\underline{\alpha jY}} \right\} e^{-\underline{\alpha jY}t} + \frac{\underline{jM_s}}{\underline{\alpha jY}} \\ P(t) &= \left\{ P(0) - \frac{jM_s}{\alpha jY} \right\} e^{-(\alpha jY)t} + \frac{jM_s}{\alpha jY} \\ \boxed{P(t) = \left\{ P(0) - \frac{M_s}{\alpha Y} \right\} e^{-(\alpha jY)t} + \frac{M_s}{\alpha Y}} \end{aligned}$$

Definite Time-path for Market price in Dynamic Sense.

Interpretation: Since, M_s appears in numerator, it has positive relationship with price level. Y is in the denominator, and has negative relationship with price level. If money-supply increases in economy, it will increase the prices in the economy. If output in the economy increases, it shall decrease the price level due to increased supply of goods/services.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (P_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (P_c). Therefore, for dynamic stability; $P_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Price level in Dynamic Sense.

$$P(t) = \left\{ P(0) - \frac{M_s}{\alpha Y} \right\} e^{-(\alpha j Y)t} + \frac{M_s}{\alpha Y}$$

$$P(t) = \underbrace{\left\{ P(0) - \frac{M_s}{\alpha Y} \right\}}_{P_c} e^{-(\alpha j Y)t} + \underbrace{\frac{M_s}{\alpha Y}}_{P_p}$$

Need for time path of price $\{P(t)\}$ to converge to equilibrium price (P^*) i.e. $P(t) \rightarrow P^*$ as $P_c \rightarrow 0$ when $t \rightarrow \infty$.

$$\begin{aligned} & P_c \rightarrow 0 \\ & \left\{ P(0) - \frac{M_s}{\alpha Y} \right\} e^{-(\alpha j Y)t} \rightarrow 0 \\ & \left[\underbrace{\left\{ P(0) - \frac{M_s}{\alpha Y} \right\}}_{\substack{\text{Deviation (+ve or -ve)}}} \underbrace{e^{-(\alpha j Y)t}}_{\substack{\text{Exponential} \\ \text{Decay} \\ \text{over time}}} \right] \rightarrow 0 \end{aligned}$$

$\{M_s, \alpha, Y > 0\}$ $\{\alpha, j, Y > 0\}$
 $\frac{M_s}{\alpha Y}$ $\alpha j Y$

Therefore,

$$P(t) = \left\{ P(0) - \frac{M_s}{\alpha Y} \right\} e^{-(\alpha j Y)t} + \frac{M_s}{\alpha Y}$$

$$P(t) = \left\{ P(0) - \frac{P_p}{P_p} \right\} e^{-(\alpha j Y)t} + P_p$$

$$P(t) = \left\{ P(0) - \frac{P^*}{P^*} \right\} e^{-(\alpha j Y)t} + P^*$$

$$\boxed{P(t) \rightarrow P^*}$$

Interpretation: It can be said that provided α, j, M_s and Y are positive, regardless of initial price $P(0)$ is greater or smaller than equilibrium price (P^*), the time-path of market price shall **converge towards equilibrium price** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.

TOPIC 085: PRICE DYNAMICS & EQUATION OF EXCHANGE USING DIFFERENTIAL EQUATIONS

Borrowing the equation of exchange:

$$M_s \times V = P \times Y$$

M_s is the Money Supply, V is the velocity of Money, P is general price level in the economy and Y shows the Aggregate Output.

Rewriting for M_s :

$$M_s = \frac{P \times Y}{V}$$

Other side of money i.e. demand for money L is assumed to have transactive component only and is considered constant.

The price level shall make adjustments due to differences between (transaction) demand and supply of the money ($L \neq M_s$):

$$\underbrace{\frac{dP}{dt}}_{\text{Price Adjustment over time}} = j \left(\underbrace{\frac{M_s - L}{\text{Money-Supply}}}_{\text{Excess Money-Supply}} \right)$$

The price adjustments over time are directly proportional to Excess Money- supply.

j is the adjustment coefficient which is positive ($j > 0$) that implies that price level rises with excess money-supply.

Time-path of the price level in the economy can be found by substituting values in the M_s .

$$\begin{aligned} \frac{dP}{dt} &= j(M_s - L) \\ \frac{dP}{dt} &= j \left(\frac{P \times Y}{V} - L \right) \\ \frac{dP}{dt} &= j \left(\frac{P \times Y}{V} - L \right) \\ \frac{dP}{dt} &= j \left(\frac{Y}{V} \right) P - jL \\ \frac{dP}{dt} - j \left(\frac{Y}{V} \right) P &= -jL \end{aligned}$$

$$\text{Here } \frac{dy}{dt} = \frac{dP}{dt}, y = P, a = -j \left(\frac{Y}{V} \right), b = -jL$$

Choice of Case of the Differential Equation:

$b = -jL$; as j is positive constant, and (transaction) demand for money (L) is fixed (some positive number), the product will be non-zero positive constant. However, $b = -jL$, so the final answer will be negative but still not zero i.e. $b \neq 0$ – **Non-Homogenous Case**.

$a = -j \left(\frac{Y}{V} \right)$; as j is positive constant. Y is Aggregate Output (some positive number) and V is velocity of money (some positive number). The ratio $\left(\frac{Y}{V} \right)$ when multiplied by j will give a positive answer. However, $a = -j \left(\frac{Y}{V} \right)$, so the final answer will be negative but still not zero.

i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$** .

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{P}(t) = \left\{ \underline{P}(0) - \frac{\underline{-jL}}{\underline{\left(-j \frac{Y}{V} \right)}} \right\} e^{-\underline{\left(-j \frac{Y}{V} \right)}t} + \frac{\underline{-jL}}{\underline{\left(-j \frac{Y}{V} \right)}}$$

$$P(t) = \left\{ P(0) - \frac{-jL}{(-j\frac{Y}{V})} \right\} e^{-(-j\frac{Y}{V})t} + \frac{-jL}{(-j\frac{Y}{V})}$$

$$P(t) = \left\{ P(0) - \frac{-L}{(-\frac{Y}{V})} \right\} e^{-(-\frac{Y}{V})t} + \frac{-L}{(-\frac{Y}{V})}$$

$$P(t) = \left\{ P(0) - \frac{L}{(\frac{Y}{V})} \right\} e^{(\frac{Y}{V})t} + \frac{L}{(\frac{Y}{V})}$$

Reciprocalizing $(\frac{Y}{V})$:

$$P(t) = \left\{ P(0) - \frac{VL}{Y} \right\} e^{(\frac{Y}{V})t} + \frac{VL}{Y}$$

$$P(t) = \left\{ P(0) - \frac{VL}{Y} \right\} e^{(\frac{Y}{V})t} + \frac{VL}{Y}$$

Definite Time-path for Market price in Dynamic Sense.

Interpretation: Since, V and L appear in numerator, they have positive relationship with price level. Y is in the denominator, and has negative relationship with price level:

Velocity of Money (V): The faster money circulates in the economy, the higher the prices; ($P \propto V$).

Transaction Demand for Money (L): The higher the transaction demand for money, the higher the price level is; ($P \propto L$).

Aggregate Output (Y): If the economy produces greater aggregate output, this will reduce prices; ($P \propto \frac{1}{Y}$).

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (P_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (P_c). Therefore, for dynamic stability; $P_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Price level in Dynamic Sense:

$$P(t) = \underbrace{\left\{ P(0) - \frac{VL}{Y} \right\}}_{P_c} e^{(\frac{Y}{V})t} + \underbrace{\frac{VL}{Y}}_{P_p}$$

Need for time path of price $\{P(t)\}$ to converge to equilibrium price (P^*) i.e. $P(t) \rightarrow P^*$ as $P_c \rightarrow 0$ when $t \rightarrow \infty$.

$$\begin{matrix} P_c \rightarrow 0 \\ \left\{ P(0) - \frac{VL}{Y} \right\} e^{(\frac{Y}{V})t} \rightarrow 0 \end{matrix}$$

However, on hair splitting, we don't get the desired result.

$$\left[\underbrace{\left\{ P(0) - \frac{VL}{Y} \right\}}_{\substack{P(0) \geq P_p \\ \text{Deviation (+ve or -ve)}}} e^{\frac{(\frac{Y}{V})}{\text{Exponential Growth over time}} t} \right] \rightarrow \infty$$

$\{V, L, Y > 0\}$ $\{Y, V > 0\}$
 $\frac{VL}{Y}$ $\frac{Y}{V}$

Therefore,

$$P(t) = \underbrace{\left\{ P(0) - \frac{VL}{Y} \right\}}_{\rightarrow \infty} e^{\left(\frac{Y}{V}\right)t} + \frac{VL}{Y}$$

$$P(t) = \underbrace{\left\{ P(0) - P_p \right\}}_{\rightarrow \infty} e^{\left(\frac{Y}{V}\right)t} + P_p$$

$$P(t) = \underbrace{\left\{ P(0) - P^* \right\}}_{\rightarrow \infty} e^{\left(\frac{Y}{V}\right)t} + P^*$$

Since addition of (P^*) in ∞ shall have no effect on infinity;

$$P(t) \rightarrow \infty$$

$$P(t) \nrightarrow P^*$$

Interpretation: It can be said that provided V , Y and L are positive, regardless of initial price $P(0)$ is greater or smaller than equilibrium price (P^*), the time-path of market price shall **diverge away from equilibrium** price as time is allowed to pass indefinitely. Hence the time-path is **dynamically unstable**.

Lesson 21

INCOME DYNAMICS USING DIFFERENTIAL EQUATIONS

TOPIC 086: INCOME DYNAMICS & MONEY MARKET USING DIFFERENTIAL EQUATIONS

On the money market, money supply M_s is fixed by the government. Furthermore, transaction demand for money L depends on income but is also positively related to its incremental increase so that:

$$L = f\left(Y, \frac{dY}{dt}\right)$$

$$L = \alpha \underbrace{Y}_{\text{National Income}} + \beta \underbrace{\frac{dY}{dt}}_{\text{Incremental increase in national income}}$$

$$L = \alpha Y + \beta \frac{dY}{dt} \quad 0 < \alpha < 1, \beta > 0$$

Find the time path of aggregate national income that ensures equilibrium on the money market, and check whether this time path is dynamically stable.

Solution: Since equilibrium in money market is to be ensured in finding the time-path for aggregate national income, we equate demand and supply of money:

$$\underline{L} = M_s$$

$$\alpha Y + \beta \frac{dY}{dt} = M_s$$

$$\alpha Y + \beta \frac{dY}{dt} = M_s$$

Rearranging for developing the first order differential equation:

$$\beta \frac{dY}{dt} + \alpha Y = M_s$$

Dividing by β to get the $\frac{dY}{dt}$ alone as found in first order differential equation.

$$\frac{dY}{dt} + \frac{\alpha}{\beta} Y = \frac{M_s}{\beta}$$

Comparing with the standard form of first order differential equation $\frac{dy}{dt} + ay = b$.

$$\frac{dY}{dt} + \frac{\alpha}{\beta} Y = \frac{M_s}{\beta}$$

$$\frac{dy}{dt} + \underbrace{\frac{\alpha}{\beta}}_a Y = \underbrace{\frac{M_s}{\beta}}_b$$

$$\text{Here } \frac{dy}{dt} = \frac{dY}{dt}, y = Y, a = \frac{\alpha}{\beta}, b = \frac{M_s}{\beta}$$

Choice of Case of the Differential Equation:

$b = \frac{M_s}{\beta}$; as money-supply (M_s) is fixed (some positive number) by the government $\beta > 0$ (positive constant), and the ratio will be non-zero positive constant. i.e. $b \neq 0$ – **Non-Homogenous Case**.

$a = \frac{\alpha}{\beta}$; as $0 < \alpha < 1$ and $\beta > 0$, both can be considered positive constants, their ratio is the value of a which will also be some positive (non-zero) value. i.e. $a \neq 0$ – **Non-Homogenous**

Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{Y}(t) = \left\{ \underline{Y}(0) - \frac{\frac{M_s}{\beta}}{\frac{\alpha}{\beta}} \right\} e^{-\frac{\alpha}{\beta}t} + \frac{\frac{M_s}{\beta}}{\frac{\alpha}{\beta}}$$

$$Y(t) = \left\{ Y(0) - \frac{\frac{M_s}{\beta}}{\frac{\alpha}{\beta}} \right\} e^{-\left(\frac{\alpha}{\beta}\right)t} + \frac{\frac{M_s}{\beta}}{\frac{\alpha}{\beta}}$$

$$Y(t) = \left\{ Y(0) - \frac{M_s}{\alpha} \right\} e^{-\left(\frac{\alpha}{\beta}\right)t} + \frac{M_s}{\alpha}$$

Definite Time-path for National Income in Dynamic Sense.

Interpretation: Since, M_s appears in numerator, it has positive relationship with aggregate national income Y .

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (Y_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (Y_c). Therefore, for dynamic stability; $Y_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Income level in Dynamic Sense.

$$Y(t) = \left\{ Y(0) - \frac{M_s}{\alpha} \right\} e^{-\left(\frac{\alpha}{\beta}\right)t} + \frac{M_s}{\alpha}$$

$$Y(t) = \underbrace{\left\{ Y(0) - \frac{M_s}{\alpha} \right\}}_{Y_c} e^{-\left(\frac{\alpha}{\beta}\right)t} + \underbrace{\frac{M_s}{\alpha}}_{Y_p}$$

Need for time path of income $\{Y(t)\}$ to converge to equilibrium income (Y^*) i.e. $Y(t) \rightarrow Y^*$ as $Y_c \rightarrow 0$ when $t \rightarrow \infty$.

$$\begin{aligned} & Y_c \rightarrow 0 \\ & \left\{ Y(0) - \frac{M_s}{\alpha} \right\} e^{-\left(\frac{\alpha}{\beta}\right)t} \rightarrow 0 \\ & \left[\underbrace{\left\{ Y(0) - \frac{M_s}{\alpha} \right\}}_{\substack{Y(0) \geq Y_p \\ \text{Deviation (+ve or -ve)}}} \underbrace{e^{-\left(\frac{\alpha}{\beta}\right)t}}_{\substack{\text{Exponential} \\ \text{Decay} \\ \text{over time}}} \right] \rightarrow 0 \end{aligned}$$

$\{M_s, \alpha > 0\}$ $\{\alpha, \beta > 0\}$

Therefore,

$$\begin{aligned} Y(t) &= \underbrace{\left\{ Y(0) - \frac{M_s}{\alpha} \right\}}_{\rightarrow 0} e^{-\left(\frac{\alpha}{\beta}\right)t} + \frac{M_s}{\alpha} \\ Y(t) &= \underbrace{\left\{ Y(0) - \frac{Y_p}{\beta} \right\}}_{\rightarrow 0} e^{-\left(\frac{\alpha}{\beta}\right)t} + Y_p \end{aligned}$$

As particular integral is also the equilibrium value of the variable $Y_p = Y^*$.

$$Y(t) = \underbrace{\{Y(0) - Y^*\}}_{\rightarrow 0} e^{-\left(\frac{\alpha}{\beta}\right)t} + Y^*$$

$$\boxed{Y(t) \rightarrow Y^*}$$

Interpretation: It can be said that provided α, β and M_s are positive, regardless of initial income $Y(0)$ is greater or smaller than equilibrium price (Y^*), the time-path of national income shall **converge towards equilibrium income level** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.

TOPIC 087: INCOME DYNAMICS & INCOME TAX USING DIFFERENTIAL EQUATIONS

If national income model is modelled by following system of equations:

$$Y = C + I$$

$$C = \alpha + \beta Y \quad \alpha > 0, 0 < \beta$$

α is the autonomous consumption and β is the marginal propensity to consume.

$$I = \delta Y + \gamma \left(\frac{dY}{dt}\right) \quad \gamma > 0, \delta < 1$$

If an income tax is introduced, the Consumption function becomes:

$$C = \alpha + \beta(1 - \tau)Y \quad \tau < 1$$

τ is the income tax rate. Find the time path and the intertemporal equilibrium level of national income.

Solution: Expenditure approach to National income is:

$$Y = \underline{C} + \underline{I}$$

Substituting values of C and I .

$$Y = \alpha + \beta(1 - \tau)Y + \delta Y + \gamma \left(\frac{dY}{dt}\right)$$

$$Y = \alpha + \beta(1 - \tau)Y + \delta Y + \gamma \left(\frac{dY}{dt}\right)$$

Rearranging to conform to a first order differential equation:

$$-\gamma \left(\frac{dY}{dt}\right) - \beta(1 - \tau)Y - \delta Y + Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt}\right) - \beta Y + \tau \beta Y - \delta Y + Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt}\right) + Y - \beta Y + \tau \beta Y - \delta Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt}\right) + (1 - \beta + \tau \beta - \delta)Y = \alpha$$

Dividing by $-\gamma$ throughout to normalize the differential equation.

$$\boxed{-\frac{\gamma}{(-\gamma)}} \left(\frac{dY}{dt}\right) \boxed{+} \frac{(1 - \beta + \tau \beta - \delta)}{(-\gamma)} Y = \frac{\alpha}{(-\gamma)}$$

$$\left(\frac{dY}{dt}\right) + \left(\frac{1 - \beta + \tau \beta - \delta}{-\gamma}\right) Y = \boxed{-\frac{\alpha}{\gamma}}$$

$$\left(\frac{dY}{dt}\right) + \left(\frac{1 - \beta + \tau \beta - \delta}{-\gamma}\right) Y = -\frac{\alpha}{\gamma}$$

Comparing with the standard form of first order differential equation; $\frac{dy}{dt} + ay = b$.

$$\left(\frac{dY}{dt}\right) + \underbrace{\left(\frac{1 - \beta + \tau \beta - \delta}{-\gamma}\right)}_a Y = \underbrace{\left(-\frac{\alpha}{\gamma}\right)}_b$$

Here $\frac{dy}{dt} = \frac{dY}{dt}$, $y = Y$, $a = \left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)$, $b = \left(-\frac{\alpha}{\gamma}\right)$

Choice of Case of the Differential Equation:

$b = \left(-\frac{\alpha}{\gamma}\right)$; As $\alpha > 0$ and $\gamma > 0$ (both positive constants), their ratio will be non-zero positive constant. However, the value of $b = \left(-\frac{\alpha}{\gamma}\right)$ has a negative sign. The answer will be negative, though still non-zero. i.e. $b \neq 0$ – **Non-Homogenous Case**.

$a = \left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)$; the numerator $(1 - \beta + \tau\beta - \delta)$ can be non-zero if $(\beta + \tau\beta - \delta) \neq 1$. Also γ is positive value. Therefore, value of a will be some positive/negative (non-zero) value. i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$** .

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\begin{aligned} \underline{Y}(t) &= \left\{ \underline{Y}(0) - \frac{\left(-\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)} \right\} e^{-\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)t} + \frac{\left(-\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)} \\ Y(t) &= \left\{ Y(0) - \frac{\left(-\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)} \right\} e^{-\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)t} + \frac{\left(-\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta+\tau\beta-\delta}{-\gamma}\right)} \\ Y(t) &= \left\{ Y(0) - \frac{(\alpha)}{(1-\beta+\tau\beta-\delta)} \right\} e^{\left(\frac{1-\beta+\tau\beta-\delta}{\gamma}\right)t} + \frac{(\alpha)}{(1-\beta+\tau\beta-\delta)} \\ Y(t) &= \left\{ Y(0) - \left(\frac{\alpha}{1-\beta+\tau\beta-\delta}\right) \right\} e^{\left(\frac{1-\beta+\tau\beta-\delta}{\gamma}\right)t} + \left(\frac{\alpha}{1-\beta+\tau\beta-\delta}\right) \end{aligned}$$

Definite Time-path for National Income in Dynamic Sense.

Interpretation: As expected the national income (Y) is negatively related to tax rate (τ). Its due to presence of τ in the denominator of the expression. Therefore, $Y \propto \frac{1}{\tau}$.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (Y_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (Y_c). Therefore, for dynamic stability; $Y_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Income level in Dynamic Sense.

$$Y(t) = \underbrace{\left\{ Y(0) - \left(\frac{\alpha}{1-\beta+\tau\beta-\delta}\right) \right\}}_{Y_c} e^{\left(\frac{1-\beta+\tau\beta-\delta}{\gamma}\right)t} + \underbrace{\left(\frac{\alpha}{1-\beta+\tau\beta-\delta}\right)}_{Y_p}$$

Need for time path of income $\{Y(t)\}$ to converge to equilibrium income (Y^*) i.e. $Y(t) \rightarrow Y^*$ as $Y_c \rightarrow 0$ when $t \rightarrow \infty$.

$$Y_c \rightarrow 0$$

$$\left\{ Y(0) - \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right) \right\} e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t} \rightarrow 0$$

However, on hair splitting, we don't get the desired result.

$$\left[\underbrace{\left\{ Y(0) - \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right) \right\}}_{\substack{Y(0) \geq Y_p \\ \text{Deviation (+ve or -ve)}}} \underbrace{e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t}}_{\substack{\text{Exponential} \\ \text{Growth} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore,

$$Y(t) = \left\{ Y(0) - \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right) \right\} e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t} + \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right)$$

$$Y(t) = \left\{ Y(0) - \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right) \right\} e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t} + \left(\frac{\alpha}{1 - \beta + \tau\beta - \delta} \right)$$

$$Y(t) = \left\{ Y(0) - Y_p \right\} e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t} + Y_p$$

As particular integral is also the equilibrium value of the variable $Y_p = Y^*$.

$$Y(t) = \left\{ Y(0) - Y^* \right\} e^{\left(\frac{1 - \beta + \tau\beta - \delta}{\gamma} \right)t} + Y^*$$

Since addition of (Y^*) in ∞ shall have no effect on infinity;

$$\frac{Y(t) \rightarrow \infty}{Y(t) \rightarrow Y^*}$$

Interpretation: It can be said that provided $\beta > 0, \delta < 1, \gamma > 0$ and $\tau < 1$, regardless of initial national income $Y(0)$ is greater or smaller than equilibrium national income (Y^*), the time-path of national income shall **diverge away from equilibrium** level of national income as time is allowed to pass indefinitely. Hence the time-path is **dynamically unstable**.

TOPIC 088: DYNAMICS OF CONSUMPTION USING DIFFERENTIAL EQUATIONS

The consumption function of an individual grows with time at a rate given by the equation: $\frac{dC}{dt} = -2C + 100$. Find the time path of this person's consumption given that $C(0) = 10$.

Solution: Given the equation for rate of consumption $\frac{dC}{dt} = -2C + 100$; we can rearrange to compare with the standard form of differential equation.

$$\frac{dC}{dt} + 2C = 100$$

$$\frac{dC}{dt} + \underbrace{2}_a C = \underbrace{100}_b$$

$$\text{Here } \frac{dy}{dt} = \frac{dC}{dt}, y = C, a = 2, b = 100, y(0) = C(0) = 10$$

Choice of Case of the Differential Equation:

As $b = 100$, so $b \neq 0$: Non-Homogenous Case.

As $a = 2$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{c}(t) = \left(\underline{c(0)} - \frac{\underline{100}}{\underline{2}} \right) e^{-\underline{2}t} + \frac{\underline{100}}{\underline{2}}$$

$$C(t) = \left\{ \underline{10} - \frac{100}{2} \right\} e^{-2t} + \frac{100}{2}$$

$$C(t) = \{10 - 50\} e^{-2t} + 50$$

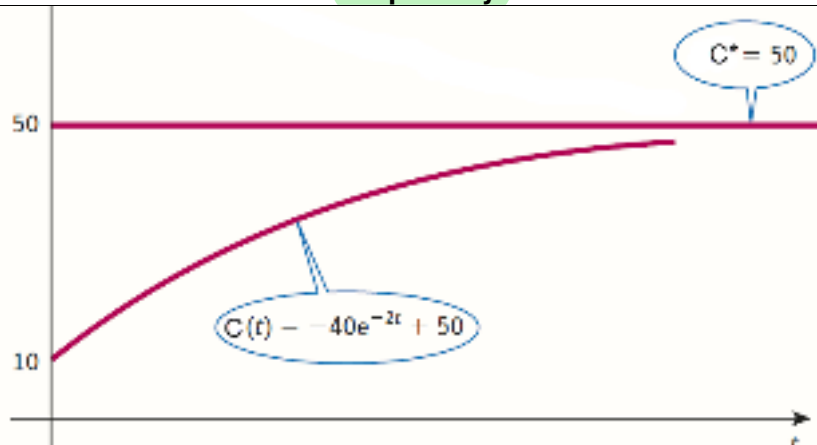
$$C(t) = \{10 - 50\} e^{-2t} + 50$$

$$\boxed{C(t) = -40e^{-2t} + 50}$$

It is the definite Time-Path of Consumption of the individual.

$$C(t) = \underbrace{-40e^{-2t}}_{\underline{c}_c} + \underbrace{50}_{\underline{c}_p}$$

Dynamic Stability: The expression; $C(t) = \boxed{-40e^{-2t}} + 50$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$C_c = -40e^{-2t}$	$C(t) = -40e^{-2t} + 50$
0	-40	10
1	-5.41341133	44.58658867
2	-0.73262556	49.26737444
3	-0.09915009	49.90084991
4	-0.01341851	49.98658149
5	-0.00181600	49.99818400
6	-0.00024577	49.99975423
7	-0.00003326	49.99996674
8	-0.00000450	49.99999550
9	-0.00000060	49.99999939
10	-0.00000008	49.99999992
...	...	...
Remarks		
Negative deviation decays over time		Time-path rises and converges to equilibrium ($C^* = 50$)
As $t \rightarrow \infty$, $C_c \rightarrow 0$		And when $C_c \rightarrow 0$, $(C_t) \rightarrow \text{Converges}(C^*)$
Graphically		
		
The straight line shows the equilibrium value of consumption (C^*), while curve shows the deviation from equilibrium (C_c), which decreases over time and leads the time-path to convergence.		

Lesson 22

NATIONAL INCOME DETERMINATION USING DIFFERENTIAL EQUATIONS

TOPIC 089: NATIONAL INCOME DETERMINATION USING DIFFERENTIAL EQUATIONS

If national income model is modelled by following system of equations:

$$Y = C + I$$

$$C = \alpha + \beta Y \quad \alpha > 0, 0 < \beta < 1$$

α is the autonomous consumption and β is the marginal propensity to consume.

$$I = \delta Y + \gamma \left(\frac{dY}{dt} \right) \quad \gamma > 0, \delta < 1$$

Find the time path and the intertemporal equilibrium level of national income.

Solution: Expenditure approach to National income is:

$$Y = \underline{C} + \underline{I}$$

Substituting values of C and I .

$$Y = \alpha + \beta Y + \delta Y + \gamma \left(\frac{dY}{dt} \right)$$

$$Y = \alpha + \beta Y + \delta Y + \gamma \left(\frac{dY}{dt} \right)$$

Rearranging to conform to a first order differential equation:

$$-\gamma \left(\frac{dY}{dt} \right) - \beta Y - \delta Y + Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt} \right) - \beta Y - \delta Y + Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt} \right) + Y - \beta Y - \delta Y = \alpha$$

$$-\gamma \left(\frac{dY}{dt} \right) + (1 - \beta - \delta)Y = \alpha$$

Dividing by $-\gamma$ throughout to normalize the differential equation.

$$\left[\frac{-\gamma}{(-\gamma)} \right] \left(\frac{dY}{dt} \right) \left[+ \right] \frac{(1 - \beta - \delta)}{(-\gamma)} Y = \frac{\alpha}{(-\gamma)}$$

$$\left(\frac{dY}{dt} \right) + \left(\frac{1 - \beta - \delta}{-\gamma} \right) Y = \frac{\alpha}{-\gamma}$$

$$\left(\frac{dY}{dt} \right) + \left(\frac{1 - \beta - \delta}{-\gamma} \right) Y = -\frac{\alpha}{\gamma}$$

Comparing with the standard form of first order differential equation; $\frac{dy}{dt} + ay = b$.

$$\left(\frac{dY}{dt} \right) + \underbrace{\left(\frac{1 - \beta - \delta}{-\gamma} \right)}_a Y = \underbrace{\left(-\frac{\alpha}{\gamma} \right)}_b$$

$$\text{Here } \frac{dy}{dt} = \frac{dY}{dt}, y = Y, a = \left(\frac{1 - \beta - \delta}{-\gamma} \right), b = \left(-\frac{\alpha}{\gamma} \right)$$

Choice of Case of the Differential Equation:

$b = \left(-\frac{\alpha}{\gamma} \right)$; As $\alpha > 0$ and $\gamma > 0$ (both positive constants), their ratio will be non-zero positive constant. However, the value of $b = \left(-\frac{\alpha}{\gamma} \right)$ has a negative sign. The answer will be negative, though still non-zero. i.e. $b \neq 0$ – **Non-Homogenous Case.**

$a = \left(\frac{1-\beta-\delta}{-\gamma}\right)$; Both β and δ are less than 1. The numerator $(1 - \beta - \delta)$ can become 0 only if $\beta + \delta = 1$. So, we assume that the $\beta + \delta \neq 1$ and $(1 - \beta - \delta \neq 0)$, so $\left(\frac{1-\beta-\delta}{-\gamma} \neq 0\right)$ i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$** .

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{Y}(t) = \left\{ \underline{Y}(0) - \frac{\left(\frac{-\alpha}{\gamma}\right)}{\left(\frac{1-\beta-\delta}{-\gamma}\right)} \right\} e^{-\left(\frac{1-\beta-\delta}{-\gamma}\right)t} + \frac{\left(\frac{-\alpha}{\gamma}\right)}{\left(\frac{1-\beta-\delta}{-\gamma}\right)}$$

$$Y(t) = \left\{ Y(0) - \frac{\left(\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta-\delta}{-\gamma}\right)} \right\} e^{\left(\frac{1-\beta-\delta}{-\gamma}\right)t} + \frac{\left(\frac{\alpha}{\gamma}\right)}{\left(\frac{1-\beta-\delta}{-\gamma}\right)}$$

$$Y(t) = \left\{ Y(0) - \frac{(\alpha)}{(1-\beta-\delta)} \right\} e^{\left(\frac{1-\beta-\delta}{\gamma}\right)t} + \frac{(\alpha)}{(1-\beta-\delta)}$$

$$Y(t) = \left\{ Y(0) - \left(\frac{\alpha}{1-\beta-\delta}\right) \right\} e^{\left(\frac{1-\beta-\delta}{\gamma}\right)t} + \left(\frac{\alpha}{1-\beta-\delta}\right)$$

Definite Time-path for National Income in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (Y_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (Y_c). Therefore, for dynamic stability; $Y_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Income level in Dynamic Sense.

$$Y(t) = \underbrace{\left\{ Y(0) - \left(\frac{\alpha}{1-\beta-\delta}\right) \right\}}_{Y_c} e^{\left(\frac{1-\beta-\delta}{\gamma}\right)t} + \underbrace{\left(\frac{\alpha}{1-\beta-\delta}\right)}_{Y_p}$$

Need for time path of income $\{Y(t)\}$ to converge to equilibrium income (Y^*) i.e. $Y(t) \rightarrow Y^*$ as $Y_c \rightarrow 0$ when $t \rightarrow \infty$.

$$\underbrace{\left\{ Y(0) - \left(\frac{\alpha}{1-\beta-\delta}\right) \right\}}_{Y_c \rightarrow 0} e^{\left(\frac{1-\beta-\delta}{\gamma}\right)t} \rightarrow 0$$

However, on hair splitting, we don't get the desired result.

$$\left[\underbrace{\left\{ Y(0) - \left(\frac{\alpha}{1-\beta-\delta}\right) \right\}}_{\substack{Y(0) \geq Y_p \\ \text{Deviation (+ve or -ve)}}} \underbrace{e^{\left(\frac{1-\beta-\delta}{\gamma}\right)t}}_{\substack{\text{Exponential} \\ \text{Growth} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore,

$$Y(t) = \left\{ Y(0) - \left(\frac{\alpha}{1 - \beta - \delta} \right) \right\} e^{\left(\frac{1 - \beta - \delta}{\gamma} \right)t} + \left(\frac{\alpha}{1 - \beta - \delta} \right)$$

$$Y(t) = \left\{ Y(0) - \left[\left(\frac{\alpha}{1 - \beta - \delta} \right) \right] \right\} e^{\left(\frac{1 - \beta - \delta}{\gamma} \right)t} + \left[\left(\frac{\alpha}{1 - \beta - \delta} \right) \right]$$

$$Y(t) = \left\{ Y(0) - \left[Y_p \right] \right\} e^{\left(\frac{1 - \beta - \delta}{\gamma} \right)t} + \left[Y_p \right]$$

As particular integral is also the equilibrium value of the variable $Y_p = Y^*$.

$$Y(t) = \left\{ Y(0) - \left[Y^* \right] \right\} e^{\left(\frac{1 - \beta - \delta}{\gamma} \right)t} + \left[Y^* \right]$$

Since addition of (Y^*) in ∞ shall have no effect on infinity;

$$\frac{Y(t) \rightarrow \infty}{Y(t) \rightarrow Y^*}$$

Interpretation: It can be said that provided $\beta > 0, \delta < 1$ and $\gamma > 0$, regardless of initial national income $Y(0)$ is greater or smaller than equilibrium national income (Y^*), the time-path of national income shall diverge away from equilibrium level of national income as time is allowed to pass indefinitely. Hence the time-path is dynamically unstable.

TOPIC 090: NATIONAL INCOME DETERMINATION: NUMERICAL

Considering a two-sector economy; we have following structural equations:

$$\frac{dY}{dt} = 0.5(C + I - Y)$$

$$C = 0.8Y + 400$$

$$I = 600$$

Find the time-path of national income of the economy; $Y(0) = 7000$.

Solution: Given the equation for rate of national income

$\frac{dY}{dt} = 0.5(C + I - Y)$; we can substitute equations of C and I .

$$\frac{dY}{dt} = 0.5(C + I - Y)$$

$$\frac{dY}{dt} = 0.5(0.8Y + 400 + 600 - Y)$$

$$\frac{dY}{dt} = 0.5(0.8Y - Y + 400 + 600)$$

$$\frac{dY}{dt} = 0.5(-0.2Y + 1000)$$

$$\frac{dY}{dt} = -0.1Y + 500$$

$$\frac{dY}{dt} + 0.1Y = 500$$

First Order Differential Equation for National Income.

$$\frac{dY}{dt} + \underbrace{0.1Y}_a = \underbrace{500}_b$$

Here $\frac{dy}{dt} = \frac{dY}{dt}$, $y = Y$, $a = 0.1$, $b = 500$, $y(0) = Y(0) = 7000$

Choice of Case of the Differential Equation:

As $b = 500$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.1$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{Y}(t) = \left(\underline{Y(0)} - \frac{\underline{500}}{\underline{0.1}} \right) e^{-\underline{0.1}t} + \frac{\underline{500}}{\underline{0.1}}$$

$$Y(t) = \left(7000 - \frac{500}{0.1} \right) e^{-0.1t} + \frac{500}{0.1}$$

$$Y(t) = (7000 - 5000)e^{-0.1t} + 5000$$

$$Y(t) = (2000)e^{-0.1t} + 5000$$

$$\boxed{Y(t) = 2000 e^{-0.1t} + 5000}$$

It is the definite Time-Path of National Income of the economy.

$$Y(t) = \underbrace{2000e^{-0.1t}}_{Y_c} + \underbrace{5000}_{Y_p}$$

Dynamic Stability: The expression; $Y(t) = \boxed{2000e^{-0.1t}} + 5000$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$Y_c = 2000e^{-0.1t}$	$Y(t) = 2000e^{-0.1t} + 5000$
0	2000	7000
1	1809.675	6809.674836
20	270.6706	5270.670566
40	36.63128	5036.631278
60	4.957504	5004.957504
80	0.670925	5000.670925
100	0.090800	5000.090800
120	0.012288	5000.012288
140	0.001663	5000.001663
160	0.000225	5000.000225
200	0.000004	5000.000004
...	...	...

	Remarks
	Positive deviation decays over time
	Time-path slides and converges to equilibrium ($Y^* = 5000$)
	As $t \rightarrow \infty$, $Y_c \rightarrow 0$
	And when $Y_c \rightarrow 0$, (Y_t) Converges (Y^*)
	Graphically
	The straight line shows the equilibrium value of national income (Y^*), while curve shows the deviation from equilibrium (Y_c), which decreases over time and leads the time-path to convergence.

TOPIC 091: INCOME DYNAMICS & INDUCED GOVERNMENT SPENDING USING DIFFERENTIAL EQUATIONS

If national income model is modelled by following system of equations:

$$Y = C + I + G$$

$$C = \alpha + \beta Y \quad \alpha > 0, 0 < \beta$$

α is the autonomous consumption and β is the marginal propensity to consume.

Following is the investment function:

$$I = \delta Y + \gamma \left(\frac{dY}{dt} \right) \quad \gamma > 0, \delta < 1$$

Government Spending is included as the 3rd sector:

$$G = gY \quad g < 1$$

g is the proportion of national income spent as government expenditure.
Find the time path and the intertemporal equilibrium level of national income.

Solution: Expenditure approach to National income is:

$$Y = \underline{C} + \underline{I} + \underline{G}$$

Substituting values of $\underline{C}$, $\underline{I}$ and $\underline{G}$.

$$Y = \alpha + \beta Y + \delta Y + \gamma \left(\frac{dY}{dt} \right) + gY$$

$$Y = \alpha + \beta Y + \delta Y + \gamma \left(\frac{dY}{dt} \right) + gY$$

Rearranging to conform to a first order differential equation:

$$-\gamma \left(\frac{dY}{dt} \right) + Y - \beta Y - \delta Y - gY = \alpha$$

$$-\gamma \left(\frac{dY}{dt} \right) + (1 - \beta - \delta - g)Y = \alpha$$

Dividing by $-\gamma$ throughout to normalize the differential equation.

$$\left[-\frac{\gamma}{(-\gamma)} \right] \left(\frac{dY}{dt} \right) \left[+ \right] \frac{(1 - \beta - \delta - g)}{(-\gamma)} Y = \frac{\alpha}{(-\gamma)}$$

$$\left(\frac{dY}{dt} \right) + \left(\frac{1 - \beta - \delta - g}{-\gamma} \right) Y = \frac{\alpha}{-\gamma}$$

$$\left(\frac{dY}{dt} \right) + \left(\frac{1 - \beta - \delta - g}{-\gamma} \right) Y = -\frac{\alpha}{\gamma}$$

Comparing with the standard form of first order differential equation; $\frac{dy}{dt} + ay = b$.

$$\left(\frac{dY}{dt} \right) + \left(\frac{1 - \beta - \delta - g}{-\gamma} \right) Y = \left(-\frac{\alpha}{\gamma} \right)$$

$$\text{Here } \frac{dy}{dt} = \frac{dY}{dt}, y = Y, a = \left(\frac{1 - \beta - \delta - g}{-\gamma} \right), b = \left(-\frac{\alpha}{\gamma} \right)$$

Choice of Case of the Differential Equation:

$b = \left(-\frac{\alpha}{\gamma} \right)$; As $\alpha > 0$ and $\gamma > 0$ (both positive constants), their ratio will be non-zero positive constant. However, the value of $b = \left(-\frac{\alpha}{\gamma} \right)$ has a negative sign. The answer will be negative, though still non-zero. i.e. $b \neq 0$ – **Non-Homogenous Case.**

$a = \left(\frac{1 - \beta - \delta - g}{-\gamma} \right)$; the numerator $(1 - \beta - \delta - g)$ can be non-zero if $(\beta + \delta + g) \neq 1$. Also γ is positive value. Therefore, value of a will be some positive/negative (non-zero) value. i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$.**

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{Y}(t) = \left\{ \underline{Y}(0) - \frac{\left(-\frac{\alpha}{\gamma} \right)}{\left(\frac{1 - \beta - \delta - g}{-\gamma} \right)} \right\} e^{-\left(\frac{1 - \beta - \delta - g}{-\gamma} \right)t} + \frac{\left(-\frac{\alpha}{\gamma} \right)}{\left(\frac{1 - \beta - \delta - g}{-\gamma} \right)}$$

$$Y(t) \rightarrow Y^*$$

Interpretation: It can be said that provided $\beta > 0, \delta < 1, \gamma > 0$ and $g < 1$, regardless of initial national income $Y(0)$ is greater or smaller than equilibrium national income (Y^*), the time-path of national income shall **diverge away from equilibrium** level of national income as time is allowed to pass indefinitely. Hence the time-path is **dynamically unstable**.

TOPIC 092: DYNAMICS OF INVESTMENT USING DIFFERENTIAL EQUATIONS

A principal of \$60 is invested. The value, $I(t)$, of the investment, t days later, satisfies the differential equation $\frac{dI}{dt} = 0.002I + 5$. Find the value of the investment if the initial level of investment $y(0)$ is 2000.

Solution: Given the equation for rate of investment $\frac{dI}{dt} = 0.002I + 5$; we can rearrange to compare with the standard form of differential equation.

$$\begin{aligned}\frac{dI}{dt} &= 0.002I + 5 \\ \frac{dI}{dt} - 0.002I &= 5 \\ \frac{dI}{dt} + \underbrace{(0.002)}_a I &= \underbrace{5}_b\end{aligned}$$

Here $\frac{dy}{dt} = \frac{dI}{dt}$, $y = I$, $a = 0.002$, $b = 5$, $y(0) = I(0) = 2000$.

Choice of Case of the Differential Equation:

As $b = 5$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.002$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{I}(t) = \left(\underline{I(0)} - \frac{\underline{5}}{\underline{0.002}} \right) e^{-\underline{0.002}t} + \frac{\underline{5}}{\underline{0.002}}$$

$$I(t) = \left(2000 - \frac{5}{0.002} \right) e^{-0.002t} + \frac{5}{0.002}$$

$$I(t) = (2000 - 2500)e^{-0.002t} + 2500$$

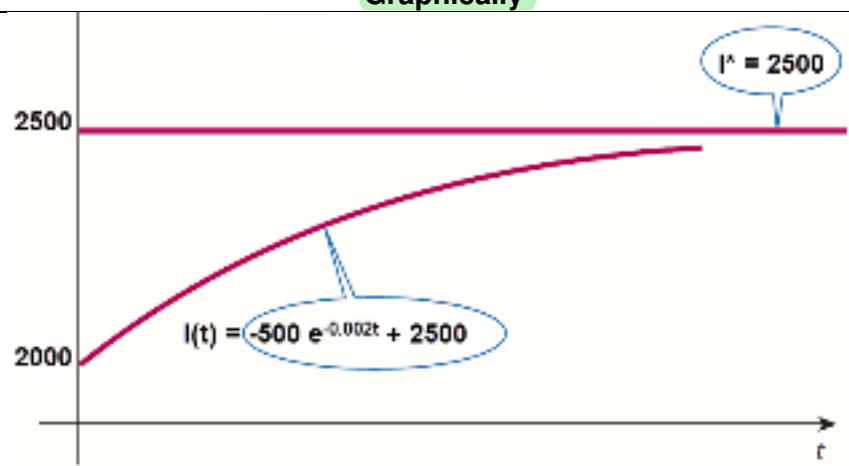
$$I(t) = -500e^{-0.002t} + 2500$$

It is the definite Time-Path of Investment.

$$I(t) = \underbrace{-500e^{-0.002t}}_{I_c} + \underbrace{2500}_{I_p}$$

Dynamic Stability: The expression; $I(t) = \boxed{-500e^{-0.002t}} + 2500$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$I_c = -500e^{-0.002t}$	$I(t) = -500e^{-0.002t} + 2500$
0	-500.0000	2000.0000
1	-499.0009	2000.9990
2	-498.0040	2001.9960
3	-497.0089	2002.9910
4	-496.0160	2003.9840
5	-495.0249	2004.9751
6	-494.0359	2005.9641
7	-493.0488	2006.9512
8	-492.0637	2007.9363
9	-491.0805	2008.9194
10	-490.0993	2009.9006
...	...	...

	Remarks
Negative deviation decays over time	Time-path rises and converges to equilibrium ($I^* = 2500$)
As $t \rightarrow \infty$, $I_c \rightarrow 0$	And when $I_c \rightarrow 0$, $(I_t) \rightarrow \overrightarrow{Converges}(I^*)$
	Graphically
	
The straight line shows the equilibrium value of consumption (I^*), while curve shows the deviation from equilibrium (I_c), and this deviation decreases over time and leads the time-path to convergence.	

TOPIC 093: DYNAMICS OF INTEREST RATE IN CAPITAL MARKET USING DIFFERENTIAL EQUATIONS

The demand and supply for borrowed capital in the capital market are given by the investment and savings functions, respectively:

$$I = \alpha - \beta i + \rho \frac{di}{dt} \quad S = -\gamma + \delta i + \sigma \frac{di}{dt} \quad (\alpha, \beta, \gamma, \delta)$$

Determine the intertemporal equilibrium level and time path of interest rate $i(t)$. What values of the parameters ρ and σ ensure dynamic stability?

Solution: Equilibrium in the market requires equality of Investment and Savings: ($S = I$).

$$\alpha - \beta i + \rho \frac{di}{dt} = -\gamma + \delta i + \sigma \frac{di}{dt}$$

Rearranging to compare with the standard form of differential equation.

$$-\beta i - \delta i + \rho \frac{di}{dt} - \sigma \frac{di}{dt} = -\alpha - \gamma$$

$$\rho \frac{di}{dt} - \sigma \frac{di}{dt} - \beta i - \delta i = -\alpha - \gamma$$

$$(\rho - \sigma) \frac{di}{dt} - (\beta + \delta) i = -(\alpha + \gamma)$$

Dividing by $(\rho - \sigma)$ for normalizing the differential equation.

$$\frac{(\rho - \sigma) \frac{di}{dt} - (\beta + \delta) i}{(\rho - \sigma)} = \frac{-(\alpha + \gamma)}{(\rho - \sigma)}$$

$$\frac{di}{dt} - \frac{(\beta + \delta)}{(\rho - \sigma)} i = -\frac{(\alpha + \gamma)}{(\rho - \sigma)}$$

Comparing with the standard form of the differential equation, $\frac{dy}{dt} + ay = b$.

$$\frac{di}{dt} + \underbrace{\left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}}_a i = \underbrace{-\frac{(\alpha + \gamma)}{(\rho - \sigma)}}_b$$

$$\text{Here } \frac{dy}{dt} = \frac{di}{dt}, y = i, a = \left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}, b = \left\{ -\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}$$

Choice of Case of the Differential Equation:

$b = \left\{ -\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}$; As $\alpha > 0$ and $\gamma > 0$ (both positive constants), their ratio will be non-zero positive constant. However, the value of $b = \left\{ -\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}$ whose denominator should be non-zero to avoid undefined answer (∞). The answer will be negative, though still non-zero. i.e. $b \neq 0$ – **Non-Homogenous Case**.

$a = \left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}$; Both β and δ are less than 1. The numerator $(\beta + \delta)$ shall be a non-zero positive constant. However, $a = \left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}$ so value of numerator $(\rho - \sigma)$ should be non-zero to avoid undefined answer (∞). So $\left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\} \neq 0$ i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$** .

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\underline{i}(t) = \left\{ \underline{i}(0) - \frac{\left\{ -\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}}{\left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}} \right\} e^{-\left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}t} + \frac{\left\{ -\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}}{\left\{ -\frac{(\beta + \delta)}{(\rho - \sigma)} \right\}}$$

$$\underline{i}(t) = \left[\underline{i}(0) - \frac{\left[\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right]}{\left[\frac{(\beta + \delta)}{(\rho - \sigma)} \right]} \right] e^{\left[\frac{(\beta + \delta)}{(\rho - \sigma)} \right]t} + \frac{\left[\frac{(\alpha + \gamma)}{(\rho - \sigma)} \right]}{\left[\frac{(\beta + \delta)}{(\rho - \sigma)} \right]}$$

$$i(t) = \left[i(0) - \frac{\left\{ \frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}}{\left\{ \frac{(\beta + \delta)}{(\rho - \sigma)} \right\}} \right] e^{\left\{ \frac{(\beta + \delta)}{(\rho - \sigma)} \right\} t} + \frac{\left\{ \frac{(\alpha + \gamma)}{(\rho - \sigma)} \right\}}{\left\{ \frac{(\beta + \delta)}{(\rho - \sigma)} \right\}}$$

$$i(t) = \left\{ i(0) - \left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right\} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \left(\frac{\alpha + \gamma}{\beta + \delta} \right)$$

$$i(t) = \left\{ i(0) - \left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right\} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \left(\frac{\alpha + \gamma}{\beta + \delta} \right)$$

Definite Time-path for Interest rate in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (i_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (i_c). Therefore, for dynamic stability; $i_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for interest rate in Dynamic Sense.

$$i(t) = \underbrace{\left\{ i(0) - \left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right\}}_{i_c} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{i_p}$$

Need for time path of income $\{i(t)\}$ to converge to equilibrium level of interest rate (i^*) i.e. $i(t) \rightarrow i^*$ as $i_c \rightarrow 0$ when $t \rightarrow \infty$.

$$i_c \rightarrow 0$$

$$\left\{ i(0) - \left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right\} e^{\left[\left(\frac{\beta + \delta}{\rho - \sigma} \right) \right] t} \rightarrow 0$$

$\left(\frac{\beta + \delta}{\rho - \sigma} \right)$ is the rate of growth/decay of the exponential expression. Therefore, its negativity shall ensure its decay. i.e. $\left(\frac{\beta + \delta}{\rho - \sigma} \right) < 0$. It's already known that β and δ are positive and their sum (Numerator; $\beta + \delta$) will also be positive. So, denominator ($\rho - \sigma$) must be negative. i.e. $(\rho - \sigma) < 0$. In comparative terms, $(\rho < \sigma)$. Therefore, the condition for dynamic stability of the time-path of interest rate $i(t)$ is $(\rho < \sigma)$:

$$\left[\underbrace{\left\{ i(0) - \frac{\left\{ \frac{(\alpha + \gamma)}{(\beta + \delta)} \right\}}{\left\{ \frac{(\beta + \delta)}{(\rho - \sigma)} \right\}} \right\}}_{\substack{\text{Deviation (+ve or -ve)}}}_{\substack{\{ \alpha, \beta, \delta, \gamma > 0 \} \\ \{ \beta, \delta > 0 \} \& \{ \rho < \sigma \} \\ \left(\frac{\beta + \delta}{\rho - \sigma} \right)}}} e^{\left[\left(\frac{\beta + \delta}{\rho - \sigma} \right) \right] t} \rightarrow 0$$

Exponential Decay over time

Therefore,

$$i(t) = \left\{ i(0) - \left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right\} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \left(\frac{\alpha + \gamma}{\beta + \delta} \right)$$

$$i(t) = \left\{ i(0) - \left[\left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right] \right\} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \left[\left(\frac{\alpha + \gamma}{\beta + \delta} \right) \right]$$

$$i(t) = \left\{ i(0) - \left[(i_p) \right] \right\} e^{\left(\frac{\beta + \delta}{\rho - \sigma} \right) t} + \left[(i_p) \right]$$

$$i(t) = \underbrace{\{i(0) - i_p\}}_{\rightarrow 0} e^{\left(\frac{\beta+\delta}{\rho-\sigma}\right)t} + i_p$$

As particular integral is also the equilibrium value of the variable $i_p = i^*$.

$$i(t) = \underbrace{\{i(0) - i^*\}}_{\rightarrow 0} e^{\left(\frac{\beta+\delta}{\rho-\sigma}\right)t} + i^*$$

$$\boxed{i(t) \rightarrow i^*}$$

Interpretation: It can be said that provided $\beta > 0, \delta > 0$ with $\rho < \sigma$, regardless of initial interest rate $i(0)$ is greater or smaller than equilibrium interest rate (i^*), the time-path of interest rate shall **converge towards equilibrium** level of interest rate as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.

Lesson 23

VARIOUS ECONOMIC DYNAMICS USING DIFFERENTIAL EQUATIONS

TOPIC 094: DYNAMICS OF VALUE OF OIL WELL USING DIFFERENTIAL EQUATIONS

The value of an oil well, which is explored, initially is $V(0) = \$2700$, and is decaying at the rate $\frac{dV}{dt} + 0.04 V(t) = 28$.

a) What is the general time path that describes the value of this well?

b) What is the future value of the well at time $t = 10$?

Solution: Given the equation for rate of value of oil-well $\frac{dV}{dt} + 0.04 V(t) = 28$; we can rewrite to compare with the standard form of differential equation:

$$\frac{dV}{dt} + 0.04V = 28$$

$$\frac{dV}{dt} + \underbrace{(0.04)}_a \underbrace{V}_y = \underbrace{28}_b$$

Here $\frac{dy}{dt} = \frac{dV}{dt}$, $y = V$, $a = 0.04$, $b = 28$, $y(0) = V(0) = 2700$.

Choice of Case of the Differential Equation:

As $b = 28$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.04$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{V}(t) = \left\{ \underline{V(0)} - \frac{\underline{28}}{\underline{0.04}} \right\} e^{-\underline{0.04}t} + \frac{\underline{28}}{\underline{0.04}}$$

$$V(t) = \left(2700 - \frac{28}{0.04} \right) e^{-0.04t} + \frac{28}{0.04}$$

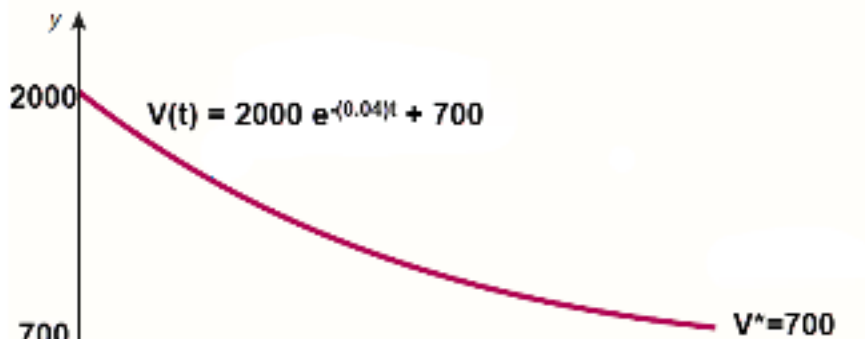
$$V(t) = (2700 - 700)e^{-0.04t} + 700$$

$$\underline{V(t) = (2000)e^{-0.04t} + 700}$$

It is the definite Time-Path of Value of Oil-Well.

$$V(t) = \underbrace{2000 e^{-0.04t}}_{V_c} + \underbrace{700}_{V_p}$$

Dynamic Stability: The expression; $V(t) = \underline{2000 e^{-0.04t}} + 700$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$V_c = 2000 e^{-0.04t}$	$V(t) = 2000 e^{-0.04t} + 700$
0	2000	2700
1	1921.58	2621.58
40	403.793	1103.79
80	81.5244	781.524
120	16.4595	716.459
160	3.32311	703.323
200	0.67093	700.671
240	0.13546	700.135
280	0.02735	700.027
320	0.00552	700.006
340	0.00248	700.002
...	...	...
Remarks		
	Positive deviation decays over time	Time-path slides and converges to equilibrium ($V^* = 700$)
	As $t \rightarrow \infty$, $V_c \rightarrow 0$	And when $V_c \rightarrow 0$, $(V_t) \rightarrow (V^*)$
Graphically		
		
The straight line shows the equilibrium value of Value of Oil-Well (V^*), while curve shows the deviation from equilibrium (V_c), and this deviation decreases over time and leads the time-path to convergence.		

TOPIC 095: DYNAMICS OF VALUE OF GOLD MINE USING DIFFERENTIAL EQUATIONS

The value of a gold mine is initially $G(0) = \$1,400$ and is decaying at the rate $\frac{dG}{dt} + 0.04 G(t) =$

16.

- What is the general time path that describes the value of the gold mine?
- What is the future value of the gold mine at time $t = 10$?

Solution: Given the rate of change of value of Gold mine $\frac{dG}{dt} + 0.04 G(t) = 16$, we can rewrite to compare with the standard form of differential equation:

$$\frac{dG}{dt} + 0.04 G = 16$$

$$\frac{dG}{dt} + \underbrace{(0.04)}_a \underbrace{G}_y = \underbrace{16}_b$$

Here $\frac{dy}{dt} = \frac{dG}{dt}$, $y = G$, $a = 0.04$, $b = 16$, $y(0) = G(0) = 1400$.

Choice of Case of the Differential Equation:

As $b = 16$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.04$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{G}(t) = \left\{ \underline{G(0)} - \frac{\underline{16}}{\underline{0.04}} \right\} e^{-\underline{0.04}t} + \frac{\underline{16}}{\underline{0.04}}$$

$$G(t) = \left(1400 - \frac{16}{0.04} \right) e^{-0.04t} + \frac{16}{0.04}$$

$$G(t) = (1400 - 400)e^{-0.04t} + 400$$

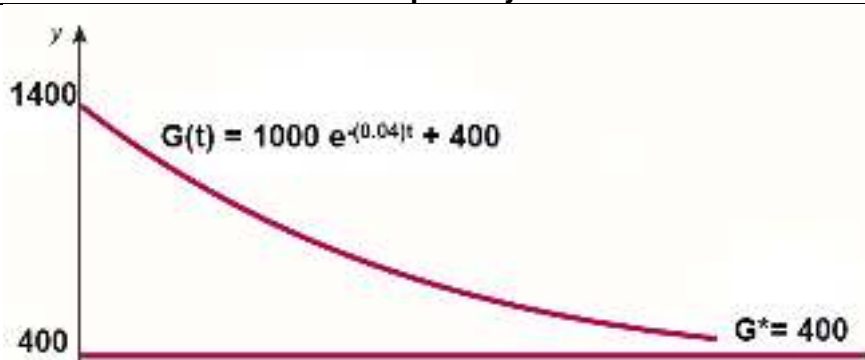
$$\underline{G(t) = (1000)e^{-0.04t} + 400}$$

It is the definite Time-Path of Value of Gold-Mine.

$$G(t) = \underbrace{1000 e^{-0.04t}}_{G_c} + \underbrace{400}_{G_p}$$

Dynamic Stability: The expression; $G(t) = \underline{1000 e^{-0.04t}} + 400$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$G_c = 1000 e^{-0.04t}$	$G(t) = 1000 e^{-0.04t} + 400$
0	1000	1400
1	960.79	1360.789
40	201.90	601.8965
80	40.762	440.7622
120	8.2297	408.2297
160	1.6616	401.6615
200	0.3355	400.3354
240	0.0677	400.0677
280	0.0137	400.0136
320	0.0028	400.0027
360	0.0006	400.0006
...	...	...

	Remarks
Positive deviation decays over time	Time-path slides and converges to equilibrium ($G^* = 400$)
As $t \rightarrow \infty$, $G_c \rightarrow 0$	And when $G_c \rightarrow 0$, $(G_t) \xrightarrow{\text{Converges}} (G^*)$
	Graphically
	
	The straight line shows the equilibrium value of Value of Oil-Well (G^*), while curve shows the deviation from equilibrium (G_c), and this deviation decreases over time and leads the time-path to convergence.

TOPIC 096: HOTELLING'S RULE USING DIFFERENTIAL EQUATIONS

Hotelling's rule is named after statistician and economic theorist Harold Hotelling.

It studies the net revenue of extraction from a nonrenewable natural resource as a function of time $\pi(t)$.

Hotelling found the optimum point to be where net revenue grows exactly at the rate of interest r . Thus, the simple rule is:

$$\frac{\pi'(t)}{\pi(t)} = \underbrace{r}_{\text{rate of Interest}}$$

Net revenue

Rewriting the Hotelling's Rule:

$$\frac{\pi'(t)}{\pi(t)} = r$$

$$\pi'(t) = r \times \pi(t)$$

$$\frac{d\{\pi(t)\}}{dt} = r \times \pi(t)$$

In simple terms:

$$\frac{d\pi}{dt} = r\pi$$

Rewriting to compare with standard form:

$$\frac{d\pi}{dt} - r\pi = 0$$

$$\frac{d\pi}{dt} - \underbrace{r}_{a} \underbrace{\pi}_{y} = \underbrace{0}_{b}$$

$$\text{Here } \frac{dy}{dt} = \frac{d\pi}{dt}, y = \pi, a = r, b = 0$$

Choice of Case of the Differential Equation:

$b = 0$ – Homogenous Case.

$a = r$; interest rate is usually a positive number $r > 0$ i.e. $a \neq 0$ – **Homogenous Case with $a \neq 0$.**

Therefore, using the definite Solution of time-path of **HC**:

$$y(t) = \{y(0)\}e^{-at}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \{\underline{y}(0)\}e^{-\underline{a}t}$$

Substituting values of y and a , respectively:

$$\underline{\pi}(t) = \{\underline{\pi}(0)\}e^{-\underline{r}t}$$

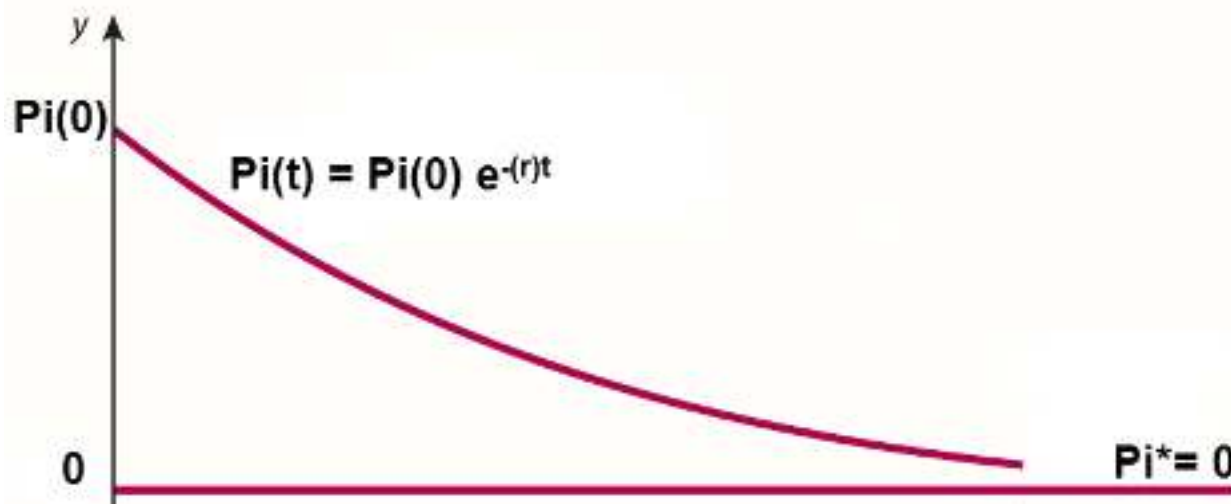
$$\pi(t) = \{\pi(0)\}e^{-rt}$$

Definite Time-path for net revenue in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, the deviation expression should fade. In homogenous case (**HC**), the particular solution is zero; $\pi_p = 0$. Therefore, the convergence should take place towards zero (**0**). In this case, we have a negative exponent $\pi(t) = \{\pi(0)\}e^{-rt}$ which shall approach to zero (**0**), when $t \rightarrow \infty$. In a differential equation, deviation component is measured by (π_c). Therefore, for dynamic stability; $\pi_c \rightarrow 0$ as $t \rightarrow \infty$.

$$\left[\underbrace{\{\pi(0)\}}_{\text{Deviation}} \underbrace{e^{\overbrace{-\overbrace{(r)}^{(r>0)}}t}}_{\substack{\text{Exponential} \\ \text{Decay} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore, the time-path of net revenue shall **converge towards equilibrium** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.



TOPIC 097: FUTURE VALUE OF INVESTMENT USING DIFFERENTIAL EQUATIONS

An initial amount $D(0) = \$1,300$ deposited in a bank is growing according to the equation:

$$\frac{dD}{dt} - 0.05 D(t) = 15$$

- What is the general time path that describes the growth of this account?
- What is the future value of this account at time $t = 6$?

Solution: Given the rate of change of value of deposit $\frac{dD}{dt} - 0.05 D(t) = 15$, we can rewrite to compare with the standard form of differential equation:

$$\begin{aligned} \frac{dD}{dt} + 0.05 D &= 15 \\ \frac{dD}{dt} + \underbrace{(0.05)}_a \underbrace{D}_y &= \underbrace{15}_b \end{aligned}$$

Here $\frac{dy}{dt} = \frac{dD}{dt}$, $y = D$, $a = 0.05$, $b = 15$, $y(0) = D(0) = 1300$.

Choice of Case of the Differential Equation:

As $b = 15$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.05$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{D}(t) = \left\{ \underline{D(0)} - \frac{\underline{15}}{\underline{0.05}} \right\} e^{-\underline{0.05}t} + \frac{\underline{15}}{\underline{0.05}}$$

$$D(t) = (1300 - 300)e^{-0.05t} + 300$$

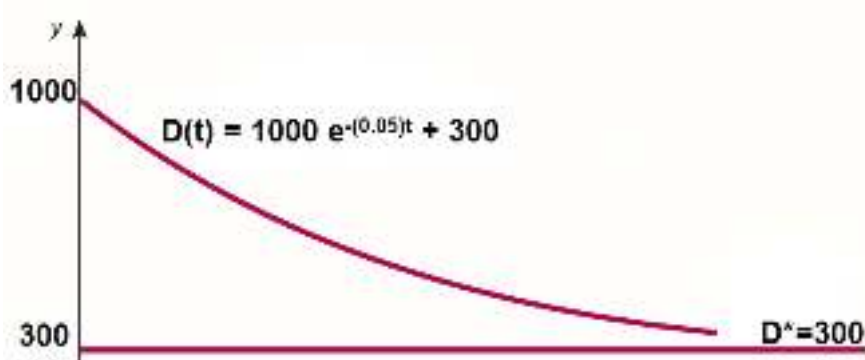
$$D(t) = (1300 - 300)e^{-0.05t} + 300$$

$$\underline{D(t) = (1000)e^{-0.05t} + 300}$$

It is the definite Time-Path of Value of Deposit.

$$D(t) = \underbrace{1000 e^{-0.05t}}_{D_c} + \underbrace{300}_{D_p}$$

Dynamic Stability: The expression; $D(t) = \boxed{1000 e^{-0.05t}} + 300$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$D_c = \boxed{1000 e^{-0.05t}}$	$D(t) = \boxed{1000 e^{-0.05t}} + 300$
0	1000	1300
1	951.2	1251.22
30	223.1	523.130
60	49.78	349.787
90	11.10	311.108
120	2.478	302.478
150	0.553	300.553
180	0.123	300.123
210	0.027	300.027
240	0.006	300.006
270	0.001	300.001
...	...	...
Remarks		
	Positive deviation decays over time	Time-path slides and converges to equilibrium ($D^* = 400$)
	As $t \rightarrow \infty$, $D_c \rightarrow 0$	And when $D_c \rightarrow 0$, $(D_t) \xrightarrow{\text{Converges}} (D^*)$
Graphically		
		
	The straight line shows the equilibrium value of Value of Deposit (D^*), while curve shows the deviation from equilibrium (D_c), and this deviation decreases over time and leads the time-path to convergence.	

TOPIC 098: DYNAMICS OF MONTHLY SALES USING DIFFERENTIAL EQUATIONS

A model predicts that monthly sales, $S(t)$, of a new product after t months satisfy the differential equation:

$$\frac{dS}{dt} = k(M - S) \text{ with initial condition } S(0) = C.$$

Where k , M and C are positive constants.

Solution: Given the rate of change of monthly sales; $\frac{dS}{dt} = k(M - S)$, we can rewrite it for the sake of comparison with first order differential equation.

$$\begin{aligned}\frac{dS}{dt} &= k(M - S) \\ \frac{dS}{dt} &= kM - kS \\ \frac{dS}{dt} + \underbrace{k}_{a} \underbrace{S}_{y} &= \underbrace{kM}_{b}\end{aligned}$$

Here $\frac{dy}{dt} = \frac{dS}{dt}$, $y = S$, $a = k$, $b = kM$, $y(0) = S(0) = C$

Choice of Case of the Differential Equation:

$b = kM$; As $k > 0$ and $M > 0$ (both positive constants), their product ($k \times M$) will be non-zero positive constant. i.e. $b \neq 0$ – **Non-Homogenous Case**.

$a = k$; $k > 0$ (non-zero positive constants). i.e. $a \neq 0$ – **Non-Homogenous Case with $a \neq 0$** .

Therefore, using the definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a and b , respectively:

$$\begin{aligned}\underline{S}(t) &= \left\{ \underline{S(0)} - \frac{\underline{kM}}{\underline{k}} \right\} e^{-\underline{k}t} + \frac{\underline{kM}}{\underline{k}} \\ S(t) &= \left\{ \underline{S(0)} - \frac{kM}{k} \right\} e^{-kt} + \frac{kM}{k} \\ S(t) &= (\underline{C} - M)e^{-kt} + M \\ \underline{S(t)} &= (C - M)e^{-kt} + M\end{aligned}$$

Definite Time-path for Monthly Sales in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, it should converge towards equilibrium. We know that equilibrium is particular solution (S_p). Whereas convergence happens when deviation from equilibrium fades to zero. In a differential equation, deviation component is measured by (S_c). Therefore, for dynamic stability; $S_c \rightarrow 0$ as $t \rightarrow \infty$.

Using the Definite Time-path for Income level in Dynamic Sense.

$$S(t) = \underbrace{(C - M)e^{-kt}}_{S_c} + \underbrace{M}_{S_p}$$

Need for time path of Monthly Sales $\{S(t)\}$ to converge to equilibrium income (S^*) i.e. $S(t) \rightarrow S^*$ as $S_c \rightarrow 0$ when $t \rightarrow \infty$.

$$\begin{aligned}& S_c \rightarrow 0 \\ & (C - M)e^{-kt} \rightarrow 0 \\ & \left[\underbrace{\left\{ C - \underbrace{M}_{\substack{\{M>0\} \\ \{S(0)=C\} \geq S_p}} \right\}}_{\substack{\text{Deviation} \\ (+ve \text{ or } -ve)}} \underbrace{e^{-\underbrace{k}_{\{k>0\}}t}}_{\substack{\text{Exponential} \\ \text{Decay} \\ \text{over time}}} \right] \rightarrow 0\end{aligned}$$

Therefore,

$$\begin{aligned}
 S(t) &= (C - M)e^{-kt} + M \\
 S(t) &= \underbrace{(C - \overset{\rightarrow 0}{M})}_{\rightarrow 0} e^{-kt} + \overset{\rightarrow 0}{M} \\
 S(t) &= \underbrace{(C - \overset{\rightarrow 0}{S_p})}_{\rightarrow 0} e^{-kt} + \overset{\rightarrow 0}{S_p} \\
 S(t) &= \underbrace{(C - \overset{\rightarrow 0}{S^*})}_{\rightarrow 0} e^{-kt} + \overset{\rightarrow 0}{S^*} \\
 S(t) &\rightarrow S^*
 \end{aligned}$$

As particular integral is also the equilibrium value of the variable $S_p = S^*$.

Interpretation: It can be said that provided $C > 0, k > 0$, regardless of initial national income $S(0)$ is greater or smaller than equilibrium monthly sales (S^*), the time-path of monthly sales shall **converge towards equilibrium** level of monthly sales as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.

TOPIC 099: DYNAMICS OF POPULATION USING DIFFERENTIAL EQUATIONS

The population, N , of a small country is currently 5 million. It is expected that this population will grow according to the differential equation.

$$\frac{dN}{dt} = 0.015N$$

where t is measured in years.

Solution: Current population of the country is 5 million. It implies; $N(0) = 5,000,000$.

Given the rate of change of population; $\frac{dN}{dt} = 0.015N$, rewrite to compare with standard form:

$$\begin{aligned}
 \frac{dN}{dt} &= 0.015N \\
 \frac{dN}{dt} - 0.015N &= 0 \\
 \frac{dN}{dt} + (-0.015)N &= 0 \\
 \frac{dN}{dt} + \underbrace{(-0.015)}_a \underbrace{N}_y &= \underbrace{0}_b
 \end{aligned}$$

Here $\frac{dy}{dt} = \frac{dN}{dt}$, $y = N$, $a = -0.015$, $b = 0$, $y(0) = N(0) = 5000000$

Choice of Case of the Differential Equation:

$b = 0$; – Homogenous Case.

$a = -0.015$; i.e. $a \neq 0$ – Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of **HC**:

$$y(t) = \{y(0)\}e^{-at}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \{\underline{y(0)}\}e^{-\underline{a}t}$$

Substituting values of y and a , respectively:

$$\underline{N}(t) = \{\underline{N(0)}\}e^{-\underline{a}t} = \{\underline{N(0)}\}e^{-\underline{(-0.015)}t}$$

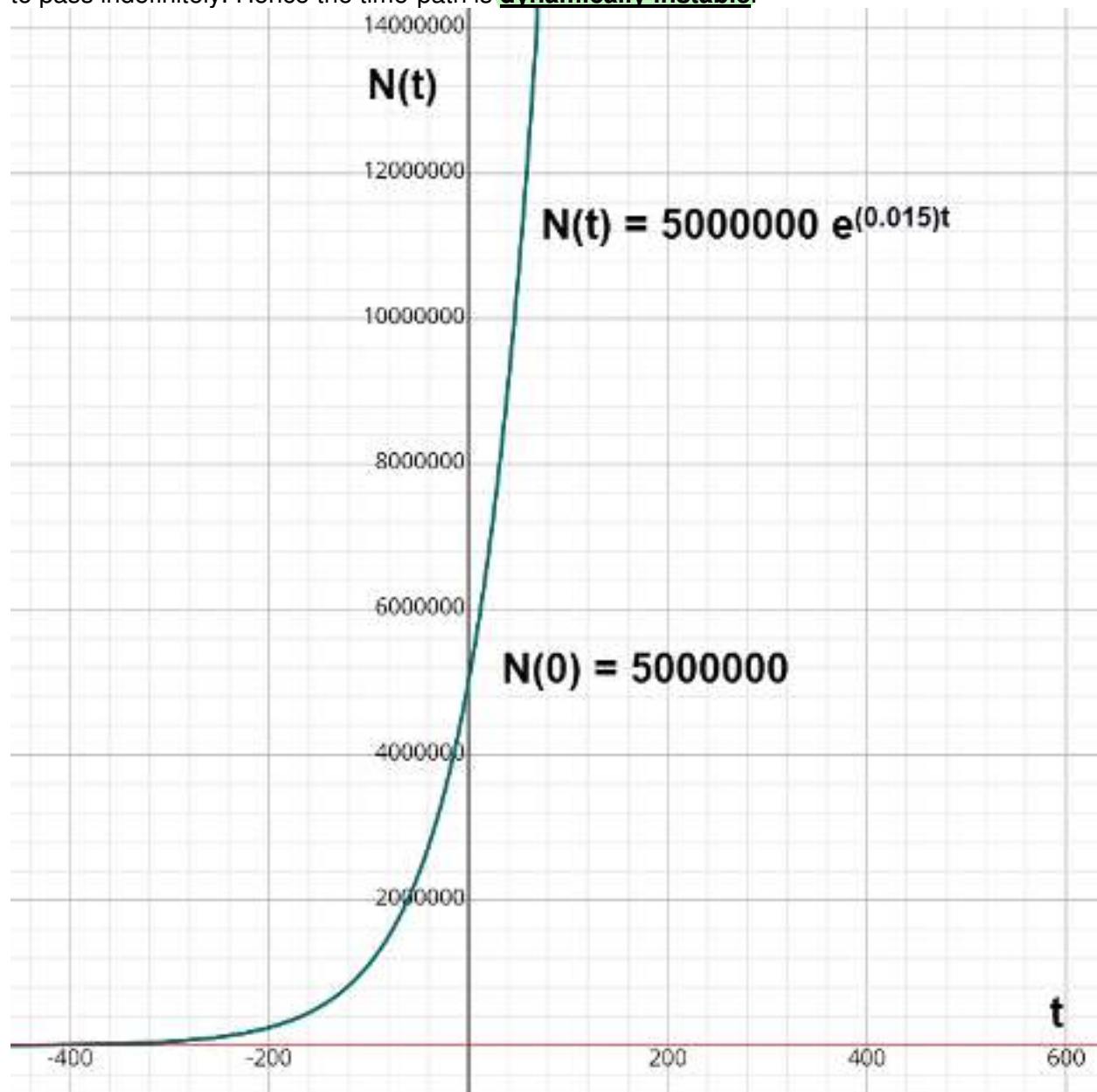
$$\underline{N(t)} = (5000000)e^{0.015t}$$

Definite Time-path for net revenue in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, the deviation expression should fade. In homogenous case (**HC**), the particular solution is zero; $N_p = 0$. Therefore, the convergence should take place towards zero (0). In this case, we have a positive exponent $N(t) = (5000000)e^{0.015t}$ which shall approach to infinity (∞), when $t \rightarrow \infty$. In a differential equation, deviation component is measured by (N_c). Here, there is dynamic instability; $N_c \rightarrow \infty$ as $t \rightarrow \infty$.

$$\left[\underbrace{\{5000000\}}_{\text{Deviation}} \underbrace{e^{\overbrace{(0.015)}^{(r>0)} t}}_{\substack{\text{Exponential} \\ \text{Growth} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore, the time-path of population shall **diverge away from equilibrium** as time is allowed to pass indefinitely. Hence the time-path is **dynamically instable**.



Policy Problem: From policy perspective, its important to know about the time the population will take to get double. i.e. $\{N(T) = 10,000,000\}_{T=?}$

Using the solution $N(t) = (5000000)e^{0.015t}$ for substitution, we get:

$$\begin{aligned} N(T) &= (5000000)e^{0.015(T)} \\ 10000000 &= (5000000)e^{0.015(T)} \\ \frac{10000000}{5000000} &= e^{0.015(T)} \\ e^{0.015(T)} &= 2 \end{aligned}$$

Raising both sides to exponent:

$$\begin{aligned} \ln(e^{0.015(T)}) &= \ln(2) \\ 0.015(T) &= 0.6931471805599 \\ T &= \frac{0.6931471805599}{0.015} \\ T &= 46.2 \end{aligned}$$

Interpretation: It will take **46.2 years** approximately for the population of the country to get double i.e. from 5,000,000 to 10,000,000.

Lesson 24

VARIOUS ECONOMIC DYNAMICS USING DIFFERENTIAL EQUATIONS (CONTINUED)

TOPIC 100: DYNAMICS OF GOVERNMENT SPENDING USING DIFFERENTIAL EQUATIONS

Government expenditure, G , satisfies the differential equation:

$$\frac{dG}{dt} = -0.05G$$

Where, G is measured in billions of dollars and time t is in years. The initial condition is $G(0) = 500$.

(a) Solve this equation to express G in terms of t .

(b) Find the value of G when $t = 6$. Round your answers to three significant figures.

Solution: Given the rate of change of Government Expenditure; $\frac{dG}{dt} = -0.05G$, rewrite to compare with standard form:

$$\begin{aligned}\frac{dG}{dt} &= -0.05G \\ \frac{dG}{dt} + 0.05G &= 0 \\ \frac{dG}{dt} + 0.05G &= 0 \\ \frac{dG}{dt} + \underbrace{(0.05)}_a \underbrace{G}_y &= \underbrace{0}_b\end{aligned}$$

$$\text{Here } \frac{dy}{dt} = \frac{dG}{dt}, y = G, a = 0.05, b = 0, y(0) = G(0) = 500$$

Choice of Case of the Differential Equation:

$b = 0$; – Homogenous Case.

$a = 0.05$; i.e. $a \neq 0$ – Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of HC :

$$y(t) = \{y(0)\}e^{-at}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \{\underline{y}(0)\}e^{-\underline{a}t}$$

Substituting values of y and a , respectively:

$$\underline{G}(t) = \{\underline{G}(0)\}e^{-\underline{(0.05)}t}$$

$$G(t) = (500)e^{-0.05t}$$

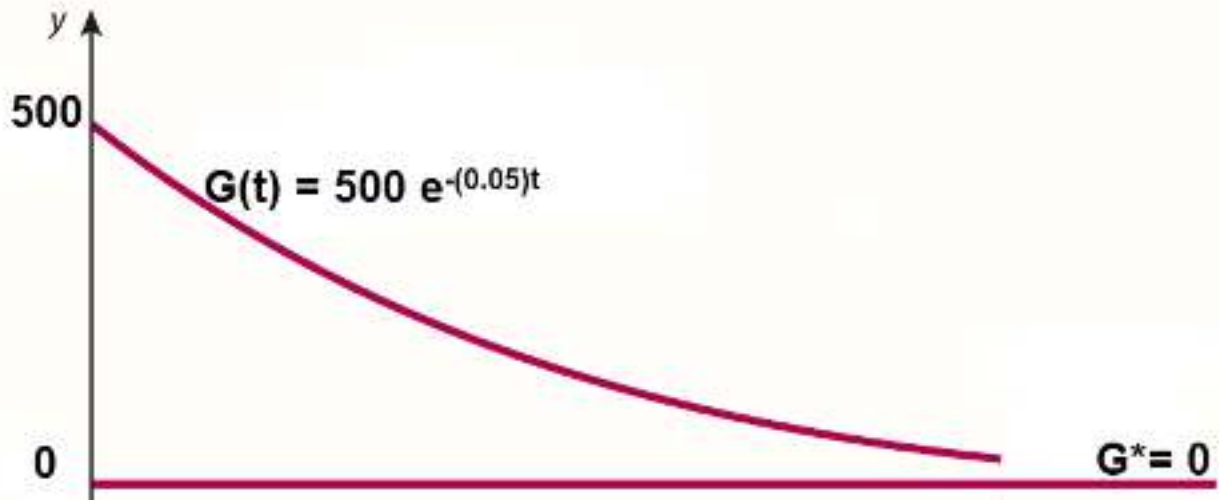
$$\boxed{G(t) = (500)e^{-0.05t}}$$

Definite Time-path for Government Expenditure in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, the deviation expression should fade. In homogenous case (HC), the particular solution is zero; $G_p = 0$. Therefore, the convergence should take place towards zero (0). In this case, we have a negative exponent $G(t) = (500)e^{-0.05t}$ which shall approach to zero (0), when $t \rightarrow \infty$. In a differential equation, deviation component is measured by (G_c). Therefore, for dynamic stability; $G_c \rightarrow 0$ as $t \rightarrow \infty$.

$$\left[\underbrace{\{500\}}_{\text{Deviation}} \underbrace{e^{-(0.05)t}}_{\substack{\text{Exponential} \\ \text{Decay} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore, the time-path of Government Expenditure shall **converge towards equilibrium** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.



Policy Problem: From policy perspective, it's important to know about the level of government expenditure after some time.

For example; after 6 years: $\{G(6) = ?\}$

Using the solution $G(t) = (500)e^{-0.05t}$ for substitution, we get:

$$G(6) = (500)e^{-0.05[6]}$$

$$G(6) = (500)e^{-0.3}$$

$$G(6) = \$ 370.409 \text{ billion}$$

TOPIC 101: DYNAMICS OF FUEL CONSUMPTION USING DIFFERENTIAL EQUATIONS

Let $F(x)$ denote the number of litre(s) of fuel left in an aircraft's fuel tank if it has flown x km. Suppose that $F(x)$ satisfies the following differential equation: $F'(x) = -aF(x) - b$. Here, the fuel consumption per km is a constant $b > 0$. The term $-aF(x)$, with $a > 0$, is due to the weight of the fuel.

Numerically speaking, the differential equation is: $F'(x) = -0.001F(x) - 8$. The initial level of fuel in the plane is $F(0) = 12000$:

Find the solution of the equation with $F(0) = F_0$.

Solution: Rate of change of remaining fuel; $F'(x) = -0.001F(x) - 8$.

Rewriting equation to compare with the standard form:

$$F'(x) = -0.001F(x) - 8$$

$$F'(x) + 0.001F(x) = -8$$

$$\frac{dF}{dx} + \underbrace{(0.001)}_a \underbrace{F}_y = \underbrace{-8}_b$$

$$\text{Here } \frac{dy}{dt} = \frac{dF}{dx}, y = F, a = 0.001, b = -8, y(0) = F(0) = 12000.$$

Choice of Case of the Differential Equation:

As $b = -8$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.001$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{F}(t) = \left\{ \underline{F}(0) - \frac{\underline{(-8)}}{\underline{0.001}} \right\} e^{-\underline{0.001}t} + \frac{\underline{(-8)}}{\underline{0.001}}$$

$$F(t) = \left\{ \underline{12000} - \frac{\underline{(-8)}}{\underline{0.001}} \right\} e^{-(\underline{0.001})t} + \frac{\underline{(-8)}}{\underline{0.001}}$$

$$F(t) = \left\{ \underline{12000} + \frac{\underline{8}}{\underline{0.001}} \right\} e^{-(\underline{0.001})t} - \frac{\underline{8}}{\underline{0.001}}$$

$$F(t) = \{ \underline{12000} + \underline{8000} \} e^{-(\underline{0.001})t} - \underline{8000}$$

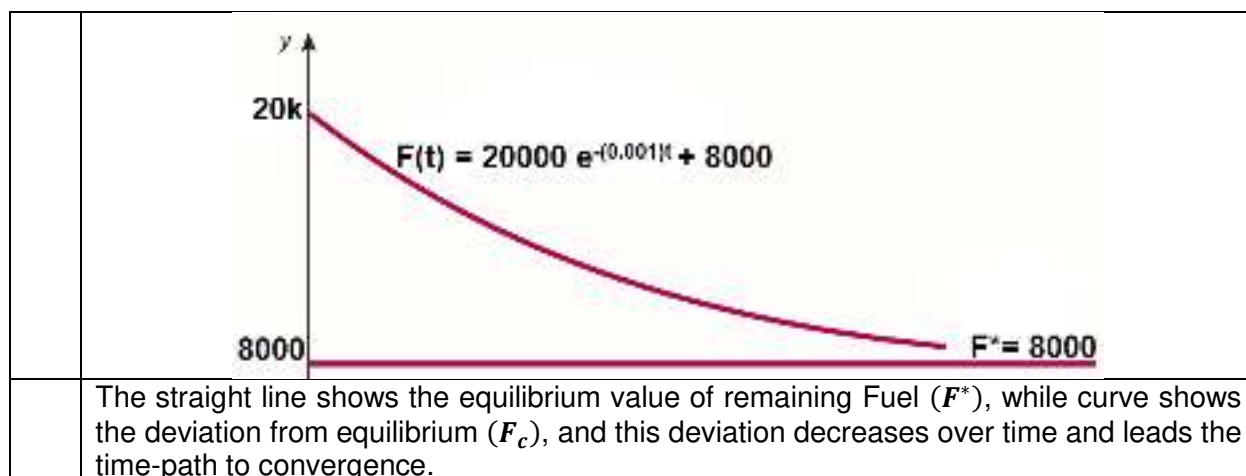
$$\underline{F(t)} = (\underline{20000})e^{-(\underline{0.001})t} - \underline{8000}$$

It is the definite Time-Path of remaining Fuel.

$$F(t) = \underbrace{20000 e^{-0.001t}}_{F_c} - \underbrace{8000}_{F_p}$$

Dynamic Stability: The expression; $F(t) = \underbrace{20000 e^{-0.001t}}_{F_c} + 8000$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$F_c = 20000 e^{-0.001t}$	$F(t) = 20000 e^{-0.001t} + 8000$
0	20000	28000
1	19980.0	27980.0
2	19960.0	27960.0
3	19940.0	27940.0
4	19920.1	27920.1
5	19900.2	27900.2
6	19880.3	27880.3
7	19860.4	27860.4
8	19840.6	27840.6
9	19820.8	27820.8
10	19800.9	27800.9
...	...	...
Remarks		
	Positive deviation decays over time	Time-path slides and converges to equilibrium ($F^* = 8000$)
	As $t \rightarrow \infty$, $F_c \rightarrow 0$	And when $F_c \rightarrow 0$, $(F_t) \rightarrow \overrightarrow{Converges}(F^*)$
	Graphically	



TOPIC 102: DYNAMICS OF UNEMPLOYMENT USING DIFFERENTIAL EQUATIONS

If unemployment data is measured in thousands, unemployment in a small economy is measured by following differential equation:

$$\frac{dU}{dt} = -3U + 180$$

Here U is the unemployment level (in thousands) and t is time in years. If unemployment is **150** (in thousands) at the beginning of analysis, find the time-path of unemployment in the economy.

Solution: Given the rate of unemployment in the economy;

$$\frac{dU}{dt} = -3U + 180$$

we can rewrite compare it with the standard form of differential equation.

$$\frac{dU}{dt} + 3U = 180$$

$$\frac{dU}{dt} + \underbrace{3}_{a} \underbrace{U}_{y} = \underbrace{180}_{b}$$

Here $\frac{dy}{dt} = \frac{dU}{dt}$, $y = U$, $a = 3$, $b = 180$, $y(0) = U(0) = 150$.

Choice of Case of the Differential Equation:

As $b = 180$, so $b \neq 0$: Non-Homogenous Case.

As $a = 3$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y(0)} - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{U}(t) = \left\{ \underline{U(0)} - \frac{\underline{180}}{\underline{3}} \right\} e^{-\underline{3}t} + \frac{\underline{180}}{\underline{3}}$$

$$U(t) = \left\{ 150 - \frac{180}{3} \right\} e^{-3t} + \frac{180}{3}$$

$$U(t) = (150 - 60)e^{-3t} + 60$$

$$U(t) = (90)e^{-3t} + 60$$

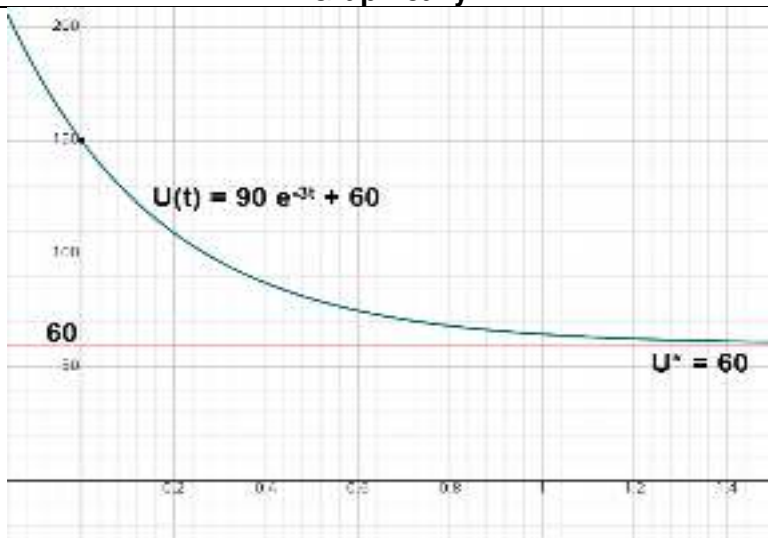
$$\boxed{U(t) = (90)e^{-3t} + 60}$$

It is the definite Time-Path of Unemployment.

$$U(t) = \underbrace{90 e^{-3t}}_{U_c} + \underbrace{60}_{U_p}$$

Dynamic Stability: The expression; $U(t) = \boxed{90 e^{-3t}} + 60$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$U_c = \boxed{90 e^{-3t}}$	$U(t) = \boxed{90 e^{-3t}} + 60$
0	90	150
1	4.480836153108	64.480836153108
2	0.223087695900	60.223087695900
3	0.011106882368	60.011106882368
4	0.000552979112	60.000552979112
5	0.000027531209	60.000027531209
6	0.000001370698	60.000001370698
7	0.000000068243	60.000000068243
8	0.000000003398	60.000000003398
9	0.000000000169	60.000000000169
10	0.000000000008	60.000000000008
...	...	...

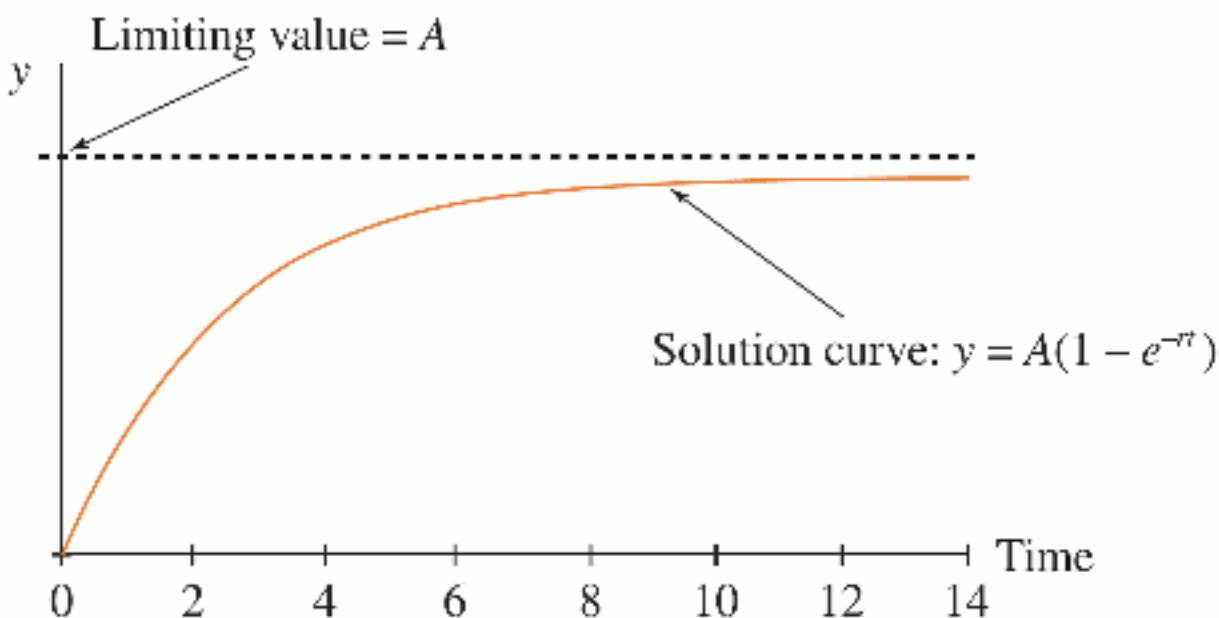
	Remarks
Positive deviation decays over time	Time-path slides and converges to equilibrium ($U^* = 60$)
As $t \rightarrow \infty$, $U_c \rightarrow 0$	And when $U_c \rightarrow 0$, $(U_t) \text{ Converges } (U^*)$
Graphically	
	

The straight line shows the equilibrium value of Unemployment (U^*), while curve shows the deviation from equilibrium (U_c), and this deviation decreases over time and leads the time-path to convergence.

TOPIC 103: LIMITED GROWTH USING DIFFERENTIAL EQUATIONS

Situations where growth of a variable is fast initially but levels-off (stabilize) just short of a maximum limit. Examples include mobile adoption, wireless internet etc.

Using differential equations of the type $\frac{dy}{dt} = r(A - y)$ represent a similar situation where A is the maximum limit. Graphically it is:



The rate at which the volume of sales (Q) for a new type of printer increases after an advertising campaign is given by the equation $\frac{dQ}{dt} = 0.04(700 - Q)$, given that $Q = 0$ at $t = 0$. Q is the number of printers sold, t is time in years.

(a) Solve the differential equation to obtain an expression for Q in terms of t .

(b) Sketch the solution curve, hence describe verbally how the volume of sales should increase in time.

Solution: Given the rate of change of Volume of Sales; $\frac{dQ}{dt} = 0.04(700 - Q)$, we can rewrite it to compare with standard form of differential equation.

$$\begin{aligned}\frac{dQ}{dt} &= 0.04(700 - Q) \\ \frac{dQ}{dt} &= 28 - 0.04Q \\ \frac{dQ}{dt} + \underbrace{0.04}_{a} \underbrace{Q}_{y} &= \underbrace{28}_{b}\end{aligned}$$

Here $\frac{dy}{dt} = \frac{dQ}{dt}$, $y = Q$, $a = 0.04$, $b = 28$, $y(0) = Q(0) = 0$.

Choice of Case of the Differential Equation:

As $b = 28$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.04$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.
Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{Q}(t) = \left\{ \underline{Q}(0) - \frac{\underline{28}}{\underline{0.04}} \right\} e^{-\underline{0.04}t} + \frac{\underline{28}}{\underline{0.04}}$$

$$Q(t) = \left\{ 0 - \frac{28}{0.04} \right\} e^{-0.04t} + \frac{28}{0.04}$$

$$Q(t) = (0 - 700)e^{-0.04t} + 700$$

$$Q(t) = (-700)e^{-0.04t} + 700$$

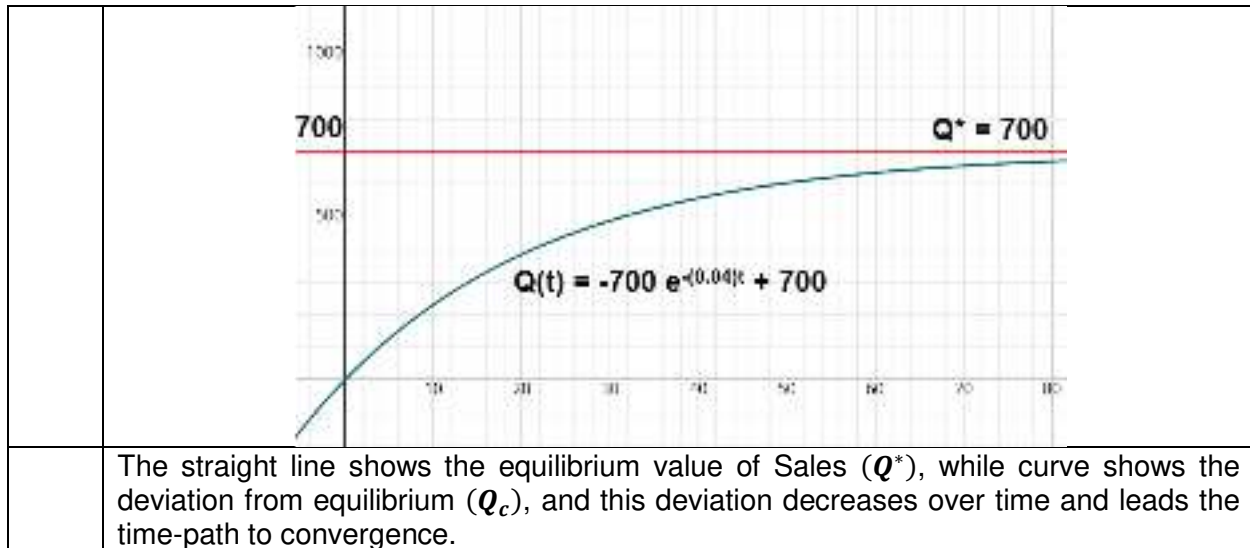
It is the definite Time-Path of Quantity Supplied.

$$Q(t) = \underbrace{-700 e^{-0.04t}}_{Q_c} + \underbrace{700}_{Q_p}$$

Dynamic Stability: The expression; $Q(t) = \underbrace{-700 e^{-0.04t}}_{Q_c} + 700$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$Q_c = -700 e^{-0.04t}$	$Q(t) = -700 e^{-0.04t} + 700$
0	-700	0
1	-672.552	27.447392
50	-94.7346	605.26530
100	-12.8209	687.17905
150	-1.73512	698.26487
200	-0.23482	699.76517
250	-0.03177	699.96822
300	-0.00430	699.99569
350	-0.00058	699.99941
400	-0.00007	699.99992
450	-0.00001	699.99998
...	...	...

	Remarks
Negative deviation decays over time	Time-path rises and converges to equilibrium ($Q^* = 700$)
As $t \rightarrow \infty$, $Q_c \rightarrow 0$	And when $Q_c \rightarrow 0$, $(Q_t) \xrightarrow{\text{Converges}} (Q^*)$
	Graphically



TOPIC 104: DYNAMICS OF ORE MINING USING DIFFERENTIAL EQUATIONS

The rate at which an ore is mined is given by the differential equation $\frac{dM}{dt} = 0.02M$. Solve the differential equation, if the total reserves of ore in 1989 were 400 million tons.

Solution: Given the rate of change of Ore Mining ($\frac{dM}{dt} = 0.02M$), we can rearrange it to compare with the standard form of differential equations.

$$\begin{aligned}\frac{dM}{dt} &= 0.02M \\ \frac{dM}{dt} - 0.02M &= 0 \\ \frac{dM}{dt} - \underbrace{0.02}_a \underbrace{M}_y &= \underbrace{0}_b\end{aligned}$$

$$\text{Here } \frac{dy}{dt} = \frac{dM}{dt}, y = M, a = -0.02, b = 0, y(0) = M(0) = 400$$

Choice of Case of the Differential Equation:

$b = 0$; – Homogenous Case.

$a = -0.02$; i.e. $a \neq 0$ – Homogenous Case with $a \neq 0$.

Therefore, using the definite Solution of time-path of HC:

$$y(t) = \{y(0)\}e^{-at}$$

Emphasizing on the values to be substituted:

$$y(t) = \{y(0)\}e^{-at}$$

Substituting values of y and a , respectively:

$$M(t) = \{M(0)\}e^{-(0.02)t}$$

$$M(t) = (400)e^{0.02t}$$

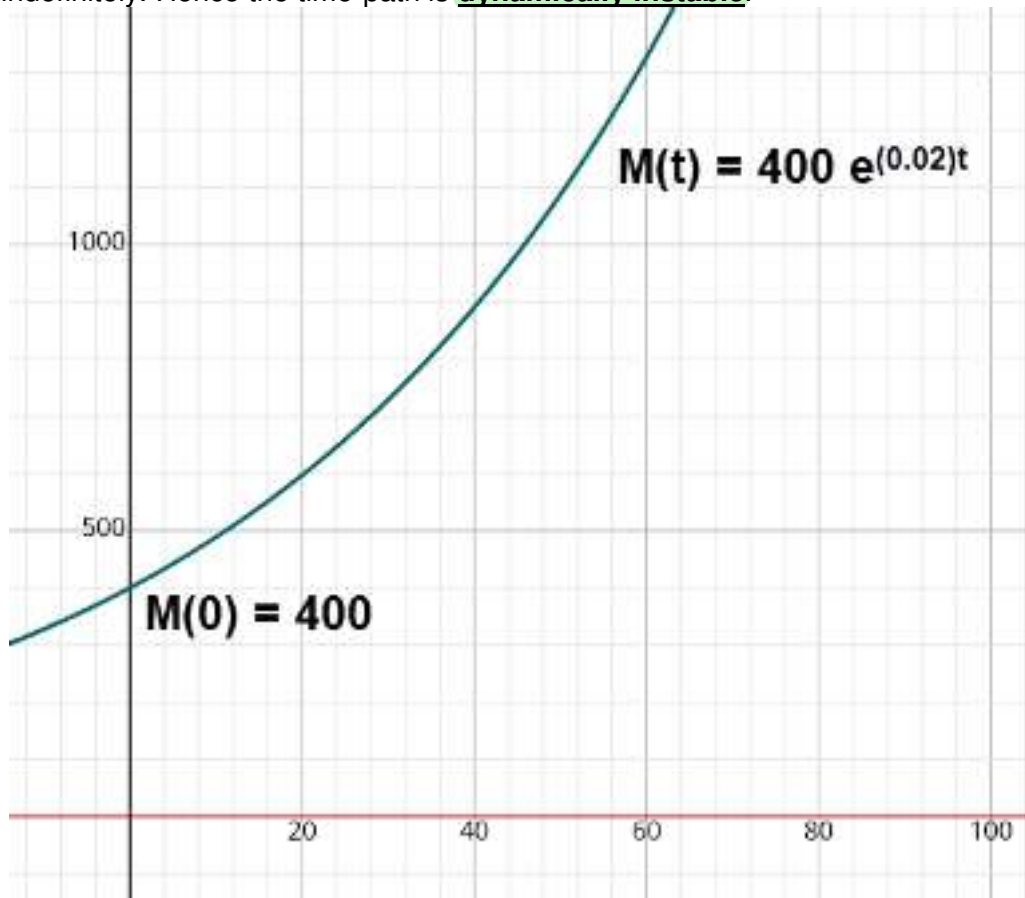
Definite Time-path for Ore Mining in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, the deviation expression should fade. In homogenous case (HC), the particular solution is zero; $M_p = 0$. Therefore, the convergence should take place towards zero (0). In this case, we have a positive exponent $M(t) = (400)e^{0.02t}$ which shall approach to infinity (∞), when $t \rightarrow \infty$. In a

differential equation, deviation component is measured by (M_c). Here, there is dynamic instability; $M_c \rightarrow \infty$ as $t \rightarrow \infty$.

$$\left[\underbrace{\{400\}}_{\text{Deviation}} \underbrace{e^{(0.02)t}}_{\substack{\text{Exponential} \\ \text{Growth} \\ \text{over time}}} \right] \rightarrow 0$$

Therefore, the time-path of Ore Mining shall **diverge away from equilibrium** as time is allowed to pass indefinitely. Hence the time-path is **dynamically instable**.



TOPIC 105: DYNAMICS OF POLLUTANT USING DIFFERENTIAL EQUATIONS

The amount of sulphur dioxide (in tons) present in the atmosphere is given by the equation $\frac{d(P)}{dt} = -0.2P$ (t is in hours). If 50 ton(s) is released into the atmosphere (i.e., $P = 50$ at $t = 0$), derive an expression for the amount of sulphur dioxide present at any time t .

Solution: Given the rate of change of Sulphur Dioxide (SO_2): $\frac{d(P)}{dt} = -0.2P$, we can rewrite it to compare it with the standard form of differential equation.

$$\frac{d(P)}{dt} = -0.2P$$

$$\frac{dP}{dt} + 0.2P = 0$$

$$\frac{dP}{dt} + \underbrace{0.2}_a \underbrace{P}_y = \underbrace{0}_b$$

Here $\frac{dy}{dt} = \frac{dP}{dt}$, $y = P$, $a = 0.2$, $b = 0$, $y(0) = P(0) = 50$

Choice of Case of the Differential Equation:

$b = 0$; – **Homogenous Case.**

$a = 0.2$; i.e. $a \neq 0$ – **Homogenous Case with $a \neq 0$.**

Therefore, using the definite Solution of time-path of **HC**:

$$y(t) = \{y(0)\}e^{-at}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \{\underline{y}(0)\}e^{-\underline{a}t}$$

Substituting values of y and a , respectively:

$$\underline{P}(t) = \{\underline{P}(0)\}e^{-\underline{(0.2)}t}$$

$$P(t) = (50)e^{-0.2t}$$

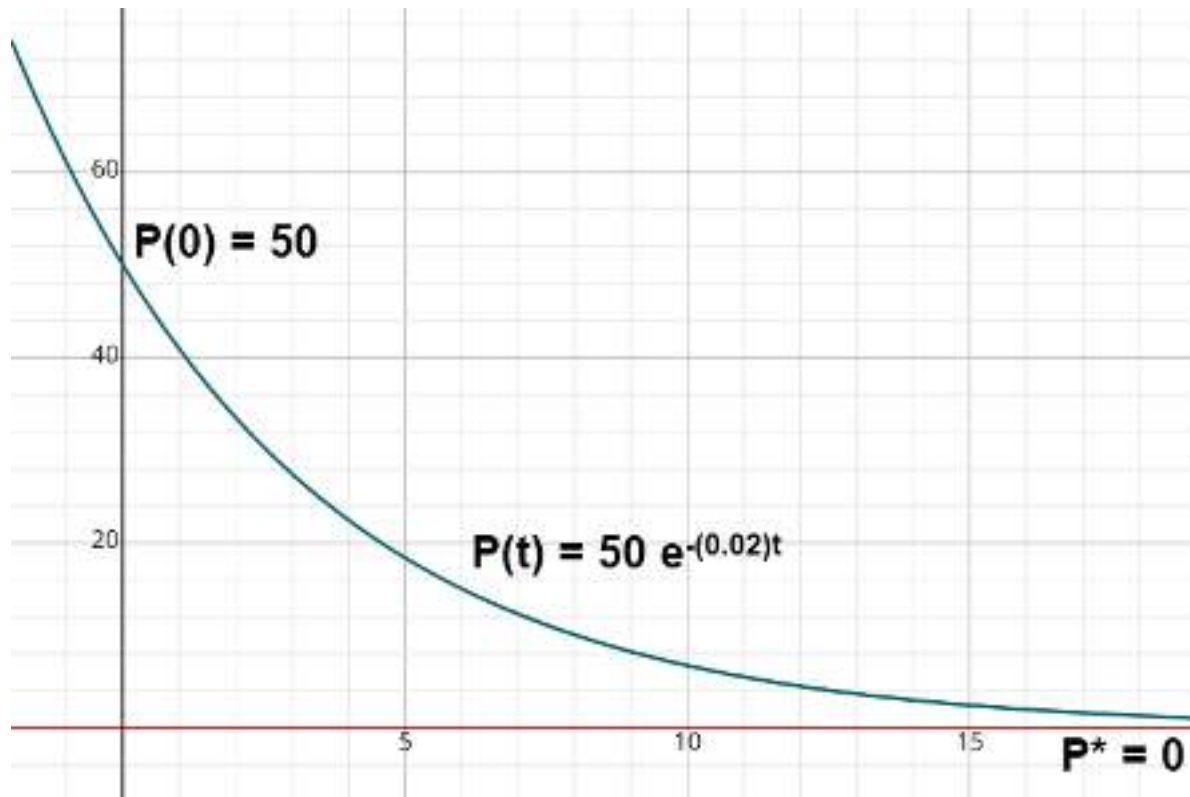
$$\boxed{P(t) = (50)e^{-0.2t}}$$

Definite Time-path for Government Expenditure in Dynamic Sense.

Dynamic Stability: For the dynamic stability of the time-path of First Order Differential Equation, the deviation expression should fade. In homogenous case (**HC**), the particular solution is zero; $P_p = 0$. Therefore, the convergence should take place towards zero (**0**). In this case, we have a negative exponent $P(t) = (50)e^{-0.2t}$ which shall approach to zero (**0**), when $t \rightarrow \infty$. In a differential equation, deviation component is measured by (P_c). Therefore, for dynamic stability; $P_c \rightarrow 0$ as $t \rightarrow \infty$.

$$\left[\begin{array}{cc} \{50\} & e^{-\underline{(0.2)}t} \\ \text{Deviation} & \text{Exponential} \\ & \text{Decay} \\ & \text{over time} \end{array} \right] \rightarrow 0$$

Therefore, the time-path of Pollutant shall **converge towards equilibrium** as time is allowed to pass indefinitely. Hence the time-path is **dynamically stable**.



TOPIC 106: DYNAMICS OF FISH STOCK USING DIFFERENTIAL EQUATIONS

The rate at which the fish stocks increase in an inland lake is given by the equation $\frac{dF}{dt} = 0.12(80 - F)$, where F is in thousands, t is in years. (a) Derive an expression for the total number of fish present at any time t given $F = 0$ at $t = 0$.

(b) Graph the total fish stock as a function of time from $t = 0$ to $t = 30$. Does the lake appear to have a maximum carrying capacity?

(c) Calculate the number of years taken for the fish stocks to reach 40000.

Solution: Given the rate of change of fish stock; $\frac{dF}{dt} = 0.12(80 - F)$, we can rearrange it to compare with the standard form of differential equation.

$$\begin{aligned}\frac{dF}{dt} &= 0.12(80 - F) \\ \frac{dF}{dt} &= 9.6 - 0.12F \\ \frac{dF}{dt} + \underbrace{0.12}_a F &= \underbrace{9.6}_b\end{aligned}$$

Here $\frac{dy}{dt} = \frac{dF}{dt}$, $y = F$, $a = 0.12$, $b = 9.6$, $y(0) = F(0) = 0$.

Choice of Case of the Differential Equation:

As $b = 9.6$, so $b \neq 0$: Non-Homogenous Case.

As $a = 0.12$, so $a \neq 0$: Non-Homogenous Case with $a \neq 0$.

Therefore, using the Definite Solution of time-path of $NHC_{a \neq 0}$:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

Emphasizing on the values to be substituted:

$$\underline{y}(t) = \left\{ \underline{y}(0) - \frac{\underline{b}}{\underline{a}} \right\} e^{-\underline{a}t} + \frac{\underline{b}}{\underline{a}}$$

Substituting values of y , a , b and $y(0)$ respectively:

$$\underline{F}(t) = \left\{ \underline{F}(0) - \frac{\underline{9.6}}{\underline{0.12}} \right\} e^{-\underline{0.12}t} + \frac{\underline{9.6}}{\underline{0.12}}$$

$$F(t) = \left\{ 0 - \frac{9.6}{0.12} \right\} e^{-0.12t} + \frac{9.6}{0.12}$$

$$F(t) = (0 - 80)e^{-0.12t} + 80$$

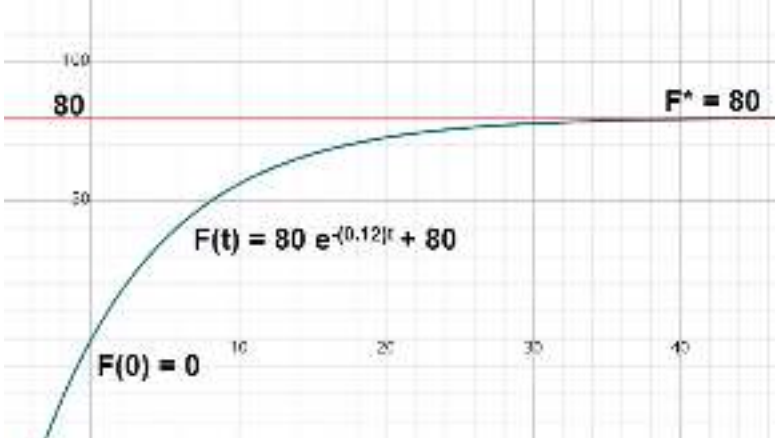
$$\boxed{F(t) = (-80)e^{-0.12t} + 80}$$

It is the definite Time-Path of Fish Stock.

$$F(t) = \underbrace{-80 e^{-0.12t}}_{F_c} + \underbrace{80}_{F_p}$$

Dynamic Stability: The expression; $F(t) = \boxed{-80 e^{-0.12t}} + 80$ has an exponential decay term in the box. i.e.:

	Complementary Function (Deviation)	Time-Path
t	$F_c = \boxed{-80 e^{-0.12t}}$	$F(t) = \boxed{-80 e^{-0.12t}} + 80$
0	-80	0
1	-70.95	9.0464
10	-24.10	55.904
20	-7.257	72.743
30	-2.186	77.814
40	-0.658	79.342
50	-0.198	79.802
60	-0.060	79.940
70	-0.018	79.982
80	-0.005	79.995
90	-0.002	79.998
...	...	...

		Remarks
	Negative deviation decays over time	Time-path rises and converges to equilibrium ($F^* = 80$)
	As $t \rightarrow \infty$, $F_c \rightarrow 0$	And when $F_c \rightarrow 0$, $(F_t) \xrightarrow{\text{Converges}} (F^*)$
		Graphically
		
	The straight line shows the equilibrium value of Fish Stock (F^*), while curve shows the deviation from equilibrium (F_c), and this deviation decreases over time and leads the time-path to convergence.	

Lesson 25

EXACT DIFFERENTIAL EQUATIONS

TOPIC 107: FIRST-ORDER DIFFERENTIAL EQUATIONS: VARIABLE TERM & VARIABLE COEFFICIENT: HOMOGENEOUS CASE

Standard Form of First-Order Differential Equation

$$\frac{dy}{dt} + u(t) \cdot y = w(t)$$

Here, $u(t)$ and $w(t)$ are functions of time and not constants i.e. t^n when $n \neq 0$.

Depending on the values of $u(t)$ and $w(t)$, we get 2 possible cases:

Case – I: $u(t) \neq 0, w(t) = 0$ [Homogenous Case]

Case – II: $u(t) \neq 0, w(t) \neq 0$ [Non-Homogenous Case]

Case – I: Homogenous Case: $u(t) \neq 0, w(t) = 0$

Standard form of First-Order Differential Equation $\frac{dy}{dt} + u(t)y = w(t)$, becomes:

$$\frac{dy}{dt} + \underbrace{u(t)}_{\substack{\text{variable} \\ \text{coefficient}}} \underbrace{y}_{\substack{\text{dependent} \\ \text{variable}}} = 0$$

$$\frac{dy}{dt} = -u(t)y$$

$$\frac{1}{y} \frac{dy}{dt} = -u(t)$$

$$\frac{\left(\frac{dy}{dt}\right)}{y} = -u(t)$$

Integrating w.r.t. t

$$\int \frac{\left(\frac{dy}{dt}\right)}{y} dt = - \int u(t) dt$$

$$\int \frac{(dy)}{y} = - \int u(t) dt$$

$$\int \left(\frac{1}{y}\right) dy = - \int u(t) dt$$

$$\ln|y| + c = - \int u(t) dt$$

$$\ln|y| = - \int u(t) dt - c$$

$$\ln|y| = - \int u(t) dt - c$$

To get rid of natural logarithm, raise exponent on both sides:

$$e^{\ln|y|} = e^{\{- \int u(t) dt - c\}}$$

$$y = e^{- \int u(t) dt} e^{-c}$$

$$y(t) = e^{- \int u(t) dt} A \quad [\because e^{-c} = A] \text{ yet another constant}$$

$$\boxed{y(t) = Ae^{- \int u(t) dt}} \quad \text{(General Solution)}$$

Finding definite solution can be easier once $\int u(t) dt$ has a specific form of $u(t)$.

Numerical Example: Find the general solution of the equation $\frac{dy}{dt} + 3t^2y = 0$

Solution: Comparing the general form of first order differential equation (variable coefficient) with given specific form:

$$\frac{dy}{dt} + \underline{u(t)} \times y = \underline{w(t)}$$

$$\frac{dy}{dt} + \underline{3t^2} \times y = 0$$

Here, $u(t) = 3t^2$, $w(t) = 0$.

Using the formula for general solution:

$$y(t) = Ae^{-\int u(t) dt}$$

Solving the exponent $\int u(t) dt$, separately before substituting, will avoid clutter:

$$\begin{aligned} &= \int \underline{u(t)} dt \\ &= \int \underline{3t^2} dt \\ &= 3 \int t^2 dt \\ &= 3 \left(\frac{t^3}{3} \right) + c \\ &\underline{\int u(t) dt = t^3 + c} \end{aligned}$$

Substituting the value of exponent back into formula of General Solution:

$$\begin{aligned} y(t) &= Ae^{-\int u(t) dt} \\ y(t) &= Ae^{-(t^3+c)} \\ y(t) &= A \times e^{-t^3-c} \\ y(t) &= A \times \underline{e^{-t^3} e^{-c}} \\ y(t) &= A \times \underline{e^{-c} e^{-t^3}} \\ y(t) &= \underline{A \times e^{-c}} e^{-t^3} \\ y(t) &= \underline{A \times A'} \times e^{-t^3} \end{aligned}$$

$[\because A \times A' = B]$

$$\underline{y(t) = Be^{-t^3}}$$

Interpretation: $y(t) = Be^{-t^3}$ is the general solution of first order differential equation with variable coefficient $\frac{dy}{dt} + 3t^2y = 0$.

TOPIC 108: FIRST-ORDER DIFFERENTIAL EQUATIONS: VARIABLE TERM & VARIABLE COEFFICIENT: NON-HOMOGENEOUS CASE

Standard Form of First-Order Differential Equation

$$\frac{dy}{dt} + u(t) \cdot y = w(t)$$

Here, $u(t)$ and $w(t)$ are functions of time and not constants i.e. t^n when $n \neq 0$.

Depending on the values of $u(t)$ and $w(t)$, we get 2 possible cases:

Case – I: $u(t) \neq 0, w(t) = 0$ [Homogenous Case]

Case – II: $u(t) \neq 0, w(t) \neq 0$ [Non-Homogenous Case]

Case – II: Non-Homogenous Case: $u(t) \neq 0, w(t) \neq 0$

$$\frac{dy}{dt} + u(t)y = w(t)$$

General Solution of this case can be derived using Exact Differential Equations which is yet to be studied in this course. Therefore, we borrow the General Solution:

$$y(t) = e^{-\int u(t) dt} \left\{ A + w(t) \int e^{\int u(t) dt} dt \right\}$$

For simplicity, we may use u and w instead of $u(t)$ and $w(t)$, respectively.

$$y(t) = e^{-\int u dt} \left\{ A + w \int e^{\int u dt} dt \right\}$$

$$y(t) = e^{-\int u dt} \left\{ A + w \int e^{\int u dt} dt \right\}$$

Definite Solution can be found by substituting initial condition.

Numerical Example (p.# 485): Find the general solution of the equation $\frac{dy}{dt} + 2ty = t$.

Solution: Comparing the general form of first order differential equation (variable coefficient & variable constant) with given specific form:

$$\frac{dy}{dt} + \underline{u(t)} \times y = \underline{w(t)}$$

$$\frac{dy}{dt} + \underline{2t} \times y = \underline{t}$$

Here, $u(t) = 2t, w(t) = t$.

Using the formula for general solution:

$$y(t) = e^{-\int u(t) dt} \left\{ A + \underline{w(t)} \int e^{\int u(t) dt} dt \right\}$$

Solving the exponent $\int u(t) dt$, separately before substituting, will avoid clutter:

$$= \int \underline{u(t)} dt$$

$$= \int \underline{2t} dt$$

$$= 2 \int \underline{t} dt$$

$$= 2 \left(\frac{t^2}{2} \right) + c$$

$$\underline{\int u(t) dt = t^2 + c_1}$$

Substituting the value of exponent back into formula of General Solution:

$$y(t) = e^{-\int u(t) dt} \left\{ A + \underline{w(t)} \int e^{\int u(t) dt} dt \right\}$$

$$y(t) = e^{-(t^2+c_1)} \left\{ A + \underline{t} \int e^{(t^2+c_1)} dt \right\}$$

$$y(t) = \underline{e^{-t^2-c_1}} \left\{ A + \underline{t} \int \underline{e^{t^2+c_1}} dt \right\}$$

$$y(t) = \underline{e^{-t^2} e^{-c_1}} \left\{ A + \underline{t} \int \underline{e^{t^2} e^{c_1}} dt \right\}$$

$$y(t) = \underline{e^{-t^2} e^{-c_1}} \left\{ A + \underline{t} \int \underline{e^{t^2} e^{c_1}} dt \right\}$$

$$y(t) = \underline{e^{-t^2} e^{-c_1}} \left\{ A + \underline{e^{c_1}} \times t \times \int \underline{e^{t^2} dt} \right\}$$

Recalling Natural Exponential Function Rule: $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)}$

Here $f(x) = t^2$, so $f'(x) = 2t$. Therefore, $\int e^{f(x)} dx = \frac{e^{t^2}}{2t}$

$$\begin{aligned}
 y(t) &= e^{-t^2} e^{-c_1} \left\{ A + e^{c_1} \times t \times \left(\frac{e^{t^2}}{2t} \right) + c_2 \right\} \\
 y(t) &= \frac{e^{-t^2} e^{-c_1}}{1} \left\{ A + e^{c_1} \times \left(\frac{e^{t^2}}{2} \right) + c_2 \right\} \\
 y(t) &= A e^{-t^2} e^{-c_1} + \frac{e^{-t^2} e^{-c_1}}{1} \times e^{c_1} \times \left(\frac{e^{t^2}}{2} \right) + \frac{e^{-t^2} e^{-c_1}}{1} \times c_2 \\
 y(t) &= A e^{-t^2} e^{-c_1} + \frac{e^{-c_1} e^{c_1}}{1} \times \frac{e^{-t^2}}{1} \times \frac{e^{t^2}}{2} + e^{-t^2} e^{-c_1} \times c_2 \\
 y(t) &= A e^{-t^2} e^{-c_1} + 1 \times 1 \times \left(\frac{1}{2} \right) + e^{-t^2} \times e^{-c_1} c_2 \\
 y(t) &= A e^{-t^2} e^{-c_1} + \left(\frac{1}{2} \right) + e^{-t^2} \times e^{-c_1} c_2 \\
 y(t) &= A e^{-c_1} e^{-t^2} + \frac{e^{-c_1} c_2}{1} \times e^{-t^2} + \frac{1}{2} \\
 y(t) &= A e^{-c_1} e^{-t^2} + \frac{e^{-c_1} c_2}{1} \times e^{-t^2} + \frac{1}{2} \\
 y(t) &= (A e^{-c_1} + c_3) \times e^{-t^2} + \frac{1}{2} \\
 y(t) &= (A e^{-c_1} + c_3) \times e^{-t^2} + \frac{1}{2} \\
 y(t) &= (B) \times e^{-t^2} + \frac{1}{2} \\
 y(t) &= B e^{-t^2} + \frac{1}{2}
 \end{aligned}$$

Interpretation: $y(t) = B e^{-t^2} + \frac{1}{2}$ is the General Solution of the first order differential equation $\frac{dy}{dt} + 2ty = t$.

TOPIC 109: NON-LINEAR DIFFERENTIAL EQUATION: EXACT DIFFERENTIAL EQUATIONS

Nonlinear differential equation can have power of dependent variable [here, (y)] greater/other than (1). Nonlinear differential equations can be solved using either of three (3) methods:

- I- **Exact Differential Equation Method**
- II- **Separable Variables Method**
- III- **Bernoulli Equation Method**

Caveat: Even though our immediate purpose is to use it to solve & linear differential equation, an exact differential equation can be either linear or nonlinear by itself.

I – Exact Differential Equations

Given a function of two variables $F(y, t) = k$, its total differential is:

$$\overline{d\{F(y, t)\}} = \underline{d(k)}$$

$$\overline{\left(\frac{\delta F}{\delta y}\right) dy + \left(\frac{\delta F}{\delta t}\right) dt} = \underline{0}$$

Since both variables (y & t) are on one side of equation, the differential of other side (some constant) will be equal to zero:

$$\underbrace{\left(\frac{\delta F}{\delta y}\right) dy + \left(\frac{\delta F}{\delta t}\right) dt}_{\text{total differential of the function } \{F(y,t)\}} = \underbrace{0}_{\text{differential of a constant (k)}}$$

Interpretation: The function $\{F(y, t) = k\}$ represents an Exact Differential Equation (EDE) as its total differential satisfies the equation, $\left\{\left(\frac{\delta F}{\delta y}\right) dy + \left(\frac{\delta F}{\delta t}\right) dt = 0\right\}$:

Development of Exactness Test: A test can be developed to check if a differential equation is an Exact Differential Equation or not.

Borrowing the result of total differential of $F(y, t) = k$:

$$\begin{aligned} d\{F(y, t)\} &= 0 \\ \left(\frac{\delta F}{\delta y}\right) dy + \left(\frac{\delta F}{\delta t}\right) dt &= 0 \\ \boxed{\left(\frac{\delta F}{\delta y}\right) dy} + \boxed{\left(\frac{\delta F}{\delta t}\right) dt} &= 0 \end{aligned}$$

As $\left(\frac{\delta F}{\delta y}\right)$ and $\left(\frac{\delta F}{\delta t}\right)$ are given in every specific form of EDE, we develop condition around these derivative terms.

Resorting to Young's Theorem for a function; $F(x, y)$:

$$\begin{aligned} F_{xy} &= F_{yx} \\ \frac{\delta}{\delta x} \left(\frac{\delta F}{\delta y}\right) &= \frac{\delta}{\delta y} \left(\frac{\delta F}{\delta x}\right) \end{aligned}$$

However, in current case, Young's Theorem is for function; $F(y, t)$:

$$\begin{aligned} F_{ty} &= F_{yt} \\ \frac{\delta}{\delta t} \left(\frac{\delta F}{\delta y}\right) &= \frac{\delta}{\delta y} \left(\frac{\delta F}{\delta t}\right) \\ \frac{\delta}{\delta t} \boxed{\left(\frac{\delta F}{\delta y}\right)} &= \frac{\delta}{\delta y} \boxed{\left(\frac{\delta F}{\delta t}\right)} \end{aligned}$$

Substituting $\left(\frac{\delta F}{\delta y}\right) = M$ and $\left(\frac{\delta F}{\delta t}\right) = N$ in two of the following equations:

Young's Theorem	Exact Differential Equation
$\frac{\delta}{\delta t} \boxed{\left(\frac{\delta F}{\delta y}\right)} = \frac{\delta}{\delta y} \boxed{\left(\frac{\delta F}{\delta t}\right)}$ $\frac{\delta}{\delta t} \boxed{M} = \frac{\delta}{\delta y} \boxed{N}$ $\frac{\delta}{\delta t} (M) = \frac{\delta}{\delta y} (N)$	$\boxed{\left(\frac{\delta F}{\delta y}\right) dy} + \boxed{\left(\frac{\delta F}{\delta t}\right) dt} = 0$ $\boxed{(M) dy} + \boxed{(N) dt} = 0$ $\boxed{(M) dy + (N) dt} = 0$
Interpretation:	
Following Young's theorem, the Exactness Test is developed for Exact Differential Equation.	Exact Differential Equation is molded for Exactness Test.

TOPIC 110: EXACT DIFFERENTIAL EQUATIONS: NUMERICAL

Given a differential equation: $2yt \, dy + y^2 \, dt = 0$.

Comparing with the standard form of exact differential equation:

$$\underline{M} \, dy + \underline{N} \, dt = 0$$

$$2yt \, dy + y^2 \, dt = 0$$

Here, $M = 2yt$ and $N = y^2$, we can employ Exactness Test: $\frac{d}{dt}(M) = \frac{d}{dy}(N)$:

$$\frac{d}{dt}(M) = \frac{d}{dy}(N)$$

$$\frac{d}{dt}(2yt) = \frac{d}{dy}(y^2)$$

$$2y \frac{d}{dt}(t) = \frac{d}{dy}(y^2)$$

$$2y(1) = 2y$$

$$2y = 2y$$

Interpretation: Since Exactness Test is met on the differential equation $2yt \, dy + y^2 \, dt = 0$, it is Exact Differential Equation.

Solution (4-Step Procedure): Exact Differential Equation requires a 4-step procedure for its solution:

Step – I: Writing the Preliminary function $F(y, t)$ in terms of derivative and integral w.r.t y , with a merger of possible constant of integration and variable t :

$$F(y, t) = \int \overbrace{\frac{\delta\{F(y, \bar{t})\}}{\delta y} dy}^{\text{integral}} + \overbrace{\tilde{c}}^{\text{integration constant}} + \underbrace{f(t)}_{\substack{\text{term(s)} \\ \text{containing "t"}}}$$

Combining c and $f(t)$ into one term $\psi(t)$ and substituting:

$$F(y, t) = \int \overbrace{\frac{\delta\{F(y, \bar{t})\}}{\delta y} dy}^{\text{integral}} + \underbrace{\psi(t)}_{\substack{\text{term(s)} \\ \text{containing "t"}}} \quad \text{integration constant}$$

$\psi(t)$ can have two components:

1) Additive constant(s) or terms containing t only reduce to zero (0) when function $F(y, \bar{t})$ gets differentiated w.r.t. y .

2) Constant of integration generated due to $\int \frac{\delta\{F(y, \bar{t})\}}{\delta y} dy$.

Recalling that $\frac{\delta\{F(y, \bar{t})\}}{\delta y} = M$, therefore:

$$F(y, t) = \int M \, dy + \psi(t)$$

Substituting value of M :

$$F(y, t) = \int 2yt \, dy + \psi(t)$$

$$F(y, t) = 2t \int y \, dy + \psi(t)$$

$$F(y, t) = 2t \left(\frac{y^2}{2} \right) + c + \psi(t)$$

To avoid excessive terms, merge c in $\psi(t)$, as $\psi(t)$ already contains an integration constant from **Step – I**:

$$F(y, t) = y^2 t + \psi(t)$$

$$\boxed{F(y, t) = y^2 t + \psi(t)} \text{ A Preliminary Result}$$

Step – II: Differentiate $F(y, t)$ w.r.t t :

$$\begin{aligned} \frac{\delta\{F(\bar{y}, t)\}}{\delta t} &= \frac{\delta}{\delta t} \{y^2 t + \psi(t)\} \\ \frac{\delta\{F(\bar{y}, t)\}}{\delta t} &= y^2 \left\{ \frac{\delta}{\delta t} (t) \right\} + \frac{\delta}{\delta t} \{\psi(t)\} \\ \frac{\delta\{F(\bar{y}, t)\}}{\delta t} &= y^2 \{1\} + \psi'(t) \\ \frac{\delta F}{\delta t} &= y^2 + \psi'(t) \end{aligned}$$

Since value of $\psi'(t)$ is not known, so we try to substitute $\left(\frac{\delta F}{\delta t}\right)$ to find it.

We know that $\boxed{\frac{\delta F}{\delta t}} = N$, therefore, we put it in $\boxed{\frac{\delta F}{\delta t}} = y^2 + \psi'(t)$.

$$\boxed{\frac{\delta F}{\delta t}} = y^2 + \psi'(t)$$

$$\boxed{N} = y^2 + \psi'(t)$$

Further $\boxed{N} = \boxed{y^2}$, so putting this value in the equation above:

$$\boxed{y^2} = y^2 + \psi'(t)$$

$$\cancel{y^2} = \cancel{y^2} + \psi'(t)$$

$$0 = \psi'(t)$$

$$\boxed{\psi'(t) = 0}$$

Step – III: We have found the value of derivative $\psi'(t)$ and not the value of original function $\psi(t)$. Therefore, taking integral to restore the original function $\psi(t)$ from $\psi'(t)$:

$$\psi'(t) = 0$$

zero (0) can be derivative
of a constant (k)

Integrating on both sides w.r.t t :

$$\int \psi'(t) dt = \underbrace{k}_{\substack{\text{a constant is primitive of} \\ \text{a derivative, } \psi'(t)=0}}$$

$$\int \frac{d}{dt} \{\psi(t)\} dt = k$$

$$\boxed{\psi(t) = k}$$

Step – IV: Substitute the result from **Step – III** in **Step – I**.

Result from **Step – I:** $F(y, t) = y^2 t + \psi(t)$

Result from **Step – III:** $\psi(t) = k$

$$\overline{F(y, t)} = y^2 t + \underline{k}$$

Recalling definition of an Exact Differential Equation:

$d\{F(y, t)\} = 0$, implies $F(y, t) = c_F$ (a constant)

0 is differential of a constant (c_F).

$$\overline{c_F} = y^2 t + \underline{k}$$

$$c_F = y^2 t + k$$

$$\frac{c_F - k}{y^2 t} = \frac{y^2 t}{y^2 t}$$

Since difference of two constants is yet another constant: $c_F - k = c'$

$$\frac{y^2 t}{y^2 t} = \frac{c'}{y^2 t}$$

This implicit function is synonymous to $F(y, t) = k$, and can be considered as the solution.

However, its conventional to find the time path, i.e. $y(t)$, therefore, we convert it into an explicit function:

$$\begin{aligned} y^2 &= \frac{c'}{t} \\ y(t) &= \sqrt{\frac{c'}{t}} \\ y(t) &= \frac{\sqrt{c'}}{\sqrt{t}} \\ y(t) &= \frac{c}{t^{\frac{1}{2}}} \quad [\because \sqrt{c'} = c] \\ y(t) &= c t^{-\frac{1}{2}} \end{aligned}$$

Interpretation: General Solution of Exact Differential Equation $2yt \, dy + y^2 \, dt = 0$ is

$$y(t) = c t^{-\frac{1}{2}}.$$

TOPIC 111: EXACT DIFFERENTIAL EQUATIONS: INTEGRATING FACTOR

What if Exactness Test fails and we have an **Inexact** Differential Equation:

$$\frac{\delta}{\delta t}(M) \neq \frac{\delta}{\delta y}(N)$$

Since Exactness Test depends on given values of M and N :

$$\frac{\delta}{\delta t}(M) \geq \frac{\delta}{\delta y}(N)$$

As per definitional requirement, **L.H.S** of should be **exact total differential** of the given function $F(y, t) = k$. However, in Inexact Differential Equation, this condition is violated:

$$\underbrace{(M)dy + (N)dt}_{\text{Not equal to total differential of the function } \{F(y, t)\}} = \underbrace{0}_{\text{differential of a constant } (k)}$$

We can try to change the values of M and N to get the equality in Exactness Test. This can be done simply by multiplying a term (F_{Int}) throughout the equation.

$$(F_{Int}) \times \{(M)dy + (N)dt\} = (F_{Int}) \times 0$$

This will have a dual effect:

1. Change values of M and N .
2. Equality, in differential equation, remains; i.e. **L.H.S = R.H.S**

$$\underbrace{\{(F_{Int}) \times (M)\}}_{\text{Value of } M \text{ changes by a factor } (F_{Int})} dy + \underbrace{\{(F_{Int}) \times (N)\}}_{\text{Value of } N \text{ changes by a factor } (F_{Int})} dt \stackrel{\text{equality remains}}{=} 0$$

This term is called Integrating Factor (F_{int}) as it's a 'multiplicative **factor**' that converts **L.H.S** of the differential equation $F(y, t)$ into Exact Differential $d\{F(y, t)\} = \left(\frac{\delta F}{\delta y}\right) dy + \left(\frac{\delta F}{\delta t}\right) dt$. After which, the differential equation becomes **integrable** – hence, **Integrating Factor**. Integrating Factor, if any, can be found by using two methods:

I – Integrating Factor by Trial & Error Method.

II – Integrating Factor by Formula.

I – Integrating Factor by Trial & Error Method:

Integrating Factor can be found by experimenting different values of Integrating Factor in the Inexact Differential Equation and selecting one that makes Exactness Test Valid on post-multiplication differential equation – **Trial and Error method**:

Numerical Example: Given the differential equation; $2t \, dy + y \, dt = 0$.

We employ Exactness Test:

$$\frac{\delta}{\delta t}(M) = \frac{\delta}{\delta y}(N)$$

Comparing the given differential equation: $2t \, dy + y \, dt = 0$

with standard form of differential equation: $(M)dy + (N)dt = 0$.

$$\underbrace{2t}_M dy + \underbrace{y}_N dt = 0$$

Here, $M = 2t$ and $N = y$, so substituting these values:

$$\frac{\delta}{\delta t}(2t) \geq \frac{\delta}{\delta y}(y)$$

$$2 > 1$$

Exactness test doesn't get satisfied, so we have an **Inexact Differential Equation**.

To convert it into Exact Differential Equation, we employ **Trial and Error method**: multiply throughout by 'y':

$$y(2t \, dy + y \, dt) = y \times 0$$

$$2ty \, dy + y^2 \, dt = 0$$

To check if it is converted into an Exact Differential Equation, employ Exactness Test again:

$$\underbrace{2ty}_M dy + \underbrace{y^2}_N dt = 0$$

After introducing the potential integrating factor, $M = 2ty$ and $N = y^2$, so substituting these values again:

$$\frac{\delta}{\delta t}(2ty) \geq \frac{\delta}{\delta y}(y^2)$$

$$2y = 2y$$

Interpretation: After introducing 'y' as a factor, Exactness Test is met. So, we can call it integrating factor ($F_{int} = y$) of the Inexact Differential Equation $2t \, dy + y \, dt = 0$. The Exact Differential Equation form of it is $2ty \, dy + y^2 \, dt = 0$.

Note: If 'y' doesn't work, then 't' can be experimented as a potential integrating factor. Higher powers of 'y' and 't' ($y^2, t^2, y^3, t^3, y^4, t^4, \dots$) can be used if both 'y' and 't' fail to serve as Integrating Factor.

II – Integrating Factor by Formula:

In addition to Trial and Error method, (F_{int}), if any, can also be found by using these rules of formula:

Rule 1. If $\frac{1}{N} \left(\frac{\delta M}{\delta t} - \frac{\delta N}{\delta y} \right) = f(y)$ alone, then $e^{\int f(y) dy}$ is the integrating factor (F_{int}).

Rule 2. If $\frac{1}{M} \left(\frac{\delta N}{\delta y} - \frac{\delta M}{\delta t} \right) = g(t)$ alone, then $e^{\int g(t) dt}$ is the integrating factor (F_{int}).

Numerical Example: Given $5yt \, dy + (5y^2 + 8t) \, dt = 0$, We employ Exactness Test:

$$\frac{\delta}{\delta t}(M) = \frac{\delta}{\delta y}(N)$$

Comparing the given differential equation: $5yt \, dy + (5y^2 + 8t) \, dt = 0$

with standard form of differential equation: $(M)dy + (N)dt = 0$.

$$\underbrace{5yt}_{M} dy + \underbrace{(5y^2 + 8t)}_{N} dt = 0$$

Here, $M = 5yt$ and $N = (5y^2 + 8t)$, so substituting these values:

$$\frac{\delta}{\delta t}(5yt) \geq \frac{\delta}{\delta y}(5y^2 + 8t)$$

$$\underline{5y \neq 10y}$$

Exactness test doesn't get satisfied, so we have an **Inexact Differential Equation**.

Applying Rule 1:

If $\frac{1}{N} \left(\frac{\delta M}{\delta t} - \frac{\delta N}{\delta y} \right) = f(y)$ alone, then $e^{\int f(y) dy}$ is the integrating factor (F_{int}):

$$\begin{aligned} &= \frac{1}{(5y^2 + 8t)} \times \left\{ \frac{\delta(5yt)}{\delta t} - \frac{\delta(5y^2 + 8t)}{\delta y} \right\} \\ &= \frac{1}{(5y^2 + 8t)} \times \{5y - 10y\} \\ &= \frac{1}{(5y^2 + 8t)} \times (-5y) \\ &= \boxed{\frac{-5y}{5y^2 + 8t} \neq f(y)} \text{ alone.} \end{aligned}$$

Since **Rule 1** could not be satisfied, we resort to **Rule 2**:

If $\frac{1}{M} \left(\frac{\delta N}{\delta y} - \frac{\delta M}{\delta t} \right) = g(t)$ alone, then $e^{\int g(t) dt}$ is the integrating factor (F_{int}).

$$\begin{aligned} &= \frac{1}{M} \left(\frac{\delta N}{\delta y} - \frac{\delta M}{\delta t} \right) \\ &= \frac{1}{5yt} \left\{ \frac{\delta(5y^2 + 8t)}{\delta y} - \frac{\delta(5yt)}{\delta t} \right\} \\ &= \frac{1}{5yt} \{10y - 5y\} \\ &= \frac{1}{5yt} (5y) \\ &= \boxed{\frac{1}{t} = g(t)} \text{ alone.} \end{aligned}$$

Since **Rule 2** is met, the integrating factor (F_{int}) is $e^{\int g(t) dt}$.

$$F_{int} = e^{\int \frac{1}{t} dt}$$

$$F_{int} = e^{\ln|t| + c}$$

$$F_{int} = e^{(\ln|t|)}$$

Constant of integration is dropped c for the sake of convenience:

$$F_{int} = e^{\ln|t|}$$

$$F_{int} = e^{\ln|t|}$$

$$F_{int} = t$$

Interpretation: Integrating factor (F_{int}) for the Inexact Differential Equation $5yt \, dy + (5y^2 + 8t) \, dt = 0$ is ' t '. Multiplying it throughout the given equation, it becomes exact differential equation: $5yt^2 \, dy + (5y^2t + 8t^2) \, dt = 0$. Test of Exact can be applied to verify the answer.

Caveat: Once the Inexact Differential Equation is converted into Exact Differential Equation, it can be solved using the 4-step procedure already explained before.

Lesson 26

ECONOMIC APPLICATIONS OF EXACT DIFFERENTIAL EQUATIONS

TOPIC 112: PRODUCTION FUNCTION USING EXACT DIFFERENTIAL EQUATIONS

A firm uses L number of workers and K number of machines as two inputs in its production process. The marginal products of the two inputs are given as:

$$MP_K = 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \text{ and } MP_L = 4 \left(\frac{K}{L} \right)^{\frac{1}{3}}$$

Express the total product function of this firm assuming it stays on the same isoquant. What kind of production function obtains?

Solution: A Production process is mathematized via a production function:

$$Q(K, L) = k$$

Its Total Differential:

$$\begin{aligned} d\{Q(K, L)\} &= 0 \\ \left(\frac{\partial Q}{\partial K} \right) dK + \left(\frac{\partial Q}{\partial L} \right) dL &= 0 \\ (MP_K) dK + (MP_L) dL &= 0 \\ \left\{ 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \right\} dK + \left\{ 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} \right\} dL &= 0 \end{aligned}$$

Comparing with the standard form:

$$\begin{aligned} (M) dy + (N) dt &= 0 \\ \left\{ 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \right\} \frac{dK}{dy} + \left\{ 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} \right\} \frac{dL}{dt} &= 0 \\ \underbrace{\left\{ 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \right\}}_M \underbrace{\frac{dK}{dy}}_{dy} + \underbrace{\left\{ 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} \right\}}_N \underbrace{\frac{dL}{dt}}_{dt} &= 0 \end{aligned}$$

It implies: $M = 2 \left(\frac{L}{K} \right)^{\frac{2}{3}}$, $N = 4 \left(\frac{K}{L} \right)^{\frac{1}{3}}$, $dy = dK$ and $dt = dL$.

Exactness Test: $\frac{\partial}{\partial L} (M) = \frac{\partial}{\partial K} (N)$

$$\begin{aligned} \frac{\partial}{\partial L} \left\{ 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \right\} &= \frac{\partial}{\partial K} \left\{ 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} \right\} \\ 2 \left(\frac{1}{K} \right)^{\frac{2}{3}} \frac{\partial}{\partial L} \left\{ (L)^{\frac{2}{3}} \right\} &= 4 \left(\frac{1}{L} \right)^{\frac{1}{3}} \frac{\partial}{\partial K} \left\{ (K)^{\frac{1}{3}} \right\} \\ 2 \left(\frac{1}{K} \right)^{\frac{2}{3}} \left\{ \frac{2}{3} \times (L)^{\frac{2}{3}-1} \times \frac{\partial}{\partial L} (L) \right\} &= 4 \left(\frac{1}{L} \right)^{\frac{1}{3}} \left\{ \frac{1}{3} \times (K)^{\frac{1}{3}-1} \times \frac{\partial}{\partial K} (K) \right\} \\ 2 \left(\frac{2}{3} \right) \left(\frac{1}{K} \right)^{\frac{2}{3}} \left\{ (L)^{-\frac{1}{3}} \times 1 \right\} &= 4 \left(\frac{1}{3} \right) \left(\frac{1}{L} \right)^{\frac{1}{3}} \left\{ (K)^{-\frac{2}{3}} \times 1 \right\} \\ \frac{4}{3} \left(\frac{1}{K^{\frac{2}{3}}} \right) \left(\frac{1}{L^{\frac{1}{3}}} \right) &= \frac{4}{3} \left(\frac{1}{L^{\frac{1}{3}}} \right) \left(\frac{1}{K^{\frac{2}{3}}} \right) \end{aligned}$$

$$\frac{4}{3} \left(\frac{1}{K^{\frac{2}{3}} L^{\frac{1}{3}}} \right) = \frac{4}{3} \left(\frac{1}{K^{\frac{2}{3}} L^{\frac{1}{3}}} \right)$$

Exactness test is met, so the given equation is an Exact Differential Equation.

Solution (4-Step Procedure): Exact Differential Equation requires a 4-step procedure for its solution:

Step – I: Writing the function $Q(K, L)$ in terms of derivative and integral w.r.t K , with a merger of possible constant of integration and variable L :

$$Q(K, L) = \overbrace{\int \frac{\delta\{Q(K, \bar{L})\}}{\delta K} dK}^{\text{integral}} + \overbrace{\tilde{c}}^{\text{integration constant}} + \underbrace{f(L)}_{\text{term(s) containing "L"}}$$

Combining c and $f(L)$ into one term $\psi(L)$ and substituting:

$$Q(K, L) = \overbrace{\int \frac{\delta\{Q(K, \bar{L})\}}{\delta K} dK}^{\text{integral}} + \underbrace{\psi(L)}_{\text{term(s) containing "L"}}$$

$\psi(L)$ can have two components:

- 1) Additive constant(s) or terms containing L only reduce to zero (0) when function

$Q(K, \bar{L})$ gets differentiated w.r.t. K .

- 2) Constant of integration generated due to $\int \frac{\delta\{Q(K, \bar{L})\}}{\delta K} dK$.

Recalling that $\frac{\delta\{Q(K, \bar{L})\}}{\delta K} = M$, therefore:

$$Q(K, L) = \int M dy + \psi(L)$$

Substituting value of M :

$$Q(K, L) = \int \left\{ 2 \left(\frac{L}{K} \right)^{\frac{2}{3}} \right\} dK + \psi(L)$$

$$Q(K, L) = 2(L)^{\frac{2}{3}} \int \left(\frac{1}{K} \right)^{\frac{2}{3}} dK + \psi(L)$$

$$Q(K, L) = 2(L)^{\frac{2}{3}} \int (K)^{-\frac{2}{3}} dK + \psi(L)$$

Solving the integral while merging the constant of integration into $\psi(L)$.

$$Q(K, L) = 2(L)^{\frac{2}{3}} \left\{ \frac{(K)^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} \right\} + \psi(L)$$

$$Q(K, L) = 2(L)^{\frac{2}{3}} \left\{ \frac{(K)^{\frac{1}{3}}}{\frac{1}{3}} \right\} + \psi(L)$$

$$Q(K, L) = [2](L)^{\frac{2}{3}} \times \left[\frac{3}{1} \right] \times (K)^{\frac{1}{3}} + \psi(L)$$

$$Q(K, L) = [6](L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + \psi(L)$$

$$Q(K, L) = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + \psi(L)$$

Step – II: Differentiate $Q(K, L)$ w.r.t L {to evaluate $\psi(L)$ }:

$$\frac{\delta\{Q(K, L)\}}{\delta L} = \frac{\delta}{\delta L} \left\{ 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + \psi(L) \right\}$$

$$\begin{aligned} \frac{\delta\{Q(K, L)\}}{\delta L} &= (K)^{\frac{1}{3}} \times \frac{\delta}{\delta L} \left\{ 6(L)^{\frac{2}{3}} \right\} + \frac{\delta}{\delta L} \{ \psi(L) \} \\ \frac{\delta\{Q\}}{\delta L} &= 6 \times (K)^{\frac{1}{3}} \left\{ \frac{2}{3} \times (L)^{\frac{2}{3}-1} \times \frac{\delta}{\delta L} (L) \right\} + \psi'(L) \\ \frac{\delta Q}{\delta L} &= 6 \times \frac{2}{3} \times (K)^{\frac{1}{3}} \left\{ (L)^{-\frac{1}{3}} \times \frac{\delta}{\delta L} (L) \right\} + \psi'(L) \end{aligned}$$

Since we already know; $\frac{\delta Q}{\delta L} = N$, therefore **L.H.S:**

$$N = 4 \times (K)^{\frac{1}{3}} \left\{ \frac{1}{L^{\frac{1}{3}}} \times 1 \right\} + \psi'(L)$$

Moreover, $N = 4 \left(\frac{K}{L} \right)^{\frac{1}{3}}$, so we substitute:

$$\begin{aligned} 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} &= 4 \frac{(K)^{\frac{1}{3}}}{L^{\frac{1}{3}}} + \psi'(L) \\ 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} &= 4 \left(\frac{K}{L} \right)^{\frac{1}{3}} + \psi'(L) \\ 0 &= \psi'(L) \\ \psi'(L) &= 0 \end{aligned}$$

Step – III: We have found the value of derivative $\psi'(L)$ and not the value of original function $\psi(L)$. Therefore, taking integral to restore the original function $\psi(L)$ from $\psi'(L)$:

$$\psi'(L) = 0$$

zero (0) can be derivative
of a constant (k)

Integrating on both sides w.r.t L :

$$\begin{aligned} \int \psi'(L) dL &= \int 0 dL \\ &= \text{a constant is primitive of a derivative, } \psi'(L)=0 \\ \int \frac{d}{dL} \{ \psi(L) \} dL &= k \\ \psi(L) &= k \end{aligned}$$

Step – IV: Substitute the result from **Step – III** in **Step – I**.

Result from **Step – I:** $Q(K, L) = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + \psi(L)$

Result from **Step – III:** $\psi(L) = k$

$$\overline{Q(K, L)} = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + k$$

Recalling definition of an Exact Differential Equation:

$d\{Q(K, L)\} = 0$, implies $Q(K, L) = c_Q$ (a constant)

0 is differential of a constant (c_Q).

$$\overline{c_Q} = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + k$$

$$c_Q = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} + k$$

$$c_Q - k = 6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}}$$

$$6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} = c_Q - k$$

Since difference of two constants is yet another constant: $c_Q - k = c$

$$6(L)^{\frac{2}{3}}(K)^{\frac{1}{3}} = c$$

$$6(K)^{\frac{1}{3}}(L)^{\frac{2}{3}} = c$$

Developing Functional expression on R.H.S:

$$Q(K, L) = 6(K)^{\frac{1}{3}}(L)^{\frac{2}{3}} - c$$

This expression is synonymous to Cobb-Douglas production function, with the exception of addition term 'c'.

In absence of either or both of inputs (K, L), the output will be zero:

$Q(0, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}} - c$	$Q(K, 0) = 6(K)^{\frac{1}{3}}(0)^{\frac{2}{3}} - c$	$Q(0, 0) = 6(0)^{\frac{1}{3}}(0)^{\frac{2}{3}} - c$
$Q(0, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}} - c$ $Q(0, L) = -c \neq 0$	$Q(K, 0) = 6(K)^{\frac{1}{3}}(0)^{\frac{2}{3}} - c$ $Q(K, 0) = -c \neq 0$	$Q(0, 0) = 6(0)^{\frac{1}{3}}(0)^{\frac{2}{3}} - c$ $Q(0, 0) = -c \neq 0$
Which is a paradox. Q should be equal to zero if either or both K or L are zero.		
Therefore, c should be equated to 0, for making 'economic' sense of Production Function.		

$$Q(K, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}} - 0$$

$$Q(K, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}} - 0$$

$$Q(K, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}}$$

Interpretation: General Solution of Exact Differential Equation $2\left(\frac{L}{K}\right)^{\frac{2}{3}}dK + 4\left(\frac{K}{L}\right)^{\frac{1}{3}}dL = 0$ is

$Q(K, L) = 6(0)^{\frac{1}{3}}(L)^{\frac{2}{3}}$. It makes economic sense of a Cobb-Douglas Production Function.

TOPIC 113: UTILITY FUNCTION USING EXACT DIFFERENTIAL EQUATIONS

A simple utility function of a consumer buying two goods x and y is denoted by the differential equation $4xy dx + x^2 dy = 0$. Express this consumer's total utility function using integrating factors.

Solution: Given Utility function, in general form is:

$$U(x, y) = k$$

$$(4xy)dx + (x^2)dy = 0$$

Comparing with the standard form:

$$(M)dy + (N)dt = 0$$

$$\underbrace{(4xy)}_M \underbrace{dx}_{dy} + \underbrace{(x^2)}_N \underbrace{dy}_{dt} = 0$$

It implies: $M = 4xy$, $N = x^2$, $dy = dx$ and $dt = dy$.

Exactness Test: $\frac{\delta}{\delta y}(M) = \frac{\delta}{\delta x}(N)$

$$\frac{\delta}{\delta y}(4xy) = \frac{\delta}{\delta x}(x^2)$$

$$4x \left\{ \frac{\delta}{\delta y} (y) \right\} = \frac{\delta}{\delta x} (x^2)$$

$$4x \{1\} = 2x$$

$$4x \neq 2x$$

Exactness test doesn't get satisfied, so we have an **Inexact Differential Equation**.

Applying Rule 1:

If $\frac{1}{N} \left(\frac{\delta M}{\delta t} - \frac{\delta N}{\delta y} \right) = f(x)$ alone, then $e^{\int f(x) dx}$ is the integrating factor (F_{int}):

$$= \frac{1}{(x^2)} \times \left\{ \frac{\delta(4xy)}{\delta y} - \frac{\delta(x^2)}{\delta x} \right\}$$

$$= \frac{1}{(x^2)} \times \{4x - 2x\}$$

$$= \frac{1}{(x^2)} \times 2x$$

$$\boxed{\frac{1}{N} \left(\frac{\delta M}{\delta y} - \frac{\delta N}{\delta x} \right) = \frac{2}{x} = f(x) \text{ alone}}$$

Since **Rule 1** is met, the integrating factor (F_{int}) is $e^{\int f(x) dx}$.

$$F_{int} = e^{\int f(x) dx}$$

$$F_{int} = e^{\int \left(\frac{2}{x}\right) dx}$$

$$F_{int} = e^{2(\ln|x|) + c}$$

Constant of integration is dropped c for the sake of convenience:

$$F_{int} = e^{(\ln|x^2|)}$$

$$F_{int} = e^{(\ln|x^2|)}$$

$$\boxed{F_{int} = x^2}$$

Interpretation: Integrating factor (F_{int}) for the Inexact Differential Equation $4xy \, dx + x^2 \, dy = 0$ is (x^2) . Multiplying it throughout the given equation, it becomes exact differential equation: $4x^3y \, dx + x^4 \, dy = 0$.

After introducing the integrating factor (x^2), $M = 4x^3y$ and $N = x^4$, so substituting these values in Exactness Test again: $\frac{\delta}{\delta y} (M) = \frac{\delta}{\delta x} (N)$

$$\frac{\delta}{\delta y} (4x^3y) = \frac{\delta}{\delta x} (x^4)$$

$$4x^3 \left\{ \frac{\delta}{\delta y} (y) \right\} = \frac{\delta}{\delta x} (x^4)$$

$$4x^3 \{1\} = 4x^3$$

$$\boxed{4x^3 = 4x^3}$$

Exactness test is met, so the given equation is an Exact Differential Equation which can be solved using 4-Step Procedure.

Solution (4-Step Procedure): Exact Differential Equation requires a 4-step procedure for its solution:

Step – I: Writing the function $U(x, y)$ in terms of derivative and integral w.r.t x , with a merger of possible constant of integration and variable y :

$$U(x, y) = \int \underbrace{\frac{\delta\{U(x, \bar{y})\}}{\delta x}}_{\text{derivative}} dx + \underbrace{\tilde{c}}_{\text{integration constant}} + \underbrace{f(y)}_{\substack{\text{term(s)} \\ \text{containing "y"}}}$$

Combining c and $f(y)$ into one term $\psi(y)$ and substituting:

$$U(x, y) = \int \underbrace{\frac{\delta\{U(x, \bar{y})\}}{\delta x}}_{\text{derivative}} dx + \underbrace{\psi(y)}_{\substack{\text{term(s)} \\ \text{containing "y"}}}$$

$\psi(y)$ can have two components:

- 1) Additive constant(s) or terms containing y only reduce to zero (0) when function $U(x, \bar{y})$ gets differentiated w.r.t. x .
- 2) Constant of integration generated due to $\int \frac{\delta\{U(x, \bar{y})\}}{\delta x} dx$.

Recalling that $\frac{\delta\{U(x, \bar{y})\}}{\delta x} = M$, therefore:

$$U(x, y) = \int M dx + \psi(y)$$

Substituting value of M :

$$U(x, y) = \int (4x^3 y) dx + \psi(y)$$

$$U(x, y) = 4y \int (x^3) dx + \psi(y)$$

Solving the integral while merging the constant of integration into $\psi(y)$.

$$U(x, y) = 4y \left\{ \frac{(x)^{3+1}}{3+1} \right\} + \psi(y)$$

$$U(x, y) = 4y \left(\frac{x^4}{4} \right) + \psi(y)$$

$$\boxed{U(x, y) = x^4 y + \psi(y)}$$

Step – II: Differentiate $U(x, y)$ w.r.t y {to evaluate $\psi(y)$ }:

$$\frac{\delta\{U(x, y)\}}{\delta y} = \frac{\delta}{\delta y} \{x^4 y + \psi(y)\}$$

$$\frac{\delta\{U(x, y)\}}{\delta y} = x^4 \frac{\delta}{\delta y} (y) + \frac{\delta}{\delta y} \{\psi(y)\}$$

$$\frac{\delta\{U\}}{\delta y} = x^4 (1) + \psi'(y)$$

$$\boxed{\frac{\delta U}{\delta y}} = x^4 + \psi'(y)$$

Since we already know; $\boxed{\frac{\delta U}{\delta y} = N}$, therefore **L.H.S:**

$$\boxed{N} = x^4 + \psi'(y)$$

Moreover, $\boxed{N = x^4}$, so we substitute:

$$\boxed{x^4} = x^4 + \psi'(y)$$

$$0 = \psi'(y)$$

$$\boxed{\psi'(y) = 0}$$

Step – III: We have found the value of derivative $\psi'(y)$ and not the value of original function $\psi(y)$. Therefore, taking integral to restore the original function $\psi(y)$ from $\psi'(y)$:

$$\psi'(y) = 0$$

zero (0) can be derivative of a constant (k)

Integrating on both sides w.r.t y :

$$\int \psi'(y) dy = \int k dy$$

a constant is primitive of a derivative, $\psi'(y)=0$

$$\int \frac{d}{dy} \{\psi(y)\} dy = k$$

$$\psi(y) = k$$

Step – IV: Substitute the result from **Step – III** in **Step – I**.

Result from **Step – I:** $U(x, y) = x^4 y + \psi(y)$

Result from **Step – III:** $\psi(y) = k$

$$\overline{U(x, y)} = x^4 y + k$$

Recalling definition of an Exact Differential Equation:

$d\{U(x, y)\} = 0$, implies $U(x, y) = c_U$ (a constant)

0 is differential of a constant (c_U).

$$\overline{c_U} = x^4 y + k$$

$$c_U = x^4 y + k$$

$$c_U - k = x^4 y$$

$$x^4 y = c_U - k$$

Since difference of two constants is yet another constant: $c_U - k = c$

$$x^4 y = c$$

$$x^4 y = c$$

Developing Functional expression on **R.H.S**:

$$\overline{U(x, y)} = x^4 y - c$$

Interpretation: General Solution of Exact Differential Equation $4x^3 y dx + x^4 dy = 0$ is

$\boxed{U(x, y) = x^4 y - c}$. It makes economic sense of a Utility Function.

TOPIC 114: POINT ELASTICITY FORMULA USING EXACT DIFFERENTIAL EQUATIONS

The point elasticity of demand a firm is faced with is $E_d = -2$. Use the exact differential equation approach to find the demand function of this firm.

Solution: Since no differential equation is given, we develop one from given data:

$$\begin{aligned} E_d &= -2 \\ \frac{dQ_d}{dP} \div \frac{Q_d}{P} &= -2 \\ \frac{dQ_d}{dP} \times \frac{P}{Q_d} &= -2 \\ \frac{dQ_d}{dP} &= -2 \frac{Q_d}{P} \end{aligned}$$

Cross Multiplying, we get:

$$\frac{dQ_d}{Q_d} = -2 \frac{dP}{P}$$

$$\frac{dQ_d}{Q_d} + 2 \frac{dP}{P} = 0$$

$$\left(\frac{1}{Q_d}\right) dQ_d + \left(\frac{2}{P}\right) dP = 0$$

Comparing with the standard form:

$$(M) dy + (N) dt = 0$$

$$\underbrace{\left(\frac{1}{Q_d}\right)}_M \underbrace{\frac{dQ_d}{dy}}_{dy} + \underbrace{\left(\frac{2}{P}\right)}_N \underbrace{\frac{dP}{dt}}_{dt} = 0$$

It implies: $M = \frac{1}{Q_d}$, $N = \frac{2}{P}$, $dy = dQ_d$ and $dt = dP$.

Exactness Test: $\frac{\delta}{\delta P}(M) = \frac{\delta}{\delta Q_d}(N)$

$$\frac{\delta}{\delta P}\left(\frac{1}{Q_d}\right) = \frac{\delta}{\delta Q_d}\left(\frac{2}{P}\right)$$

$$0 = 0$$

Exactness test gets satisfied, so we have an **Exact Differential Equation**.

After verification of Exactness test, given Exact Differential Equation can be solved using 4-Step Procedure.

Solution (4-Step Procedure): Exact Differential Equation requires a 4-step procedure for its solution:

Step – I: Writing the function $F(Q_d, P)$ in terms of derivative and integral w.r.t Q_d , with a merger of possible constant of integration and variable P :

$$F(Q_d, P) = \int \underbrace{\frac{\delta\{F(Q_d, P)\}}{\delta Q_d}}_{\text{derivative}} dQ_d + \underbrace{\tilde{c}}_{\text{integration constant}} + \underbrace{f(P)}_{\substack{\text{term(s)} \\ \text{containing "P"}}}$$

Combining c and $f(P)$ into one term $\psi(P)$ and substituting:

$$F(Q_d, P) = \int \underbrace{\frac{\delta\{F(Q_d, P)\}}{\delta Q_d}}_{\text{derivative}} dQ_d + \underbrace{\psi(P)}_{\substack{\text{term(s) containing "P"} \\ \text{integration constant}}}$$

$\psi(P)$ can have two components:

- 1) Additive constant(s) or terms containing P only reduce to zero (0) when function

$F(Q_d, \bar{P})$ gets differentiated w.r.t. Q_d .

- 2) Constant of integration generated due to $\int \frac{\delta\{F(Q_d, \bar{P})\}}{\delta Q_d} dQ_d$.

Recalling that $\frac{\delta\{F(Q_d, \bar{P})\}}{\delta Q_d} = M$, therefore:

$$F(Q_d, P) = \int M dQ_d + \psi(P)$$

Substituting value of M :

$$F(Q_d, P) = \int \left(\frac{1}{Q_d}\right) dQ_d + \psi(P)$$

Solving the integral while merging the constant of integration into $\psi(P)$.

$$\boxed{F(Q_d, P) = \ln|Q_d| + \psi(P)}$$

Step – II: Differentiate $F(Q_d, P)$ w.r.t P {to evaluate $\psi(P)$ }:

$$\begin{aligned}\frac{\delta\{F(Q_d, P)\}}{\delta P} &= \frac{\delta}{\delta P} \{\ln|Q_d| + \psi(P)\} \\ \frac{\delta\{F(Q_d, P)\}}{\delta P} &= \frac{\delta}{\delta P} (\ln|Q_d|) + \frac{\delta}{\delta y} \{\psi(P)\} \\ \frac{\delta\{U\}}{\delta y} &= 0 + \psi'(P) \\ \boxed{\frac{\delta U}{\delta y}} &= \psi'(P)\end{aligned}$$

Since we already know; $\boxed{\frac{\delta U}{\delta y} = N}$, therefore *L.H.S*:

$$\boxed{N} = \psi'(P)$$

Moreover, $\boxed{N = \frac{2}{P}}$, so we substitute:

$$\begin{aligned}\boxed{\frac{2}{P}} &= \psi'(P) \\ \boxed{\psi'(P) = \frac{2}{P}}\end{aligned}$$

Step – III: We have found the value of derivative $\psi'(P)$ and not the value of original function $\psi(P)$.

Therefore, taking integral to restore the original function $\psi(P)$ from $\psi'(P)$:

Integrating on both sides w.r.t P :

$$\begin{aligned}\int \psi'(P) dP &= \int \left(\frac{2}{P}\right) dP \\ \int \frac{d}{dP} \{\psi(P)\} dP &= 2 \times \int \left(\frac{1}{P}\right) dP \\ \boxed{\psi(P) = 2 \times \ln|P| + c_1}\end{aligned}$$

Step – IV: Substitute the result from **Step – III** in **Step – I**.

Result from **Step – I:** $F(Q_d, P) = \ln|Q_d| + \psi(P)$

Result from **Step – III:** $\psi(P) = \psi(P) = 2 \times \ln|P| + c_1$

$$\boxed{F(Q_d, P) = \ln|Q_d| + 2 \times \ln|P| + c_1}$$

Recalling definition of an Exact Differential Equation:

$d\{F(Q_d, P)\} = 0$, implies $F(Q_d, P) = c_D$ (a constant)

0 is differential of a constant (c_D).

$$\begin{aligned}\overline{c_D} &= \ln|Q_d| + \boxed{2} \times \ln|P| + c_1 \\ \underline{c_D - c_1} &= \ln|Q_d| + \ln|P^{\boxed{2}}| \\ \underline{c_2} &= \ln|Q_d| + \ln|P^2| \\ \ln|Q_d \times P^2| &= c_2\end{aligned}$$

Raising to exponent on both sides, to get rid of natural logarithm.

$$\begin{aligned}e^{\ln|Q_d \times P^2|} &= e^{c_2} \\ Q_d \times P^2 &= \underline{c} \\ \boxed{Q_d = \frac{c}{P^2}}\end{aligned}$$

Interpretation: The Differential Equation $\left(\frac{1}{Q_d}\right) dQ_d + \left(\frac{2}{P}\right) dP = 0$ has General solution $Q_d = \frac{c}{P^2}$. In economic terms, the solution $Q_d = \frac{c}{P^2}$ represents a demand function; $Q_d = f(P)$.

TOPIC 115: NON-LINEAR DIFFERENTIAL EQUATION: SEPARABLE VARIABLES

The Concept: If $f(y, t) dy + g(y, t) dt = 0$ has the convenient property that function f and g can be written in terms of y and t **only**, i.e. $f(y) dy + g(t) dt = 0$. Variables y and t are said to be **separable**, as $f(y, t)$ is consolidated into $f(y)$ and $g(y, t)$ is consolidated into $g(t)$.

y is mathematically separated from t .

Solve using (simple) integration.

Example # 1: (p. 492): Solve the following differential equation:

$$3y^2 dy - t dt = 0$$

Rearranging we get: $3y^2 dy = t dt$

Introducing integral on both sides:

$$\begin{aligned}\int 3y^2 dy &= \int t dt \\ y^3 + \underline{c_1} &= \frac{t^2}{2} + \underline{c_2} \\ y^3 &= \frac{t^2}{2} + \underline{c_2 - c_1} \\ y^3 &= \frac{t^2}{2} + \underline{c'}, [\because c_2 - c_1 = c'] \\ y &= \left(\frac{t^2}{2} + c'\right)^{\frac{1}{3}}\end{aligned}$$

In functional form of a time-path:

$$y(t) = \left(\frac{t^2}{2} + c\right)^{\frac{1}{3}}$$

Example # 2: (p. 493): $2t dy + y dt = 0$ is a differential equation.

Rearranging it, we get:

$$2t dy = -y dt,$$

Needs more rearrangement, therefore cross-multiplying;

$$\begin{aligned}\frac{dy}{y} &= -\frac{dt}{2t} \\ \frac{dy}{y} &= -\frac{1}{2} \frac{dt}{t}\end{aligned}$$

Introducing integrals on both sides:

$$\begin{aligned}\int \frac{dy}{y} &= -\frac{1}{2} \int \frac{dt}{t} \\ \ln|y| + \underline{c_1} &= -\frac{1}{2} \ln|t| + \underline{c_2} \\ \ln|y| + \frac{1}{2} \ln|t| &= \underline{c_2 - c_1} \\ \ln|y| + \ln\left|t^{\frac{1}{2}}\right| &= \underline{c'}\end{aligned}$$

$$\ln \left| y \times t^{\frac{1}{2}} \right| = c'$$

Raising to exponent on both sides, to get rid of natural logarithm:

$$e^{\ln \left| y \times t^{\frac{1}{2}} \right|} = e^{c'}$$

$e^{c'} = c$: exponent of a constant c' shall be yet another constant c .

$$y \times t^{\frac{1}{2}} = c$$

$$y = ct^{-\frac{1}{2}}$$

In functional form of a time-path:

$$y(t) = ct^{-\frac{1}{2}}$$

TOPIC 116: DYNAMICS OF CAPITAL USING SEPARABLE VARIABLES METHOD

Let $X = X(t)$ denote the national product, $K = K(t)$ the capital stock, and $L = L(t)$ the number of workers in a country at time t . Suppose that, for all $t \geq 0$ (a) $X = \sqrt{K}\sqrt{L}$, (b) $\dot{K} = 0.4X$, (c) $L = e^{0.04t}$. Derive from these equations a single differential equation for $K = K(t)$ and find the solution of that equation when $K(0) = 10000$.

$X = \sqrt{K}\sqrt{L}$; is a Cobb-Douglas production function.

$\dot{K} = 0.4X$; shows that investment is proportional to output

$L = e^{0.04t}$; depicts the exponential growth of labor.

Solution: Starting with the equation of rate of change of capital; $\dot{K} = 0.4X$

$$\frac{dK}{dt} = 0.4X$$

Borrowing value of X from equation (a): $\dot{K} = 0.4X$

$$\frac{dK}{dt} = 0.4 \times \sqrt{K}\sqrt{L}$$

$$\frac{dK}{dt} = 0.4 \times \sqrt{K}\sqrt{L}$$

Borrowing value of L from equation (c): $L = e^{0.04t}$

$$\frac{dK}{dt} = 0.4 \times \sqrt{K}\sqrt{e^{0.04t}}$$

$$\frac{1}{\sqrt{K}} \times \frac{dK}{dt} = 0.4 \times \sqrt{e^{0.04t}}$$

$$K^{-\frac{1}{2}} \times \frac{dK}{dt} = 0.4 \times \sqrt{(e^{0.02t})^2}$$

$$K^{-\frac{1}{2}} \times \frac{dK}{dt} = 0.4 \times e^{0.02t}$$

Cross-multiplying for separable variable situation:

$$\underbrace{\left(K^{-\frac{1}{2}}\right) \times dK}_{\text{in terms of } (K) \text{ only}} = \underbrace{0.4 (e^{0.02t}) dt}_{\text{in terms of } (t) \text{ only}}$$

$$\int \left(K^{-\frac{1}{2}}\right) dK = \int 0.4 (e^{0.02t}) dt$$

$$\left(\frac{K^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + c_K = 0.4 \left(\frac{e^{0.02t}}{\frac{d}{dt}(0.02t)} \right) + c_t$$

$$\left(\frac{K^{\frac{1}{2}}}{\frac{1}{2}} \right) = 0.4 \left(\frac{e^{0.02t}}{0.02} \right) + c_t - c_K$$

$$2K^{\frac{1}{2}} = \frac{0.4}{0.02} (e^{0.02t}) + c_1$$

$$K(t)^{\frac{1}{2}} = \frac{0.4}{0.02} \times 2 (e^{0.02t}) + \frac{c_1 \times 2}{2}$$

$$K(t)^{\frac{1}{2}} = \left(\frac{0.4}{0.02 \times 2} \right) (e^{0.02t}) + c_2$$

$$\sqrt{K(t)} = (10)(e^{0.02t}) + c_2$$

Introducing the initial condition; $t = 0$, in the equation above:

$$\sqrt{K(0)} = (10)\{e^{0.02(0)}\} + c_2$$

$$\sqrt{10000} = (10)(1) + c_2$$

$$100 = 10 + c_2$$

$$100 - 10 = c_2$$

$$c_2 = 90$$

Substituting the value of c_2 in $\sqrt{K(t)} = (10)(e^{0.02t}) + c_2$.

$$\sqrt{K(t)} = (10)(e^{0.02t}) + 90$$

$$K(t) = \{10(e^{0.02t}) + 90\}^2$$

$$K(t) = [10\{e^{0.02t} + 9\}]^2$$

$$K(t) = 10^2(e^{0.02t} + 9)^2$$

$$K(t) = 100(e^{0.02t} + 9)^2$$

Interpretation: Non-linear differential equation for capital $\left[\left(K^{-\frac{1}{2}} \right) \times dK = 0.4(e^{0.02t})dt \right]$ has solution $K(t) = 100(e^{0.02t} + 9)^2$ which represents the time-path of capital (K) over time (t).

Lesson 27

ECONOMIC APPLICATIONS OF EXACT DIFFERENTIAL EQUATIONS (CONTINUED)

TOPIC 117: NON-LINEAR DIFFERENTIAL EQUATION: UNDERSTANDING BERNOULLI'S EQUATION

A certain type of nonlinear differential equation, attributed to Jacob Bernoulli who was a Swiss Mathematician in 17th century (1654 – 1705).

If the differential equation $\frac{dy}{dt} = h(y, t)$ has the specific nonlinear form:

$$\frac{dy}{dt} + Ry = Ty^m$$

Salient Feature of this form:

$\frac{dy}{dt}$	+	$\underbrace{R}_{\substack{\text{function of } (t)}}$	$\underbrace{y}_{\substack{\text{(y) appears on L.H.S with power (1)}}}$	=	$\underbrace{T}_{\substack{\text{function of } (t)}}$	×	$\underbrace{y^m}_{\substack{\text{(y) appears on R.H.S with power other than 0 or 1}}}$
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R : function of t .

T : function of t .

m : ($m = 0$) $\bar{\vee}$ ($m = 1$).

With these conditions fulfilled we get a **Bernoulli Differential Equation**.

Caveat: what if $m = 0$ or $m = 1$?

$m = 0$	$m = 1$
$\frac{dy}{dt} + Ry = Ty^{[0]}$ $\frac{dy}{dt} + Ry = T \times 1$ $\frac{dy}{dt} + R[y] = T$	$\frac{dy}{dt} + Ry = Ty^{[1]}$ $\frac{dy}{dt} + Ry = Ty$ $\frac{dy}{dt} + (R - T)[y] = 0$
Differential equation doesn't remain nonlinear, rather becomes linear differential equation.	Differential equation doesn't remain nonlinear, rather becomes linear differential equation.

Solution: Bernoulli Equation (nonlinear differential equation) can be reduced to a linear differential equation and be solved as such.

Reduction to linear form: Standard form of Bernoulli Equation: $\frac{dy}{dt} + Ry = T[y^m]$

Since y^m is the cause of nonlinearity, we reduce it to 1 by self-dividing:

$$\begin{aligned}\frac{\left(\frac{dy}{dt} + Ry\right)}{y^m} &= \frac{T y^m}{y^m} \\ \frac{1}{y^m} \times \frac{dy}{dt} + R \frac{y}{y^m} &= \frac{T y^m}{y^m} \\ y^{-m} \times \frac{dy}{dt} + R(y \times y^{-m}) &= T \\ \underbrace{(y^{-m}) \frac{dy}{dt}}_{\text{Substitution-II}} + R \underbrace{y^{1-m}}_{\text{Substitution-I}} &= T\end{aligned}$$

Boxed terms, if substituted will free equation from nonlinear term of y .

Substitution – I:
Introducing a shorthand notation z for y^{1-m} :

$$z = y^{1-m}$$

Nonlinear term of y will be substituted by linear term of z .

Substitution – II: z depends on y , which further depends on t : $z = f(y)$ and $y = g(t)$; combining the two function: $z = f\{y(t)\}$

We get a composite function, which if differentiated, calls for Chain

Rule: $\frac{dz}{dt} = \frac{dz}{dy} \times \frac{dy}{dt}$

differential term of new equation $z=f(t)$	differentiate $z=y^{1-m}$ w.r.t y	differential term of new equation $y=f(t)$
$\frac{dz}{dt}$	$\left(\frac{dz}{dy}\right)$	$\times \left(\frac{dy}{dt}\right)$

Differentiate $z = y^{1-m}$ w.r.t y :

$$\frac{dz}{dy} = \frac{d}{dy}(y^{1-m}) = (1-m) \times (y^{1-m-1}) \times \frac{d}{dy}(y)$$

$$\frac{dz}{dy} = (1-m) \times (y^{-m})$$

Substituting value of $\frac{dz}{dy}$ in chain rule; $\frac{dz}{dt} = \frac{dz}{dy} \times \frac{dy}{dt}$

$$\frac{dz}{dt} = [(1-m) \times (y^{-m})] \times \frac{dy}{dt}$$

$$\frac{1}{(1-m)} \times \frac{dz}{dt} = (y^{-m}) \frac{dy}{dt}$$

Combining the **Substitution – I & Substitution – II:**

Now we can surrogate z instead of y in

$$\underbrace{(y^{-m}) \frac{dy}{dt}}_{\text{Substitution-II}} + R \underbrace{y^{1-m}}_{\text{Substitution-I}} = T$$

$$\underbrace{\frac{1}{(1-m)} \times \frac{dz}{dt}}_{\text{Substitution-II}} + R \underbrace{z}_{\text{Substitution-I}} = T$$

$$\frac{1}{(1-m)} \times \frac{dz}{dt} + Rz = T$$

$$\frac{dz + Rz(1-m)dt}{(1-m)dt} = T$$

$$dz + Rz(1-m)dt = T \times (1-m)dt$$

$$dz + Rz(1-m)dt - T(1-m)dt = 0$$

$$dz + \{(1-m) \times (Rz - T)\}dt = 0$$

Objective of replacing a nonlinear term y^m with linear term $z^{(1)}$.

$$dz + \left\{ (1-m) \times \left(R \underbrace{\overset{\substack{z \text{ instead} \\ \text{of } y}}{\hat{z}}}_{\text{linear term}} - T \right) \right\} dt = 0$$

TOPIC 118: NON-LINEAR DIFFERENTIAL EQUATION: BERNOULLI'S EQUATION NUMERICALLY

Solve the equation $\frac{dy}{dt} + ty = 3ty^2$.

Solution: It is Bernoulli equation as we compare it with standard from:

$$\frac{dy}{dt} + Ry = Ty^{\overline{m}}$$

$$\frac{dy}{dt} + \underbrace{t}_R y = \underbrace{3t}_T \underbrace{y^{\overline{2}}}_{m=2}$$

Here; $R = t$, $T = 3t$, $m = 2$.

Substitution to replace z instead y^{1-m} :

$$z = y^{1-m}$$

$$z = y^{1-2}$$

$$z = y^{-1}$$

Recalling the linearized form of Bernoulli Equation, in terms of z instead of y :

$$dz + \{(1-m) \times (Rz - T)\}dt = 0$$

Substituting the given values of m , R , and T .

$$dz + \{(1-2) \times (tz - 3t)\}dt = 0$$

$$dz + \{(-1) \times t(z - 3)\}dt = 0$$

$$dz + (-zt + 3t)dt = 0$$

Transforming into standard form of first order differential equation $\frac{dy}{dt} + u(t) \times y = w(t)$:

$$\frac{\{dz + (-zt + 3t)dt\}}{dt} = \frac{0}{dt}$$

$$\frac{dz}{dt} + (-t)z + 3t = 0$$

$$\frac{d\tilde{z}}{dt} + \underbrace{(-t)}_{u(t)} \tilde{z} = \underbrace{3t}_{w(t)}$$

Using the General Solution of FODE-variable coefficient & variable term:

$$\boxed{y}(t) = e^{-\int u dt} \left(A + \int \boxed{w} e^{\int u dt} dt \right)$$

Solving the $\int u dt$ separately to avoid the clutter:

$$\int u dt = \int (-t) dt$$

*dropping c
for ease*

$$\int u dt = -\frac{\widetilde{t^2}}{2}$$

Returning the value of $\int u dt$ in general solution equation:

$$z(t) = e^{-\left(\frac{t^2}{2}\right)} \left(A + \int \boxed{-3t} e^{\frac{t^2}{2}} dt \right)$$

$$z(t) = e^{\left(\frac{t^2}{2}\right)} \left\{ A + 3 \int (-t) \times e^{\left(-\frac{t^2}{2}\right)} dt \right\}$$

Since here $f(x) = -\frac{t^2}{2}$, $f'(x) = -t$:

$$z(t) = e^{\left(\frac{t^2}{2}\right)} \left\{ A + 3 \underbrace{\int \boxed{-t} \times e^{\frac{t^2}{2}} dt}_{\int \{f'(x) \times e^{f(x)}\} dx} \right\}$$

$$z(t) = e^{\left(\frac{t^2}{2}\right)} \left\{ A + 3 \underbrace{e^{\left(-\frac{t^2}{2}\right)}}_{e^{f(x)}} \right\}$$

$$z(t) = Ae^{\left(\frac{t^2}{2}\right)} + 3 \boxed{e^{\left(\frac{t^2}{2}\right)} e^{\left(-\frac{t^2}{2}\right)}}$$

$$\boxed{z(t)} = Ae^{\left(\frac{t^2}{2}\right)} + 3$$

Time-path of z in terms of time, however, given information was in terms of y . So, we restore the y to find its time-path $y(t)$.

$$\boxed{z} = y^{-1}$$

After substitution:

$$\boxed{y^{-1}} = Ae^{\left(\frac{t^2}{2}\right)} + 3$$

$$\frac{1}{y} = Ae^{\left(\frac{t^2}{2}\right)} + 3$$

$$\frac{1}{Ae^{\left(\frac{t^2}{2}\right)} + 3} = y$$

$$y(t) = \frac{1}{Ae^{\left(\frac{t^2}{2}\right)} + 3}$$

Interpretation: Using Bernoulli equation method, we substituted z in place of y to linearize the nonlinear differential equation. Then used procedure of solving FODE to find time-path (however, of z). Finally, we restored y from z and found time-path $y(t)$.

TOPIC 119: AVERAGE COST FUNCTION USING EXACT DIFFERENTIAL EQUATIONS

The unit cost of a production firm is AC , while the number of units produced within the firm is Q such that the following relationship exists:

$$\frac{d(AC)}{dQ} + \frac{AC}{Q} = (AC)^3$$

$$\frac{d(AC)}{dQ} + \left(\frac{1}{Q}\right)AC = (AC)^3$$

Solution: It is Bernoulli equation as we compare it with standard form:

$$\frac{dy}{dt} + Ry = Ty^m$$

$$\frac{d(AC)}{dQ} + \left(\frac{1}{Q}\right)AC = \frac{1}{T} \frac{(AC)^3}{y^m}$$

Here; $y = AC$, $t = Q$, $\frac{dy}{dt} = \frac{d(AC)}{dQ}$, $R = \frac{1}{Q}$, $T = 1$, $m = 3$.

Substitution to replace z instead $(AC)^{1-m}$:

$$z = (AC)^{1-3}$$

$$z = (AC)^{1-3}$$

$$z = (AC)^{-2}$$

Recalling the linearized form of Bernoulli Equation, in terms of z instead of (AC) :

$$dz + \{(1 - m) \times (Rz - T)\}dQ = 0$$

Substituting the given values of m , R , and T .

$$dz + \left[(1 - 3) \times \left\{\left(\frac{1}{Q}\right)z - 1\right\}\right]dQ = 0$$

$$dz + \left[-2 \left\{\left(\frac{1}{Q}\right)zdQ + dQ\right\}\right] = 0$$

$$dz - 2\left(\frac{1}{Q}\right)zdQ - 2dQ = 0$$

Dividing by dQ throughout, to get the standard term of $\frac{dz}{dQ}$:

$$\frac{dz}{dQ} - \frac{2\left(\frac{1}{Q}\right)z dQ}{dQ} - \frac{2dQ}{dQ} = \frac{0}{dQ}$$

$$\frac{dz}{dQ} + (-2) \left(\frac{1}{Q} \right) z - 2 = 0$$

Transforming into standard form of first order differential equation $\frac{dy}{dt} + u(t) \times y = w(t)$:

$$\underbrace{\frac{dz}{dQ}}_{\frac{dy}{dt} = \frac{dz}{dQ}} + \underbrace{\left\{ (-2) \left(\frac{1}{Q} \right) \right\}}_{u(t)=u(Q)} z = \underbrace{2}_{w(t)=w(Q)}$$

Here; $\frac{dy}{dt} = \frac{dz}{dQ}$, $y = z$, $t = Q$, $u(t) = u(Q) = (-2) \left(\frac{1}{Q} \right)$, $w(t) = w(Q) = 2$.

Using the General Solution of FODE-variable coefficient & variable term:

$$y(t) = e^{-\int u dt} \left(A + \int w e^{\int u dt} dt \right)$$

Here the symbols are economic in nature, $y = AC$ & $t = Q$.

$$AC(Q) = e^{-\int u(Q) dQ} \left(A + \int w(Q) e^{\int u(Q) dQ} dQ \right)$$

Formula General Solution is also expressed in economic variable:

$$\underbrace{\frac{dz}{dQ}}_{\frac{dy}{dt} = \frac{dz}{dQ}} - \left\{ (-2) \left(\frac{1}{Q} \right) \right\} z = \underbrace{2}_{w(Q)}$$

Solving the $\int u(Q) dQ$ separately to avoid the clutter:

$$\begin{aligned} \int u(Q) dQ &= \int (-2) \left(\frac{1}{Q} \right) dQ \\ \int u(Q) dQ &= (-2) \int \left(\frac{1}{Q} \right) dQ \\ &\quad \text{dropping c for ease} \\ \int u(Q) dQ &= (-2) \times \ln|Q| \\ \int u(Q) dQ &= (-2) \times \ln|Q| \end{aligned}$$

Returning the value of $\int u(Q) dQ$ in general solution equation:

$$AC(Q) = e^{-\int u(Q) dQ} \left(A + \int w(Q) e^{\int u(Q) dQ} dQ \right)$$

$$z(Q) = e^{-(-2) \times \ln|Q|} \left(A + \int 2 e^{(-2) \times \ln|Q|} dQ \right)$$

$$z(Q) = e^{(2 \times \ln|Q|)} \left(A + \int 2 \times e^{(-2) \times \ln|Q|} dQ \right)$$

$$z(Q) = e^{\ln|Q|^2} \left(A + 2 \times \int e^{\ln|Q|^{-2}} dQ \right)$$

$$z(Q) = Q^2 \left(A + 2 \times \int Q^{-2} dQ \right)$$

$$z(Q) = Q^2 \times A + Q^2 \times 2 \times \underbrace{\frac{Q^{-1}}{(-1)}}_{\text{omitting integration constant}}$$

$$z(Q) = Q^2 \times A + \frac{2}{(-1)} \times Q^2 \times Q^{-1}$$

$$z(Q) = AQ^2 - 2 \times Q^1$$

$$\boxed{z(Q)} = AQ^2 - 2Q$$

It is the solution of z in terms of Q , however, we started the analysis in terms of (AC) ; returning value of z in terms of (AC) : $\boxed{z(Q) = z = (AC)^{-2}}$

$$\boxed{(AC)^{-2}} = AQ^2 - 2Q$$

$$(AC)^{-2} = AQ^2 - 2Q$$

$$(AC) = (AQ^2 - 2Q)^{-\frac{1}{2}}$$

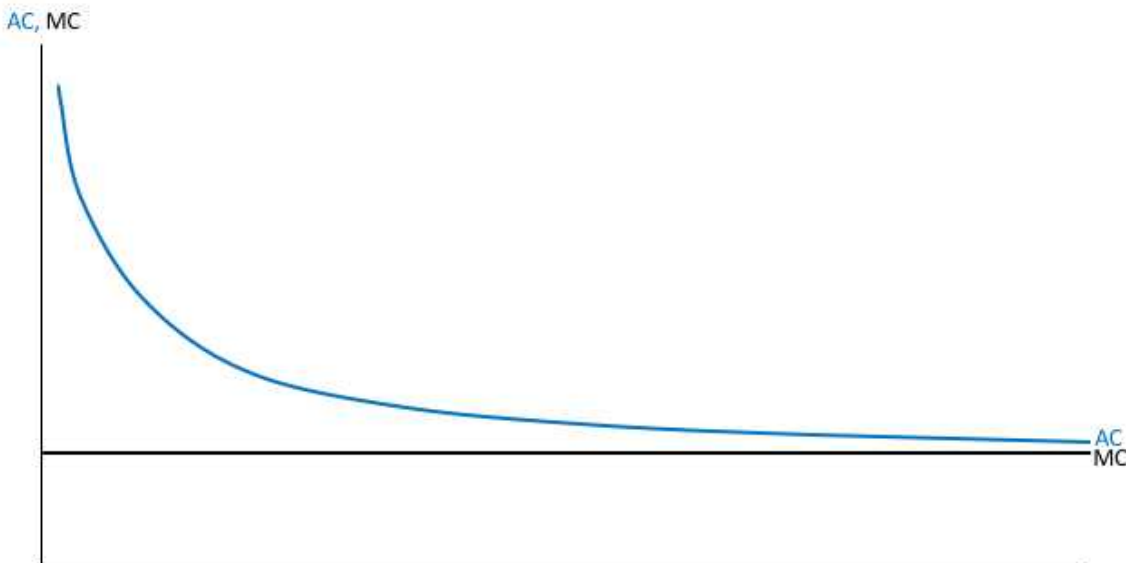
$$\boxed{AC(Q) = \frac{1}{(AQ^2 - 2Q)^{\frac{1}{2}}}}$$

Interpretation: $AC(Q) = \frac{1}{(AQ^2 - 2Q)^{\frac{1}{2}}}$ is the General solution of the Bernoulli type differential

equation $\frac{d(AC)}{dQ} + \left(\frac{1}{Q}\right) AC = (AC)^3$.

$AC(Q) = \frac{1}{(AQ^2 - 2Q)^{\frac{1}{2}}}$ depicts a negative relationship between average cost and output, provided $A > 0$.

In economic sense, AC is declining over the range of Q . This shows that AC must be predominantly in the form of fixed cost (FC). This usually happens in case of Natural monopoly, where the FC constitute a large share of overall cost and declining trend of its average (AFC) dominates the shape of AC curve. A long-run situation is plotted here:



TOPIC 120: QUALITATIVE ANALYSIS: PHASE DIAGRAM

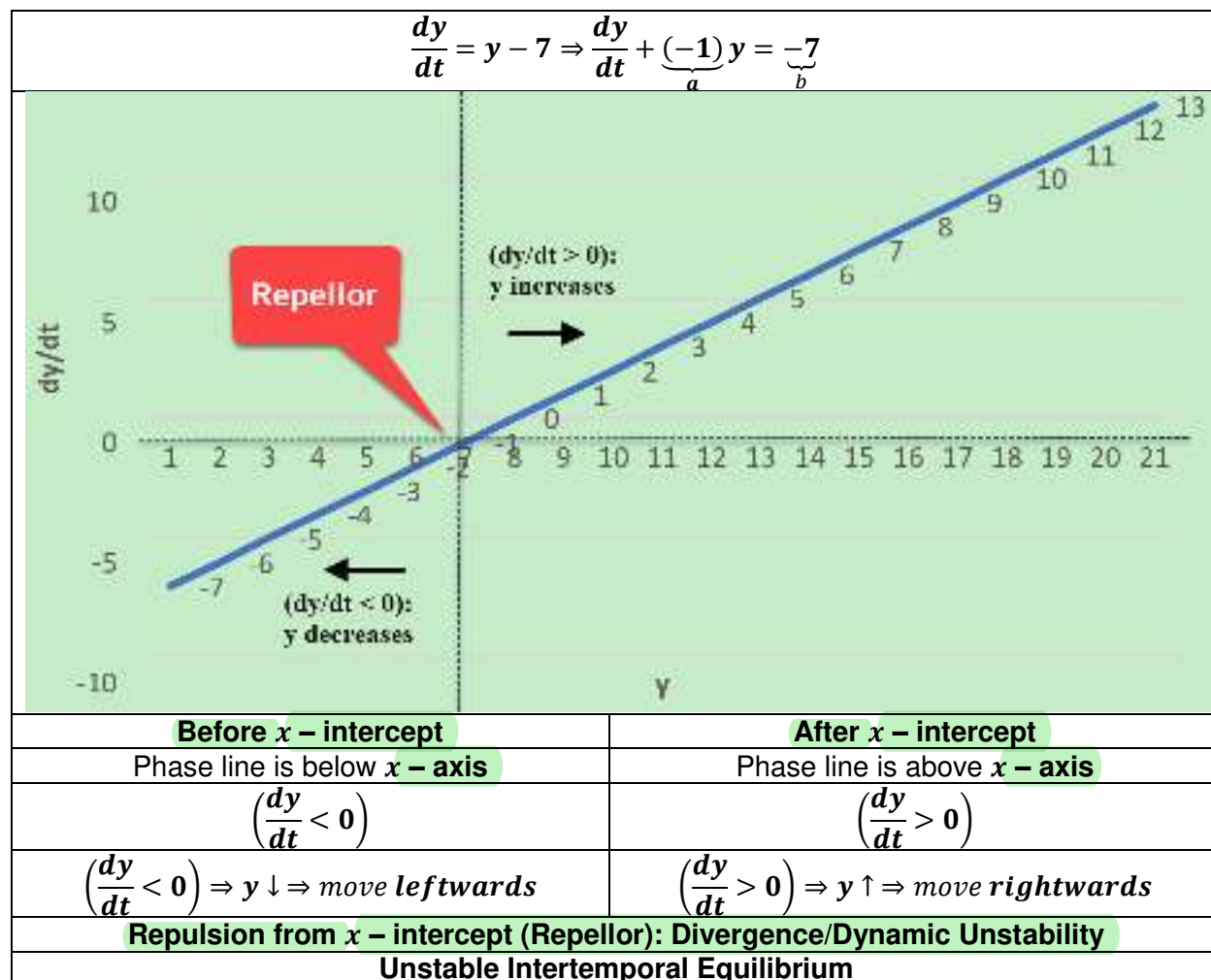
There is a Qualitative aspect of a time-path – Phase Diagram.

Phase Diagrams convey the dynamic properties of differential equations.

Let $y(t)$ denote an autonomous differential equation, so that $y'(t)$ is just a function of y and independent of t .

Mathematically its functional form is: $\frac{dy}{dt} = f(y)$

Similar but not the Same		
Differential Equation	Phase Diagram (Autonomous) Differential Equation	Time-Path
$\frac{dy}{dt} = f(y, t)$	$\frac{dy}{dt} = f(y)$	$y = f(t)$

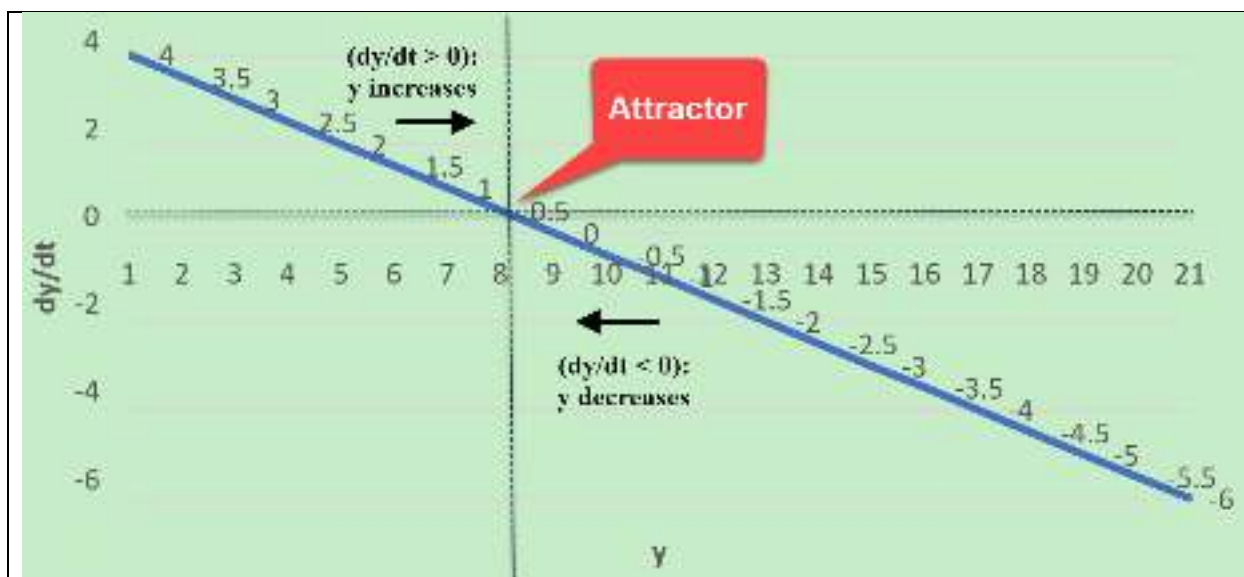


Verification: To check if qualitative (graphical) analysis has correct diagnosis regarding the time-path, we recall the qualitative method.

$$\frac{dy}{dt} + ay = b \xrightarrow{\text{convergent if } (a>0)} \frac{dy}{dt} + \underbrace{(-7)}_{\substack{a<0 \\ \text{divergent}}} y = \underbrace{4}_b$$

Hence verified that qualitative method has the correct diagnosis.

$$\frac{dy}{dt} = 4 - \frac{y}{2} \Rightarrow \frac{dy}{dt} + \underbrace{\left(\frac{1}{2}\right)}_a y = \underbrace{4}_b$$



Before x – intercept	After x – intercept
Phase line is above x – axis	Phase line is below x – axis
$\left(\frac{dy}{dt} > 0\right)$	$\left(\frac{dy}{dt} < 0\right)$
$\left(\frac{dy}{dt} > 0\right) \Rightarrow y \uparrow \Rightarrow \text{move rightwards}$	$\left(\frac{dy}{dt} < 0\right) \Rightarrow y \downarrow \Rightarrow \text{move leftwards}$
Attraction towards x – intercept (Attractor): Convergence/ Dynamic Stability	
Stable Intertemporal Equilibrium OR Steady-State Equilibrium	

Verification: To check if qualitative (graphical) analysis has correct diagnosis regarding the time-path, we recall the qualitative method.

$$\frac{dy}{dt} + ay = b \xrightarrow{\text{convergent if } (a>0)} \frac{dy}{dt} + \underbrace{\left(\frac{1}{2}\right)}_{\substack{a>0 \\ \text{convergent}}} y = \underbrace{4}_{b}$$

Hence verified that qualitative method has the correct diagnosis.

TOPIC 121: SOLOW GROWTH MODEL: QUALITATIVE ANALYSIS

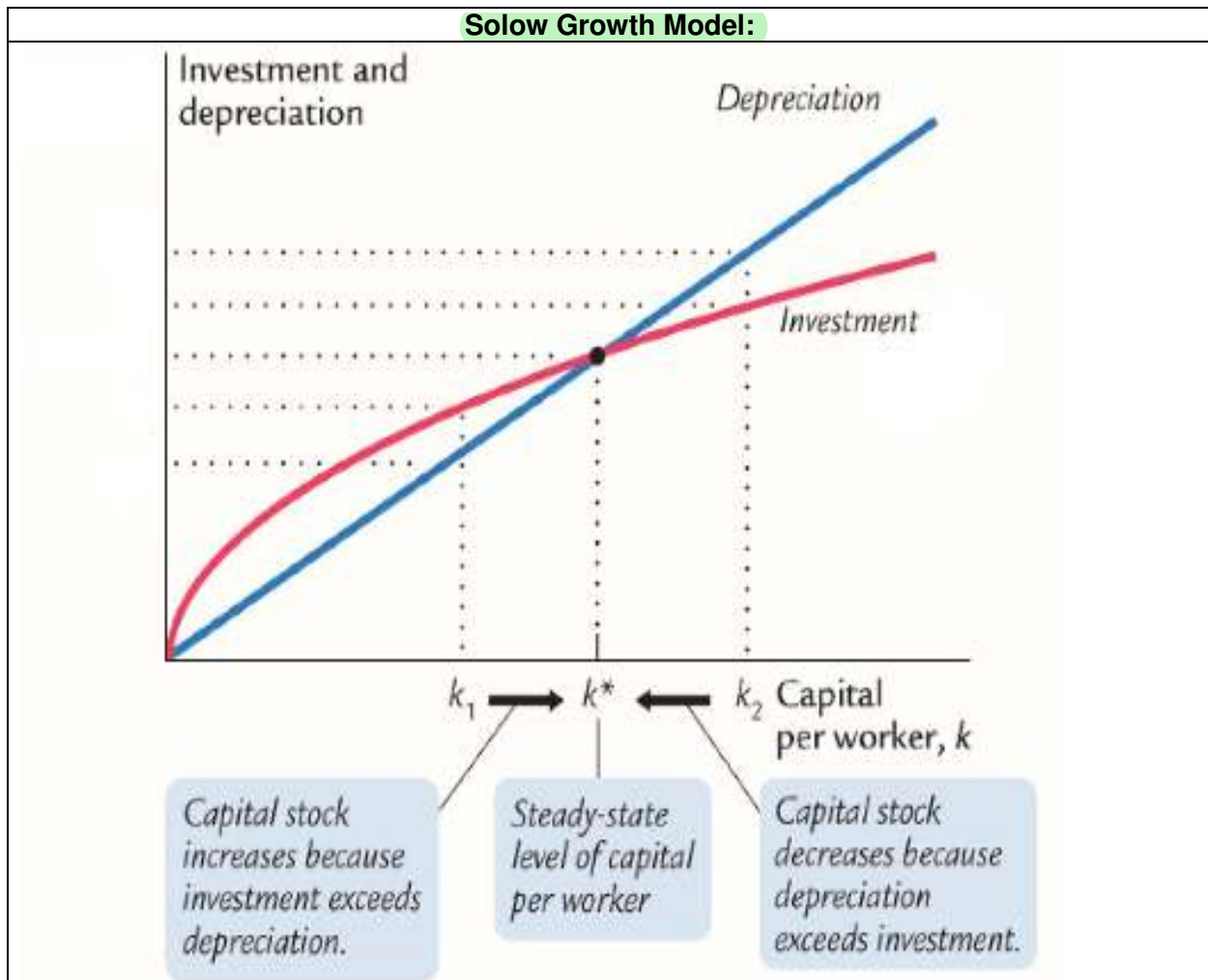
Attributed to Robert M. Solow – a Nobel Laureate.
Model was built on Domar's growth model questioning the need for 'delicate balancing'.

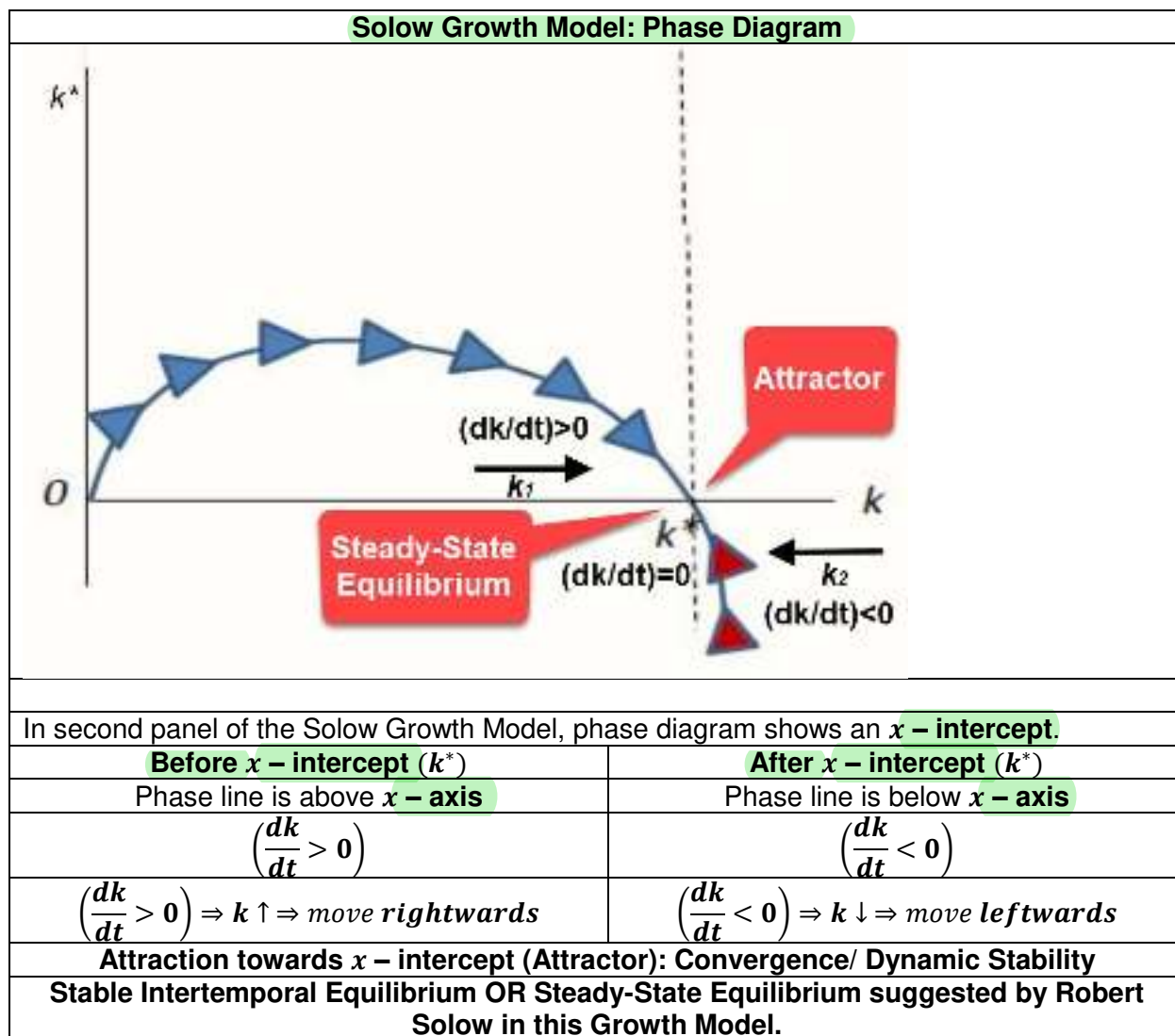


Given the qualitative result $\dot{k} = s\phi(k) - \lambda k$ we can qualitatively analyze the Solow growth model.

$$\underbrace{\dot{k}}_{\substack{\text{rate of} \\ \text{change of} \\ \text{capital}}} = \underbrace{s}_{\text{MPS}} \underbrace{\phi(k)}_{\substack{\text{production} \\ \text{function}}} - \underbrace{\lambda k}_{\text{depreciation}}$$

$\underbrace{\dot{k} = s\phi(k) - \lambda k}_{\dot{k} > 0 \quad s\phi(k) > \lambda k}$ $k \uparrow \left(k_1 \text{ to } k^* \right)$ (steady state equilibrium)	$\underbrace{\dot{k} = s\phi(k) - \lambda k}_{\dot{k} = 0 \quad s\phi(k) = \lambda k}$ $\bar{k} \left(\text{already at } k^* \right)$ (steady state equilibrium)	$\underbrace{\dot{k} = s\phi(k) - \lambda k}_{\dot{k} < 0 \quad s\phi(k) < \lambda k}$ $k \downarrow \left(k_2 \text{ to } k^* \right)$ (steady state equilibrium)
This equation shows that investment-depreciation gap leads to investment. This causes capital to grow and remove the gap. Investment (& hence capital) shall decrease if there is depreciation is greater than investment.		





TOPIC 122: SOLOW GROWTH MODEL: QUANTITATIVE ANALYSIS

After understanding the qualitative analysis of the model, we conduct a quantitative analysis by assuming a linearly homogeneous Cobb-Douglas function:

$Q = f(K, L)$ Taking L as common: $Q = L \left\{ f\left(\frac{K}{L}, 1\right) \right\}$ As $\left(\frac{K}{L} = k\right)$, f in terms of K and L while ϕ in terms of k . $Q = L\{\phi(k)\}$	$Q = K^\alpha L^{1-\alpha}$ $Q = K^\alpha L L^{-\alpha}$ $Q = L \frac{K^\alpha}{L^\alpha}$ $Q = L \left(\frac{K}{L}\right)^\alpha$ $Q = L k^\alpha$
Comparing the two results: $Q = L \overset{\phi(k)}{\widetilde{k}^\alpha} \Rightarrow \boxed{\phi(k) = k^\alpha}$	

First Order Differential Equation of Solow Growth Model in terms of k is:

$$\dot{k} = s \phi(k) - \lambda k$$

$$\dot{k} = s \underbrace{\phi(k)}_{k^\alpha} - \lambda k$$

$$\dot{k} = s k^\alpha - \lambda k$$

$$\dot{k} + \lambda k = s k^\alpha$$

$$\frac{dk}{dt} + \lambda k = s k^\alpha$$

We have a First Order Differential of nonlinear nature due to k^α with $\alpha \neq 1$. This makes a Bernoulli equation:

$$\left(\frac{dy}{dt} + R y = T y^m \right)$$

$$\underbrace{\frac{dk}{dt}}_{\frac{dy}{dt}} + \underbrace{\lambda k}_{R y} = \underbrace{s k^\alpha}_{T y^m}$$

Solution: Bernoulli Equation (nonlinear differential equation) can be reduced to a linear differential equation and be solved as such.

Here; $\frac{dy}{dt} = \frac{dk}{dt}$, $y = k$, $R = \lambda$, $T = s$, $m = \alpha$.

Reduction to Linear Form: Standard form of Bernoulli Equation:

$$\boxed{\frac{dy}{dt} + R y = T y^m} \xrightarrow{\text{linearization}} \boxed{dz + [(1-m)Rz - (1-m)T]dt = 0}$$

Linearization of Solow's Growth Differential Equation can also be done by recalling the linearized form of Bernoulli's equation and putting the values:

$$dz + [(1-m)Rz - (1-m)T]dt = 0$$

$$dz + [(1-\underline{m})\underline{R}\underline{z} - (1-\underline{m})\underline{T}]dt = 0$$

$$dz + [(1-\underline{\alpha})\underline{\lambda}\underline{z} - (1-\underline{\alpha})\underline{s}]dt = 0$$

$$\underbrace{dz + [(1-\alpha)\lambda z - (1-\alpha)s]dt = 0}_{\text{Linearized form of Solow's Growth Differential equation}}$$

Dividing throughout by dt and rearranging to develop the standard form of First Order Differential equation.

$$\frac{dz}{dt} + (1-\alpha)(\lambda z - s) = 0$$

$$\frac{dz}{dt} + (1-\alpha)\lambda z - (1-\alpha)s = 0$$

$$\frac{dz}{dt} + (1 - \alpha)\lambda z = (1 - \alpha)s$$

Comparable with $\frac{dy}{dt} + a y = b$

$a = (1 - \alpha)\lambda$ and $b = (1 - \alpha)s$

$$\frac{dz}{dt} + \underbrace{(1 - \alpha)\lambda}_a z = \underbrace{(1 - \alpha)s}_b$$

Definite Solution:

$$y(t) = \left\{ y(0) - \frac{b}{a} \right\} e^{-at} + \frac{b}{a}$$

$$z(t) = \left\{ z(0) - \frac{(1 - \alpha)s}{(1 - \alpha)\lambda} \right\} e^{-(1 - \alpha)\lambda t} + \frac{(1 - \alpha)s}{(1 - \alpha)\lambda}$$

$$z(t) = \left\{ z(0) - \frac{s}{\lambda} \right\} e^{-(1 - \alpha)\lambda t} + \frac{s}{\lambda}$$

Time-path of z . However, the original function was k .

Therefore, recalling $z = k^{1-\alpha}$ and substituting in $\underline{z(t)} = \left\{ \underline{z(0)} - \frac{s}{\lambda} \right\} e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$

$$k^{1-\alpha} = \left[\{k(0)\}^{1-\alpha} - \frac{s}{\lambda} \right] e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$$

Here $k(0)$ is the initial value of capital-labor ratio.

Dynamic Stability:

$$k^{1-\alpha} = \underbrace{\left[\{k(0)\}^{1-\alpha} - \frac{s}{\lambda} \right]}_{\substack{\text{Arbitrary constant} \\ \text{in terms of} \\ \text{initial condition}}} \underbrace{e^{-(1-\alpha)\lambda t}}_{\substack{\text{exponential} \\ \text{expression}}} + \frac{s}{\lambda}$$

k_c k_p

$\underbrace{\{e^{-(1-\alpha)\lambda t}\}}_{\substack{\text{Time dependent} \\ \text{exponential expression}}}$ explains the dynamic stability of time path.

For dynamic stability, $e^{-(1-\alpha)\lambda t} \rightarrow 0$, as $t \rightarrow \infty$

Unraveling the Exponent $(1 - \alpha)\lambda > 0$:

As already known, that $(0 < \alpha < 1)$ and $\lambda > 0$ (assumed in the model).

$$\underbrace{\left(\frac{1 - \tilde{\alpha}}{(>0, <1)} \right)}_{>0} \underbrace{\tilde{\lambda}}_{(>0)} > 0$$

$$k^{1-\alpha} = \underbrace{\left[\{k(0)\}^{1-\alpha} - \frac{s}{\lambda} \right]}_{\substack{\text{multiplicand of} \\ \text{exponential decay term}}} \underbrace{e^{-(1-\alpha)\lambda t}}_{\substack{\text{exponential} \\ \text{decay} \\ (>0)}} + \frac{s}{\lambda}$$

$\rightarrow 0 \text{ as } t \rightarrow \infty$

$$k^{1-\alpha} \rightarrow \frac{s}{\lambda} \text{ as } t \rightarrow \infty$$

Since we need the Original value of z , we shift its power to R.H.S

$$k(t) \rightarrow \left(\frac{s}{\lambda} \right)^{\frac{1}{1-\alpha}} \text{ as } t \rightarrow \infty.$$

Interpretation: Capital-labor (k) ratio assumes a constant value at equilibrium. Which varies directly w.r.t. s and indirectly w.r.t. λ .

Appendix: Linearization via Derivation:

However, linearization can be derived via following steps:

Since k^α is the cause of nonlinearity, we reduce it to 1 by self-dividing:

$$\begin{aligned} \frac{\left(\frac{dk}{dt} + \lambda k\right)}{k^\alpha} &= \frac{s k^\alpha}{k^\alpha} \\ \frac{1}{k^\alpha} \times \frac{dk}{dt} + \lambda \frac{k}{k^\alpha} &= \frac{s k^\alpha}{k^\alpha} \\ k^{-\alpha} \times \frac{dk}{dt} + \lambda(k \times k^{-\alpha}) &= s \\ \underbrace{(k^{-\alpha}) \frac{dz}{dt}}_{\text{Substitution-II}} + \lambda \underbrace{k^{1-\alpha}}_{\text{Substitution-I}} &= s \end{aligned}$$

Boxed terms, if substituted will free equation from nonlinear term of z .

Substitution – I:

Introducing a shorthand notation z for $k^{1-\alpha}$:

$$\underbrace{z}_{\text{linear term}} = \underbrace{k^{1-\alpha}}_{\text{nonlinear term}}$$

Nonlinear term $k^{1-\alpha}$ will be substituted by linear term of z .

Substitution – II: z depends on k , which further depends on t : $z = f(k)$ and $k = g(t)$; combining the two function: $z = f\{g(t)\}$

We get a composite function, which if differentiated, calls for Chain

Rule: $\frac{dz}{dt} = \frac{dz}{dk} \times \frac{dk}{dt}$

differential term of new equation $z=f(t)$	differentiate $\underbrace{z=k^{1-\alpha}}_{\text{w.r.t } k}$	differential term of new equation $k=g(t)$
$\frac{dz}{dt}$	$= \left(\frac{dz}{dk}\right)$	$\times \left(\frac{dk}{dt}\right)$

Differentiate $z = k^{1-\alpha}$ w.r.t k :

$$\frac{dz}{dk} = \frac{d}{dk} (k^{1-\alpha}) = (1-\alpha) \times (k^{1-\alpha-1}) \times \frac{d}{dk} (k)$$

$$\frac{dz}{dk} = (1-\alpha) \times (k^{-\alpha})$$

Substituting value of $\frac{dz}{dk}$ in chain rule; $\frac{dz}{dt} = \frac{dz}{dk} \times \frac{dk}{dt}$

$$\frac{dz}{dt} = (1-\alpha) \times (k^{-\alpha}) \times \frac{dk}{dt}$$

$$\frac{1}{(1-\alpha)} \times \frac{dz}{dt} = \boxed{(k^{-\alpha}) \frac{dk}{dt}}$$

Combining the **Substitution – I & Substitution – II:**Now we can surrogate z instead of k in

$$\underbrace{(k^{-m}) \frac{dk}{dt}}_{\text{Substitution-II}} + \lambda \underbrace{k^{1-\alpha}}_{\text{Substitution-I}} = s$$

$$\underbrace{\frac{1}{(1-\alpha)} \times \frac{dz}{dt}}_{\text{Substitution-II}} + \lambda \underbrace{z}_{\text{Substitution-I}} = s$$

$$\frac{1}{(1-\alpha)} \times \frac{dz}{dt} + \lambda z = s$$

$$\frac{dz + \lambda z(1-\alpha)dt}{(1-\alpha)dt} = s$$

$$dz + \lambda z(1-\alpha)dt = s \times (1-\alpha)dt$$

$$dz + \lambda z(1-\alpha)dt - s(1-\alpha)dt = 0$$

$$dz + \{(1-\alpha) \times (\lambda z - s)\}dt = 0$$

Objective of replacing a nonlinear term k^α with linear term $z^{(1)}$

$$dz + \left\{ (1-\alpha) \times \left(\lambda \underbrace{\frac{z \text{ instead of } k}{\hat{z}}}_{\text{linear term}} - s \right) \right\} dt = 0$$

Numerical Evidence $k(0) = 100, s = 0.2, \alpha = 0.7, \lambda = 2.5\% (= 0.025)$

t	$e^{-(1-\alpha)\lambda t}$	$\left[\{k(0)\}^{1-\alpha} - \frac{s}{\lambda} \right] e^{-(1-\alpha)\lambda t}$	$k^{1-\alpha} = \left[\{k(0)\}^{1-\alpha} - \frac{s}{\lambda} \right] e^{-(1-\alpha)\lambda t} + \frac{s}{\lambda}$
0	1	-4.019	3.981
1	0.993	-3.989	4.011
100	0.472	-1.898	6.102
200	0.223	-0.897	7.103
300	0.105	-0.424	7.576
400	0.050	-0.200	7.800
500	0.024	-0.095	7.905
600	0.011	-0.045	7.955
700	0.005	-0.021	7.979
800	0.002	-0.010	7.990
900	0.001	-0.005	7.995
998	0.001	-0.002	7.998
	Exponential decay	k_c fading to 0 as $t \rightarrow \infty$	Nearing $k^* = \frac{s}{\lambda} = 8$

Exponential Decay causes k_c to decay which converges the $k^{1-\alpha}(t)$ to $k_p (= k^*) = 8$.

$$k^{1-\alpha} \rightarrow \frac{s}{\lambda} \text{ or } k \rightarrow \left(\frac{s}{\lambda}\right)^{\frac{1}{1-\alpha}} \text{ as } t \rightarrow \infty.$$

In economic terms, (k^*) represents the Steady-State equilibrium suggested in Solow's Growth Model. So, the model shows Convergence towards Steady-State equilibrium.

Lesson 28

SECOND ORDER DIFFERENTIAL EQUATIONS

TOPIC 123: HIGHER (SECOND) ORDER DIFFERENTIAL EQUATION

In economic analysis, **Rate of change of the rate of change** can be question regarding economic variables e.g. income, profits etc.

Such situations can be mathematized in second (higher) order differential equations; e.g.

$$\frac{d^2 y}{dt^2} = ky$$

$$\underbrace{\frac{d}{dt}}_{\text{rate of change}} \times \underbrace{\frac{d}{dt}}_{\text{rate of change}} \underbrace{(y)}_{\text{dependent variable}} = ky$$

Since derivatives can have any order higher than second order, we generalize the higher order differential equations:

Linear Differential Equation with n^{th} Order:

Leibniz Notation:

$$\underbrace{\left(\frac{d^n y}{dt^n}\right)}_{n^{\text{th}} \text{ Order Derivative}} + \underbrace{a_1}_{1^{\text{st}} \text{ constant coefficient}} \times \underbrace{\left(\frac{d^{n-1} y}{dt^{n-1}}\right)}_{(n-1)^{\text{th}} \text{ Order Derivative}} + \underbrace{a_2}_{2^{\text{nd}} \text{ constant coefficient}} \times \underbrace{\left(\frac{d^{n-2} y}{dt^{n-2}}\right)}_{(n-2)^{\text{th}} \text{ Order Derivative}} + \dots + a_{n-1} \times \underbrace{\left(\frac{dy}{dt}\right)}_{1^{\text{st}} \text{ Order Derivative}} + \underbrace{a_n}_{n^{\text{th}} \text{ constant coefficient}} \underbrace{(y)}_{\text{dependent variable}} = \underbrace{b}_{\text{constant}}$$

Lagrange Notation:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + a_2 y^{(n-2)}(t) + \dots + a_{n-1} y^{(1)}(t) + a_n y = b$$

Order: n^{th} (due to n^{th} derivative)

Degree: Linear/First degree (due to highest power '1' of the n^{th} derivative)

Constant Coefficients (a 's) and constant term (b).

Delimitations:

1. For simplicity variable coefficients and variable term are not considered.
 2. This course deals with linear cases only.
 3. Cross terms – product of function ' $y(t)$ ' and its derivative $y'(t)$ – are not included.
- ($n = 2$) $\Rightarrow$ **Second Order Differential Equation (SODE):**

For ($n = 2$), Standard Form of Second-Order Differential Equation reduces to:

$$\frac{d^2 y}{dt^2} + a_1 \cdot \frac{dy}{dt} + a_2 \cdot y = b$$

$$y''(t) + a_1 \cdot y'(t) + a_2 \cdot y = b$$

Possible Cases: Depending on the values of b , we get 2 possible broad cases:

Main Cases based on value of (b)						
Identification	$b = 0$			$b \neq 0$		
Terminology	Homogenous Case			Non-Homogenous Case		
Components of Solution	$y_p = 0$			$y_p \neq 0$		
Sub-Cases based on values of (a_1) and/or (a_2)						
Identification	$a_2 \neq 0,$ $\left\{ \begin{matrix} (a_1 \neq 0) \\ \vee \\ (a_1 = 0) \end{matrix} \right\}$	$a_2 = 0,$ $a_1 \neq 0$	$a_2 = 0,$ $a_1 = 0$	$a_2 \neq 0,$ $\left\{ \begin{matrix} (a_1 \neq 0) \\ \vee \\ (a_1 = 0) \end{matrix} \right\}$	$a_2 = 0,$ $a_1 \neq 0$	$a_2 = 0,$ $a_1 = 0$

Particular Solution (y_p)	$y_p = \frac{(0)}{a_2}$	$y_p = \frac{(0)}{a_1} t$	$y_p = \frac{(0)}{2} t^2$	$y_p = \frac{b}{a_2}$	$y_p = \frac{b}{a_1} t$	$y_p = \frac{b}{2} t^2$
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TOPIC 124: SECOND-ORDER DIFFERENTIAL EQUATION: PARTICULAR INTEGRAL

Particular Integral (y_p): Any particular solution (containing no arbitrary constants) of complete equation. y_p provides equilibrium value of $y(t)$, in intertemporal sense:

$$\underbrace{y(t)}_{\text{Time Path}} = \underbrace{y_c}_{\text{Deviation from equilibrium value of } y(t)} + \underbrace{y_p}_{\text{Equilibrium value of } y(t)}$$

Subcase – I: Since the particular integral (y_p) can be any solution, i.e., any value of y that satisfies this nonhomogeneous equation, we should always try the simplest possible type; namely, $y = a \text{ constant}$. If $y = k$, it follows that:

$$\begin{aligned} y &= y(t) = k \\ y'(t) &= 0 \\ y''(t) &= 0 \end{aligned}$$

Substituting the values of $y(t)$, $y'(t)$ and $y''(t)$ in the general form $y''(t) + a_1 \cdot y'(t) + a_2 \cdot y(t) = b$:

$$\begin{aligned} 0 + a_1 \cdot 0 + a_2 \cdot k &= b \\ a_2 \cdot k &= b \\ k &= \frac{b}{a_2} \end{aligned}$$

Since k is a particular value of y , that satisfies the differential equation, we can call it Particular Solution (y_p):

$$\begin{aligned} y_p &= \frac{b}{a_2} \\ y_p &= \frac{b}{a_2} \end{aligned}$$

Interpretation: Particular Solution has actually a constant value as b and a_2 are constants and so will be their ratio.

Subcase – II: If $a_2 = 0$, the formula of particular solution ($y_p = \frac{b}{a_2}$) shall give undefined answer. i.e. ($y_p = \frac{b}{0} = \infty$). Therefore, we need some other formula for (y_p). For another formula, we need another assumption regarding solution value of $y(t)$. Instead of considering constant a solution y , we try a nonconstant (variable); i.e. $y = kt$.

$$\begin{aligned} y(t) &= kt \\ y'(t) &= k \\ y''(t) &= 0 \end{aligned}$$

Substituting the values of $y(t)$, $y'(t)$, $y''(t)$ and a_2 in the general form $y''(t) + a_1 \cdot y'(t) + a_2 \cdot y(t) = b$:

$$\begin{aligned} 0 + a_1 \cdot k + 0 \cdot kt &= b \\ a_1 \cdot k + 0 &= b \end{aligned}$$

$$\boxed{k} = \frac{b}{a_1}$$

Since the assumed solution value of y was $y = \boxed{k}t$, we substitute the value of k to get the solution value of y :

$$y = \boxed{\frac{b}{a_1}}t$$

Since solution value is actually the Particular Solution (y_p):

$$y_p = \left(\frac{b}{a_1}\right)t$$

Interpretation: Particular Solution has actually a nonconstant value regardless of b and a_1 being constants. It is due to t that gets multiplied with their ratio $\left(\frac{b}{a_1}\right)$.

Subcase – III: If $a_1 = 0$ and already $a_2 = 0$, we check if **Subcase – I** and **Subcase – II** formula of particular solution are applicable:

Subcase – I	Subcase – II
$y_p = \frac{b}{a_1}$	$y_p = \frac{b}{a_2}t$
Substituting values of Subcase – III ($a_1 = 0, a_2 = 0$)	
$y_p = \frac{b}{0} = \infty$	$y_p = \frac{b}{0}t = \infty$
Therefore, the particular integral using both Subcase – III give an undefined answer. We need another formula to find the value of y_p .	

Building on the assumptions $y = k$ and $y = kt$, we use $y = kt^2$ in **Subcase – III**.

$$y(t) = kt^2$$

$$y'(t) = 2kt$$

$$y''(t) = 2k$$

Substituting the values of $y(t)$, $y'(t)$, $y''(t)$, $\underline{a_1}$ and $\underline{a_2}$ in the standard form: $y''(t) + \underline{a_1} \cdot$

$$y'(t) + \underline{a_2} \cdot y(t) = b:$$

$$\underline{2k} + \underline{0} \cdot \underline{2kt} + \underline{0} \cdot \underline{kt} = b$$

$$\underline{2k} + 0 + 0 = b$$

$$\underline{2k} = b$$

$$\boxed{k} = \frac{b}{2}$$

Since the assumed solution value of y was $y = \boxed{k}t^2$, we substitute the value of k to get the solution value of y :

$$y = \boxed{\frac{b}{2}}t^2$$

Since solution value is actually the Particular Solution (y_p):

$$y_p = \left(\frac{b}{2}\right)t^2$$

Interpretation: Particular Solution has actually a nonconstant value regardless of b being constant. It is due to t^2 that gets multiplied with their ratio $\left(\frac{b}{2}\right)$.

TOPIC 125: SECOND-ORDER DIFFERENTIAL EQUATION: PARTICULAR INTEGRAL NUMERICAL

Second-Order Differential Equation: Particular Integral Numerical Non-Homogenous Subcases

Non-Homogenous Subcase – I: $b \neq 0, a_2 \neq 0, \begin{cases} (a_1 \neq 0) \\ \vee \\ (a_1 = 0) \end{cases}$.

Example: Consider a Second Order Differential Equation:

$$y''(t) + y'(t) - 2y = -10$$

Comparing with the given specific equation with the standard form:

$$y''(t) + \underbrace{a_1}_{1} y'(t) + \underbrace{a_2}_{-2} y = \underbrace{b}_{-10}$$

Subcase-I *Non-Homogenous Case*

Here; $(a_1 = 1 \neq 0), (a_2 = -2 \neq 0), b = -10 \neq 0$.

Suitable Formula for Particular Solution: $y_p = \frac{b}{a_2}$

$$y_p = \frac{-10}{-2}$$

$$\boxed{y_p = 5}$$

Interpretation: $y_p = 5$ is particular solution of the **Non-Homogenous Subcase – I** of Second

Order Differential Equation $y''(t) + y'(t) - 2y = -10$.

Non-Homogenous Case – II: $b \neq 0, a_2 = 0, a_1 \neq 0$.

Example: Consider a Second Order Differential Equation:

$$y''(t) + y'(t) = -10$$

Comparing with the given specific equation with the standard form:

$$y''(t) + \underbrace{a_1}_{1} y'(t) + \underbrace{a_2}_{0} y = \underbrace{b}_{-10}$$

Subcase-II *Non-Homogenous Case*

Here; $(a_1 = 1 \neq 0), (a_2 = 0), b = -10 \neq 0$.

Suitable Formula for Particular Solution: $y_p = \frac{b}{a_1} t$

$$y_p = \frac{-10}{1} \times t$$

$$\boxed{y_p = -10t}$$

Interpretation: $y_p = -10t$ is particular solution of the **Non-Homogenous Case – II** of Second

Order Differential Equation $y''(t) + y'(t) = -10$.

Non-Homogenous Case – III: $b \neq 0, a_2 = 0, a_1 = 0$,

Example: Consider a Second Order Differential Equation:

$$y''(t) = -10$$

Comparing with the given specific equation with the standard form:

$$y''(t) + a_1 y'(t) + a_2 y = b$$

$$y''(t) + \overbrace{\underbrace{(0)}_{a_1} \times \underbrace{y'(t)}_{\boxed{y'(t)}} + \underbrace{(0)}_{a_2} \times \underbrace{y}_{\boxed{y}}}^{\text{self generated}} = \underbrace{-10}_b$$

Subcase-III Non-Homogenous Case

Here; $\overbrace{(a_1 = 0), (a_2 = 0)}$, $\overbrace{b = -10 \neq 0}$.

Suitable Formula for Particular Solution: $y_p = \frac{b}{2} t^2$

$$y_p = \frac{-10}{2} \times t^2$$

$$\boxed{y_p = -5t^2}$$

Interpretation: $y_p = -5t^2$ is particular solution of the **Non-Homogenous Case – III** of Second Order Differential Equation $y''(t) = -10$.

Summary: Non-Homogenous Case		
Subcase – I	Subcase – II	Subcase – III
$y''(t) + y'(t) - 2y = \boxed{-10}$	$y''(t) + y'(t) = \boxed{-10}$	$y''(t) = \boxed{-10}$
$y_p = \frac{\boxed{b}}{a_2}$	$y_p = \frac{\boxed{b}}{a_1} t$	$y_p = \frac{\boxed{b}}{2} t^2$
In Non-Homogenous case, $\boxed{b = -10}$		
$y_p = \frac{\boxed{-10}}{-2}$	$y_p = \frac{\boxed{-10}}{1} t$	$y_p = \frac{\boxed{-10}}{2} t^2$
$y_p = 5$	$y_p = -10t$	$y_p = -5t^2$

HOMOGENOUS SUBCASES

Since formulae of all subcases of y_p have b in numerator:

Summary: Homogenous Case		
Subcase – I	Subcase – II	Subcase – III
$y''(t) + y'(t) - 2y = \boxed{0}$	$y''(t) + y'(t) = \boxed{0}$	$y''(t) = \boxed{0}$
$y_p = \frac{\boxed{b}}{a_2}$	$y_p = \frac{\boxed{b}}{a_1} t$	$y_p = \frac{\boxed{b}}{2} t^2$
In Homogenous case, $\boxed{b = 0}$		
$y_p = \frac{\boxed{0}}{-2}$	$y_p = \frac{\boxed{0}}{1} t$	$y_p = \frac{\boxed{0}}{2} t^2$
$y_p = 0$	$y_p = 0$	$y_p = 0$

All subcases have a zero value of particular solution; i.e. ($y_p = 0$).

TOPIC 126: SECOND-ORDER DIFFERENTIAL EQUATION: STANDARD FORM & COMPLEMENTARY FUNCTION

Complementary Function (y_c): General solution (containing arbitrary constants) of reduced equation. y_c provides deviation of time path $y(t)$ from equilibrium value, at every point in time.

$y(t)$	=	y_c	+	y_p
<i>Time Path</i>		<i>Deviation from equilibrium value of $y(t)$</i>		<i>Equilibrium value of $y(t)$</i>

This is why, for dynamic stability, we want y_c term to disappear. So that $y(t)$ reaches y_p : $y(t) \rightarrow y_p$ when $y_c \rightarrow 0$.

Derivation of the Complementary Function (y_c):

Exponential expressions (Ae^{rt}) served well in complementary functions for differential equations as first order differential equations.

Experimenting with a comparison numerical instance:

$y(t) = Ae^{rt}$, For simplicity; $A = 1$ $y(t) = e^{rt}$ $y(t) = re^{rt}$ $y(t) = r^2 e^{rt}$	$y(t) = t^2$ $y'(t) = 2t$ $y''(t) = 2$
Assuming a second order differential equation: $y''(t) + a_1 y'(t) + a_2 y(t) = 0$	
$r^2 e^{rt} + a_1 r e^{rt} + a_2 e^{rt} = 0$	$2 + a_1 2t + a_2 t^2 = 0$
Reduction of L.H.S to zero (0) is possible as all variable expression e^{rt} are same. This, off course, will happen with the help of certain values of r , a_1 and a_2 .	Reduction of L.H.S to zero (0) is not possible as there are two different functions i.e. linear ($2t$) and quadratic (t^2). Here, values of a_1 and a_2 cannot help in reducing of R.H.S. to zero (0).

Therefore, we can develop Characteristic Equation, by substituting the value of $y(t)$, $y'(t)$ and $y''(t)$:

$$\begin{aligned}
 y(t) &= Ae^{rt} \\
 y'(t) &= rAe^{rt} \\
 y''(t) &= r^2 Ae^{rt} \\
 y''(t) + a_1 y'(t) + a_2 y(t) &= 0 \\
 r^2 Ae^{rt} + a_1 rAe^{rt} + a_2 Ae^{rt} &= 0 \\
 Ae^{rt}(r^2 + a_1 r + a_2) &= 0 \\
 r^2 + a_1 r + a_2 &= 0
 \end{aligned}$$

Comparing $ax^2 + bx + c = 0$ we get $x = r$, $a = 1$, $b = a_1$, $c = a_2$.

Quadratic Formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ becomes:

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \text{ and } r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

Two Relationships to verify the Roots (r_1, r_2):

Relationship – I: Sum of the two roots equals ($-a_1$).

$$\begin{aligned}
 r_1 + r_2 &= \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \right) + \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \right) \Rightarrow -\frac{a_1}{2} + \frac{\sqrt{a_1^2 - 4a_2}}{2} + -\frac{a_1}{2} - \frac{\sqrt{a_1^2 - 4a_2}}{2} \Rightarrow -\frac{a_1}{2} - \frac{a_1}{2} \Rightarrow \\
 &\frac{-a_1 - a_1}{2} \Rightarrow \frac{-2a_1}{2} \Rightarrow -a_1
 \end{aligned}$$

Relationship – II: Product of the two roots equals (a_2).

$$r_1 \times r_2 = \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \right) \times \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \right) \Rightarrow \left(-\frac{a_1}{2} \right)^2 - \left(\frac{\sqrt{a_1^2 - 4a_2}}{2} \right)^2$$

$$\Rightarrow \frac{a_1^2}{4} - \frac{a_1^2 - 4a_2}{4} \Rightarrow \frac{a_1^2 - a_1^2 + 4a_2}{4} \Rightarrow \frac{4a_2}{4} \Rightarrow a_2$$

TOPIC 127: SECOND-ORDER DIFFERENTIAL EQUATION: DISTINCT REAL ROOTS

Characteristic Equation of second order differential equation is (r_1, r_2) :

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \text{ and } r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

Nature of characteristic roots depends on value of discriminant $(a_1^2 - 4a_2)$:

It can have three possibilities: $(a_1^2 - 4a_2) \gtrless 0$

Case – I	Case – II	Case – III
$(a_1^2 - 4a_2) > 0$ $a_1^2 > 4a_2$	$(a_1^2 - 4a_2) = 0$ $a_1^2 = 4a_2$	$(a_1^2 - 4a_2) < 0$ $a_1^2 < 4a_2$
Distinct Real Roots	Repeated Real Roots	Complex Roots

Here, we consider the Distinct Real Roots $(a_1^2 - 4a_2) > 0$:

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1, r_2 = \frac{\overbrace{-a_1}^{\text{added}} \pm \sqrt{a_1^2 - 4a_2}}{\underbrace{2}_{\text{subtracted}}} \Rightarrow r_1 \neq r_2$$

The complementary function in case of Distinct Real Roots shall have **two** and **distinct** exponential terms:

$$y_c = y_1 + y_2$$

$$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t} \quad (r_1 \neq r_2)$$

Numerical Example: Solve the differential equation: $y''(t) + y'(t) - 2y = -10$, if $y(0) = 12$ and $y'(0) = -2$.

Comparing with the standard of second order differential equation:

$$y''(t) + a_1 y'(t) + a_2 y = b$$

$$y''(t) + \underbrace{(1)}_{a_1} y'(t) + \underbrace{(-2)}_{a_2} y = \underbrace{-10}_b$$

Nature of Roots: $(a_1^2 - 4a_2) \gtrless 0$

$$\underbrace{9 > 0}_{\substack{\text{Distinct} \\ \text{Real Roots}}}$$

$$\{(1)^2 - 4(-2)\} \geq 0$$

$$\underbrace{9 > 0}_{\substack{\text{Distinct} \\ \text{Real Roots}}}$$

Calculation of Roots:

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1, r_2 = \frac{-(1) \pm \sqrt{(1)^2 - 4(-2)}}{2}$$

$$r_1, r_2 = \frac{-1 \pm \sqrt{9}}{2}$$

$$r_1 = \frac{-1 + 3}{2}, r_2 = \frac{-1 - 3}{2}$$

$$\boxed{r_1 = 1} \text{ or } \boxed{r_2 = -2}$$

These roots can be substituted in the formula of Complementary Function (y_c):

Complementary Function: $y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$

$$y_c = A_1 e^{1t} + A_2 e^{-2t}$$

$$\boxed{y_c = A_1 e^t + A_2 e^{-2t}}$$

General Solution: $y(t) = y_c + y_p$

$$\boxed{y(t) = \underbrace{A_1 e^t + A_2 e^{-2t}}_{y_c} + \underbrace{5}_{y_p}}$$

Definite Solution: For definite solution, we substitute $t = 0$ in the General Solution; $y(t) = A_1 e^t + A_2 e^{-2t} + 5$:

$$\underbrace{y(0)}_{\text{Given}} = A_1 e^{(0)} + A_2 e^{-2(0)} + 5$$

$$\underbrace{y(0)}_{12} = A_1(1) + A_2(1) + 5$$

$$12 = A_1 + A_2 + 5$$

$$7 = A_1 + A_2$$

As there are two unknowns A_1 and A_2 , we need two (simultaneous) equations to evaluate them. Second equation can be the derivative of $y(t)$, as $y'(0)$ is given.

$$y(t) = A_1 e^t + A_2 e^{-2t} + 5$$

$$y'(t) = A_1 e^t + A_2(-2)e^{-2t} + 0$$

$$y'(t) = A_1 e^t - 2A_2 e^{-2t}$$

Introducing the initial condition $t = 0$ in $y'(t) = A_1 e^t - 2A_2 e^{-2t}$:

$$\underbrace{y'(0)}_{\text{Given}} = A_1 e^{(0)} - 2A_2 e^{-2(0)}$$

$$\underbrace{y'(0)}_{(-2)} = A_1(1) - 2A_2(1)$$

Solving simultaneously: $7 = A_1 + A_2$ and $-2 = A_1 - 2A_2$

$$7 = A_1 + A_2 \Rightarrow \boxed{7 - A_2 = A_1}$$

$$-2 = \boxed{A_1} - 2A_2 \Rightarrow -2 = \boxed{7 - A_2} - 2A_2$$

$$A_2 + 2A_2 = 7 + 2$$

$$3A_2 = 9$$

$$\boxed{A_2 = 3}$$

For evaluating A_1 , put $A_2 = 3$ in $7 - A_2 = A_1$

$$A_1 = 7 - A_2$$

$$A_1 = 7 - 3$$

$$\boxed{A_1 = 4}$$

Returning value of A_1 and A_2 in General Solution $y(t) = \boxed{A_1}e^t + \boxed{A_2}e^{-2t} + 5$ to get the Definite Solution:

$$y(t) = \boxed{4}e^t + \boxed{3}e^{-2t} + 5$$

Interpretation: Definite Solution of Second Order Differential Equation $y''(t) + y'(t) - 2y = -10$ with case of Distinct Real Roots is $y(t) = 4e^t + 3e^{-2t} + 5$.

Lesson 29

SECOND ORDER DIFFERENTIAL EQUATIONS (CONTINUED 1)

TOPIC 128: SECOND-ORDER DIFFERENTIAL EQUATION: REPEATED REAL ROOTS

Characteristic Equation of second order differential equation is (r_1, r_2) :

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \text{ and } r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

Nature of characteristic roots depends on value of discriminant $(a_1^2 - 4a_2)$:

It can have three possibilities: $(a_1^2 - 4a_2) \geq 0$

Case – I	Case – II	Case – III
$(a_1^2 - 4a_2) > 0$ $a_1^2 > 4a_2$	$(a_1^2 - 4a_2) = 0$ $a_1^2 = 4a_2$	$(a_1^2 - 4a_2) < 0$ $a_1^2 < 4a_2$
Distinct Real Roots	Repeated Real Roots	Complex Roots

Here, we consider the Repeated Real Roots $(a_1^2 - 4a_2) = 0$:

$$r = \frac{-a_1 \pm \sqrt{0}}{2}$$

$$r_1, r_2 = \frac{-a_1}{2} \Rightarrow r_1 = r_2 = r$$

The complementary function in case of Repeated Real Roots shall have **two** but **Repeated** exponential terms:

$$y_c = A_3 e^{r_1 t} + A_4 t e^{r_2 t}$$

Numerical Example: Solve the differential equation: $y''(t) + 6y'(t) + 9y = 27$, if $y(0) = 5$ and $y'(0) = -5$.

Comparing with the standard of second order differential equation:

$$y''(t) + a_1 y'(t) + a_2 y = b$$

$$y''(t) + \underbrace{(6)}_{a_1} y'(t) + \underbrace{(9)}_{a_2} y = \underbrace{27}_b$$

Nature of Roots: $(a_1^2 - 4a_2) \geq 0$

$$36 - 36 = 0$$

$$\{(6)^2 - 4(9)\} \geq 0$$

$$36 - 36 = 0$$

$$0 > 0$$

*Repeated
Real Roots*

Calculation of Roots:

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1, r_2 = \frac{-(6) \pm \sqrt{(6)^2 - 4(9)}}{2}$$

$$r_1, r_2 = \frac{-6 \pm \sqrt{0}}{2}$$

$$r_1 = \frac{-6}{2}, r_2 = \frac{-6}{2}$$

$$r_1 = -3 \text{ and } r_2 = -3$$

$$r = -3$$

These roots can be substituted in the formula of Complementary Function (y_c):

Complementary Function: $y_c = A_3 e^{rt} + A_4 t e^{rt}$

$$y_c = A_3 e^{-3t} + A_4 t e^{-3t}$$

$$y_c = A_3 e^{-3t} + A_4 t e^{-3t}$$

General Solution: $y(t) = y_c + y_p$

$$y(t) = \underbrace{A_3 e^{-3t} + A_4 t e^{-3t}}_{y_c} + \underbrace{3}_{y_p}$$

Definite Solution: For definite solution, we substitute $t = 0$ in the General Solution; $y(t) = A_3 e^{-3t} + A_4 t e^{-3t} + 3$:

$$\underbrace{y(0)}_{\text{Given}} = A_3 e^{-3(0)} + A_4(0) e^{-3(0)} + 3$$

$$\underbrace{y(0)}_5 = A_3(1) + 0 + 3$$

$$5 = A_3 + 3$$

$$A_3 = 2$$

There is another unknown constant (A_4). Value of $y'(0)$ is given, so take derivative of $y(t) = A_3 e^{-3t} + A_4 t e^{-3t} + 3$:

$$y(t) = A_3 e^{-3t} + A_4 t e^{-3t} + 3$$

$$y'(t) = A_3(-3)e^{-3t} + A_4\{(1)e^{-3t} + t(-3)e^{-3t}\} + 0$$

$$y'(t) = -3 \underbrace{A_3}_{\text{given}} e^{-3t} + A_4(e^{-3t} - 3te^{-3t})$$

$$y'(t) = -3 \underbrace{A_3}_2 e^{-3t} + A_4 e^{-3t} - 3A_4 t e^{-3t}$$

$$y'(t) = -3(2)e^{-3t} + A_4 e^{-3t} - 3A_4 t e^{-3t}$$

Introducing initial condition $t = 0$ in $y'(t) = -3(2)e^{-3t} + A_4 e^{-3t} - 3A_4 t e^{-3t}$:

$$\underbrace{y'(0)}_{\text{given}} = -3(2)(1) + A_4(1) - 0$$

$$\underbrace{y'(0)}_{-5} = -6 + A_4$$

$$-5 = -6 + A_4$$

$$A_4 = 1$$

Returning value of A_3 and A_4 in General Solution $y(t) = A_3 e^{-3t} + A_4 t e^{-3t} + 3$ to get the Definite Solution:

$$y(t) = 2e^{-3t} + 1te^{-3t} + 3$$

$$y(t) = 2e^{-3t} + te^{-3t} + 3$$

Interpretation: Definite Solution of Second Order Differential Equation $y''(t) + 6y'(t) + 9y = 27$ with case of Repeated Real Roots is $y(t) = 2e^{-3t} + te^{-3t} + 3$.

TOPIC 129: SECOND-ORDER DIFFERENTIAL EQUATION: COMPLEX ROOTS

Characteristic Equation of second order differential equation is (r_1, r_2) :

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \text{ and } r_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

Nature of characteristic roots depends on value of discriminant $(a_1^2 - 4a_2)$:

It can have three possibilities: $(a_1^2 - 4a_2) \geq 0$

Case – I	Case – II	Case – III
$(a_1^2 - 4a_2) > 0$	$(a_1^2 - 4a_2) = 0$	$(a_1^2 - 4a_2) < 0$
$a_1^2 > 4a_2$	$a_1^2 = 4a_2$	$a_1^2 < 4a_2$
Distinct Real Roots	Repeated Real Roots	Complex Roots

Here, we consider the Complex Roots $(a_1^2 - 4a_2) < 0$:

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{(4a_2 - a_1^2) \times (-1)}}{2}$$

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{(4a_2 - a_1^2) \times \sqrt{(-1)}}}{2}$$

$$r_1, r_2 = \frac{-a_1 \pm \sqrt{(4a_2 - a_1^2) \times i}}{2}$$

$$r_1, r_2 = \frac{-a_1}{2} \pm \frac{\sqrt{4a_2 - a_1^2}}{2} \times i$$

$$r_1 = \frac{-a_1}{2} + \frac{\sqrt{4a_2 - a_1^2}}{2} \times i \text{ and } r_2 = \frac{-a_1}{2} - \frac{\sqrt{4a_2 - a_1^2}}{2} \times i$$

Using shorthand notations h and v .

$$r_1 = h + vi \text{ and } r_2 = h - vi$$

In the form of Conjugate Complex Roots:

$$r_1, r_2 = \underbrace{h}_{\text{real part}} \pm \underbrace{vi}_{\text{imaginary part}}$$

Borrowing these roots to put in the Complementary Function:

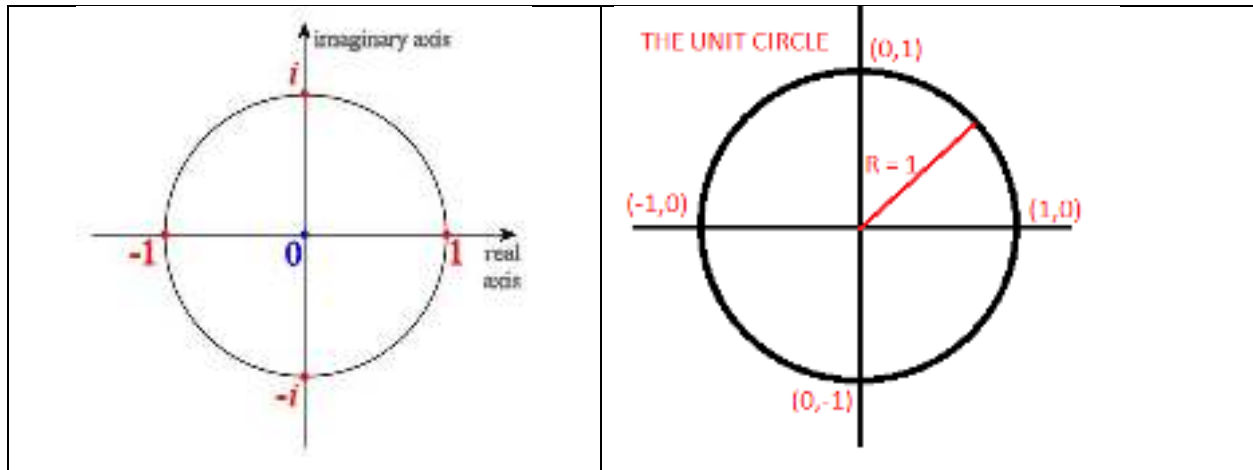
$$\begin{aligned} y_c &= y_1 + y_2 \\ y_c &= A_5 e^{r_1 t} + A_6 e^{r_2 t} \\ y_c &= A_5 e^{(h+vi)t} + A_6 e^{(h-vi)t} \\ y_c &= A_5 e^{ht+vit} + A_6 e^{ht-vit} \\ y_c &= A_5 e^{ht} e^{vit} + A_6 e^{ht} e^{-vit} \\ y_c &= e^{ht} (A_5 e^{vit} + A_6 e^{-vit}) \end{aligned}$$

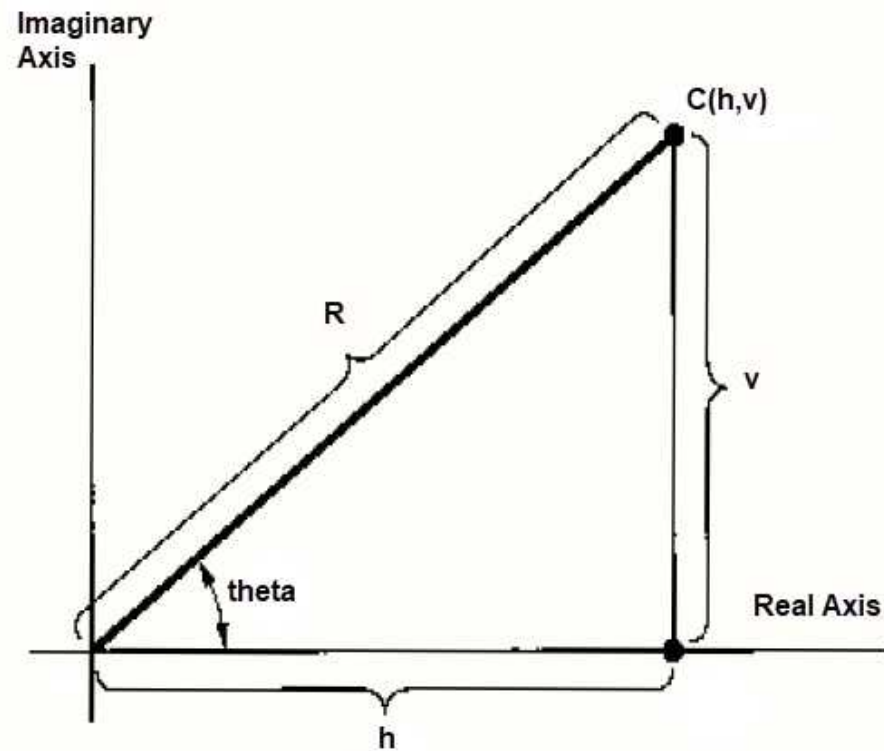
The complementary function in case of Complex Roots shall have **two** but **Complex** exponential terms:

$$y_c = e^{ht} (A_5 e^{vit} + A_6 e^{-vit})$$

Involvement of imaginary component (i) in the expression requires attention. An alternative form in terms of trigonometric ratios is:

$$y_c = e^{ht} (A_5 \cos vt + A_6 \sin vt)$$





Argand Diagram: Graphical Representation of a Complex number

$$R^2 = h^2 + v^2 \text{ (Pythagorean Theorem)}$$

$$R = \sqrt{h^2 + v^2}$$

Numerical Example: Solve the differential equation: $y''(t) + 2y'(t) + 17y = 34$, if $y(0) = 3$ and $y'(0) = 11$.

Comparing with the standard of second order differential equation:

$$y''(t) + a_1 y'(t) + a_2 y = b$$

$$y''(t) + \underbrace{(2)}_{a_1} y'(t) + \underbrace{(17)}_{a_2} y = \underbrace{34}_b$$

Here, $a_1 = 2$, $a_2 = 17$, $b = 34$

Nature of Roots: $(a_1^2 - 4a_2) \gtrless 0$

$$\frac{-64 < 0}{\text{Complex Roots}}$$

$$-64 \geq 0$$

$$\frac{-64 < 0}{\text{Complex Roots}}$$

Calculation of Roots:

$$\begin{aligned} r_1, r_2 &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \\ r_1, r_2 &= \frac{-(2) \pm \sqrt{(2)^2 - 4(17)}}{2} \\ r_1, r_2 &= \frac{-2 \pm \sqrt{-64}}{2} \\ r_1, r_2 &= \frac{-2 \pm \sqrt{64} \times \sqrt{-1}}{2} \\ r_1, r_2 &= \frac{-2 \pm 8i}{2} \\ r_1 &= \frac{-2 + 8i}{2}, r_2 = \frac{-2 - 8i}{2} \\ r_1 &= -\frac{2}{2} + \frac{8}{2}i, r_2 = -\frac{2}{2} - \frac{8}{2}i \\ \boxed{r_1 = \underbrace{-1}_h + \underbrace{4i}_v}, \boxed{r_2 = \underbrace{-1}_h - \underbrace{4i}_v} \end{aligned}$$

From the roots, we extract: $h = -1$ and $v = 4$.

Substituting in the formula of Complementary Function $y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$:

Complementary Function: $y_c = e^{ht}(A_5 \cos vt + A_6 \sin vt)$
 $y_c = e^{-1t}(A_5 \cos 4t + A_6 \sin 4t)$
 $\boxed{y_c = e^{-t}(A_5 \cos 4t + A_6 \sin 4t)}$

After developing complementary function, we get general solution by adding particular integral $y_p = 2$.

$$\boxed{y(t) = e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2}$$

After attaining the General Solution, we get Definite Solution by definitizing the arbitrary constants A_5 and A_6 .

Introducing initial condition; $t = 0$ in the General Solution:

$$\begin{aligned} y(0) &= e^{-(0)}\{A_5 \cos 4(0) + A_6 \sin 4(0)\} + 2 \\ \underbrace{y(0)}_{\text{given}} &= 1 \times (A_5 \cos 0 + A_6 \sin 0) + 2 \\ \underbrace{y(0)}_3 &= \{A_5(1) + A_6(0)\} + 2 \\ 3 &= (A_5) + 2 \\ \boxed{A_5} &= 1 \end{aligned}$$

To substitute $y'(0)$, we take the derivative of general solution $y(t)$:

$$\frac{d}{dt}\{y(t)\} = \frac{d}{dt}\{e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2\}$$

$$y'(t) = \frac{d}{dt} \left\{ \underbrace{e^{-t}}_{g(t)} \underbrace{(A_5 \cos 4t + A_6 \sin 4t)}_{h(t)} \right\} + \frac{d}{dt} (2)$$

Product Rule of differentiation: $\frac{d}{dt} \{f(t)\} = \frac{d}{dt} \{g(t)\} \times h(t) + \frac{d}{dt} \{h(t)\} \times g(t)$

$$y'(t) = \frac{d}{dt} \{(e^{-t})\} \times (A_5 \cos 4t + A_6 \sin 4t) + \frac{d}{dt} \{(A_5 \cos 4t + A_6 \sin 4t)\} \times (e^{-t})$$

$$y'(t) = (-1)(e^{-t}) \times (A_5 \cos 4t + A_6 \sin 4t) + \left\{ \frac{d}{dt} (A_5 \cos 4t) + \frac{d}{dt} (A_6 \sin 4t) \right\} \times (e^{-t})$$

$$y'(t) = (-e^{-t}) \times (A_5 \cos 4t + A_6 \sin 4t) + \left\{ A_5 \frac{d}{dt} (\cos 4t) + A_6 \frac{d}{dt} (\sin 4t) \right\} \times (e^{-t})$$

$$y'(t) = (e^{-t}) \times (-A_5 \cos 4t - A_6 \sin 4t) + \{A_5(-4)(\sin 4t) + A_6 4(\cos 4t)\} \times (e^{-t})$$

$$y'(t) = (e^{-t}) \times (-A_5 \cos 4t - A_6 \sin 4t) + \{-4A_5(\sin 4t) + 4A_6(\cos 4t)\} \times (e^{-t})$$

Introducing the initial condition ($t = 0$) in this derivative:

$$y'(0) = \{e^{-(0)}\} \times \{-A_5 \cos 4(0) - A_6 \sin 4(0)\} + \{-4A_5 \sin 4(0) + 4A_6 \cos 4(0)\} \times \{e^{-(0)}\}$$

$$\underbrace{y'(0)}_{\text{given}} = (1) \times (-A_5 \cos 0 - A_6 \sin 0) + \{-4A_5(\sin 0) + 4A_6(\cos 0)\} \times (1)$$

$$\underbrace{y'(0)}_{11} = \{-A_5(1) - A_6(0)\} - 4A_5(0) + 4A_6(1)$$

$$11 = -A_5 - 0 - 0 + 4A_6$$

$$11 = -A_5 + 4A_6$$

already
found

$$11 = -A_5 + 4A_6$$

$$11 = -1 + 4A_6$$

$$12 = 4A_6$$

$$A_6 = 3$$

Returning the values of A_5 and A_6 in the general solution $y(t) = e^{-t}(A_5 \cos 4t + A_6 \sin 4t) + 2$:

$$y(t) = e^{-t}(\cos 4t + 3 \sin 4t) + 2$$

$$y(t) = \underbrace{e^{-t}(\cos 4t + 3 \sin 4t)}_{y_c} + \underbrace{2}_{y_p}$$

Interpretation: Definite Solution of Second Order Differential Equation $y''(t) + 2y'(t) + 17y = 34$ with case of Complex Roots is $y(t) = e^{-t}(\cos 4t + 3 \sin 4t) + 2$.

TOPIC 130: SECOND-ORDER DIFFERENTIAL EQUATION: SUMMARY OF COMPLEMENTARY FUNCTION

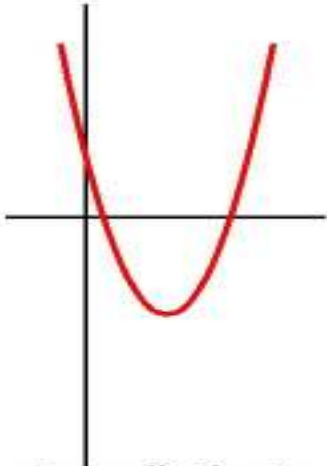
As we have developed the characteristic roots before:

$$r = r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

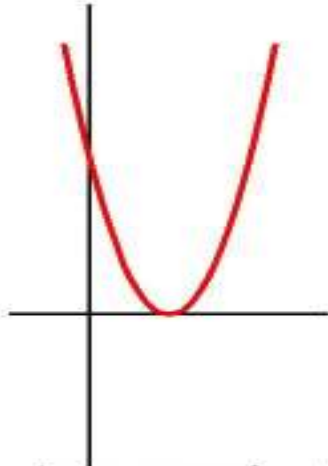
Discriminant ($a_1^2 - 4a_2$) determines the nature of roots.

Possible Cases for Complementary Solution (y_c) of (SODE)			
Condition	$a_1^2 - 4a_2 > 0, a_1^2 > 4a_2$	$a_1^2 - 4a_2 = 0, a_1^2 = 4a_2$	$a_1^2 - 4a_2 < 0, a_1^2 < 4a_2$
Remarks	+ve Discriminant.	zero Discriminant.	-ve Discriminant.

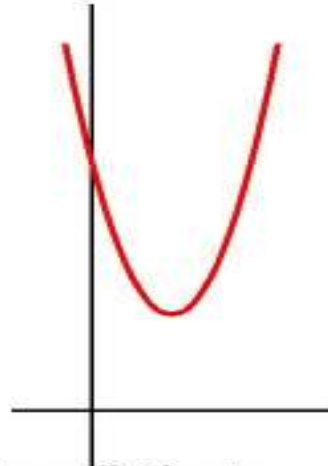
	Distinct Real Roots Distinct: $\pm(+ve \text{ Discriminant})$ $\Rightarrow r_1 \neq r_2$	Repeated/Equal Real Roots Repeated/Equal: $\pm(\text{zero Discriminant})$ $\Rightarrow r_1 = r_2 \Rightarrow r$	Complex Roots Complex: $\pm(-ve \text{ Discriminant})$ $r_1 = r_2$
Root(s)' Formula	$r_1, r_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$	$r_1 = r_2 = r = \frac{-a_1}{2}$	(Conjugate Roots) $r_1, r_2 = h \pm vi$ Where $h = \frac{-a_1}{2}$ $v = \frac{\sqrt{4a_2 - a_1^2}}{2}$
Verification	$r_1 + r_2 = -a_1$ $r_1 \times r_2 = a_2$	$r_1 + r_2 = -a_1$ $r_1 \times r_2 = a_2$	$r_1 + r_2 = -a_1$ $r_1 \times r_2 = a_2$
(y_c) Formula	$y_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$	$y_c = A_3 e^{rt} + A_4 t e^{rt}$	$y_c = A_5 e^{(h+vi)t} + A_6 e^{(h-vi)t}$ $y_c = e^{ht} (A_5 e^{vit} + A_6 e^{-vit})$ $y_c = e^{ht} (A_5 \cos vt + A_6 \sin vt)$
Dynamic Stability	$r_1 \& r_2 < 0$	$r < 0$	$h < 0$



two distinct
real roots



two equal
real roots



two distinct
complex roots

TOPIC 131: SECOND-ORDER DIFFERENTIAL EQUATION: DYNAMIC STABILITY OF DISTINCT REAL ROOTS

Recalling the under Distinct Real Roots case:

$$y(t) = \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{y_c} + \underbrace{\frac{b}{a_2} \frac{b}{a_1} t \frac{b}{2} t^2}_{y_p}$$

$$y(t) = \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{\text{deviation from equilibrium}} + \underbrace{\frac{b}{a_2} \frac{b}{a_1} t \frac{b}{2} t^2}_{\text{equilibrium}}$$

$$\begin{array}{c}
 \text{Try to mitigate it} \\
 \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{\substack{\text{deviation from} \\ \text{equilibrium}}} \\
 \begin{array}{ccccccc}
 \text{arbitrary} & & \text{independent} & & \text{arbitrary} & & \text{independent} \\
 \text{constant} & & \text{variable} & & \text{constant} & & \text{variable} \\
 \widehat{A}_1 & e^{\widehat{r}_1 t} & & & \widehat{A}_2 & e^{\widehat{r}_2 t} & \\
 & & & + & & & \\
 \underbrace{\hspace{10em}}_{\text{deviation from equilibrium}}
 \end{array}
 \end{array}$$

A_1 and A_2 are constant and their values depend on initial conditions. t is independent variable spanning $(0 - \infty)$. Deviation from equilibrium will depend on roots (r_1, r_2) .

Hair Splitting:

$$A_1 e^{\widehat{r}_1(t)} + A_2 e^{\widehat{r}_2(t)}$$

r_1 & r_2 , determine the dynamic stability.

If both r_1 & r_2 are $-ve$, then the exponential terms ($A_1 e^{r_1 t}$ & $A_2 e^{r_2 t}$) will fade away due to exponential decay. Automatically their sum also fades away:

$$\left\{ A_1 \times \underbrace{e^{\widehat{r}_1 t}}_{\substack{\text{exponential} \\ \text{decay}}} + A_2 \times \underbrace{e^{\widehat{r}_2 t}}_{\substack{\text{exponential} \\ \text{decay}}} \right\} \rightarrow 0 \text{ when } t \rightarrow \infty, \therefore y_c \rightarrow 0.$$

Caveat: Both r_1 and r_2 should be negative, else convergence may not take place.

Numerical Example:

$y(t) = 4e^t + 3e^{2t} + 5$			$y(t) = 4e^t + 3e^{-2t} + 5$			$y(t) = 4e^{-t} + 3e^{-2t} + 5$			
$4e^t + 3e^{2t}$		y_c	$4e^t + 3e^{-2t}$		y_c	$4e^{-t} + 3e^{-2t}$		y_c	
$A_1 e^{r_1 t}$	$A_2 e^{r_2 t}$		$A_1 e^{r_1 t}$	$A_2 e^{r_2 t}$		$A_1 e^{r_1 t}$	$A_2 e^{r_2 t}$		
t	$4e^{(+ve)\hat{(1)}t}$		$3e^{(+ve)\hat{(2)}t}$	$4e^{(+ve)\hat{(1)}t}$		$3e^{(-ve)\hat{(-2)}t}$	$4e^{(-ve)\hat{(-1)}t}$		$3e^{(-ve)\hat{(-2)}t}$
0	4	3	7	4	3	7	4	3	7
1	10.9	22.2	33.0	10.9	0.4060	11.3	1.4715	0.4060	1.8775
2	29.6	163.8	193.4	29.6	0.0549	29.6	0.5413	0.0549	0.5963
3	80.3	1210.3	1290.6	80.3	0.0074	80.3	0.1991	0.0074	0.2066
4	218.4	8942.9	9161.3	218.4	0.0010	218.4	0.0733	0.0010	0.0743
5	593.7	66079.4	66673.1	593.7	0.0001	593.7	0.0270	0.0001	0.0271
6	1613.7	488264.4	489878.1	1613.7	0.0000	1613.7	0.0099	0.0000	0.0099
↑↓	$(A_1 e^{r_1 t})$	$(A_2 e^{r_2 t})$	(y_c)	$(A_1 e^{r_1 t})$	$(A_2 e^{r_2 t})$	(y_c)	$(A_1 e^{r_1 t})$	$(A_2 e^{r_2 t})$	(y_c)
	↑	↑	↑	↑	↓	↑	↓	↓	↓
	Both exponential terms increase		Deviation increases	1 st exponential term increases, 2 nd one decreases		Deviation increases	Both exponential terms decrease		Deviation decreases
	Divergence		Divergence			Convergence			
	Dynamically Unstable		Dynamically Unstable			Dynamically Stable			
Hence it is numerically confirmed that for Dynamic Stability, BOTH the roots must be $-ve: (r_1 \wedge r_2) < 0$									

Lesson 30

SECOND ORDER DIFFERENTIAL EQUATIONS (CONTINUED 2)

TOPIC 132: SECOND-ORDER DIFFERENTIAL EQUATION: DYNAMIC STABILITY OF REPEATED REAL ROOTS

Recalling the under Repeated Real Roots case:

$$y(t) = \underbrace{A_3 e^{rt} + A_4 t e^{rt}}_{y_c} + \underbrace{\frac{b}{a_2} \frac{b}{a_1} \frac{b}{2} t^2}_{y_p}$$

$$y(t) = \underbrace{A_3 e^{rt} + A_4 t e^{rt}}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{Try to mitigate it}}} + \underbrace{y_p}_{\text{equilibrium}}$$

$$\underbrace{A_3 e^{rt} + A_4 t e^{rt}}_{\substack{\text{deviation from} \\ \text{equilibrium}}}$$

$$\underbrace{\widehat{A}_3}_{\substack{\text{arbitrary} \\ \text{constant}}} e^{\widehat{r} t} \underbrace{\widehat{t}}_{\substack{\text{independent} \\ \text{variable}}} + \underbrace{\widehat{A}_4}_{\substack{\text{arbitrary} \\ \text{constant}}} \underbrace{\widehat{t}}_{\substack{\text{independent} \\ \text{variable}}} e^{\widehat{r} t} \underbrace{\widehat{t}}_{\substack{\text{independent} \\ \text{variable}}}$$

deviation from equilibrium

A_3 and A_4 are constant and their values depend on initial conditions. t is independent variable spanning $(0 - \infty)$. Deviation from equilibrium will depend on root (r).

Hair Splitting:

$$A_3 e^{\widehat{r} t} + A_4 t e^{\widehat{r} t}$$

r , determine the dynamic stability.

If r is $-ve$, then the exponential terms ($A_1 e^{r_1 t}$ & $A_2 t e^{r_2 t}$) will fade away due to exponential decay. Though, the second term ($A_2 t e^{r_2 t}$) has an additional term t being multiplied, yet it doesn't reverse the dynamics of this exponential term. Eventually, sum of the two exponential terms also fades away:

$$\left\{ A_3 \times \underbrace{e^{\widehat{r} t}}_{\substack{-ve \\ \text{exponential} \\ \text{decay}}} + A_4 \times \underbrace{\widehat{t}}_{\substack{(0,1,2,\dots,\infty) \\ \text{Neutral} \\ \text{multiplier}}} \underbrace{e^{\widehat{r} t}}_{\substack{-ve \\ \text{exponential} \\ \text{decay}}} \right\} \rightarrow 0 \text{ when } t \rightarrow \infty, \therefore y_c \rightarrow 0.$$

Numerical Example:

	$y(t) = 2e^{3t} + te^{3t} + 3$			$y(t) = 2e^{-3t} + te^{-3t} + 3$		
	$2e^{3t} + te^{3t}$		y_c	$2e^{-3t} + te^{-3t}$		y_c
	$A_3 e^{rt}$	$A_4 t e^{rt}$		$A_3 e^{rt}$	$A_4 t e^{rt}$	
t	$2e^{(+ve)(3)t}$	$te^{(+ve)(3)t}$		$2e^{(-ve)(-3)t}$	$te^{(-ve)(-3)t}$	
0	2	0	2	2	0	2
1	40.2	20.1	60.3	0.0996	0.0498	0.149361
2	806.9	806.9	1613.7	0.0050	0.0050	0.009915
3	16206.2	24309.3	40515.4	0.0002	0.0004	0.000617
4	325509.6	651019.2	976528.7	0.00001	0.00002	0.000037
5	6538034.7	16345086.9	22883121.6	0.0000006	0.000002	0.000002
↓↑	$(A_3 e^{rt}) \uparrow$	$(A_4 t e^{rt}) \uparrow$	$(y_c) \uparrow$	$(A_3 e^{rt}) \downarrow$	$(A_4 t e^{rt}) \downarrow$	$(y_c) \downarrow$

	Both exponential terms increase	Deviation increases	Both exponential terms decrease	Deviation decreases
	Divergence		Convergence	
	Dynamically Unstable		Dynamically Stable	
Hence it is numerically confirmed that for Dynamic Stability, the repeated root (r) must be $-ve$; $r < 0$				

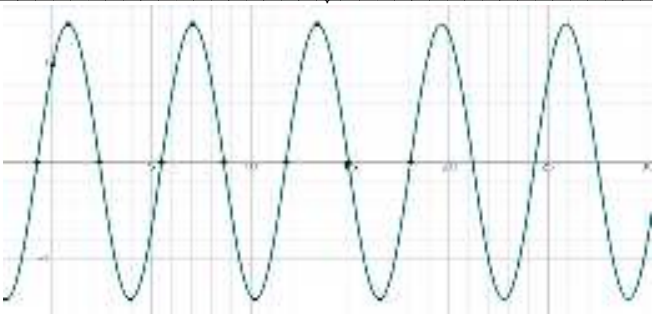
TOPIC 133: SECOND-ORDER DIFFERENTIAL EQUATION: DYNAMIC STABILITY OF COMPLEX ROOTS

Recalling the under Complex Roots case:

$$y(t) = \underbrace{e^{ht}(A_5 \cos vt + A_6 \sin vt)}_{y_c} + \underbrace{y_p}_{\frac{b}{a_2}, \frac{b}{a_1}t, \frac{b}{2}t^2}$$

Hair Splitting:

$$y(t) = \underbrace{e^{ht}(A_5 \cos vt + A_6 \sin vt)}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{Try to mitigate it}}} + \underbrace{y_p}_{\text{equilibrium}}$$

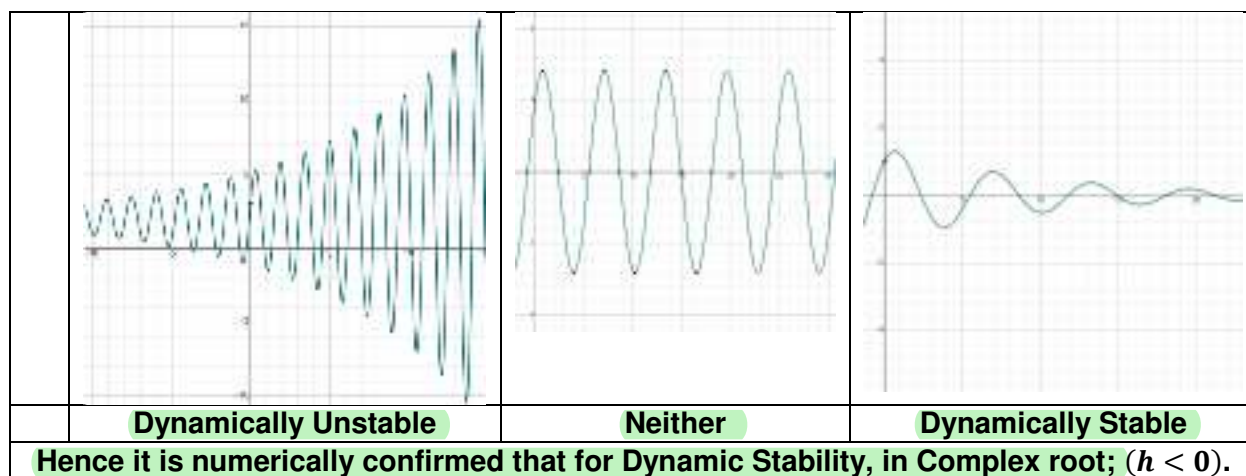
$$\underbrace{e^{ht}}_{\substack{\text{arbitrary} \\ \text{constant}}} \underbrace{(A_5 \cos vt + A_6 \sin vt)}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{multiplicand}}} = \underbrace{e^{ht}}_{\substack{\text{multiplier}}} \left(\underbrace{\widehat{A}_5}_{\substack{\text{arbitrary} \\ \text{constant}}} \underbrace{\cos v}_{\substack{\text{independent} \\ \text{variable}}} \underbrace{\widehat{t}}_{\substack{\text{arbitrary} \\ \text{constant}}} + \underbrace{\widehat{A}_6}_{\substack{\text{arbitrary} \\ \text{constant}}} \underbrace{\sin v}_{\substack{\text{independent} \\ \text{variable}}} \underbrace{\widehat{t}}_{\substack{\text{arbitrary} \\ \text{constant}}} \right)$$


deviation from equilibrium

The multiplicand $(A_5 \cos vt + A_6 \sin vt)$ has the knack of having a wave-like graph as shown above. This shall have uniform amplitude above and below the equilibrium. So, convergence shall not take place. However, multiplier e^{ht} can mitigate the amplitude of the graph if it's value declines with time (t): i.e. $e^{ht} \rightarrow 0$ as $t \rightarrow \infty$.

Numerical Example:

	$y(t)$								
	$e^t(\cos 4t + 3 \sin 4t) + 2$			$e^{(0)t}(\cos 4t + 3 \sin 4t) + 2$			$e^{-t}(\cos 4t + 3 \sin 4t) + 2$		
	$e^{(1)t}(\cos 4t + 3 \sin 4t)$		y_c	$e^{(0)t}(\cos 4t + 3 \sin 4t)$		y_c	$e^{(-1)t}(\cos 4t + 3 \sin 4t)$		y_c
	e^{ht}	$(A_5 \cos vt + A_6 \sin vt)$		e^{ht}	$(A_5 \cos vt + A_6 \sin vt)$		e^{ht}	$(A_5 \cos vt + A_6 \sin vt)$	
t	$e^{(1)t}$	$(\cos 4t + 3 \sin 4t)$		$e^{(0)t}$	$(\cos 4t + 3 \sin 4t)$		$e^{(-1)t}$	$(\cos 4t + 3 \sin 4t)$	
0	1	1	1	1	1	1	1	1	1
1	2.7	-2.9	-7.9	1	-2.9	-2.9	0.3678	-2.9	-1.0757
2	7.4	2.8	20.9	1	2.8	2.8	0.1353	2.8	0.3820
3	20.1	-0.8	-15.4	1	-0.8	-0.8	0.0497	-0.8	-0.0381
4	54.6	-1.8	-99.4	1	-1.8	-1.8	0.0183	-1.8	-0.0334
5	148.4	3.1	467.0	1	3.1	3.1	0.0067	3.1	0.0212
6	403.4	-2.3	-924.9	1	-2.3	-2.3	0.0024	-2.3	-0.0057
7	1096.6	-0.1	-164.4	1	-0.1	-0.1	0.0009	-0.1	-0.0001
8	2981.0	2.5	7418.1	1	2.5	2.5	0.0003	2.5	0.0008
9	8103.1	-3.1	-25146.3	1	-3.1	-3.1	0.0001	-3.1	-0.0004
10	22026.5	1.6	34546.3	1	1.6	1.6	0.0000	1.6	0.00007
$\downarrow \uparrow$	$(e^{ht}) \uparrow$	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \downarrow \uparrow$	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \uparrow$	(e^{ht})	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \downarrow \uparrow$	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \downarrow \uparrow$	$(e^{ht}) \downarrow$	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \downarrow \uparrow$	$\begin{matrix} (+ve) \\ (-ve) \end{matrix} \downarrow$
	Increasing	Fluctuation	Explosive	constant	Fluctuation	Uniform	Decreasing	Fluctuation	Damped
	Explosive Fluctuation & Divergent			Uniform Fluctuation & Neither Divergent nor Convergent			Damped Fluctuation & Convergent		



TOPIC 134: ALTERNATIVE FORMS OF COMPLEX ROOTS

Complex Roots are:

$$r_1, r_2 = h \pm vi$$

However, two more alternative forms exist: $r_1, r_2 = R(\cos\theta \pm i \sin\theta) = Re^{\pm i\theta}$

Therefore, three forms are there:

1. **Cartesian Form ($h \pm vi$):** Based on Quadrant – I of cartesian coordinates, with h on x -axis and v on y -axis.

Examples:

$$\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$4(\sqrt{3} + i)$$

2. **Polar Form $\{R(\cos\theta \pm i \sin\theta)\}$:** Based on Polar Functions which have angles (θ) involved in them. Aka Triangular Form.

Example:

$$2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

3. **Exponential Form $\{Re^{\pm i\theta}\}$:** Based on exponential function $\{e^{f(x)}\}$.

Examples:

$$4e^{\frac{i\pi}{3}}$$

$$\sqrt{2}e^{-\frac{i\pi}{4}}$$

De Moivre's Theorem: If complex number has a power n other than 1, then all three forms can be rewritten as follows:

Cartesian Form with ($n \neq 1$): It is easy as we simply raise the power of $(h \pm vi)$ to n :

$$(h \pm vi)^n$$

Exponential Form with ($n \neq 1$): It is also easy as we simply raise the power of $(Re^{\pm i\theta})$ to n :

$$= (R e^{\pm i\theta})^n$$

$$= (R)^n (e^{\pm i\theta})^n$$

$$R^n e^{\pm(i\theta)n} = R^n e^{\pm in\theta}$$

Observation: In Exponential Form, via De Moivre's theorem, R is raised to R^n and $e^{\pm(i\theta)}$ is raised to $e^{\pm(i\theta)n}$.

Polar Form with ($n \neq 1$): Using observations from Exponential form, raise R to R^n and θ becomes $n\theta$:

$$\tilde{R}^n \left(\cos \tilde{\theta} \pm i \sin \tilde{\theta} \right)$$

$$R^n \{ \cos (n\theta) \pm i \sin (n\theta) \}$$

Summarizing: Gathering all forms of Complex Number(Roots) under De Moivre's theorem:

$$\underbrace{(h \pm vi)^n}_{\text{Cartesian form}} = \underbrace{R^n e^{\pm in\theta}}_{\text{Exponential form}} = \underbrace{R^n \{ \cos (n\theta) \pm i \sin (n\theta) \}}_{\text{Polar form}}$$

TOPIC 135: ALTERNATIVE FORMS OF COMPLEX ROOTS: INTERCONVERSION

Inter-conversion should be easy while remembering the basic trigonometric ratios:

$\frac{\text{perp}}{\text{hyp}} = \sin \theta$ $\frac{v}{R} = \sin \theta$	$\frac{\text{base}}{\text{hyp}} = \cos \theta$ $\frac{h}{R} = \cos \theta$	
Rearranging; $\frac{v}{R} = \sin \theta, \frac{h}{R} = \cos \theta$ $v = R \sin \theta, h = R \cos \theta$ Extracting value of θ from both equations $\theta = \sin^{-1} \left(\frac{v}{R} \right), \theta = \cos^{-1} \left(\frac{h}{R} \right)$ $\theta = \sin^{-1} \left(\frac{v}{R} \right) = \cos^{-1} \left(\frac{h}{R} \right)$		

Example 6 (p-520): Find the Cartesian form of the following Exponential form $5e^{i\frac{3\pi}{2}}$.

Solution: Rewriting the given Exponential form to compare with the standard form:

$$5e^{i\left(\frac{3\pi}{2}\right)}$$

$$Re^{\pm i\theta}$$

Here $R = 5, \theta = \frac{3\pi}{2}$.

Cartesian Form: $(h \pm vi)$; however, we don't have values of h and v .

As already known: $v = R \sin \theta$ & $h = R \cos \theta$

$v = R \sin \theta$ & $h = R \cos \theta$; putting values:

$$v = 5 \sin \left(\frac{3\pi}{2} \right) \text{ \& } h = 5 \cos \left(\frac{3\pi}{2} \right)$$

$$v = 5(-1) \text{ \& } h = 5(0)$$

$$v = -5 \text{ \& } h = 0$$

Returning values of v & h in $(h \pm vi)$:

$$= 0 \pm -5i$$

$$\text{Cartesian Form} = -5i$$

In addition to Cartesian form, Polar form can also be found:

$$= \left\{ \underset{\text{given}}{R} \left(\underset{\text{given}}{\cos \theta} \pm i \underset{\text{given}}{\sin \theta} \right) \right\}$$

$$= \left\{ \underset{5}{R} \left(\underset{\frac{3\pi}{2}}{\cos \theta} \pm i \underset{\frac{3\pi}{2}}{\sin \theta} \right) \right\}$$

$$\text{Polar Form} = \left[5 \left\{ \cos \left(\frac{3\pi}{2} \right) \pm i \sin \left(\frac{3\pi}{2} \right) \right\} \right]$$

Note: This equation can be further simplified to two numerical values, due to $(\pm)$.

Example 7 (p-520): Find Polar and Exponential forms of the following Cartesian form $(1 + \sqrt{3}i)$.

Solution: Labelling the given Cartesian form and comparing with the standard form:

$$(\boxed{1} + \boxed{\sqrt{3}}i)$$

$$(h \pm vi)$$

Here $h = 1$, $v = \sqrt{3}$.

Polar Form: $\left\{ \underset{\text{unknown}}{R} \left(\underset{\text{unknown}}{\cos \theta} \pm i \underset{\text{unknown}}{\sin \theta} \right) \right\}$

we have values of h and v to find value of $R = \sqrt{h^2 + v^2}$:

$$R = \sqrt{(1)^2 + (\sqrt{3})^2}$$

$$\boxed{R = 2}$$

$$\left\{ \underset{\text{calculated}}{R} \left(\underset{\text{unknown}}{\cos \theta} \pm i \underset{\text{unknown}}{\sin \theta} \right) \right\}$$

It's possible to find value of θ as well using $\theta = \sin^{-1} \left(\frac{v}{R} \right) = \cos^{-1} \left(\frac{h}{R} \right)$

$$\theta = \sin^{-1} \left(\frac{\boxed{\sqrt{3}}}{\boxed{2}} \right) = \cos^{-1} \left(\frac{\boxed{1}}{\boxed{2}} \right)$$

$$\theta = \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \cos^{-1} \left(\frac{1}{2} \right)$$

$$\theta = \boxed{60^\circ} = \boxed{60^\circ}$$

$$\left\{ \underset{\text{calculated}}{R} \left(\underset{\text{calculated}}{\cos \theta} \pm i \underset{\text{calculated}}{\sin \theta} \right) \right\}$$

$$= \left\{ \underset{2}{R} \left(\underset{60^\circ}{\cos \theta} \pm i \underset{60^\circ}{\sin \theta} \right) \right\}$$

$$\text{Polar Form} = \{ 2 \times (\cos 60^\circ \pm i \sin 60^\circ) \}$$

Note: This equation can be further simplified to two numerical values, due to $(\pm)$.

In addition to Polar form, Exponential form can also be found:

Exponential Form ($Re^{\pm i\theta}$):

$$\underset{\text{calculated}}{\hat{R}} e^{\pm i \underset{\text{calculated}}{\hat{\theta}}}$$

$$\underset{2}{\hat{R}} e^{\pm i \underset{60^\circ}{\hat{\theta}}}$$

$$\begin{aligned} &= 2e^{\pm i(60^\circ)} \\ \theta \text{ can also be written in terms of } \pi, \text{ as } \left(60^\circ = \frac{180^\circ}{3} = \left[\frac{\pi}{3}\right]\right): \\ &= 2e^{\pm i\left(\frac{\pi}{3}\right)} \end{aligned}$$

$$\boxed{\text{Exponential Form} = 2e^{\pm i\left(\frac{\pi}{3}\right)}}$$

Hence, interconversion among the three forms (Cartesian, Polar & Exponential) is possible using the formula given above.

Lesson 31

A MARKET MODEL WITH PRICE EXPECTATIONS

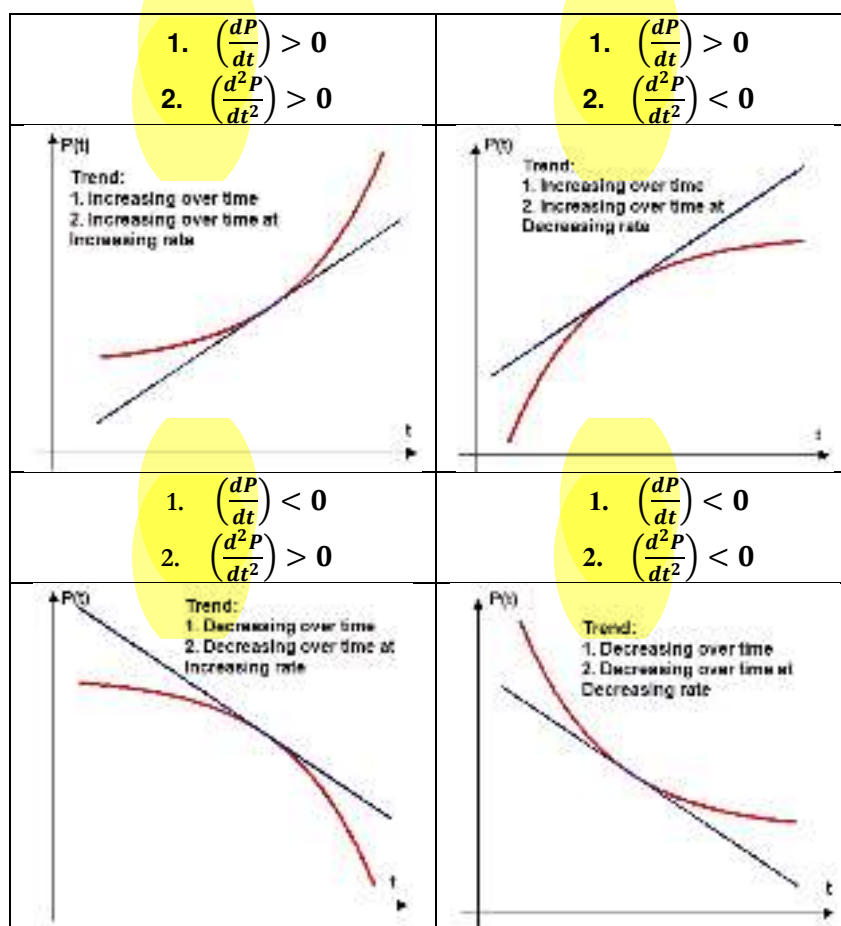
TOPIC 136: A MARKET MODEL WITH PRICE EXPECTATIONS: FRAMEWORK

In a simple market model:

$Q_d = f(P)$ Current quantity demanded depends only on Current Price.	$Q_s = f(P)$ Current quantity supplied depends only on Current Price.
--	--

However, in real-life, **trend** of price-change can affect the Q_d . Trends help in forming (price) expectations which affect the current Q_d .

Trends show how the changes in variable takes place. i.e.



So, the Trend can be understood from 2 points.

1. Is price increasing or decreasing over time – **Rate of change of price** $\left(\frac{dP}{dt}\right)$.
2. Is **Rate of change of price** increasing or decreasing over time – **Slope of Rate of change of price** $\left\{\frac{d}{dP}\left(\frac{dP}{dt}\right)\right\}$.

Mathematically, the trend is measured can be measured via 1st order and 2nd order derivatives and the new demand and supply functions will be:

Q_d Current quantity demand	$= f \left\{ \begin{array}{l} \text{Price-Trend} \\ \underbrace{P(t)}_{\text{current price}}, \underbrace{\frac{dP(t)}{dt}}_{\text{Rate of price-change over time}}, \underbrace{\frac{d^2P(t)}{dt^2}}_{\text{Slope of Rate of price-change over time}} \end{array} \right\}$
Q_s Current quantity supplied	$= f \left\{ \begin{array}{l} \text{Price-Trend} \\ \underbrace{P(t)}_{\text{current price}}, \underbrace{\frac{dP(t)}{dt}}_{\text{Rate of price-change over time}}, \underbrace{\frac{d^2P(t)}{dt^2}}_{\text{Slope of Rate of price-change over time}} \end{array} \right\}$
Current quantity demanded and quantity demanded depend not only on Current Price, but also the Price-Trend.	

Writing the general equations of the **price expectations augmented** Q_d and Q_s functions:

$$Q_d = \underbrace{\alpha}_{\text{Simple Demand function}} \underbrace{-\beta P}_{\text{effect of current price}} + \underbrace{mP' + nP''}_{\text{price expectations augmented}} \underbrace{+}_{\text{effect of Price-Trend}} \quad (\alpha, \beta > 0)$$

$$Q_s = \underbrace{-\gamma}_{\text{Simple Supply function}} \underbrace{+\delta P}_{\text{effect of Current Price}} + \underbrace{uP' + wP''}_{\text{price expectations augmented}} \underbrace{+}_{\text{effect of Price-Trend}} \quad (\gamma, \delta > 0)$$

$\alpha, \beta, \gamma, \delta$ are the conventional parameters, however, m, n, u and w are parameters generated by terms of price expectations.

Assuming price expectations in demand function only and not in supply function:

$$Q_d = \underbrace{\alpha - \beta P}_{\text{simple demand function}} + \underbrace{mP' + nP''}_{\substack{\text{buyers have} \\ \text{price expectations}}} \quad \begin{matrix} \neq 0 \\ \neq 0 \end{matrix}$$

Effect of m and n : If $m > 0$, $Q_d \uparrow$, as buyer is expecting the price to increase over time and consider it cheaper to buy now.

Similarly, if $n > 0$, Q_d will $\uparrow$ even faster as rate of price-increase is increasing over time.

$$Q_s = \underbrace{-\gamma + \delta P}_{\text{simple supply function}} + \underbrace{uP' + wP''}_{\substack{\text{suppliers have no} \\ \text{price expectations}}} \quad \begin{matrix} = 0 \\ = 0 \end{matrix}$$

Therefore, we have modified market model with price expectations:

$$\frac{Q_d = \alpha - \beta P + mP' + nP''}{Q_s = -\gamma + \delta P}$$

TOPIC 137: A MARKET MODEL WITH PRICE EXPECTATIONS: SOLUTION

Assuming Market-Clearing at all points in time ($Q_d = Q_s$).

$$\alpha - \beta P + mP' + nP'' = -\gamma + \delta P$$

$$\overline{n}P'' + mP'[-\beta P - \delta P] = -\alpha - \gamma$$

Normalizing the equation by dividing by n .

$$P'' + \left(\frac{m}{n}\right)P' + \left\{-\frac{(\beta+\delta)}{n}\right\}P = \frac{-(\alpha+\gamma)}{n}$$

Comparing with the standard form of First Order Differential Equation

$$y'' + a_1 y' + a_2 y = b$$

$$\underbrace{P''}_{y''} + \underbrace{\left(\frac{m}{n}\right)}_{a_1} \underbrace{P'}_{y'} + \underbrace{\left\{-\frac{(\beta+\delta)}{n}\right\}}_{a_2} \underbrace{P}_y = \underbrace{\left(-\frac{\alpha+\gamma}{n}\right)}_b \text{ comparable with}$$

Here; $y = P$, $y' = P'$, $y'' = P''$, $a_1 = \frac{m}{n}$, $a_2 = -\frac{(\beta+\delta)}{n}$, $b = -\left(\frac{\alpha+\gamma}{n}\right)$

Particular Solution: Particular solution of a Second Order Differential Equation y_p has three cases. Starting from 1st case; $a_2 \neq 0$.

Here; $a_2 = -\left\{\frac{\left(\overset{>0}{\beta} + \overset{>0}{\delta}\right)}{\underset{\neq 0}{n}}\right\}$; β and δ are already known to be positive. n is considered to be non-zero

($n \neq 0$), since buyer is assumed to have price-expectations. Therefore, the expression $-\frac{(\beta+\delta)}{n} \neq 0$. Therefore, we use the Case – I of the $y_p = \frac{b}{a_2}$.

$$y_p = \frac{-\left(\frac{\alpha+\gamma}{n}\right)}{-\frac{(\beta+\delta)}{n}}$$

$$\boxed{y_p = \frac{\alpha+\gamma}{\beta+\delta}}$$

Interpretation: Since all parameters; α , β , γ and δ are positive constants, so y_p will also be a constant value. Therefore, a **Stationary Equilibrium** will be there.

TOPIC 138: A MARKET MODEL WITH PRICE EXPECTATIONS: POSSIBLE CASES

Depending on the value of discriminant, the nature of roots of characteristic equation gets defined. This gives rise to three cases of complementary functions:

$$r = r_1, r_2 = \frac{-a_1 \pm \sqrt{\frac{a_1^2 - 4a_2}{\text{discriminant}}}}{2}$$

$$\frac{a_1^2 - 4a_2 \geq 0 \text{ OR } a_1^2 \geq 4a_2}{2}$$

1. Distinct Real Roots ($a_1^2 > 4a_2$): Here; $a_1 = \frac{m}{n}$, $a_2 = -\frac{(\beta+\delta)}{n}$, $b = -\left(\frac{\alpha+\gamma}{n}\right)$, so substituting the values, we get:

$$\boxed{a_1^2} > 4\boxed{a_2}$$

$$\boxed{\left(\frac{m}{n}\right)^2} > 4\boxed{-\frac{(\beta+\delta)}{n}}$$

$$\boxed{\left(\frac{m}{n}\right)^2} > -4\frac{(\beta+\delta)}{n}$$

Forming the complementary function with Distinct Real Roots:

$$\boxed{P_c = A_1 e^{\boxed{r_1}t} + A_2 e^{\boxed{r_2}t}}$$

Boxes avoid clutter:

$$\text{Where, } \boxed{r_1}, \boxed{r_2} = \frac{1}{2} \left\{ -\frac{m}{n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\left(\frac{\beta+\delta}{n}\right)} \right\}$$

Accordingly, General Solution is:

$$P(t) = \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta}\right)}_{P_p}$$

2. Repeated Real Roots ($a_1^2 = 4a_2$): Here; $a_1 = \frac{m}{n}$, $a_2 = -\frac{(\beta+\delta)}{n}$, $b = -\left(\frac{\alpha+\gamma}{n}\right)$, so substituting the values, we get:

$$\begin{aligned} \boxed{a_1^2} &= 4\boxed{a_2} \\ \left(\frac{m}{n}\right)^2 &= 4\left[-\frac{(\beta+\delta)}{n}\right] \\ \left(\frac{m}{n}\right)^2 &= -4\frac{(\beta+\delta)}{n} \end{aligned}$$

Forming the complementary function with Repeated Real Roots:

$$P_c = A_3 e^{\boxed{r}t} + A_4 t e^{\boxed{r}t}$$

$$\text{Where, } r = \frac{1}{2} \left\{ -\frac{m}{n} \pm \sqrt{\frac{\left(\frac{m}{n}\right)^2 + 4\left(\frac{\beta+\delta}{n}\right)}{0}} \right\}$$

$$r = \frac{1}{2} \left(-\frac{m}{n} \pm \sqrt{0} \right)$$

$$\boxed{r} = -\frac{m}{2n}$$

Substituting the value of r in the complementary function (y_c):

$$\begin{aligned} P_c &= A_3 e^{\boxed{-\frac{m}{2n}}t} + A_4 t e^{\boxed{-\frac{m}{2n}}t} \\ P_c &= A_3 e^{\left(-\frac{mt}{2n}\right)} + A_4 t e^{\left(-\frac{mt}{2n}\right)} \end{aligned}$$

Accordingly, General Solution is:

$$P(t) = \underbrace{A_3 e^{\left(-\frac{mt}{2n}\right)} + A_4 t e^{\left(-\frac{mt}{2n}\right)}}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta}\right)}_{P_p}$$

3. Complex Roots ($a_1^2 < 4a_2$): Here; $a_1 = \frac{m}{n}$, $a_2 = -\frac{(\beta+\delta)}{n}$, $b = -\left(\frac{\alpha+\gamma}{n}\right)$, so substituting the values, we get:

$$\begin{aligned} \boxed{a_1^2} &< 4\boxed{a_2} \\ \left(\frac{m}{n}\right)^2 &< 4\left[-\frac{(\beta+\delta)}{n}\right] \\ \left(\frac{m}{n}\right)^2 &< -4\frac{(\beta+\delta)}{n} \end{aligned}$$

Forming the complementary function with Complex Roots:

$$P_c = e^{\boxed{h}t} (A_5 \cos \boxed{v}t + A_6 \sin \boxed{v}t)$$

$$\text{Where, } h = -\frac{\boxed{a_1}}{2}, v = \frac{1}{2} \sqrt{4\boxed{a_2} - (\boxed{a_1})^2}$$

$$h = -\frac{\boxed{\frac{m}{n}}}{2}, v = \frac{1}{2} \sqrt{-4\frac{\boxed{\beta+\delta}}{n} - \left(\frac{\boxed{m}}{n}\right)^2}$$

$$h = -\frac{m}{2n}, \quad v = \frac{1}{2} \sqrt{-4 \left(\frac{\beta + \delta}{n} \right) - \left(\frac{m}{n} \right)^2}$$

Accordingly, General Solution is:

$$P(t) = \underbrace{e^{\frac{h}{P_c} t} (A_5 \cos \frac{v}{P_c} t + A_6 \sin \frac{v}{P_c} t)}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{P_p}$$

$$P(t) = e^{\frac{-\frac{m}{2n}}{P_c} t} \underbrace{(A_5 \cos \frac{v}{P_c} t + A_6 \sin \frac{v}{P_c} t)}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{P_p}$$

Parametric Interpretation for Dynamic Stability:

Parameters involved: β , δ , m and n

Given stipulations: $\beta > 0$, $\delta > 0$

Therefore, the dynamic stability depends on values of m and n .

As n is in all of the terms i.e. $\left(\frac{m}{n}\right)$, $\left(\frac{m}{n}\right)^2$ and $\left\{-4 \left(\frac{\beta + \delta}{n}\right)\right\}$, its importance is pivotal in determining the nature of roots as well as dynamic stability.

Dropping the possibility $n = 0$, as the result of terms given above will become undefined.

Pivotal Role of (n) and secondary role of (m) in Dynamic Stability	
$n > 0$	$n < 0$
Only Distinct Real Roots can exist. $(r_1 > 0, r_2 < 0)$ (Dynamically Unstable) Can be Dynamically Stable only if $(A_1 = 0)$. $P_c = A_1 e^{r_1 t} + A_2 e^{r_2 t}$ $\underbrace{P_c}_{\rightarrow 0} = \underbrace{A_1}_{=0} \times \underbrace{e^{r_1 t}}_{\substack{+ve \\ \text{exponential} \\ \text{growth}}} + \underbrace{A_2}_{\neq 0} \times \underbrace{e^{r_2 t}}_{\substack{-ve \\ \text{exponential} \\ \text{decay}}}$ $\underbrace{\hspace{10em}}_{=0} \quad \quad \quad \underbrace{\hspace{10em}}_{\rightarrow 0}$	All Cases can exist: 1. Distinct Real Roots, dynamically stable if $(m < 0)$. 2. Repeated Real Roots, dynamically stable if $(m < 0)$. 3. Complex Roots, dynamically stable if $(m < 0)$.
Conclusion: Dynamic Stability is ensured if $\{(n < 0) \wedge (m < 0)\}$ with the exception of where $(n > 0) \wedge (A_1 = 0)$.	

TOPIC 139: A MARKET MODEL WITH PRICE EXPECTATIONS: NUMERICAL

Example 1 (p. # 530): Let the Demand and Supply Functions be:

$$Q_d = 42 - 4P - 4P' + P''$$

$$Q_s = -6 + 8P$$

With initial conditions $P(0) = 6$ and $P'(0) = 4$. Assuming market clearance at every point of time, find the time path $P(t)$.

Recalling the standard form of expectations-augmented demand and supply functions:

$$Q_d = \alpha - \beta P + mP' + nP''$$

$$Q_s = -\gamma + \delta P + uP' + wP''$$

Comparison gives:

$$Q_d = \underbrace{42}_{\alpha} - \underbrace{4}_{\beta} P + \underbrace{(-4)}_m P' + \underbrace{(1)}_n P''$$

$$Q_s = -\underbrace{6}_{\gamma} + \underbrace{8}_{\delta} P + \underbrace{0}_u P' + \underbrace{0}_w P''$$

Therefore, here $\alpha = 42$, $\beta = 4$, $\gamma = 6$, $\delta = 8$, $\boxed{m = -4 < 0}$, $\boxed{n = 1 > 0}$, $u = 0$, $w = 0$.

Double Boxed value of $\boxed{n = 1 > 0}$ shows that only Distinct Real Roots are possible. It also confirms that $r_1 > 0$ and $r_2 < 0$.

So dynamic unstability is present unless $A_1 = 0$.

Therefore, we solve the system to get definite solution which evaluates A_1 and A_2 . Consequently, possibility of $A_1 = 0$ can be checked as well.

Distinct Real Roots: $r_1, r_2 = \frac{1}{2} \left\{ -\frac{m}{n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4\left(\frac{\beta+\delta}{n}\right)} \right\}$

$$r_1, r_2 = \frac{1}{2} \left\{ -\frac{(-4)}{1} \pm \sqrt{\left(\frac{(-4)}{1}\right)^2 + 4\left(\frac{4+8}{1}\right)} \right\}$$

$$r_1, r_2 = \frac{1}{2} (4 \pm \sqrt{16 + 48})$$

$$r_1, r_2 = \frac{1}{2} (4 \pm \sqrt{64})$$

$$r_1, r_2 = \frac{1}{2} (4 \pm 8)$$

$$\boxed{r_1, r_2 = 6, -2}$$

Accordingly, General Solution is:

$$P(t) = \underbrace{A_1 e^{r_1 t} + A_2 e^{r_2 t}}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta}\right)}_{P_p}$$

$$P(t) = \underbrace{A_1 e^{(6)t} + A_2 e^{(-2)t}}_{P_c} + \underbrace{\left\{\frac{42 + 6}{4 + 8}\right\}}_{P_p}$$

$$\boxed{P(t) = A_1 e^{6t} + A_2 e^{-2t} + 4}$$

Definitizing the General Solution by introducing the initial condition; ($t = 0$):

$$\underbrace{P(0)}_{\text{given}} = A_1 e^{6(0)} + A_2 e^{-2(0)} + 4$$

$$\underbrace{P(0)}_6 = A_1 e^0 + A_2 e^0 + 4$$

$$6 = A_1(1) + A_2(1) + 4$$

$$6 - 4 = A_1 + A_2$$

$$\boxed{2 = A_1 + A_2}$$

Differentiating the General Solution for introducing $P'(0)$.

$$\frac{d}{dt} \{P(t)\} = \frac{d}{dt} (A_1 e^{6t} + A_2 e^{-2t} + 4)$$

$$P'(t) = \frac{d}{dt} (A_1 e^{6t}) + \frac{d}{dt} (A_2 e^{-2t}) + \frac{d}{dt} (4)$$

$$P'(t) = A_1(6)e^{6t} + A_2(-2)e^{6t} + 0$$

$$\underline{P'(t) = 6A_1 e^{6t} - 2A_2 e^{6t}}$$

Substituting the initial condition for First Derivative of General Solution; $P'(0)$:

$$\underline{P'(0) = 6A_1 e^{6(0)} - 2A_2 e^{6(0)}}$$

$$\underbrace{P'(0)}_{\text{given}} = 6A_1 e^0 - 2A_2 e^0$$

$$\underbrace{P'(0)}_4 = 6A_1(1) - 2A_2(1)$$

$$4 = 6A_1 - 2A_2$$

Simultaneously solving the resultant equations:

$$\underbrace{2 = A_1 + A_2}_{A_1 = 2 - A_2}, \quad 4 = 6 \underbrace{A_1}_{\text{substituting value of } A_1} - 2A_2$$

$$4 = 6(2 - A_2) - 2A_2$$

$$4 = 12 - 6A_2 - 2A_2$$

$$4 - 12 = -8A_2$$

$$A_2 = 1$$

To find A_1 , plugin the value of A_2 in $A_1 = 2 - A_2$:

$$A_1 = 2 - 1$$

$$A_1 = 1$$

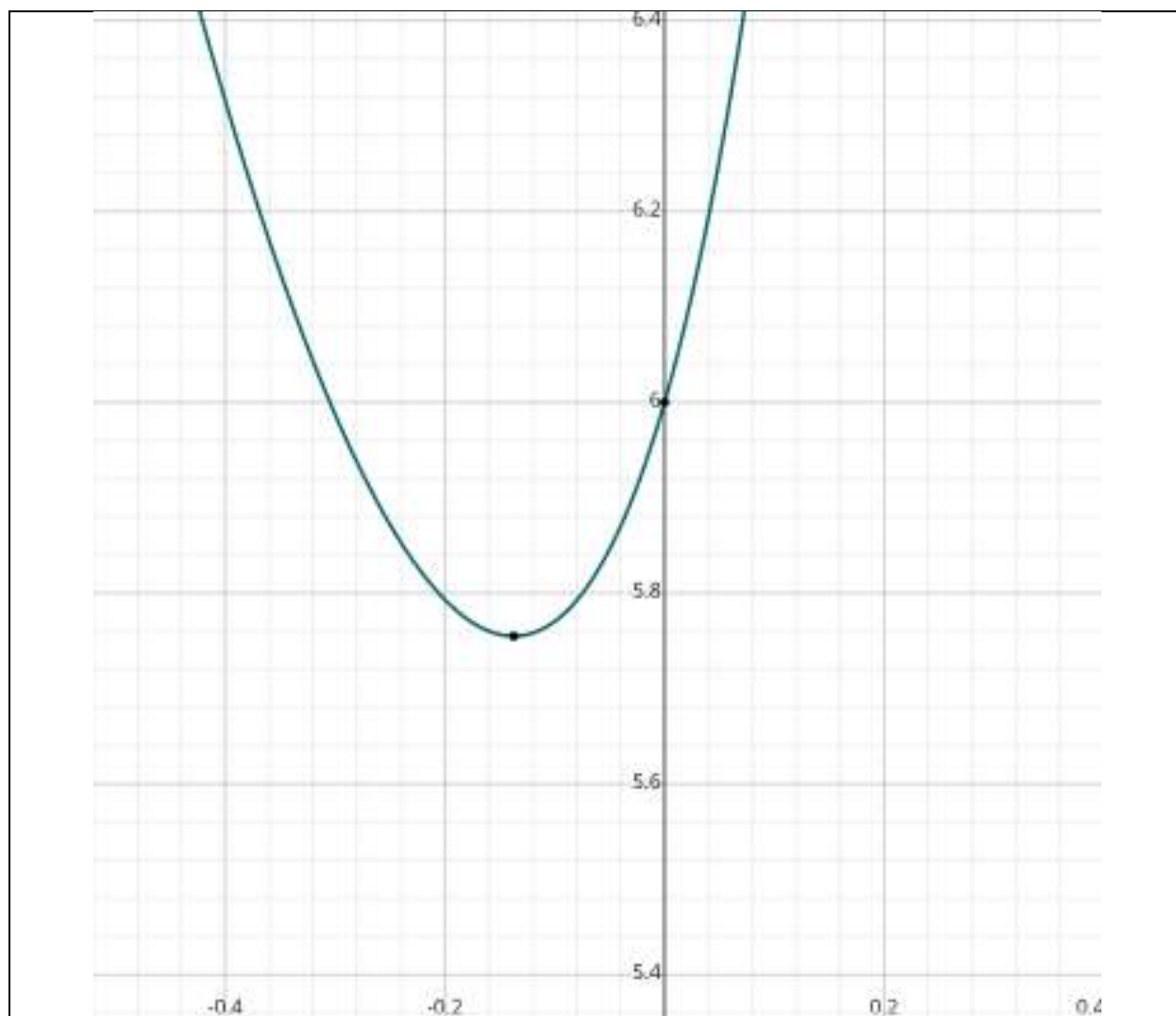
Returning the values of A_1 and A_2 in General Solution:

$$P(t) = A_1 e^{6t} + A_2 e^{-2t} + 4$$

$$P(t) = 1e^{6t} + 1e^{-2t} + 4$$

$$P(t) = e^{6t} + e^{-2t} + 4$$

Interpretation: As already observed, $(n > 0) \wedge (A_1 \neq 0)$, only dynamic unstable price time-path with Distinct Real Roots is possible. Specifically, the dynamic instability in Definite Solution is due to the presence of a +ve root ($r_1 = 6$).



Example 1 (p. # 531): Let the Demand and Supply Functions be:

$$Q_d = 40 - 2P - 2P' - P''$$

$$Q_s = -5 + 3P$$

With initial conditions $P(0) = 12$ and $P'(0) = 1$. Assuming market clearance at every point of time, find the time path $P(t)$.

Recalling the standard form of expectations-augmented demand and supply functions:

$$Q_d = \alpha - \beta P + mP' + nP''$$

$$Q_s = -\gamma + \delta P + uP' + wP''$$

Comparison gives:

$$Q_d = \overset{\alpha}{40} - \overset{\beta}{2}P + \overset{m}{(-2)}P' + \overset{n}{(-1)}P''$$

$$Q_s = -\underset{\gamma}{5} + \underset{\delta}{3}P + \underset{u}{0}P' + \underset{w}{0}P''$$

Therefore, here $\alpha = 40$, $\beta = 2$, $\gamma = 5$, $\delta = 3$, $\boxed{m = -2 < 0}$, $\boxed{n = -1 < 0}$, $u = 0$, $w = 0$.

Double Boxed value of $\boxed{n = -1 < 0}$ shows that any of three types of Roots are possible.

Single Boxed value of $\boxed{m = -2 < 0}$ in addition to $(n < 0)$ shows Dynamic Stability in any of the cases. For deciding which case is present here, we apply the test for nature of roots.

Therefore, we apply the test: $a_1^2 \geq -4a_1$. However, here the given parameters are m , n , β and δ , so the test will appear as follows:

$$\begin{aligned} \left(\frac{m}{n}\right)^2 &\geq -4 \frac{(\beta + \delta)}{n} \\ \left(\frac{-2}{-1}\right)^2 &\geq -4 \frac{(2 + 3)}{-1} \\ \boxed{16 < 20} \\ &\text{complex roots} \end{aligned}$$

Complex Roots: $r_1, r_2 = h \pm vi = \frac{1}{2} \left\{ -\frac{m}{n} \pm \sqrt{\left(\frac{m}{n}\right)^2 + 4 \left(\frac{\beta + \delta}{n}\right)} \right\}$

Substituting the values:

$$\begin{aligned} r_1, r_2 &= \frac{1}{2} \left\{ -\frac{(-2)}{(-1)} \pm \sqrt{\left(\frac{(-2)}{(-1)}\right)^2 + 4 \left(\frac{2 + 3}{-1}\right)} \right\} \\ r_1, r_2 &= \frac{1}{2} (-2 \pm \sqrt{4 - 20}) \\ r_1, r_2 &= \frac{1}{2} \left(-2 \pm \frac{\sqrt{-16}}{\frac{\sqrt{16 \times -1}}{4} i} \right) \\ \underbrace{r_1, r_2}_{h \pm vi} &= \frac{1}{2} (-2 \pm 4i) \\ \boxed{r_1, r_2 = \frac{-1 \pm 2i}{h \pm vi}} \end{aligned}$$

Extracted values: $\boxed{h = -1}$ and $\boxed{v = 2}$.

Accordingly, General Solution is:

$$\begin{aligned} P(t) &= \underbrace{e^{\boxed{h}t} (A_5 \cos \boxed{v}t + A_6 \sin \boxed{v}t)}_{P_c} + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{P_p} \\ P(t) &= \underbrace{e^{\boxed{-1}t} (A_5 \cos \boxed{2}t + A_6 \sin \boxed{2}t)}_{P_c} + \underbrace{9}_{P_p} \\ \boxed{P(t) = e^{-t} \{A_5 \cos(2t) + A_6 \sin(2t)\} + 9} \end{aligned}$$

Definitizing the General Solution by introducing the initial condition; $(t = 0)$:

$$\begin{aligned} \underbrace{P(0)}_{\text{given}} &= e^{-(0)} [A_5 \cos\{2(0)\} + A_6 \sin\{2(0)\}] + 9 \\ \underbrace{P(0)}_{12} &= 1 \times \{A_5 \cos(0) + A_6 \sin(0)\} + 9 \\ 12 &= \{A_5(1) + A_6(0)\} + 9 \\ 12 &= A_5 + 9 \\ \boxed{A_5} &= 3 \end{aligned}$$

Differentiating the General Solution for introducing $P'(0)$.

$$\frac{d}{dt} \{P(t)\} = \frac{d}{dt} \left\{ \boxed{e^{-t}} \{A_5 \cos(2t) + A_6 \sin(2t)\} + 9 \right\}$$

$$\begin{aligned}
 P'(t) &= \underbrace{\frac{d}{dt} \left[\underbrace{e^{-t}}_{h(x)} \underbrace{\{A_5 \cos(2t)\}}_{g(x)} \right]}_{\text{product rule}} + \underbrace{\frac{d}{dt} \left[\underbrace{e^{-t}}_{h(x)} \underbrace{\{A_6 \sin(2t)\}}_{g(x)} \right]}_{\text{product rule}} + \frac{d}{dt} (9) \\
 P'(t) &= \left[\underbrace{\frac{d}{dt} (e^{-t})}_{-e^{-t}} \times \{A_5 \cos(2t)\} + A_5 \underbrace{\frac{d}{dt} \{\cos(2t)\}}_{-2 \times \sin(2t)} \times (e^{-t}) \right] \\
 &\quad + \left[\underbrace{\frac{d}{dt} (e^{-t})}_{-e^{-t}} \times \{A_6 \sin(2t)\} + A_6 \underbrace{\frac{d}{dt} \{\sin(2t)\}}_{2 \times \cos(2t)} \times (e^{-t}) \right] \\
 P'(t) &= [-e^{-t} \times \{A_5 \cos(2t)\} + A_5 \{-2 \times \sin(2t)\} \times (e^{-t})] \\
 &\quad + [-e^{-t} \times \{A_6 \sin(2t)\} + A_6 \{2 \times \cos(2t)\} \times (e^{-t})]
 \end{aligned}$$

Substituting the initial condition for First Derivative of General Solution; $P'(0)$:

$$\begin{aligned}
 P'(0) &= [\underbrace{-e^{-(0)}}_{-1} \times \{A_5 \cos\{2(0)\}\} + A_5 \{-2 \times \sin\{2(0)\}\} \times \{e^{-(0)}\}] \\
 &\quad + [\underbrace{-e^{-(0)}}_{-1} \times \{A_6 \sin\{2(0)\}\} + A_6 \{2 \times \cos\{2(0)\}\} \times \{e^{-(0)}\}] \\
 \underbrace{P'(0)}_{\text{given}} &= [(\underline{-1}) \times \{A_5 \cos\{2(0)\}\} + A_5 \{-2 \times \sin\{2(0)\}\} \times (\underline{1})] \\
 &\quad + [(\underline{-1}) \times \{A_6 \sin\{2(0)\}\} + A_6 \{2 \times \cos\{2(0)\}\} \times (\underline{1})] \\
 \underbrace{P'(0)}_1 &= \left[\underline{-1} A_5 \underbrace{\cos(0)}_1 + A_5 \left\{ -2 \times \underbrace{\sin(0)}_0 \right\} \right] + \left[\left\{ \underline{-1} A_6 \underbrace{\sin(0)}_0 \right\} + A_6 \left\{ 2 \times \underbrace{\cos(0)}_1 \right\} \right] \\
 1 &= [(-A_5 \times 1) + A_5(-2 \times 0)] + [(-A_6 \times 0) + A_6(-2 \times 1)] \\
 1 &= (-A_5 + 0) + (0 - 2A_6) \\
 \underline{1} &= \underline{-A_5 - 2A_6}
 \end{aligned}$$

Simultaneously solving the resultant equations:

$$\boxed{A_5 = 3}, \quad \boxed{1 = -\underbrace{A_5}_3 - 2A_6}$$

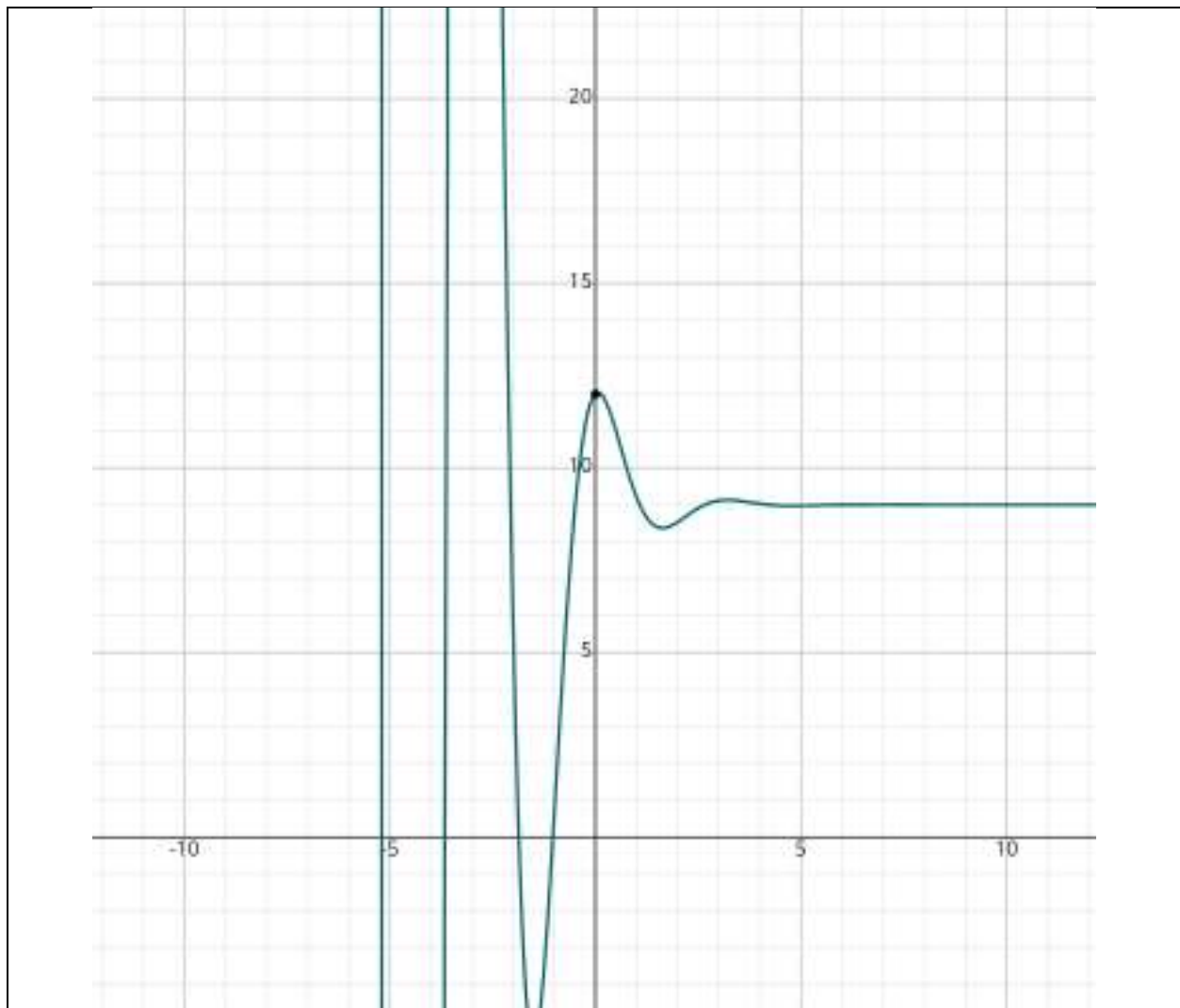
To find A_6 , plugin the value of A_5 in $1 = -A_5 - 2A_6$:

$$\begin{aligned}
 \boxed{A_5 = 3}, \quad \boxed{1 = -3 - 2A_6} \\
 -2A_6 &= -3 - 1 \\
 \underline{A_6} &= \underline{2}
 \end{aligned}$$

Returning the values of A_5 and A_6 in General Solution:

$$\begin{aligned}
 P(t) &= e^{-t} \{ \underline{A_5} \cos(2t) + \underline{A_6} \sin(2t) \} + 9 \\
 P(t) &= e^{-t} \{ \underline{3} \cos(2t) + \underline{2} \sin(2t) \} + 9 \\
 P(t) &= \underbrace{e^{-t}}_{\text{dampening}} \left\{ \underbrace{3 \cos(2t) + 2 \sin(2t)}_{\text{wave like fluctuation}} \right\} + \underbrace{9}_{\text{equilibrium}}
 \end{aligned}$$

Interpretation: As already observed, $(n < 0) \wedge (m < 0)$, dynamic stability is present in the price time-path. Specifically, the time-path will be fluctuating as well as dampening.



Lesson 32

PHILLIPS CURVE: THE PHILLIPS RELATION

TOPIC 140: PHILLIPS CURVE: THE PHILLIPS RELATION

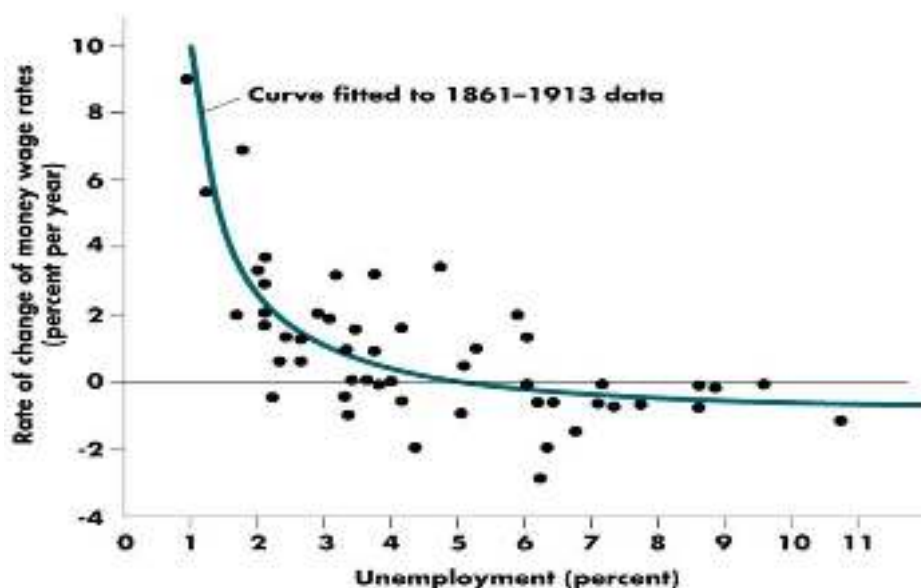
The Interaction of Inflation and Unemployment

An application of the second-order differential equation with a macro model dealing with the problem of inflation and unemployment.

Phillips relation is the most widely used concepts in the modern analysis of the problem of inflation and unemployment.

Original Phillips Relation:

Its original formulation depicts an empirically based negative relation between the rate of growth of money wage and the rate of unemployment:



$$w = f(U)$$

$$[f'(U) < 0]$$

Here, w : Rate of Growth of Money Wage (W).

$$w = \frac{\dot{W}}{W}$$

U : Rate of Unemployment.

Adapted Phillips Relation:

Unemployment pertains to labor market. However, later version of Phillips Relation uses **inflation rate instead of wage rate**.

Its intuitive as, increasing w increases the cost of production, which causes (cost push) inflation ($\dot{P}$):

$$w \uparrow \Rightarrow \uparrow (\text{cost of production}) \Rightarrow \uparrow p$$

Here; p is the rate of growth of inflation ($\dot{P} = \frac{dP}{dt}$).

$$p = \frac{\dot{P}}{P}$$

Dual Effect of Increased Money-Wages:

$$\boxed{w \uparrow}$$

↓	
Productivity $T = \left(\frac{Q}{L}\right) \uparrow$	
↓	
Per unit Cost $\left(\frac{C}{L}\right) \downarrow$	
↓	
Cost of Production $(C) \downarrow$	
↓	
$p \downarrow$	$p \uparrow$
Effect of Productivity on Inflation	Effect of money-wage on Inflation
$p_T \downarrow$	$p_w \uparrow$
$p_T \downarrow \gtrless p_w \uparrow$	
Outcome of Off-setting between inflation increase & inflation decrease	

Therefore, introducing the role of productivity (T) in the link between money-wage (w) and inflation ($\dot{P}$):

$$p = \underbrace{\left(\frac{p \propto w}{\bar{w}}\right)}_{\text{exogenous}} - \underbrace{\left(\frac{p \propto \frac{1}{T}}{\bar{T}}\right)}_{\text{exogenous}}$$

Original Phillips Relation: $w = f(U)$. Assuming a linear relationship, we get:

$$w = \alpha - \beta U$$

Substituting value of w in $p = \bar{w} - T$:

$$p = \alpha - \beta U - T$$

$$p = \alpha - T - \beta U \quad (\alpha, \beta > 0)$$

Interpretation: From the **Original Phillips Relation**, we have derived the **Adapted Phillips Relation**. Where, Productivity (T) and Unemployment rate (U) can negatively affect the inflation-growth rate (p).

TOPIC 141: PHILLIPS CURVE: EXPECTATIONS AUGMENTED PHILLIPS RELATION

More recently, economists have preferred to use the expectations-augmented version of the Phillips relation:

$$w = f(U) + g\pi \quad (0 < g \leq 1)$$

$$w = f(U) + \overbrace{g \pi}^{\substack{\text{enhanced effect} \\ \text{of inflation} \\ \text{expectations} \\ \text{on wages}}} \quad \substack{\pi \\ \text{expected rate} \\ \text{of inflation}}$$

Nobel laureate Professor Friedman, suggests that if an inflationary trend has been in effect long enough, people are apt to form certain inflation expectations which they then attempt to incorporate into their money-wage demands. Thus w should be an increasing function of π .

Inflation Expectations can also be incorporated in the Adapted Phillips Relation ($p = \alpha - T - \beta U$):

$$p = \alpha - T - \beta U$$

$$\boxed{p = \alpha - T - \beta U + \overset{\text{inflation}}{\underset{\text{expectations}}{\hat{g}\pi}}} \quad (0 < g \leq 1)$$

Formation of Expectations – Adaptive Expectations:

After introducing π , it's meaningful to assess how π are formed. Here, we assume Adaptive Expectations hypothesis instead of rational expectations hypothesis.

Note: Adaptive Expectations are when one makes forecasts of future values of a variable using only past values of the variable. **Rational expectations** are when forecasts of future values are made using all available information.

Adaptive Expectations can be assessed by rate of change of expectations. Therefore, we calculate the derivative of Expectations variable (π):

$$\underbrace{\frac{d\pi}{dt}}_{\text{Rate of change of expectations}} \propto \underbrace{\left(\overset{\text{actual rate of inflation}}{\hat{p}} - \overset{\text{expected rate of inflation}}{\hat{\pi}} \right)}_{\text{discrepancy}}$$

$$\frac{d\pi}{dt} = \underbrace{j}_{\text{adjustment coefficient}} (p - \pi) \quad (0 < j \leq 1)$$

The discrepancy term ($p - \pi$) can have following possibilities:

Actual Inflation exceeds Expected Inflation	Actual Inflation equals Expected Inflation	Expected Inflation exceeds Actual Inflation
$(p > \pi)$	$(p = \pi)$	$(p < \pi)$
Upward revision of Expected Inflation needed.	No adjustment needed.	Downward revision of Expected Inflation needed.

Interpretation: Inflation Expectations are incorporated in the form of Expected rate of inflations (π). While Adaptive Inflation Expectations is introduced using the derivative of π .

TOPIC 142: PHILLIPS CURVE: FEEDBACK FROM INFLATION TO UNEMPLOYMENT

Expectations-augmented Phillips Relation & Adaptive Expectations Hypothesis, can serve as a two-equation system with two endogenous variables ($\hat{p}$ & $\hat{\pi}$) and one exogenous (U):

$$\underbrace{\hat{p} = \alpha - T - \beta U + g\hat{\pi}}_{\text{Expectations-augmented Phillips Relation}} \quad (0 < g \leq 1)$$

$$\underbrace{\frac{d\hat{\pi}}{dt} = j(\hat{p} - \hat{\pi})}_{\text{Adaptive Expectations Hypothesis}} \quad (0 < j \leq 1)$$

U can also be explained like p and π , so the model becomes richer in behavioral characteristics.

Feedback Effect of U : Explaining U can help us understand the feedback effect of inflation on unemployment ($U \rightleftharpoons p$).

1. $U \rightarrow p$: Expectations-augmented Phillips relation is based mostly on supply side and deals with $U \rightarrow p$:

$$p = \underbrace{\overbrace{\alpha - \beta U}^{\substack{\text{w (labor cost)}}}}_{\substack{\text{supply side}}} - \underbrace{\overbrace{\bar{T}}^{\text{productivity}}}_{\substack{\text{supply side}}} + \underbrace{\overbrace{g\pi}_{\substack{\text{inflation expectations}}}}_{\substack{\text{supply side via} \\ \text{w-increase}}}$$

2. $U \leftarrow p$: However, the Feedback effect on unemployment on inflation can be through:

Feedback Effect: $U \leftarrow p$		
Demand Side	Government Intervention	
	Fiscal Policy	Monetary Policy
$p \uparrow$	$\Delta G \uparrow: (given\ p)$	$\Delta M \uparrow: (given\ p)$
$\downarrow$	$\downarrow$	$\downarrow$
$\Delta(\text{Consumption-Saving Pattern})$	$\Delta(\text{Real-Expenditure})$	$\Delta(\text{Real Money})$
$\downarrow$	$\downarrow$	$\downarrow$
$\Delta(AD)$	$\Delta(AD)$	$\Delta(AD)$
$\downarrow$	$\downarrow$	$\downarrow$
$\Delta(U)$	$\Delta(U)$	$\Delta(U)$

For simplicity sake, we consider Monetary Policy channel only.

Let M be the nominal-money balance.

$m = \frac{\dot{M}}{M}$ be the rate of growth of nominal money balance.

Recalling the Monetary Policy channel:

$$\boxed{\Delta(U)} \leftarrow \Delta(AD) \leftarrow \boxed{\Delta(\textit{Real Money})} \leftarrow \Delta(M)$$

$$\boxed{\Delta(U)} \leftarrow \boxed{\Delta(\textit{Real Money})}$$

Mathematizing the same:

$$\left(\frac{dU}{dt}\right) \propto \frac{1}{(m-p)}$$

$$\frac{dU}{dt} = \underbrace{k}_{\substack{\text{constant of} \\ \text{proportionality}}} \underbrace{r_{\frac{M}{P}}^{(M)}(m-p)}_{\substack{\text{rate of growth} \\ \text{of real money}}}$$

Expression $(m - p)$ can be explained as follows:

$$\begin{aligned}(m-p) &= \frac{\dot{M}}{M} - \frac{\dot{P}}{P} \\ &= r_M - r_P\end{aligned}$$

$$\text{Rate of growth of real money} = r\left(\frac{M}{P}\right)$$

Interpretation: Feedback effect of inflation on unemployment is developed: $\frac{du}{dt} = -k(m - p)$.
Negative relation exists between the two variables.

TOPIC 143: PHILLIPS CURVE: TIME PATH OF EXPECTED RATE OF INFLATION

After developing feedback effect (monetary policy) of inflation on unemployment, we have a system of equation, rich in behavioral characteristics:

$$\frac{dU}{dt} = -k(m - p)$$

Including Expectations-augmented Phillips Relation:

$$p = \alpha - T - \beta U + g\pi$$

we get a system of equations with the three variables π , p and U .

Time Path: Two of the three variables need to be removed to get a time-path of the remaining (single) variable. We remove p and U in step – I and step – II, respectively.

Step – I (Remove p):

Given the Adaptive Expectations Hypothesis: $\frac{d\pi}{dt} = j(\bar{p} - \pi)$, we substitute

$$\boxed{p = \alpha - T - \beta U + g\pi}$$

$$\begin{aligned}\frac{d\pi}{dt} &= j(\boxed{\alpha - T - \beta U + g\pi} - \pi) \\ \frac{d\pi}{dt} &= j(\alpha - T - \beta U + g\pi - \pi) \\ \frac{d\pi}{dt} &= j\{\alpha - T - \beta U + (g - 1)\pi\} \\ \frac{d\pi}{dt} &= j(\alpha - T - \beta \boxed{U}) + j(g - 1)\pi\end{aligned}$$

p is removed, U remains

Step – II (Remove U): Substitution of U is not straightforward, as value of U is not available.

Instead, we have $\left(\frac{dU}{dt}\right)$. So, we create $\left(\frac{dU}{dt}\right)$ by differentiating the given equation with U :

$$\begin{aligned}\frac{d^2\pi}{dt^2} &= \frac{d}{dt} \{j(\alpha - T - \beta \boxed{U}) + j(g - 1)\pi\} \\ &= j \times \frac{d}{dt} \left\{ \left(\overset{\text{parameter}}{\hat{\alpha}} - \overset{\text{exogenous}}{\hat{T}} - \beta \overset{=f(t)}{\hat{U}} \right) \right\} + j \times \frac{d}{dt} \left\{ \left(\overset{\text{parameter}}{\hat{g}} - 1 \right) \overset{=f(t)}{\hat{\pi}} \right\} \\ \frac{d^2\pi}{dt^2} &= j \times (-\beta) \frac{d}{dt} (U) + j \times (g - 1) \frac{d}{dt} (\pi) \\ \frac{d^2\pi}{dt^2} &= -\beta j \boxed{\frac{dU}{dt}} + j(g - 1) \frac{d\pi}{dt} \\ &\quad \text{feedback effect of } p \text{ on } U \\ \frac{d^2\pi}{dt^2} &= -\beta j \left\{ -k \left(m - \overset{p \text{ reintroduced}}{\hat{p}} \right) \right\} + j(g - 1) \frac{d\pi}{dt} \\ &\quad \text{U removed} \\ \frac{d^2\pi}{dt^2} &= j\beta km - j\beta k \boxed{\hat{p}} + j(g - 1) \frac{d\pi}{dt}\end{aligned}$$

To remove p again, we recall $\frac{d\pi}{dt} = j(p - \pi)$ and rearrange to get p in terms of π & $\frac{d\pi}{dt}$.

$$\begin{aligned}\frac{d\pi}{dt} \frac{1}{j} &= (p - \pi) \\ \frac{d\pi}{dt} \frac{1}{j} + \pi &= p \\ \boxed{p} &= \frac{d\pi}{dt} \frac{1}{j} + \pi\end{aligned}$$

Substituting the reintroduced p to replace it with π & $\frac{d\pi}{dt}$.

$$\begin{aligned}\frac{d^2\pi}{dt^2} &= j\beta km - j\beta k \left[\left(\frac{d\pi}{dt} \frac{1}{j} + \pi \right) \right] + j(g-1) \frac{d\pi}{dt} \\ \frac{d^2\pi}{dt^2} &= j\beta km - j\beta k \left[\left(\frac{d\pi}{dt} \frac{1}{j} \right) \right] - j\beta k (\pi) + j(g-1) \frac{d\pi}{dt} \\ \underbrace{\frac{f''(x)}{\frac{d^2\pi}{dt^2}}}_{\text{potential second order differential equation}} &= j\beta km - \beta k \left(\frac{f'(x)}{\frac{d\pi}{dt}} \right) - j\beta k \left(\frac{f(x)}{\pi} \right) + j(g-1) \left(\frac{f'(x)}{\frac{d\pi}{dt}} \right) \\ \frac{d^2\pi}{dt^2} + \beta k \left(\frac{d\pi}{dt} \right) - j(g-1) \left(\frac{d\pi}{dt} \right) + j\beta k \pi &= j\beta km \\ \frac{d^2\pi}{dt^2} + \underbrace{\{\beta k - j(g-1)\}}_{a_1} \left(\frac{d\pi}{dt} \right) + \underbrace{j\beta k}_{a_2} (\pi) &= \underbrace{j\beta km}_b\end{aligned}$$

Here, $a_1 = \{\beta k - j(g-1)\}$, $a_2 = j\beta k$, $b = j\beta km$

Particular Solution: $y_p = \frac{b}{a_2}$, if $a_2 \neq 0$.

Here $a_2 = \overset{>0}{j} \overset{>0}{\beta} \overset{>0}{k} \neq 0$. So, we can use $y_p = \frac{b}{a_2}$.

Here $\pi_p = \frac{b}{a_2} = \frac{j\beta km}{j\beta k}$

$$\boxed{\pi_p = m}$$

Interpretation: Thus, in this model, the intertemporal equilibrium value of the expected rate of inflation (π) hinges exclusively on the rate of growth of nominal money (m).

Complementary Function: For choosing among the three cases of complementary function, we test for the nature of roots.

Nature of Roots: Deploying $\boxed{a_1^2 \gtrless 4a_2}$

$$\{\beta k - j(g-1)\} \boxed{\gtrless} 4(j\beta k)$$

Perhaps it is not possible to confirm one of the three possibilities $\boxed{\gtrless}$, given the parametric restrictions of symbols β , k , g and j . So, we assume the possibility of any of the nature of roots i.e. Distinct Real, Repeated Real and Complex Roots.

Formula for Roots: $r_1, r_2 = \frac{1}{2} \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right)$

$$\text{Here, } a_1 = \left\{ \overset{+ve}{\beta k} - j \left(\overset{+ve}{g} - 1 \right) \right\} > 0 \text{ and } a_2 = 4(j\beta k) > 0$$

$\begin{matrix} <1 \\ -ve \end{matrix}$

Dynamic Stability: It all boils down to question of dynamic stability of all three cases:

i- **Distinct Real Roots:** $r_1, r_2 = \frac{1}{2} \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right)$

Since, both r_1 and r_2 are $-ve$, the time-path will be **dynamic stable**.

ii- **Repeated Real Roots:** $r_1, r_2 = \frac{1}{2} \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right)$

Since, r_1 and r_2 are equal and $-ve$, the time-path will be **dynamic stable**.

iii- **Complex Roots:** $r_1, r_2 = \left(\underbrace{-\frac{a_1}{2}}_{\text{real part}} \pm \frac{1}{2} \sqrt{\underbrace{a_1^2 - 4a_2}_{-ve}} \right)$

Since, $h = -\frac{a_1}{2} < 0$, the time-path will be **dynamic stable**.

Caveat: In this derivation, time-path of expected rate of inflation was calculated. However, one can do it for other variables ($U \& p$) as well.

TOPIC 144: PHILLIPS CURVE: TIME PATH OF EXPECTED RATE OF INFLATION - NUMERICAL

Example: Let the three equations of the model take the specific forms:

Expectations-augmented Phillips Relation: $p = \alpha - T - \beta U + g\pi$

$$p = \underbrace{\left(\frac{1}{6}\right)}_{\alpha} - \underbrace{3}_{\beta} U + \underbrace{(1)}_g \pi$$

Adaptive Expectations Hypothesis: $\frac{d\pi}{dt} = j(p - \pi)$

$$\frac{d\pi}{dt} = \underbrace{\left(\frac{3}{4}\right)}_j (p - \pi)$$

Feedback effect of Inflation on Unemployment: $\frac{dU}{dt} = -k(m - p)$

$$\frac{dU}{dt} = -\underbrace{\left(\frac{1}{2}\right)}_k (m - p)$$

The given data; $\beta = 3$, $g = 1$, $j = \frac{3}{4}$ and $k = \frac{1}{2}$. Values of a_1 , a_2 and b can be found by using:

$$\begin{aligned} a_1 &= \beta k + j(1 - g) & a_2 &= j\beta k & \text{and} & b &= j\beta k m \\ a_1 &= (3) \left(\frac{1}{2}\right) + \frac{3}{4}(1 - 1) & a_2 &= \left(\frac{3}{4}\right)(3) \left(\frac{1}{2}\right) & \text{and} & b &= \left(\frac{3}{4}\right)(3) \left(\frac{1}{2}\right) m \\ \boxed{a_1 = \frac{3}{2}} & & \boxed{a_2 = \frac{9}{8}} & & \text{and} & \boxed{b = \left(\frac{9}{8}\right) m} \end{aligned}$$

Particular Integral (π_p): $\pi_p = \frac{b}{a_2}$ as $(a_2 = \frac{9}{8} \neq 0)$

$$\pi_p = \frac{\left(\frac{9}{8}\right)m}{\left(\frac{9}{8}\right)}$$

$$\boxed{\pi_p = m}$$

Interpretation: Thus, in this model, the intertemporal equilibrium value of the expected rate of inflation (π) hinges exclusively on the rate of growth of nominal money (m).

Complementary Function: For choosing among the three cases of complementary function, we test for the nature of roots.

Nature of Roots: Deploying $\boxed{a_1^2 \gtrless 4a_2}$

$$\left(\frac{3}{2}\right)^2 \gtrless 4\left(\frac{9}{8}\right)$$

$$\boxed{\frac{9}{4} < \frac{9}{2}}$$

Complex Roots are present in this case.

Formula for Roots: $r_1, r_2 = \frac{1}{2} \left(-a_1 \pm \sqrt{a_1^2 - 4a_2} \right)$

$$r_1, r_2 = \frac{1}{2} \left\{ -\left(\frac{3}{2}\right) \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4\left(\frac{9}{8}\right)} \right\}$$

$$r_1, r_2 = -\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \pm \left(\frac{1}{2}\right) \sqrt{-\frac{9}{4}}$$

$$\frac{3}{2} \times \sqrt{i}$$

$$r_1, r_2 = -\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \pm \left(\frac{1}{2}\right) \frac{3}{2} \sqrt{i}$$

$$\boxed{r_1, r_2 = -\underbrace{\frac{3}{4}}_h \pm \underbrace{\frac{3}{4}}_v \sqrt{i}}$$

Substituting the extracted values $h = -\frac{3}{4}$ and $v = \frac{3}{4}$ in the complex root case of y_c :

$$\pi_c = e^{\boxed{h}t} (A_5 \cos \boxed{v}t + A_6 \sin \boxed{v}t)$$

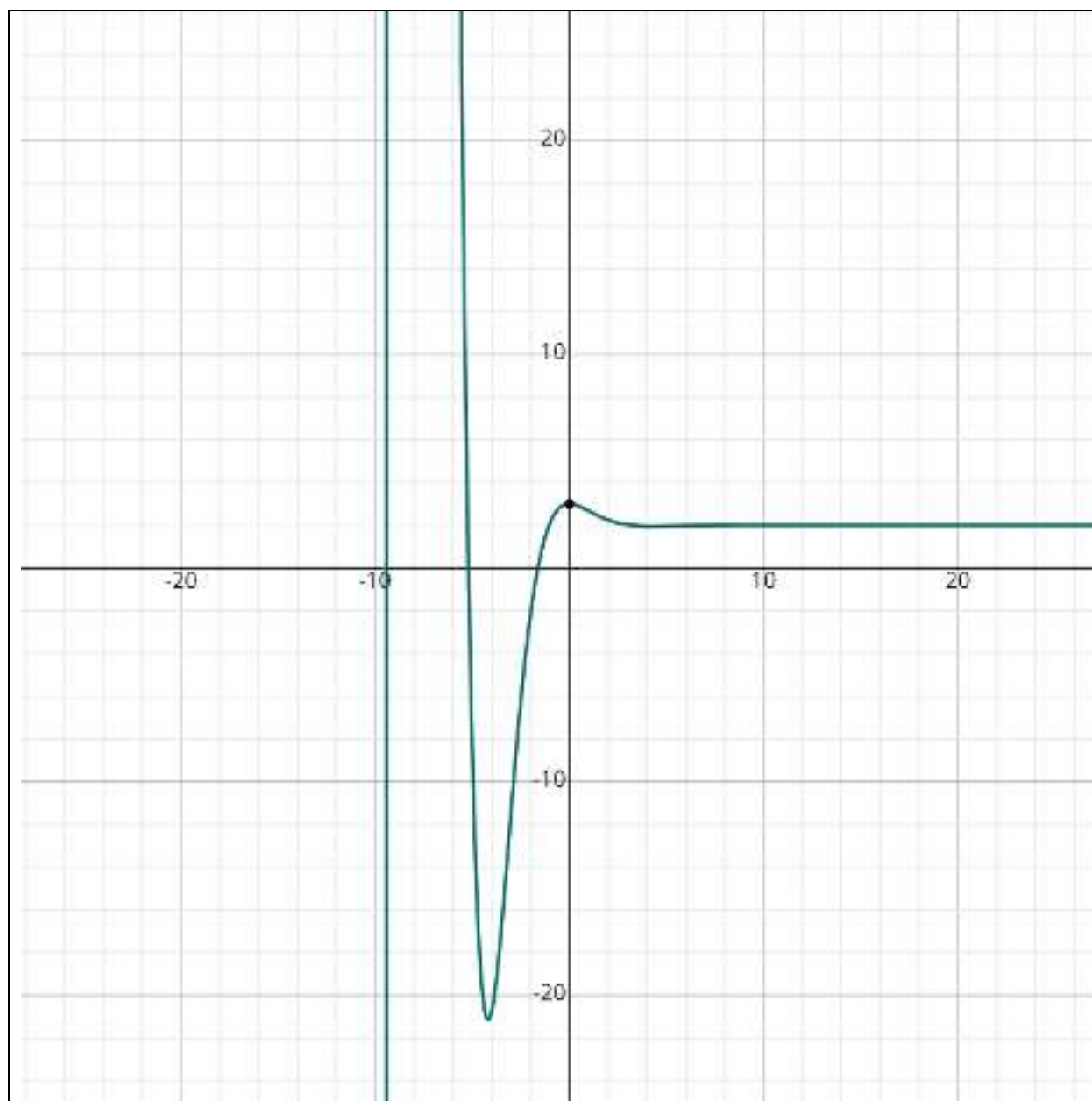
$$\pi_c = e^{\boxed{-\frac{3}{4}}t} \left(A_5 \cos \boxed{\frac{3}{4}}t + A_6 \sin \boxed{\frac{3}{4}}t \right)$$

$$\boxed{\pi_c = e^{\left(-\frac{3}{4}\right)t} \left\{ A_5 \cos \left(\frac{3}{4}t\right) + A_6 \sin \left(\frac{3}{4}t\right) \right\}}$$

General Solution $\pi(t)$: We already know; $\pi(t) = \pi_c + \pi_p$, so by substituting:

$$\pi(t) = e^{\left(-\frac{3}{4}\right)t} \left\{ A_5 \cos \left(\frac{3}{4}t\right) + A_6 \sin \left(\frac{3}{4}t\right) \right\} + \boxed{m}$$

Interpretation: The time-path of Expected Rate of Inflation-growth (π), is $\pi(t) = e^{\left(-\frac{3}{4}\right)t} \left\{ A_5 \cos \left(\frac{3}{4}t\right) + A_6 \sin \left(\frac{3}{4}t\right) \right\} + m$. It is **dynamically stable** as $h = -\frac{3}{4} < 0$. It has a fluctuating time-path converging towards equilibrium value of m .



TOPIC 145: PHILLIPS CURVE: TIME PATH OF INFLATION - NUMERICAL

Time-Path of Inflation (p): Easiest possible method of finding $p(t)$ should be based on simplest equation containing p and already solved variable (π). i.e. $\frac{d\pi}{dt} = j(p - \pi)$.

$$\frac{d\pi}{dt} \frac{1}{j} = (p - \pi)$$

$$\frac{d\pi}{dt} \frac{1}{j} + \pi = p$$

$$\underbrace{p(t)}_{\substack{\text{time-path} \\ \text{of } p}} = \underbrace{\frac{d\pi}{dt}}_{\substack{\text{derivative} \\ \text{of } \pi(t)}} \underbrace{\frac{1}{j}}_{\substack{\text{Not given} \\ \text{already} \\ \text{calculated}}} + \underbrace{\pi(t)}_{\substack{\text{time-path} \\ \text{of } \pi}}$$

Therefore, we need to calculate $\left(\frac{d\pi}{dt}\right)$, before finding $p(t)$, by differentiating $\pi(t)$ w.r.t t .

$$\begin{aligned} \frac{d}{dt}\{\pi(t)\} &= \frac{d}{dt}\left[e^{\left(-\frac{3}{4}\right)t}\left\{A_5 \cos\left(\frac{3}{4}t\right) + A_6 \sin\left(\frac{3}{4}t\right)\right\} + m\right] \\ \frac{\frac{d\pi(t)}{dt}}{\frac{d\pi}{dt}} &= \frac{d}{dt}\left[\underbrace{e^{\left(-\frac{3}{4}\right)t}}_{g(x)} \underbrace{\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\}}_{h(x)}\right] + \frac{d}{dt}\left[\underbrace{e^{\left(-\frac{3}{4}\right)t}}_{g(x)} \underbrace{\left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}}_{h(x)}\right] + \frac{d}{dt}\left(\overset{\text{devoid}}{\text{of } t} \hat{m}\right) \\ \frac{d\pi}{dt} &= \left[\frac{d}{dt}\left\{e^{\left(-\frac{3}{4}\right)t}\right\}\right] \times \left\{A_5 \cos\left(\frac{3}{4}t\right)\right\} + \left[\frac{d}{dt}\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\}\right] \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} + \\ &\quad \left[\frac{d}{dt}\left\{e^{\left(-\frac{3}{4}\right)t}\right\}\right] \times \left\{A_6 \sin\left(\frac{3}{4}t\right)\right\} + \left[\frac{d}{dt}\left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}\right] \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} + 0 \\ \frac{d\pi}{dt} &= \left[\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}\right] \times \left\{A_5 \cos\left(\frac{3}{4}t\right)\right\} + \left[A_5 \left\{\left(-\frac{3}{4}\right)\sin\left(\frac{3}{4}t\right)\right\}\right] \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} + \\ &\quad \left[\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}\right] \times \left\{A_6 \sin\left(\frac{3}{4}t\right)\right\} + \left[A_6 \left\{\left(\frac{3}{4}\right)\cos\left(\frac{3}{4}t\right)\right\}\right] \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} \\ \frac{d\pi}{dt} &= \underbrace{\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}}_{\text{exists in } \pi(t)} \times \underbrace{\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)} + A_5 \left\{\left(-\frac{3}{4}\right)\sin\left(\frac{3}{4}t\right)\right\} \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} + \\ &\quad \underbrace{\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}}_{\text{exists in } \pi(t)} \times \underbrace{\left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)} + A_6 \left\{\left(\frac{3}{4}\right)\cos\left(\frac{3}{4}t\right)\right\} \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\} \\ \frac{d\pi}{dt} &= \left[\underbrace{\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}}_{\text{exists in } \pi(t)} \times \underbrace{\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)} + \underbrace{\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}}_{\text{exists in } \pi(t)} \times \underbrace{\left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)}\right] + \left[A_5 \left\{\left(-\frac{3}{4}\right)\sin\left(\frac{3}{4}t\right)\right\} \times \right. \\ &\quad \left.\left\{e^{\left(-\frac{3}{4}\right)t}\right\} + A_6 \left\{\left(\frac{3}{4}\right)\cos\left(\frac{3}{4}t\right)\right\} \times \left\{e^{\left(-\frac{3}{4}\right)t}\right\}\right] \\ \frac{d\pi}{dt} &= \underbrace{\left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t}}_{\text{common}} \left[\underbrace{\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)} + \underbrace{\left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}}_{\text{exists in } \pi(t)}\right] \\ &\quad + \underbrace{\left\{e^{\left(-\frac{3}{4}\right)t}\right\}}_{\text{common}} \left[A_5 \left\{\left(-\frac{3}{4}\right)\sin\left(\frac{3}{4}t\right)\right\} + A_6 \left\{\left(\frac{3}{4}\right)\cos\left(\frac{3}{4}t\right)\right\}\right] \\ \frac{d\pi}{dt} &= \left(-\frac{3}{4}\right)e^{\left(-\frac{3}{4}\right)t} \left[\left\{A_5 \cos\left(\frac{3}{4}t\right)\right\} + \left\{A_6 \sin\left(\frac{3}{4}t\right)\right\}\right] + \left\{e^{\left(-\frac{3}{4}\right)t}\right\} \left[A_5 \left\{\left(-\frac{3}{4}\right)\sin\left(\frac{3}{4}t\right)\right\} + A_6 \left\{\left(\frac{3}{4}\right)\cos\left(\frac{3}{4}t\right)\right\}\right] \end{aligned}$$

Recalling the main purpose i.e. $p(t)$:

$$\underbrace{p(t)}_{\text{time-path of } p} = \underbrace{\frac{d\pi}{dt}}_{\text{derivative of } \pi(t)} \times \frac{1}{j} + \underbrace{\pi(t)}_{\text{time-path of } \pi}$$

$\frac{d\pi}{dt}$ is *recently calculated*
 $\pi(t)$ is *previously calculated*

$$p(t) = \left[\left(-\frac{3}{4} \right) e^{\left(-\frac{3}{4} \right) t} \left[\left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} + \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} \right] + \left\{ e^{\left(-\frac{3}{4} \right) t} \left[A_5 \left\{ \left(-\frac{3}{4} \right) \sin \left(\frac{3}{4} t \right) \right\} + A_6 \left\{ \left(\frac{3}{4} \right) \cos \left(\frac{3}{4} t \right) \right\} \right] \right\} \right] \times \left[\frac{1}{\left(\frac{3}{4} \right)} + \underbrace{\left[e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) + A_6 \sin \left(\frac{3}{4} t \right) \right\} + m \right]}_{\pi(t)} \right]$$

$$p(t) = \left[\left(-\frac{3}{4} \right) e^{\left(-\frac{3}{4} \right) t} \left[\left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} + \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} \right] + \left\{ e^{\left(-\frac{3}{4} \right) t} \left[A_5 \left\{ \left(-\frac{3}{4} \right) \sin \left(\frac{3}{4} t \right) \right\} + A_6 \left\{ \left(\frac{3}{4} \right) \cos \left(\frac{3}{4} t \right) \right\} \right] \right\} \right] \times \left[\frac{4}{3} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) + A_6 \sin \left(\frac{3}{4} t \right) \right\} + m \right]$$

$$p(t) = \left[-e^{\left(-\frac{3}{4} \right) t} \left[\left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} + \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} \right] + \left\{ e^{\left(-\frac{3}{4} \right) t} \left[A_5 \left\{ -\sin \left(\frac{3}{4} t \right) \right\} + A_6 \left\{ \cos \left(\frac{3}{4} t \right) \right\} \right] \right\} \right] + e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) + A_6 \sin \left(\frac{3}{4} t \right) \right\} + m$$

$$p(t) = \left[-e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} - e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} - e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \sin \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \cos \left(\frac{3}{4} t \right) \right\} \right] + e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} + m$$

$$p(t) = \left[-e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} - e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} \right] - e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \sin \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \cos \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \cos \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \sin \left(\frac{3}{4} t \right) \right\} + m$$

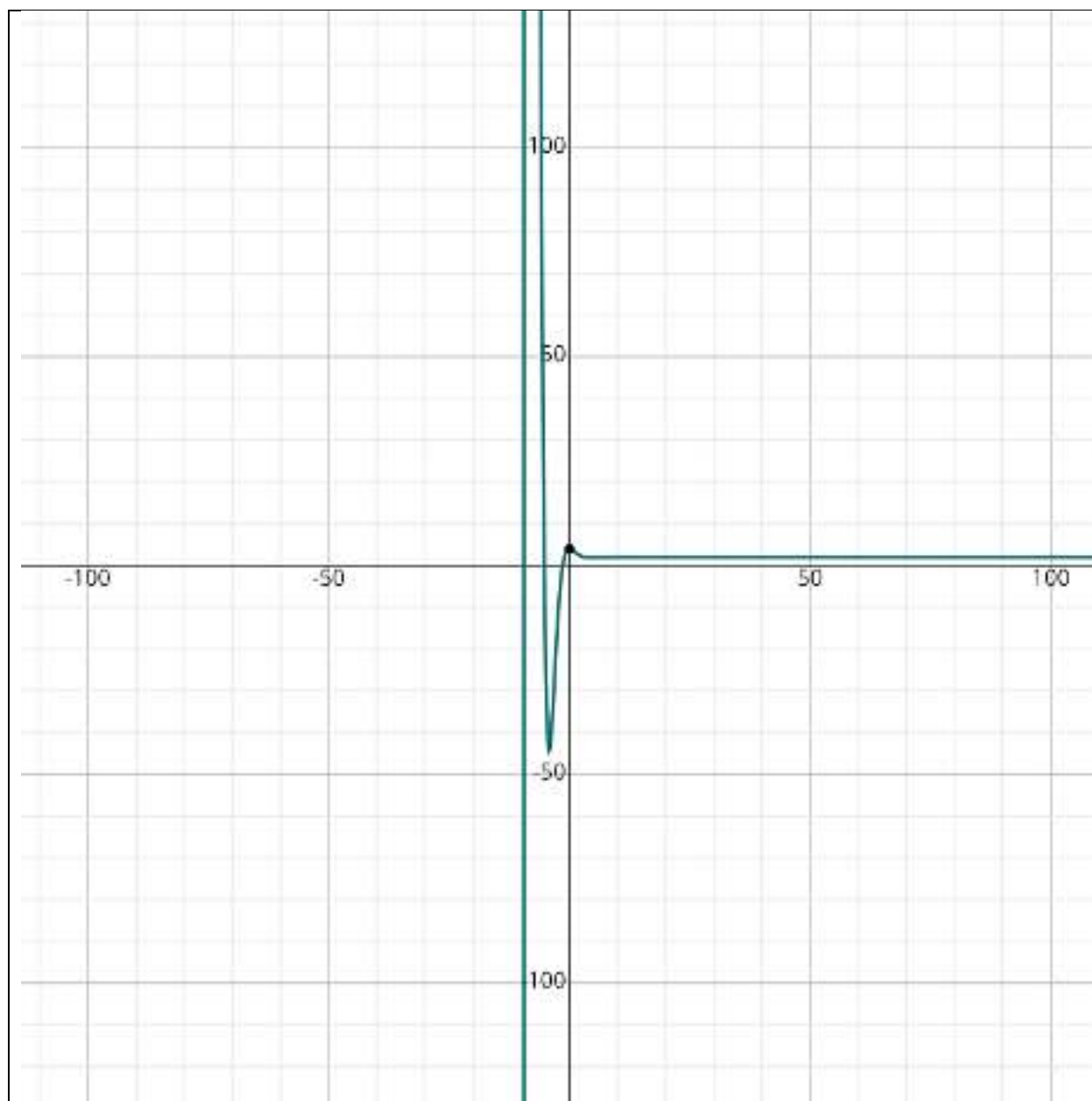
$$p(t) = \underbrace{\left[-e^{\left(-\frac{3}{4} \right) t} \left\{ A_5 \sin \left(\frac{3}{4} t \right) \right\} + e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \cos \left(\frac{3}{4} t \right) \right\} \right]}_{\text{common}} + m$$

$$p(t) = e^{\left(-\frac{3}{4} \right) t} \left[\left\{ -A_5 \sin \left(\frac{3}{4} t \right) \right\} + \left\{ A_6 \cos \left(\frac{3}{4} t \right) \right\} \right] + m$$

$$p(t) = e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \cos \left(\frac{3}{4} t \right) - A_5 \sin \left(\frac{3}{4} t \right) \right\} + m$$

Interpretation: The time-path of Actual rate of inflation-growth (p), is

$p(t) = e^{\left(-\frac{3}{4} \right) t} \left\{ A_6 \cos \left(\frac{3}{4} t \right) - A_5 \sin \left(\frac{3}{4} t \right) \right\} + m$. It is **dynamically stable** as $h = -\frac{3}{4} < 0$. Like time-path of Expected rate of inflation-growth, $p(t)$ has a fluctuating time-path converging towards equilibrium value of m as well.



TOPIC 146: PHILLIPS CURVE: TIME PATH OF UNEMPLOYMENT – NUMERICAL

Time-Path of Unemployment (U): Feedback effect of Inflation on Unemployment is: $p = \frac{1}{6} - 3U + \pi$.

Rewriting it for U , we get: $3U = \frac{1}{6} + \pi - p$

$$U = \frac{1}{18} + \frac{\pi}{3} - \frac{p}{3}$$

$$U = \frac{1}{3}(\pi - p) + \frac{1}{18}$$

Given the time-paths of π and p , we can get time-path of U .

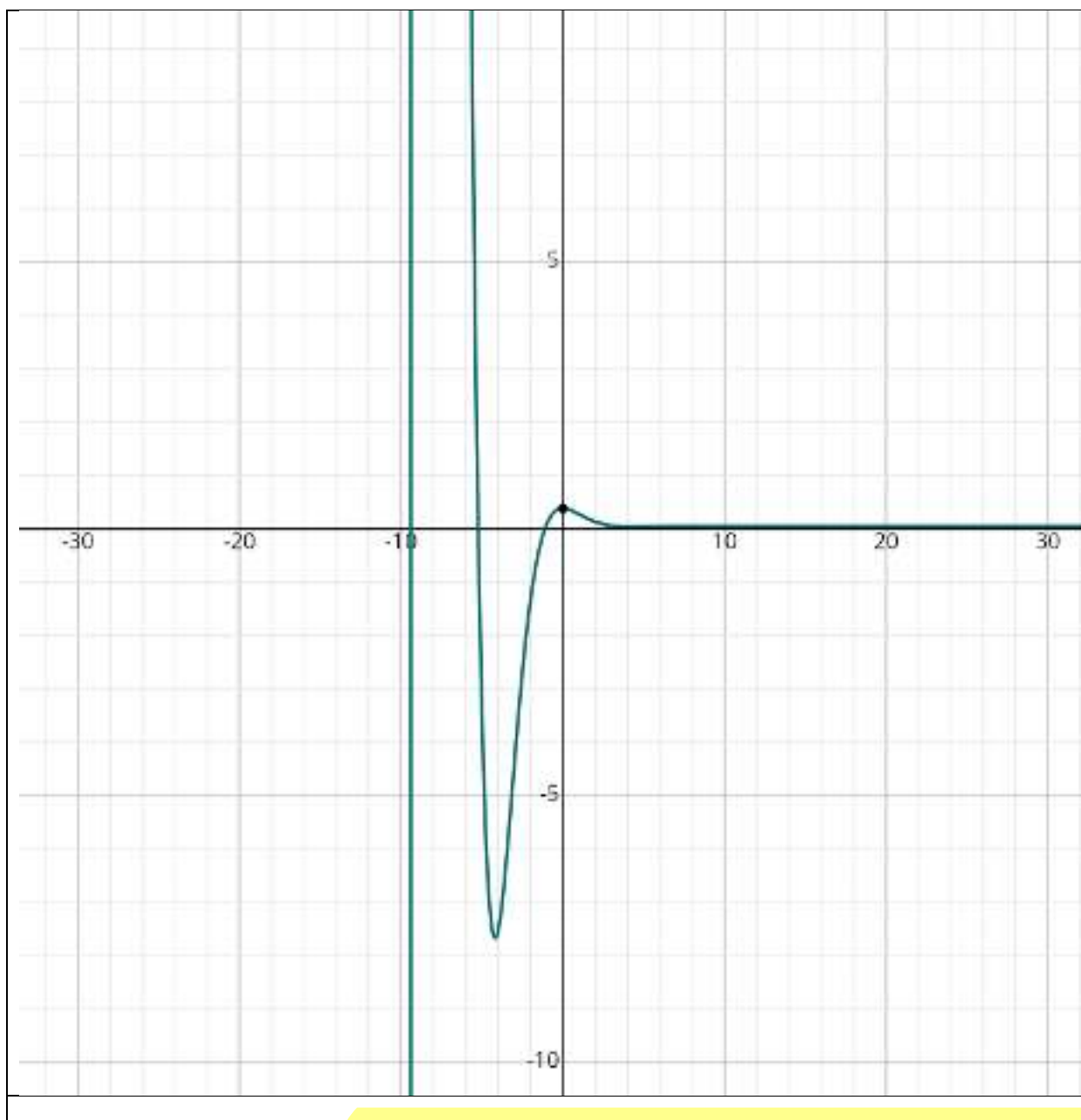
$$U = \frac{1}{3}(\bar{\pi} - \bar{p}) + \frac{1}{18}$$

converted to
time-path

$$\begin{aligned}\widehat{U}(t) &= \frac{1}{3} \left[e^{\left(-\frac{3}{4}\right)t} \left\{ A_5 \cos\left(\frac{3}{4}t\right) + A_6 \sin\left(\frac{3}{4}t\right) \right\} + m \right] - \\ &\quad \left[e^{\left(-\frac{3}{4}\right)t} \left\{ A_6 \cos\left(\frac{3}{4}t\right) - A_5 \sin\left(\frac{3}{4}t\right) \right\} + m \right] + \frac{1}{18} \\ U(t) &= \frac{1}{3} \left[e^{\left(-\frac{3}{4}\right)t} \left[A_5 \cos\left(\frac{3}{4}t\right) + A_6 \sin\left(\frac{3}{4}t\right) + m - A_6 \cos\left(\frac{3}{4}t\right) - A_5 \sin\left(\frac{3}{4}t\right) - m \right] + \frac{1}{18} \right] \\ U(t) &= \frac{1}{3} e^{\left(-\frac{3}{4}\right)t} \left[A_5 \cos\left(\frac{3}{4}t\right) - A_6 \cos\left(\frac{3}{4}t\right) + A_6 \sin\left(\frac{3}{4}t\right) - A_5 \sin\left(\frac{3}{4}t\right) \right] + \frac{1}{18} \\ U(t) &= \frac{1}{3} e^{\left(-\frac{3}{4}\right)t} \left[\overbrace{(A_5 - A_6) \cos\left(\frac{3}{4}t\right)}^{\text{common}} + \overbrace{(A_6 - A_5) \sin\left(\frac{3}{4}t\right)}^{\text{common}} \right] + \frac{1}{18} \\ U(t) &= \frac{1}{3} e^{\left(-\frac{3}{4}\right)t} \left[\underbrace{(A_5 - A_6) \left\{ \cos\left(\frac{3}{4}t\right) \right\}}_{\bar{U}_c} + \underbrace{(A_6 - A_5) \left\{ \sin\left(\frac{3}{4}t\right) \right\}}_{\bar{U}_p} \right] + \frac{1}{18}\end{aligned}$$

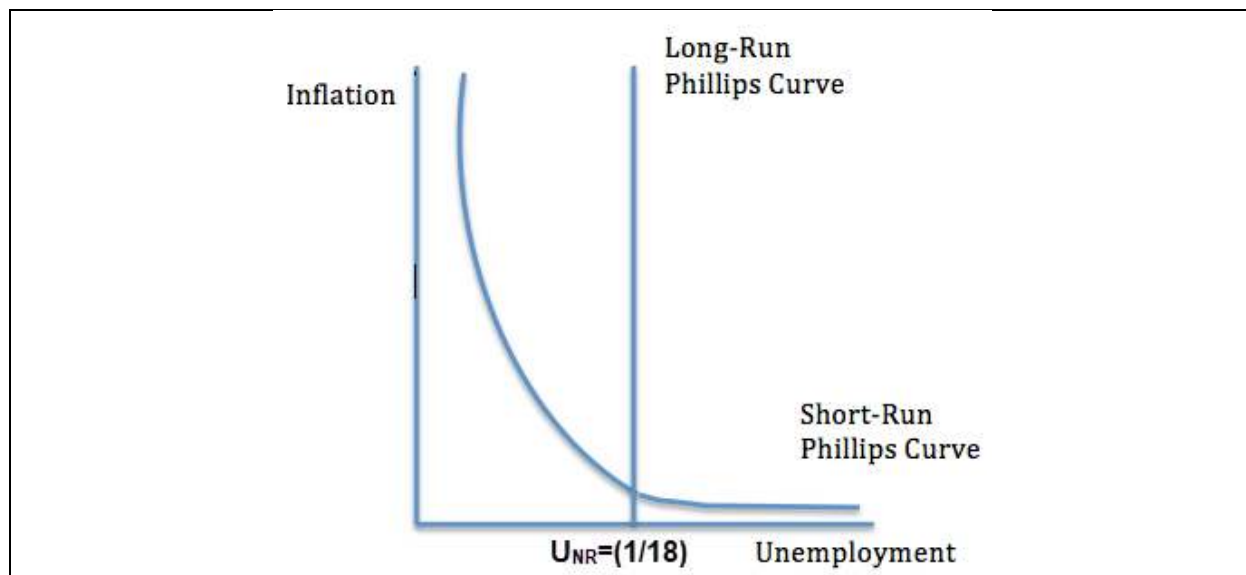
Interpretation: The time-path of Unemployment (U), is

$U(t) = \frac{1}{3} e^{\left(-\frac{3}{4}\right)t} \left[(A_5 - A_6) \left\{ \cos\left(\frac{3}{4}t\right) \right\} + (A_6 - A_5) \left\{ \sin\left(\frac{3}{4}t\right) \right\} \right] + \frac{1}{18}$. It is **dynamically stable** as $h = -\frac{3}{4} < 0$. Like time-paths of Expected & Actual rates of inflation-growth $\{\pi(t), p(t)\}$, $U(t)$ has a fluctuating time-path converging towards equilibrium value of $\left(\frac{1}{18}\right)$.



U converges to the constant $\left(\frac{1}{18}\right)$ regardless of the rate of growth of nominal money (m) and hence regardless of the equilibrium rate of inflation. This constant equilibrium value of U is referred to as the natural rate of unemployment (U_{NR}).

Replotting, the Phillips Relation using U_{NR} . We get vertical straight line parallel to the p -axis. That vertical line relating the equilibrium values of U and p to each other, is known as the long-run Phillips curve.



Lesson 33

HIGHER ORDER DIFFERENTIAL EQUATIONS

TOPIC 147: DIFFERENTIAL EQUATION WITH VARIABLE TERM

Second Order Differential Equation with constant coefficient and constant term:

$$y''(t) + \underbrace{a_1}_{\text{constant coefficient}} y'(t) + \underbrace{a_2}_{\text{constant coefficient}} y = \underbrace{b}_{\text{constant term}}$$

What if: instead of b , there is a variable term $\{f(t)\}$ in terms of t .

$$\overbrace{y''(t) + \underbrace{a_1}_{\text{constant coefficient}} y'(t) + \underbrace{a_2}_{\text{constant coefficient}} y = \underbrace{f(t)}_{\substack{t^n: n \neq 0 \\ (y_p) \text{ shall change}}}}^{\text{constant-coefficient variable-term differential equation}}$$

no change on L.H.S $\Rightarrow (y_c)$ remains unchanged, as it always depends on $r^2 + a_1 r + a_2 = 0$.

Solution: Such situation calls for a certain method, known as **Method of Undetermined Coefficients**.

Pre-condition: Method of Undermined Coefficients requires that the variable term and its successive derivatives together contain only a **finite** and distinct types of expressions:

Function & its Derivatives	Finite & Distinct Type Expressions	Infinite & Repetitive Type of Expressions	
$y(t)$	$6t^2 - t - 1$	$2e^{-4t}$	$\sin(t)$
$y'(t)$	$12t - 1$	$-8e^{-4t}$	$\cos(t)$
$y''(t)$	12	$32e^{-4t}$	$-\sin(t)$
$y'''(t)$	0	$-128e^{-4t}$	$-\cos(t)$
$y''''(t)$	$-$	$\dots$	$\dots$

Numerical Example: Find the Particular Integral of:

$$y''(t) + \underbrace{5}_{a_1} y'(t) + \underbrace{3}_{a_2} y(t) = \underbrace{6t^2 - t - 1}_{\substack{\text{variable term} \\ f(t)}}$$

Here, $a_1 = 5$, $a_2 = 3$ and $f(t) = 6t^2 - t - 1$.

Here, particular integral (y_p) is calculated differently due to change in **L.H.S**.

$$\underbrace{y''(t) + 5y'(t) + 3y(t)}_{\substack{\text{L.H.S} \\ y(t), y'(t) \& y''(t) \text{ terms of } (t)}} \stackrel{\text{must be equal}}{=} \underbrace{6t^2 - t - 1}_{\substack{\text{R.H.S} \\ \text{quadratic} \\ \text{terms of } (t)}}$$

Both sides are in terms of (t) , however, the degree of polynomial on **L.H.S** is not explicit.

Equality ($=$) requires that if **R.H.S** is quadratic in terms of (t) , then **L.H.S** should also be quadratic in terms of (t) . So, we assume a (trial) general quadratic function that could satisfy

$$[y(t), y'(t) \& y''(t)] \text{ on L.H.S to satisfy equality: } [y(t) = B_1 t^2 + B_2 t + B_3]$$

Developing $y'(t)$ and $y''(t)$ for the given $([y''(t)] + 5[y'(t)] + 3[y(t)] = 6t^2 - t - 1)$.

$$[y'(t) = 2B_1 t + B_2]$$

$$[y''(t) = 2B_1]$$

Since $y(t)$, $y'(t)$ and $y''(t)$ are now symbolized, we substitute them on **L.H.S**:

$$\begin{aligned}
 & \overbrace{y''(t) + 5y'(t) + 3y(t)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{2B_1 + 5(2B_1t + B_2) + 3(B_1t^2 + B_2t + B_3)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{2B_1 + 10B_1t + 5B_2 + 3B_1t^2 + 3B_2t + 3B_3}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{3B_1t^2 + 10B_1t + 3B_2t + 2B_1 + 5B_2 + 3B_3}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{(3B_1)t^2 + (10B_1 + 3B_2)t + (2B_1 + 5B_2 + 3B_3)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{(3B_1)t^2 + (10B_1 + 3B_2)t + (2B_1 + 5B_2 + 3B_3)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S} \\
 & \overbrace{(3B_1)t^2 + (10B_1 + 3B_2)t + (2B_1 + 5B_2 + 3B_3)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S}
 \end{aligned}$$

Comparing the coefficients of t^2 & t and the constants on both sides of equation:

$$\begin{aligned}
 (3B_1) &= 6 & (10B_1 + 3B_2) &= (-1) & (2B_1 + 5B_2 + 3B_3) &= (-1) \\
 B_1 &= 2 & 10(2) + 3B_2 &= -1 & 2(2) + 5(-7) + 3B_3 &= -1 \\
 B_1 &= 2 & 20 + 3B_2 &= -1 & 4 - 35 + 3B_3 &= -1 \\
 B_1 &= 2 & B_2 &= -7 & B_3 &= 10
 \end{aligned}$$

So, the **Undetermined** coefficients are now **determined**.

Substituting B_1 , B_2 and B_3 in $y(t) = B_1t^2 + B_2t + B_3$ assumed above as the (particular) solution.

$$y_p(t) = (2)t^2 + (-7)t + (10)$$

$$y_p = 2t^2 - 7t + 10$$

Interpretation: So, the particular integral of the 'constant-coefficient and variable-term second order differential equation' $y''(t) + 5y'(t) + 3y(t) = 6t^2 - t - 1$ is $y_p = 2t^2 - 7t + 10$.

Method used to find (y_p) is Method of Undetermined Coefficients. Whereas, (y_c) can be found using methods same as 'constant-coefficient and constant-term second order differential equation'.

TOPIC 148: DIFFERENTIAL EQUATION WITH VARIABLE TERM – A MODIFICATION

Sometimes, 'constant-coefficient and variable-term second order differential equation' $\{y''(t) + a_1y'(t) + a_2y = f(t)\}$ lacks term(s) on *L.H.S*. Under this situation, the Method of Undetermined Coefficient is applied with a **Modification**: We experiment with a numerical example:

Numerical Example: Find the Particular Integral of:

$$y''(t) + 5y'(t) = 6t^2 - t - 1$$

Labelling the given differential equation:

$$y''(t) + \underbrace{5}_{a_1} y'(t) + \underbrace{0}_{a_2} y(t) = \underbrace{6t^2 - t - 1}_{\substack{\text{variable term} \\ f(t)}}$$

Here, $a_1 = 5$, $a_2 = 0$ and $f(t) = 6t^2 - t - 1$.

However, particular integral (y_p) is differently calculated due to change in *L.H.S*.

$$\overbrace{y''(t) + 5y'(t) + 0y(t)}^{L.H.S} \stackrel{\text{must be equal}}{\cong} \overbrace{6t^2 - t - 1}^{R.H.S}$$

$y(t), y'(t) \& y''(t)$ interms of (t)
 $\text{quadratic interms of } (t)$

Both sides are in terms of (t) , however, the degree of polynomial on $L.H.S$ is not explicit. Equality ($=$) requires that if $R.H.S$ is quadratic in terms of (t) , then $L.H.S$ should also be quadratic in terms of (t) . So, we assume a general quadratic function that could satisfy $y(t)$ on $L.H.S$ to satisfy equality:

$$y(t) = B_1 t^2 + B_2 t + B_3$$

Developing $y'(t)$ and $y''(t)$ for the given $(y''(t) + 5y'(t)) = 6t^2 - t - 1$.

$$y'(t) = 2B_1 t + B_2$$

$$y''(t) = 2B_1$$

Since $y(t)$, $y'(t)$ and $y''(t)$ are now symbolized, we substitute them on $L.H.S$:

$$\begin{aligned} \overbrace{y''(t) + 5y'(t) + 0y(t)}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{2B_1 + 5(2B_1 t + B_2) + 0(B_1 t^2 + B_2 t + B_3)}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{2B_1 + 10B_1 t + 5B_2 + 0 + 0 + 0}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{10B_1 t + 2B_1 + 5B_2}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{(0)t^2 + (10B_1)t + 2B_1 + 5B_2}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{(0)t^2 + (10B_1)t + (2B_1 + 5B_2)}^{L.H.S} &= \overbrace{6t^2 + (-1)t + (-1)}^{R.H.S} \end{aligned}$$

Extracting the coefficients of t^2 & t and the constants on both sides of equation:

$$\boxed{(0) = 6} \quad \boxed{(10B_1) = (-1)} \quad \boxed{(2B_1 + 5B_2) = (-1)}$$

ILLOGICAL

Comparison gives an illogical coefficient, which rejects $y(t) = B_1 t^2 + B_2 t + B_3$ as the (Particular) Solution of differential equation – need for **Modification**.

Modification: The trick is to experiment $y(t) = t \times (B_1 t^2 + B_2 t + B_3)$ as the trial (Particular) Solution. The multiplicative term $t \times$ increase the degree of $y(t)$, $y'(t)$ and $y''(t)$ by 1. In the last comparison, t^2 was missing a coefficient. This might be fixed due to this modification.

So, we assume a (trial) general quadratic function (after modification) that could satisfy $y(t)$ on $L.H.S$ to satisfy equality:

$$y(t) = (B_1 t^3 + B_2 t^2 + B_3 t)$$

Developing $y'(t)$ and $y''(t)$ for the given $(y''(t) + 5y'(t)) = 6t^2 - t - 1$.

$$y'(t) = (3B_1 t^2 + 2B_2 t + B_3)$$

$$y''(t) = 6B_1 t + 2B_2$$

Since $y(t)$, $y'(t)$ and $y''(t)$ are now symbolized, we substitute them on $L.H.S$:

$$\begin{aligned} \overbrace{y''(t) + 5y'(t) + 0y(t)}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \\ \overbrace{6B_1 t + 2B_2 + 5(3B_1 t^2 + 2B_2 t + B_3) + 0(B_1 t^3 + B_2 t^2 + B_3 t)}^{L.H.S} &= \overbrace{6t^2 - t - 1}^{R.H.S} \end{aligned}$$

$$\overbrace{6B_1t + 2B_2 + 15B_1t^2 + 10B_2t + 5B_3}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S}$$

$$\overbrace{(15B_1)t^2 + (6B_1 + 10B_2)t + (2B_2 + 5B_3)}^{L.H.S} = \overbrace{6t^2 - t - 1}^{R.H.S}$$

The **Modification** has served its purpose. As t^2 has non-zero coefficient. Now the comparison is foolproof.

Extracting the coefficients of t^2 & t and the constants on both sides of equation:

$$\begin{array}{l} \boxed{(15B_1) = 6} \quad \boxed{(6B_1 + 10B_2) = (-1)} \quad \boxed{(2B_2 + 5B_3) = (-1)} \\ \boxed{B_1 = \frac{6}{15}} \quad \boxed{6\left[\frac{2}{5}\right] + 10B_2 = -1} \quad \boxed{2\left(-\frac{17}{50}\right) + 5B_3 = -1} \\ \boxed{B_1 = \frac{2}{5}} \quad \boxed{\frac{12}{5} + 10B_2 = -1} \quad \boxed{-\frac{34}{50} - 5B_3 = -1} \\ \boxed{B_1 = \frac{2}{5}} \quad \boxed{B_2 = -\frac{17}{50}} \quad \boxed{B_3 = -\frac{8}{125}} \end{array}$$

So, all of the **Undetermined** coefficients are now **determined**.

Substituting B_1 , B_2 and B_3 in $y(t) = B_1t^2 + B_2t + B_3$ assumed above as the (particular) solution.

$$\underbrace{y(t)}_{y_p} = \left(\frac{2}{5}\right)t^2 + \left(-\frac{17}{50}\right)t + \left(-\frac{8}{125}\right)$$

$$\boxed{y_p = \left(\frac{2}{5}\right)t^2 + \left(-\frac{17}{50}\right)t + \left(-\frac{8}{125}\right)}$$

Interpretation: So, the particular integral of the 'constant-coefficient and variable-term second-order differential equation' $\boxed{y''(t) + 5y'(t) = 6t^2 - t - 1}$ is $\boxed{y_p = \left(\frac{2}{5}\right)t^2 + \left(-\frac{17}{50}\right)t + \left(-\frac{8}{125}\right)}$.

Method of Undetermined Coefficients was applied with a Modification which was to multiply the trial (quadratic function) solution with (t) . Complementary Function (y_c) can be found using methods same as 'constant-coefficient and constant-term second order differential equation'.

TOPIC 149: HIGHER-ORDER DIFFERENTIAL EQUATION: PARTICULAR INTEGRAL

After comprehending Second-Order Differential Equation, we can extend the order of the differential equation to n^{th} (higher) order. It is synonymous to an n^{th} degree polynomial with degree of polynomial replaced with order of derivative:

$$\overbrace{a_0y^n(t) + a_1y^{(n-1)}(t) + \dots + a_{n-1}y'(t) + a_ny = b}^{order\ of\ derivative\ increases\ \{y(t), y'(t)\ to\ y^{(n)}(t)\}}$$

$$\overbrace{a_0y^n(t) + a_1y^{(n-1)}(t) + \dots + a_{n-1}y'(t) + a_ny = b}^{coefficient-subscript\ increases\ (a_0\ to\ a_n)}$$

Solution: For higher order differential equation, we need Particular Integral (y_p) and Complementary Function (y_c).

Particular Integral (y_p):

Since second order differential equation is also a higher order differential equation, we adopt the Particular Integrals from it:

	2 nd Order Differential Equation			n^{th} Order Differential Equation		
Order	$n = 2$			$n > 2$		
y_p	$a_2 \neq 0,$ $\left\{ \begin{array}{c} (a_1 \neq 0) \\ \vee \\ (a_1 = 0) \end{array} \right\}$	$a_2 = 0,$ $a_1 \neq 0$	$a_2 = 0,$ $a_1 = 0$	$a_n \neq 0,$ $\left\{ \begin{array}{c} (a_{n-1} \neq 0) \\ \vee \\ (a_{n-1} = 0) \end{array} \right\}$	$a_n = 0,$ $a_{n-1} \neq 0$	$a_n = 0, a_{n-1} = 0$

	$y_p = \frac{b}{a_2}$	$y_p = \frac{b}{a_1} t$	$y_p = \frac{b}{2} t^2$	$y_p = \frac{b}{a_n}$	$y_p = \frac{b}{a_{n-1}} t$	Use $y = kt^2$ as trial solution, as used in SODE .
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Numerical Example: Find the Particular Solution of:

$$y^{(4)}(t) + \underbrace{6}_{a_1} y'''(t) + \underbrace{14}_{a_2} y''(t) + \underbrace{16}_{a_3} y'(t) + \underbrace{8}_{a_4} y = \underbrace{24}_b$$

Here; $a_1 = 6$, $a_2 = 14$, $a_{n-1} = a_3 = 16$, $a_n = a_4 = 8 \neq 0$ and $b = 24 \neq 0$.
 $y_p = \frac{b}{a_n}$ non-homogenous case

$$y_p = \frac{b}{a_4}$$

$$y_p = \frac{24}{8}$$

$$y_p = 3$$

Interpretation: The 4th (higher) order differential equation, $y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$ has a particular integral $y_p = 3$. It is a static equilibrium as y_p is a constant value.

Numerical Example: Find the Particular Solution of:

$$y'''(t) + y''(t) + 3y'(t) = 1$$

$$y'''(t) + \underbrace{1}_{a_1} y''(t) + \underbrace{3}_{a_2} y'(t) + \underbrace{0}_{a_3} y = \underbrace{1}_b$$

Here; $a_1 = 1$, $a_{n-1} = a_2 = 3$, $a_n = a_3 = 0$ and $b = 1 \neq 0$.
 $b = \frac{b}{a_{n-1}} t$ non-homogenous case

$$y_p = \frac{b}{a_{n-1}} t$$

$$y_p = \frac{1}{3} t$$

Interpretation: The 3rd (higher) order differential equation, $y'''(t) + y''(t) + 3y'(t) = 1$ has a particular integral $y_p = \frac{1}{3} t$. It is a moving equilibrium as y_p contains variable (t).

Numerical Example: Find the Particular Solution of:

$$y'''(t) + 3y''(t) = \frac{1}{3}$$

$$y'''(t) + \underbrace{3}_{a_1} y''(t) + \underbrace{0}_{a_2} y'(t) + \underbrace{0}_{a_3} y = \underbrace{\frac{1}{3}}_b$$

Here; $a_1 = 3$, $a_{n-1} = a_2 = 0$, $a_n = a_3 = 0$ and $b = \frac{1}{3} \neq 0$.
 $b = \frac{b}{a_{n-1}} t$ non-homogenous case

Experiment with trial solution $y = kt^2$ which implies $y'(t) = 2kt$, $y''(t) = 2k$ and $y'''(t) = 0$. Substituting all of these values in given equation will give us k :

$$0 + 3(2k) + (0)2kt + (0)kt^2 = \frac{1}{3}$$

$$6k = \frac{1}{3}$$

$$k = \frac{1}{18}$$

Returning back the value of k in, $\underbrace{y}_{\substack{\text{trial} \\ \text{solution}}} = \boxed{k}t^2$:

Since trial solution has satisfied the differential equation, it can, now, be called particular solution of the differential equation.

$$\underbrace{y}_{\substack{\text{becomes} \\ \text{particular} \\ \text{solution } (y_p)}} = \boxed{\frac{1}{18}}t^2$$

$$\boxed{y_p = \frac{1}{18}t^2}$$

Interpretation: The 3rd (higher) order differential equation, $y'''(t) + 3y''(t) = \frac{1}{3}$ has a particular integral $\boxed{y_p = \frac{1}{18}t^2}$. It is a moving equilibrium as y_p contains variable (t) .

TOPIC 150: HIGHER-ORDER DIFFERENTIAL EQUATION: COMPLEMENTARY FUNCTION

For Higher Order Differential equations, the complementary functions can be built upon those of Second (Higher) Order Differential equations. We consider the following numerical example:

Numerical Example: Find the Complementary Function of:

$$y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$$

$$y^{(4)}(t) + \underbrace{6}_{a_1}y'''(t) + \underbrace{14}_{a_2}y''(t) + \underbrace{16}_{a_3}y'(t) + \underbrace{8}_{a_4}y = \underbrace{24}_b$$

Here; $a_1 = 6$, $a_2 = 14$, $a_{n-1} = a_3 = 16$, $a_n = a_4 = 8 \neq 0$ and $\underbrace{b = 24 \neq 0}_{\substack{\text{non-homogenous} \\ \text{case}}}$.

For Second Order Differential equation, the characteristic equation was:

$$r^2 + a_1r + a_2 = 0$$

Accordingly, for Fourth Order Differential equation, the characteristic equation will be:

$$r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0$$

To find the roots (r), we need to solve the characteristic equation, therefore, factorizing $r^4 + a_1r^3 + a_2r^2 + a_3r + a_4 = 0$, we get:

$$\underbrace{(r+2)}_{\substack{\text{linear} \\ \text{function}}} \times \underbrace{(r-2)}_{\substack{\text{linear} \\ \text{function}}} \times \underbrace{(r^2+2r+2)}_{\substack{\text{quadratic} \\ \text{equation}}} = 0$$

Either;

$(r+2) = 0$	$(r-2) = 0$	$\left(\underbrace{a}_{(1)}r^2 + \underbrace{b}_{2}r + \underbrace{c}_{2}\right) = 0$
$\Downarrow$	$\Downarrow$	$r_3, r_4 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$
$\Downarrow$	$\Downarrow$	$r_3, r_4 = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2}$
$\boxed{r_1 = -2}$	$\boxed{r_2 = 2}$	$\boxed{r_3, r_4 = \underbrace{-1}_h \pm \underbrace{2i}_v}$

Summarizing the (hybrid) group of roots:

$$\underbrace{r_1 = -2, r_2 = 2}_{\substack{\text{distinct real roots}}} , \underbrace{r_3 = -1 + 2i, r_4 = -1 - 2i}_{\substack{\text{conjugate complex roots}}}$$

Accordingly, the complementary function will also be hybrid:

$$y_c = \underbrace{A_1 e^{-r_1 t} + A_2 e^{-r_2 t}}_{\text{Distinct Real roots segment}} + \underbrace{e^{ht}(A_3 \cos t + A_4 \sin t)}_{\text{Complex roots segment}}$$

Hybrid nature of complementary function

Interpretation: Complementary Function in case of higher order differential equations can be hybrid. Here, it is a combination of Distinct Real roots and Complex roots.

TOPIC 151: CONVERGENCE AND THE ROUTH THEOREM

In Higher Order Differential Equations, the roots are more than 2 as well as the nature of roots can be hybrid. i.e. Distinct Real Roots, Repeated Real Roots and Complex Roots. As the number of roots increase, the complexity in the calculations also increase. A simpler method for ascertaining the dynamic stability without getting entangled in calculation of roots can be instrumental. English Mathematician Edward Routh (1831-1907) did just that by giving a qualitative method of judging the dynamic stability of higher order differential equations.



Limitation: Routh Theorem doesn't allow to plot time path, as it doesn't calculate the roots.

Formation: It is based on the Characteristic Equation of Higher Order Differential Equation:

$$\begin{aligned} & \stackrel{(=1)}{\widehat{a_0}} r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_{n-1} r + a_n = 0 \\ & \quad \quad \quad \text{power of root (r) increases} \\ & \underbrace{a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \dots + a_{n-1} r + a_n}_{\text{coefficient-subscript increases}} = 0 \end{aligned}$$

All roots will be negative ($r_1, r_2, \dots, r_n < 0$), only if first n of the following determinants are positive:

$$|a_1|, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \dots$$

all determinants should be positive

Pattern of Development:

$$|a_1|, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \dots$$

1. Number of Determinants = Order of Differential Equation. e.g. For $n = 4$, four determinants shall be evaluated.
2. **Red Arrows:** A (V) like shape occurs from principal diagonal to the last column. It contains coefficients from $a_1, \dots$.
3. **Yellow Arrows:** remaining elements are filled in such a way that vertically there exists an ascending order from bottom to top.
4. Zero(s) (0) are used where elements are below (a_0).

5. Subscript of coefficient can be greater than order of given differential equation. e.g. if order of given differential equation is 4. There will be 4 determinants. However, in these 4 determinants, we can see elements with subscripts greater than 4. i.e. a_5, a_6, a_7 . See final column of (4×4) determinant.

Numerical Example 1: Apply Routh Theorem and find the dynamic stability of the time-path (without calculating the characteristic roots):

$$y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$$

$$\underbrace{(1)}_{a_0} y^{(4)}(t) + \underbrace{6}_{a_1} y'''(t) + \underbrace{14}_{a_2} y''(t) + \underbrace{16}_{a_3} y'(t) + \underbrace{8}_{a_4} y = \underbrace{24}_b$$

Here; $\underbrace{a_0 = 1}_{\text{coefficient of } y^{(4)}(t)}$, $a_1 = 6$, $a_2 = 14$, $a_3 = 16$, $a_4 = 8$ and $b = 24$

Routh Theorem States:

$$\underbrace{\left| a_1 \right|, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \dots}_{\text{all determinants should be positive}}$$

$$\underbrace{\left| a_1 \right|, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & \boxed{a_5} \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & \boxed{a_7} \\ a_0 & a_2 & a_4 & \boxed{a_6} \\ 0 & a_1 & a_3 & \boxed{a_5} \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}}_{\text{For order (4), only four determinants shall be evaluated}}$$

Substituting the coefficient values and aligning vertically:

$$\begin{aligned} \left| a_1 \right| &= |6| = \boxed{6 > 0} \\ \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} &= \begin{vmatrix} 6 & 16 \\ 1 & 14 \end{vmatrix} = \boxed{68 > 0} \\ \begin{vmatrix} a_1 & a_3 & \boxed{a_5} \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} &= \begin{vmatrix} 6 & 16 & \boxed{0} \\ 1 & 14 & 8 \\ 0 & 6 & 16 \end{vmatrix} = \boxed{800 > 0} \end{aligned}$$

$\boxed{0}$ is substituted $\boxed{a_5}$ as this coefficient doesn't exist in the given differential equation.

$$\begin{vmatrix} a_1 & a_3 & \boxed{a_5} & \boxed{a_7} \\ a_0 & a_2 & a_4 & \boxed{a_6} \\ 0 & a_1 & a_3 & \boxed{a_5} \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} = \begin{vmatrix} 6 & 16 & \boxed{0} & \boxed{0} \\ 1 & 14 & 8 & \boxed{0} \\ 0 & 6 & 16 & \boxed{0} \\ 0 & 1 & 14 & 8 \end{vmatrix} = \boxed{6400 > 0}$$

$\boxed{0}$ is substituted $\boxed{a_5}$, $\boxed{a_6}$, $\boxed{a_7}$ as these coefficients don't exist in the given differential equation.

Interpretation: For the given **Fourth order** differential equation, **four determinants** were evaluated under Routh Theorem. All turned out to be positive (**6, 68, 800, 6400**). Hence, without finding the roots, we have found that the time-path of given differential equation $y^{(4)}(t) + 6y'''(t) + 14y''(t) + 16y'(t) + 8y = 24$, is **dynamically stable**.

Numerical Example 2: Apply Routh Theorem and find the dynamic stability of the time-path (without calculating the characteristic roots):

$$y'''(t) - 2y''(t) - y'(t) + 2y = 4$$

$$\underbrace{(1)}_{a_0} y'''(t) + \underbrace{(-2)}_{a_1} y''(t) + \underbrace{(-1)}_{a_2} y'(t) + \underbrace{2}_{a_3} y = \underbrace{4}_b$$

Here; $\underbrace{a_0 = 1}_{\text{coefficient of } y'''(t)}$, $\overbrace{a_1 = -2}^{\text{Hack}}$, $a_2 = -1$, $a_3 = 2$ and $b = 4$
 $a_1 = 1^{\text{st}} \text{ determinant}$
 $|a_1| < 0 \Rightarrow \text{Dynamically Unstable}$

Routh Theorem States: $|a_1|, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \dots$
all determinants should be positive
 $\begin{vmatrix} (-ve) \\ \widetilde{a_1} \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & \boxed{a_5} \\ a_0 & a_2 & \boxed{a_4} \\ 0 & a_1 & a_3 \end{vmatrix}$

For order (3), only **three** determinants shall be evaluated

Since a_1 gives the value of 1st determinant and it is $-ve$. One can already guess that time-path is dynamically unstable due to negativity of one (i.e. 1st) determinant.

Nevertheless, we evaluate all of the determinants to put a stamp of validity on this hack.

Substituting the coefficient values and aligning vertically:

$$\begin{aligned} |a_1| &= |-2| = \boxed{-2 < 0} \\ \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} &= \begin{vmatrix} -2 & 2 \\ 1 & -1 \end{vmatrix} = \boxed{0 \not> 0} \\ \begin{vmatrix} a_1 & a_3 & \boxed{a_5} \\ a_0 & a_2 & \boxed{a_4} \\ 0 & a_1 & a_3 \end{vmatrix} &= \begin{vmatrix} -2 & 2 & \boxed{0} \\ 1 & -1 & \boxed{0} \\ 0 & -2 & 2 \end{vmatrix} = \boxed{0 \not> 0} \end{aligned}$$

$\boxed{0}$ is substituted $\boxed{a_4}$ and $\boxed{a_5}$ as these coefficients don't exist in the given differential equation.

Interpretation: For the given **Third order** differential equation, **three determinants** were evaluated under Routh Theorem. **1st is negative**, while 2nd and 3rd are non-positive(= 0). Precisely speaking, the value of determinants are $(-2, 0, 0)$. Hence, without finding the roots, we have found that the time-path of given differential equation $\boxed{y'''(t) - 2y''(t) - y'(t) + 2y = 4}$, is **dynamically unstable**.

Lesson 34

FIRST ORDER DIFFERENCE EQUATIONS

TOPIC 152: DISCRETE TIME: FIRST ORDER DIFFERENCE EQUATION

Till now, dynamic analysis has been conducted assuming time as a continuous variable. However, there can be situations in which time might be taken as non-continuous (discrete) variable. Now, variable t will be allowed to take integer values only.

Comparison of Continuous Time and Discrete Time Analysis:

Criterion	Continuous Time Analysis	Discrete Time Analysis
Etymology (Latin Origin)	Uninterrupted	Separate
Nature of Time (t)	(t) is a continuous variable	(t) is a discrete variable
Nature of Values	Decimals as well as Integers	Integers only
Successive Gaps	Not Necessarily Equal	Equal
Possible Values of (t)	0, 0.5, 1, 1.6, 2, 2.9, ...	0, 1, 2, 3, 4, 5, ...
AKA	Point Analysis	Period Analysis
Tool of Analysis	Derivatives & Differentials	Difference, Lags & Leads
Symbolization	$d(y), \left\{\frac{d(y)}{dt}\right\}, \delta(y), \left\{\frac{\delta(y)}{\delta t}\right\}$	$\Delta(y_t), (y_{t-1}), (y_{t+1})$
Type of Equation	Differential Equations	Difference Equations
Examples	Stock Market Activity	Price of Seasonal crops.

Exclusive Features of Period Analysis:

Period doesn't necessarily relate to calendar rather than analytical sense.

A period is merely a length of time that elapses before the variable y undergoes a change.

$\left(\frac{\Delta y}{\Delta x}\right)$ is the discrete-time counterpart of $\left(\frac{dy}{dx}\right)$.

Consecutive time periods $\Rightarrow \Delta t = 1$ or First Difference of y .

$$\Delta y_t \equiv y_{t+1} - y_t$$

Symbolology:

Use of subscript t for discrete-time.

y_t is the value of y in time period t .

y_{t+1} is the value of y in time period immediately following t .

y_{t+1} is known as 'lead' variable of y_t .

y_{t-1} is known as 'lag' variable of y_t .

Common Terminology: Unlike book of A. C. Chiang, most widely used terminology is not Difference Equations, rather its Recurrence Relations.

Examples:

$\Delta y_t = 2$ (instead of $dy = 2$)	$\Delta y_t = -0.1 y_t$ (instead of $dy = -0.1 y$)
Equivalent Forms	
$\Delta y_t = 2$ $y_{t+1} - y_t = 2$ $y_{t+1} = y_t + 2$	$\Delta y_t = -0.1 y_t$ $y_{t+1} - y_t = -0.1 y_t$ $y_{t+1} - y_t + 0.1 y_t = 0$ $y_{t+1} - 0.9 y_t = 0$

Standard Form:

$$y_{t+1} + ay_t = c$$

Here, a and c are two constants.

y_t is the value of the y in current time period t .

y_{t+1} is the value of the y in next time period $(t + 1)$.

Solution Methods: Difference equations can be solved either by Iterative (Crude) Method or by General (Formula) Method.

TOPIC 153: SOLVING DIFFERENCE EQUATION USING ITERATIVE METHOD

Like differential equation, difference equations can also be solved to get time-path. A crude method of solving the first order difference equation involves repetition. It can be understood with the help of an example.

Example: Solve the following difference equations by iteration, while $y_0 = 10$:

$$y_{t+1} = y_t + 1$$

Solution: Experimenting different values of t in the given differential equation:

$y_{t+1} = y_t + 1$			
Value of t	Substitution	Simplification	Final Form
$t = 0$	$y_{0+1} = y_0 + 1$	$y_1 = y_0 + 1$	$y_1 = y_0 + 1$
$t = 1$	$y_{1+1} = \boxed{y_1} + 1$ $y_1 = y_0 + 1$	$y_2 = \boxed{y_0 + 1} + 1$	$y_2 = y_0 + 2$
$t = 2$	$y_{2+1} = \boxed{y_2} + 1$ $y_2 = y_0 + 2$	$y_3 = \boxed{y_0 + 2} + 1$	$y_3 = y_0 + 3$
$t = 3$	$y_{3+1} = \boxed{y_3} + 1$ $y_3 = y_0 + 3$	$y_4 = \boxed{y_0 + 3} + 1$	$y_4 = y_0 + 4$
...	...	...	...
$t = t$	Generalizing, we get:		$y_t = y_0 + t$

Interpretation: So, we have found the Definite Solution $y_t = y_0 + t$ of the given difference equation. y_t is the current value of y .

y_{t+1} is the future value of y . We can also find its numerical value by substituting the initial condition ($y_0 = 10$).

$$y_t = y_0 + t$$

$$y_t = 10 + t$$

$$y_t = 10 + t$$

Example: Solve the following difference equations by iteration, while $y_0 = \beta$:

$$y_{t+1} = \alpha y_t$$

Solution: Experimenting different values of t in the given differential equation:

$y_{t+1} = \alpha y_t$			
Value of t	Substitution	Simplification	Final Form
$t = 0$	$y_{0+1} = \alpha y_0$	$y_1 = \alpha y_0$	$y_1 = \alpha^1 y_0$
$t = 1$	$y_{1+1} = \alpha \boxed{y_1}$ $y_1 = \alpha y_0$	$y_2 = \alpha \boxed{\alpha y_0}$	$y_2 = \alpha^2 y_0$
$t = 2$	$y_{2+1} = \alpha \boxed{y_2}$ $y_2 = \alpha^2 y_0$	$y_3 = \alpha \boxed{\alpha^2 y_0}$	$y_3 = \alpha^3 y_0$

$t = 3$	$y_{[3]+1} = \alpha \underbrace{y_3}_{y_3 = \alpha^3 y_0}$	$y_4 = \alpha \alpha^3 y_0$	$y_{[4]} = \alpha^4 y_0$
...	...	...	...
$t = t$	Generalizing, we get:		$y_{[t]} = \alpha^t y_0$

Interpretation: So, we have found the Definite Solution $y_t = \alpha^t y_0$ of the given

$$\underbrace{y_t}_{\substack{\text{current} \\ \text{value of } (y)}} = \alpha^t y_0$$

difference equation $\underbrace{y_{t+1}}_{\substack{\text{future} \\ \text{value of } (y)}} = \alpha y_t$. We can also find its numerical value by substituting the

initial condition ($y_0 = \beta$).

$$y_t = \alpha^t y_0$$

$$y_t = \alpha^t \beta$$

$$y_t = \alpha^t \beta$$

TOPIC 154: SOLVING DIFFERENCE EQUATION USING GENERAL METHOD: $A \neq -1$

In addition to Iterative Method, we can use a General (Formula) Method to solve difference equations to get their time-paths. General Solution contains two components:

$$\underbrace{y_t}_{\text{Time Path}} = \underbrace{y_c}_{\substack{\text{Deviation from} \\ \text{equilibrium} \\ \text{value of } y_t}} + \underbrace{y_p}_{\substack{\text{Equilibrium} \\ \text{value of } y_t}}$$

Homogenous Case: Consider the following homogeneous difference equation:

$$m y_{t+1} - n y_t = 0$$

Normalizing it, we get:

$$y_{t+1} - \left(\frac{n}{m}\right) y_t = 0$$

Rearranging to apply iteration method:

$$y_{t+1} = \left(\frac{n}{m}\right) y_t$$

Experimenting different values of t in the given differential equation:

$y_{[t]+1} = \left(\frac{n}{m}\right) y_{[t]}$			
Value of t	Substitution	Simplification	Final Form
$t = 0$	$y_{[0]+1} = \left(\frac{n}{m}\right) y_0$	$y_1 = \left(\frac{n}{m}\right) y_0$	$y_{[1]} = \left(\frac{n}{m}\right)^1 y_0$
$t = 1$	$y_{[1]+1} = \left(\frac{n}{m}\right) \underbrace{y_1}_{y_1 = \left(\frac{n}{m}\right) y_0}$	$y_2 = \left(\frac{n}{m}\right) \left(\frac{n}{m}\right) y_0$	$y_{[2]} = \left(\frac{n}{m}\right)^2 y_0$
$t = 2$	$y_{[2]+1} = \left(\frac{n}{m}\right) \underbrace{y_2}_{y_2 = \left(\frac{n}{m}\right)^2 y_0}$	$y_3 = \left(\frac{n}{m}\right) \left(\frac{n}{m}\right)^2 y_0$	$y_{[3]} = \left(\frac{n}{m}\right)^3 y_0$

$t = 3$	$y_{\boxed{3}+1} = \left(\frac{n}{m}\right) \boxed{y_3}$ $y_3 = \left(\frac{n}{m}\right)^3 y_0$	$y_4 = \left(\frac{n}{m}\right) \boxed{\left(\frac{n}{m}\right)^3 y_0}$	$y_{\boxed{4}} = \left(\frac{n}{m}\right)^{\boxed{4}} y_0$
...	...	...	...
$t = t$	Generalizing, we get:		$y_{\boxed{t}} = \left(\frac{n}{m}\right)^{\boxed{t}} y_0$

Interpretation: So, we have found the Definite Solution $y_t = \left(\frac{n}{m}\right)^{\boxed{t}} y_0$ of the given difference equation

$$\boxed{y_{t+1}} = \left(\frac{n}{m}\right) y_t$$

future
value of (y)

Consolidating the $y_t = \left(\frac{n}{m}\right)^t y_0$ into $y_t = (b)^t A$ or $y_t = Ab^t$.

$y_t = Ab^t$ serves the same role as $y_t = Ae^{-rt}$ did in the differential equations.

(r) played a pivotal role in the dynamic stability of the time-path of a differential equation. In case of difference equations, (b) plays the same role.

Recalling the standard form of first order difference equation:

$$y_{t+1} + ay_t = c$$

Complementary Function: Reducing the standard form of first order difference equation to (homogeneous) Reduced form to get the complementary function:

$$\boxed{y_{t+1}} + a\boxed{y_t} = 0$$

As we found $y_t = Ab^t$, we can also guess its lead form, $y_{\boxed{t+1}} = Ab^{\boxed{t+1}}$

$$\boxed{y_{t+1}} + a\boxed{y_t} = 0$$

Substituting values of y_t and y_{t+1} :

$$\boxed{Ab^{t+1}} + a\boxed{Ab^t} = 0$$

$$Ab^{t+1} + aAb^t = 0$$

$$Ab^t(b + a) = 0$$

$$b + a = 0$$

$$\boxed{b = -a}$$

As the General Solution of reduced form of complete equation is the complementary function, i.e.

$$(y_t)_{R.F.}^{C.F.} = y_c$$

Returning the value of b in $y_t = Ab^t$, we get:

y_c

$$\boxed{y_c = A(-a)^t}$$

Particular Integral: It is a solution that satisfies the difference equation, so we experiment with the simplest (constant) trial solution, i.e. $y_t = k$.

$$y_t = k$$

Extending the time period by 1, leaves constant unchanged:

$$y_{t+1} = k$$

Substituting y_t and y_{t+1} in standard form of first order difference equation:

$$\boxed{y_{t+1}} + a\boxed{y_t} = c$$

$$\boxed{k} + a\boxed{k} = c$$

$$k(1 + a) = c$$

$$\underbrace{k}_{\substack{\text{was opted as} \\ \text{a solution of } y_t}} = \frac{c}{1+a}$$

$$y_p = \frac{c}{1+a}$$

General Solution: Summing the expressions of y_c and y_p :

$$y_t = y_c + y_p$$

$$y_t = A(-a)^t + \frac{c}{1+a}$$

Interpretation: We have derived the General Solution of First Order Difference equation, with a parametric restriction of $a \neq -1$. As if $a \neq -1$ is allowed then the general solution will become undefined:

$$y_t = A(-a)^t + \frac{c}{1+\underbrace{a}_{\neq -1}}$$

Definite Solution: To find the definite solution, we introduce the initial condition, ($t = 0$) in the general solution, $y_{[0]} = A(-a)^{[0]} + \frac{c}{1+a}$.

$$y_{[0]} = A(-a)^{[0]} + \frac{c}{1+a}$$

$$y_0 = A \times (1) + \frac{c}{1+a}$$

$$y_0 - \frac{c}{1+a} = A$$

$$A = y_0 - \frac{c}{1+a}$$

Returning the value of arbitrary constant (A) in the general solution $y_t = \boxed{A}(-a)^t + \frac{c}{1+a}$ to get the definite solution:

$$y_t = \left[y_0 - \frac{c}{1+a} \right] (-a)^t + \frac{c}{1+a}$$

$$y_t = \left(y_0 - \frac{c}{1+a} \right) (-a)^t + \frac{c}{1+a}$$

Interpretation: We have derived the Definite Solution of First Order Difference equation, with a parametric restriction of $a \neq -1$. As if $a \neq -1$ is allowed then the definite solution will become undefined:

$$\text{undefined: } y_t = \left(y_0 - \frac{c}{1+\underbrace{a}_{\neq -1}} \right) (-a)^t + \frac{c}{1+\underbrace{a}_{\neq -1}}$$

TOPIC 155: DIFFERENCE EQUATION USING GENERAL METHOD: $A \neq -1$ NUMERICAL

Solve the first-order difference equation:

$$y_{t+1} - 5y_t = 1$$

$$(y_0 = \frac{7}{4})$$

Recalling standard form of First-Order Difference Equation:

$$y_{t+1} - ay_t = c$$

Comparing the given specific form, we extract values:

$$y_{t+1} + \underbrace{(-5)}_a y_t = \underbrace{1}_c$$

Here, $a = -5 \neq -1$, $c = 1$.

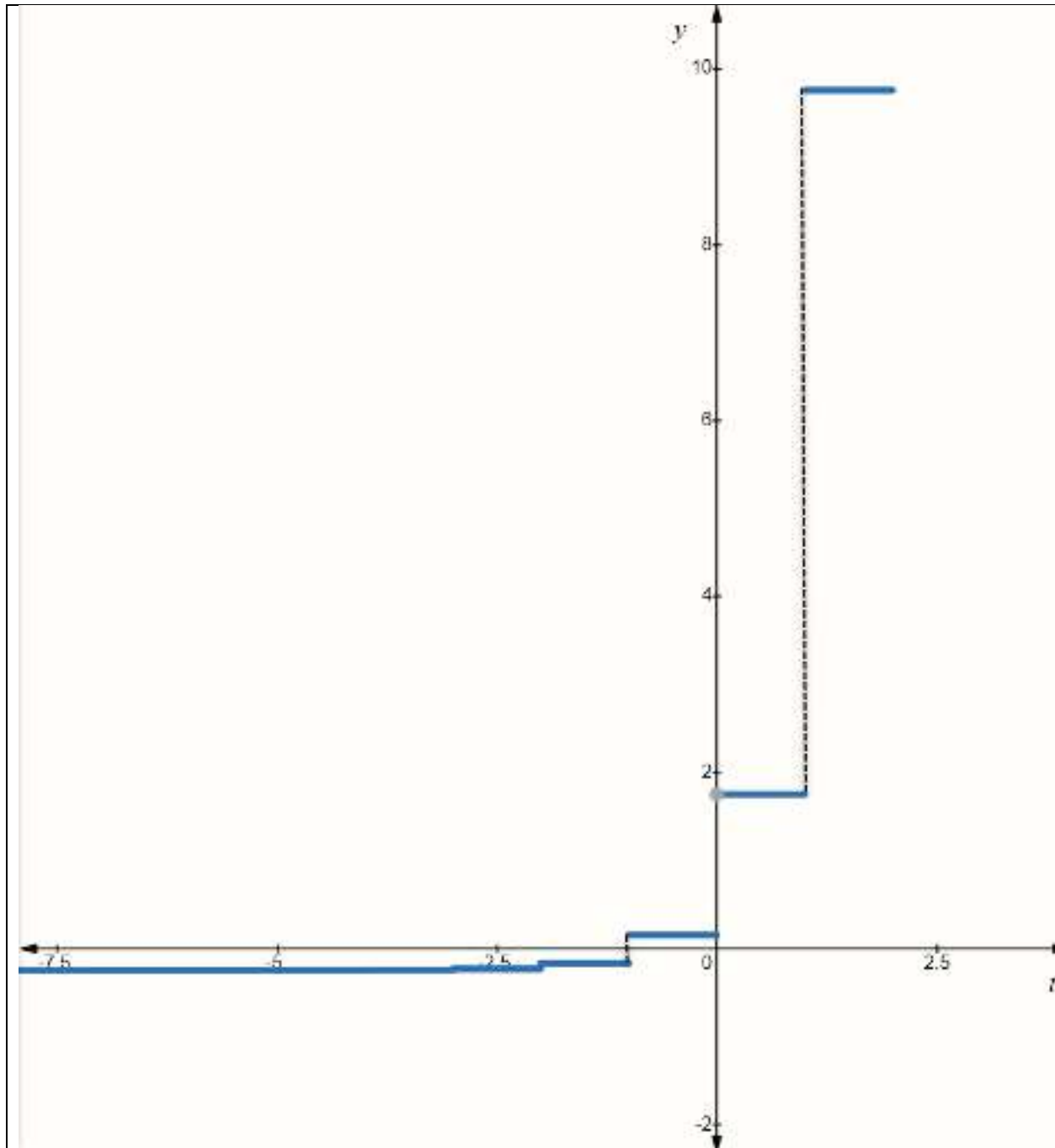
$y_p = \frac{c}{1+a}$ is suitable

Definite Solution: The equation can be solved for the definite solution of time-path by using the formula:

$$\begin{aligned}
 & \text{definitized form of} \\
 & \text{arbitrary constant (A)} \\
 y_t &= \underbrace{\left(\underbrace{y_0}_{\text{given}} - \frac{c}{1+a} \right)}_{y_c} (-a)^t + \underbrace{\frac{c}{1+a}}_{y_p} \\
 y_t &= \left\{ \frac{7}{4} - \frac{1}{1+(-5)} \right\} \{-(-5)\}^t + \frac{1}{1+(-5)} \\
 y_t &= \left(\frac{7}{4} - \frac{1}{1-5} \right) (5)^t + \frac{1}{1-5} \\
 y_t &= \left(\frac{7}{4} - \frac{1}{-4} \right) (5)^t + \frac{1}{-4} \\
 y_t &= \left(\frac{7}{4} + \frac{1}{4} \right) (5)^t - \frac{1}{4} \\
 y_t &= (2)(5)^t - \frac{1}{4} \\
 y_t &= \underbrace{2(5)^t}_{y_c} + \underbrace{\left(-\frac{1}{4} \right)}_{y_p}
 \end{aligned}$$

Interpretation: The definite solution of the first order difference equation $y_{t+1} - 5y_t = 1$ is

$y_t = 2(5)^t - \frac{1}{4}$. $y_p = \frac{1}{4}$, represents a static equilibrium.



TOPIC 156: SOLVING DIFFERENCE EQUATION USING GENERAL METHOD: $A=-1$

General Solution of First Order Difference equation, with a parametric restriction of $a \neq -1$ is

$y_t = A(-a)^t + \frac{c}{1+a}$. However, if $a = -1$ is allowed then the general solution will need revisit.

Complementary Function: As found before, $y_c = A(-a)^t$. If we allow $a = -1$ in it, y_c will not breakdown. So, we can use the same y_c even if $a = -1$ Reducing

Particular Integral: The simplest (constant) trial solution, should not be useful as it was with $a \neq -1$. Therefore, we try next possibility (a variable function). i.e. $y_t = kt$.

$$y_t = kt$$

Extending the time period by 1:

$$y_{t+1} = k(t+1)$$

Substituting y_t and y_{t+1} in standard form of first order difference equation:

$$\frac{y_{t+1}}{k(t+1)} + \frac{ay_t}{akt} = \frac{c}{c}$$

$$kt + k + akt = c$$

$$k(t+1+at) = c$$

$$k = \frac{c}{t+1+\underbrace{a}_{=-1}t}$$

$$k = \frac{c}{t+1+(-1)t}$$

$$k = \frac{c}{t+1-t}$$

$$k = c$$

$$y_t = kt$$

$$\frac{y_t}{ct} = \frac{c}{c}t$$

was opted as
a solution of y_t

$$y_p = ct$$

General Solution: Summing the expressions of y_c and y_p :

$$y_t = y_c + y_p$$

$$y_t = A(-a)^t + ct$$

Interpretation: We have derived the General Solution of First Order Difference equation, without the parametric restriction of $a \neq -1$.

Definite Solution: To find the definite solution, we introduce the initial condition, ($t = 0$) in the general solution, $y_t = A(-a)^t + ct$.

$$y_0 = A(-a)^0 + c0$$

$$y_0 = A \times (1) + 0$$

$$y_0 = A$$

$$A = y_0$$

Returning the value of arbitrary constant (A) in the general solution $y_t = A(-a)^t + ct$ to get the definite solution:

$$y_t = y_0(-a)^t + ct$$

$$y_t = (y_0)(-a)^t + ct$$

Interpretation: We have derived the Definite Solution of First Order Difference equation, without the parametric restriction of $a \neq -1$.

TOPIC 157: DIFFERENCE EQUATION USING GENERAL METHOD: A=-1 NUMERICAL

Solve the first-order difference equation:

$$y_{t+1} - y_t = 2$$

$$(y_0 = 0)$$

Recalling standard form of First-Order Difference Equation:

$$y_{t+1} - ay_t = c$$

Comparing the given specific form, we extract values:

$$y_{t+1} + \underbrace{(-1)}_a y_t = \underbrace{2}_c$$

Here, $\underbrace{a = -1}_{y_p = ct \text{ is suitable}}, c = 2.$

Definite Solution: The equation can be solved for the definite solution of time-path by using the formula:

$$y_t = \underbrace{\underbrace{(\widehat{y_0})}_{\text{given}}}_{y_c} (-a)^t + \underbrace{ct}_{y_p}$$

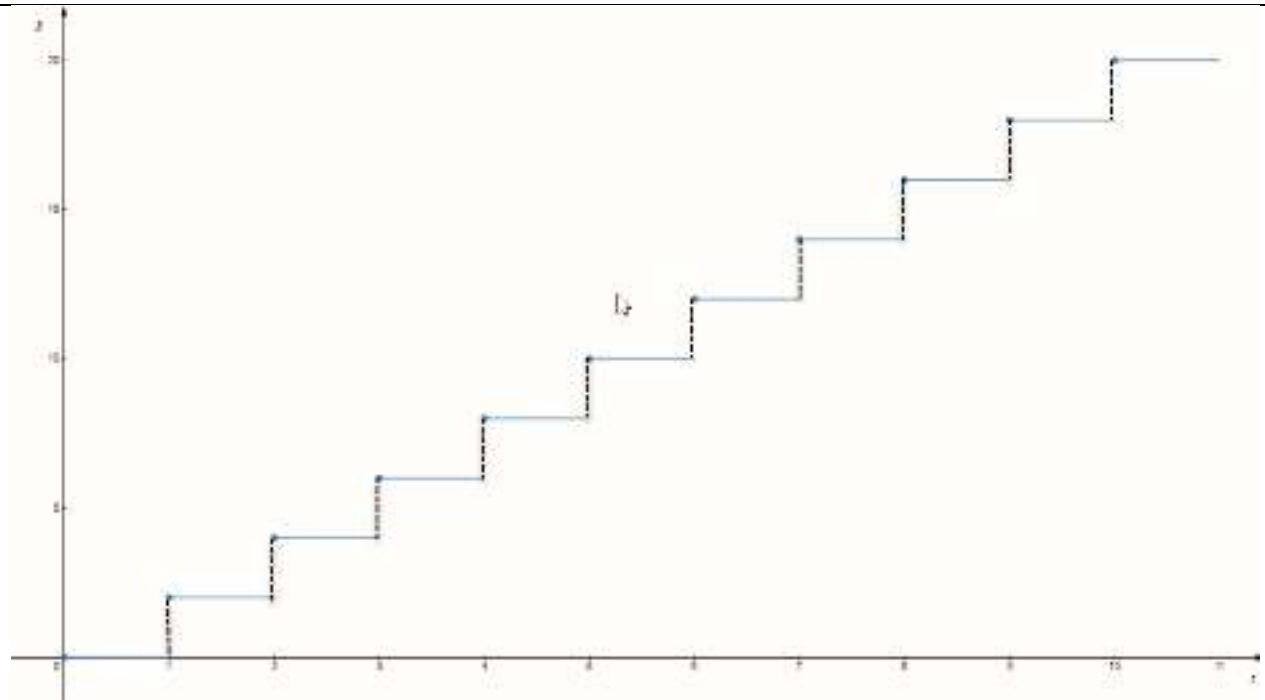
$$y_t = (0)\{-(-1)\}^t + 2t$$

$$y_t = (0)\{1\}^t + 2t$$

$$y_t = \underbrace{\underbrace{\widehat{0}}_{y_c}}_{\text{no deviation from equilibrium}} + \underbrace{\underbrace{\widehat{2t}}_{y_p}}_{\text{time-path already at equilibrium}}$$

$$\boxed{y_t = 2t}$$

Interpretation: The definite solution of the first order difference equation $y_{t+1} - y_t = 2$ is $y_t = 2t$. $y_p = 2t$, represents a moving (jumping/stepped) equilibrium.



Lesson 35

DYNAMIC STABILITY OF FIRST ORDER DIFFERENCE EQUATIONS

TOPIC 158: DYNAMIC STABILITY OF FIRST ORDER DIFFERENCE EQUATION: NON-OSCILLATORY CASES

Like differential equations, the dynamic stability of difference equations also depends on complementary function (y_c). Here, $y_c = A(b)^t$, we consider $(A = 1) \Rightarrow y_c = (b)^t$. This means that expression $(b)^t$ shall determine the dynamics of y_c .

Understanding $(b)^t$ Expression: $(b)^t$ contains b which further is composed of a ($b = -a$).

b	a
Base in Solution of First Order Differential Equation: $y_t = A(\boxed{b})^t + y_p$ $y_t = A(\boxed{-a})^t + y_p$	Coefficient in First Order Differential Equation: $y_t + \boxed{a}y_{t-1} = c$
Determines Dynamic Stability	Determines formula for y_c .

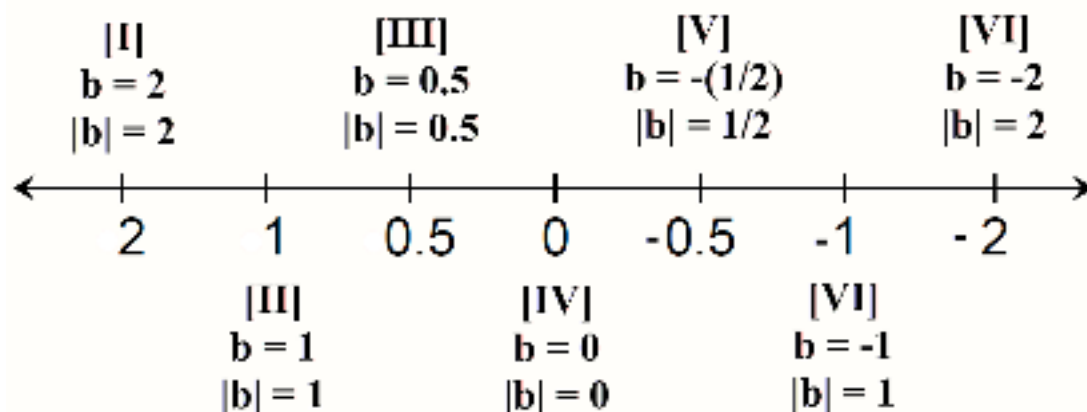
$(b)^t$ is an exponential expression, behavior of which is tabulated below by broadly using +ve, zero and -ve values:

t	$(b)^t$		
	$b = 2 > 0$	$b = 0$	$b = -2 < 0$
	Non-negative		Negative
0	1	0	1
1	0.5	0	-2
2	4	0	4
3	8	0	-8
4	16	0	16
5	32	0	-32
6	64	0	64
7	128	0	-128
8	256	0	256
9	512	0	-512
10	1024	0	1024
...	...	...	...
Remarks	Exponentially growth with +ve signs.	Neither Growth nor Decay.	Exponential change with Alternative (+ve/-ve) signs.
Outcomes	1 st Quadrant, Smooth graph	Horizontal Straight line	1 st & 4 th Quadrants, Ups & Downs in the graph
Terminology	Non-oscillatory when ($b \neq 0$)		Oscillatory when ($b < 0$)

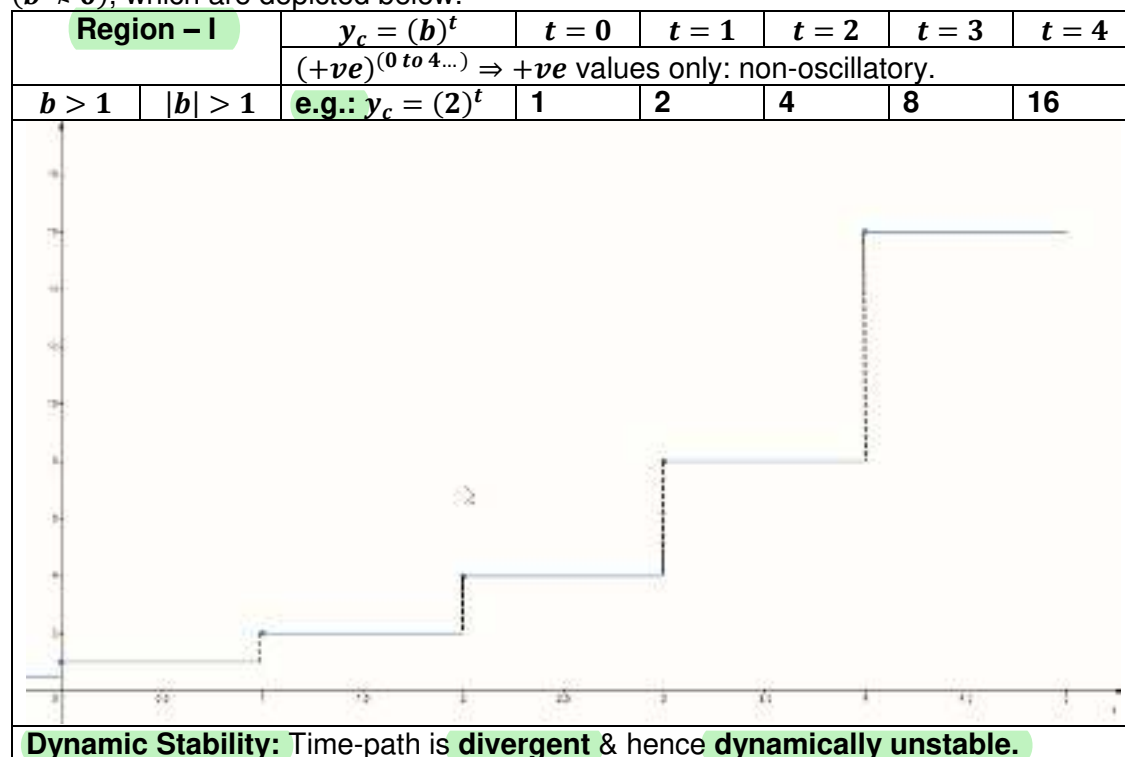
Interpretation: We observe that First Order Difference Equation, have an additional feature of pattern (i.e. Non-oscillatory or Oscillatory):

Time-path	Pattern	Nature
First Order Differential Equation	Non-oscillatory or Oscillatory	Convergent or Divergent

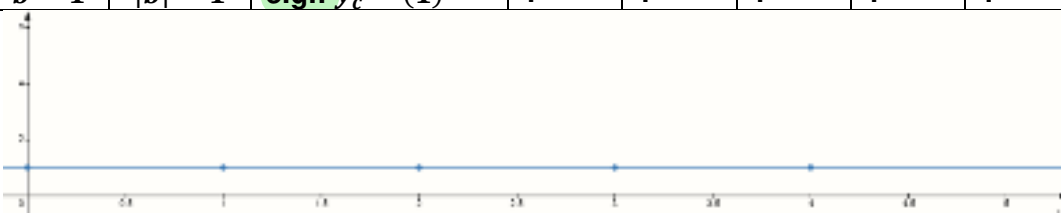
Narrowing down, we focus on the role of b only. Using number line to visualize various regions of b :



The above-mentioned seven values of b : $(-1 > b > 1)$ include all possible patterns of graph of (b^t) . Therefore, for First Order Difference Equations, these values are used to portray cases(regions) of time-paths. Out of seven cases (VII) cases, first Four (I-IV) are **non-oscillatory** ($b \neq 0$), which are depicted below:



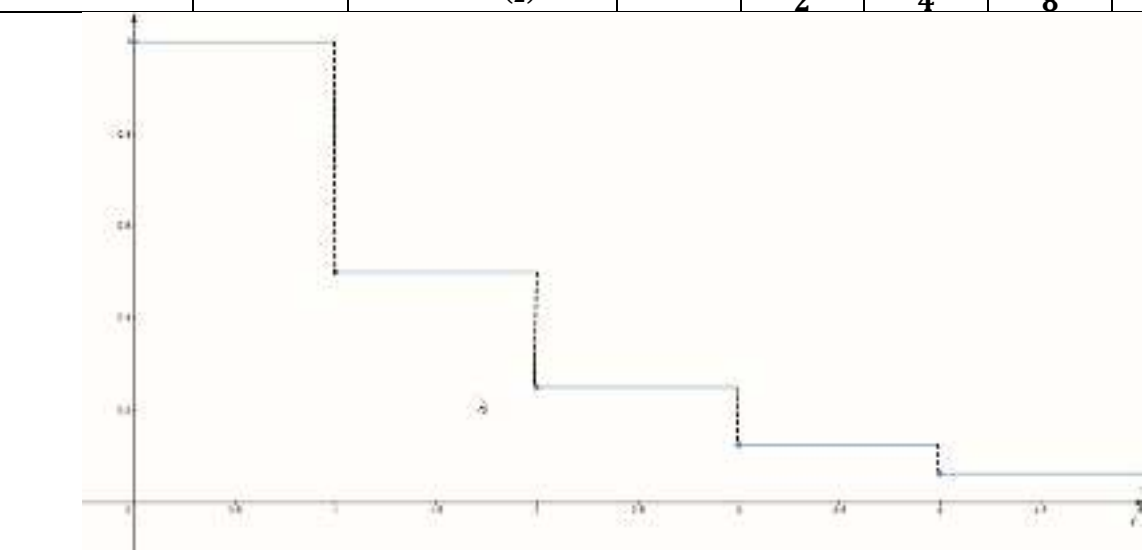
Region – II		$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
		$(+ve)^{(0 \text{ to } 4...)} \Rightarrow +ve$ values only: non-oscillatory.					
$b = 1$	$ b = 1$	e.g.: $y_c = (1)^t$	1	1	1	1	1



The graph shows a horizontal line at $y = 1$ for t values from 0 to 4. The x-axis is labeled t and has tick marks at 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5. The y-axis has tick marks at 0, 2, 4.

Dynamic Stability: Time-path is **neither divergent nor convergent**.

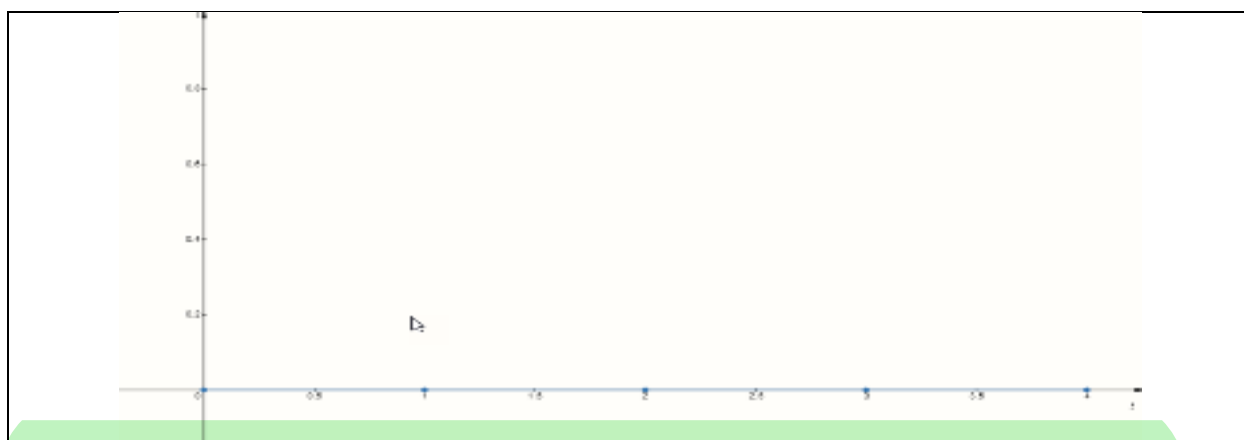
Region – III		$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
		$(+ve)^{(0 \text{ to } 4...)} \Rightarrow +ve$ values only: non-oscillatory.					
$0 < b < 1$	$ b < 1$	e.g.: $y_c = \left(\frac{1}{2}\right)^t$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$



The graph shows a step-down function for $y_c = (1/2)^t$. The x-axis is labeled t and has tick marks at 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5. The y-axis has tick marks at 0, 0.2, 0.4, 0.6, 0.8, 1.0. The function starts at $y = 1$ for $t = 0$, drops to $y = 0.5$ at $t = 1$, drops to $y = 0.25$ at $t = 2$, drops to $y = 0.125$ at $t = 3$, and drops to $y = 0.0625$ at $t = 4$.

Dynamic Stability: Time-path is **convergent** & hence **dynamically stable**.

Region – IV		$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
		$(+ve)^{(0 \text{ to } 4...)} \Rightarrow +ve$ values only: non-oscillatory.					
$b = 0$	$ b = 0$	e.g.: $y_c = (0)^t$	0	0	0	0	0



Dynamic Stability: Time-path is **neither divergent nor convergent**. Infact it is pre-converged at equilibrium. However, since y_c is 0, it means that time-path is already on equilibrium.

Summary: Cases (Regions) I-IV have non-negative b . A non-negative number when raised to any power generates a **positive number**, which gives rise to a graph (time-path, here) in 1st Quadrant only – hence **non-oscillatory**.

$$b \neq 0 \Rightarrow \text{Non-oscillatory}$$

Only when $|b| < 1$, the time-path is convergent.

$$|b| < 1 \Rightarrow \text{Convergent}$$

Tabulating the observations:

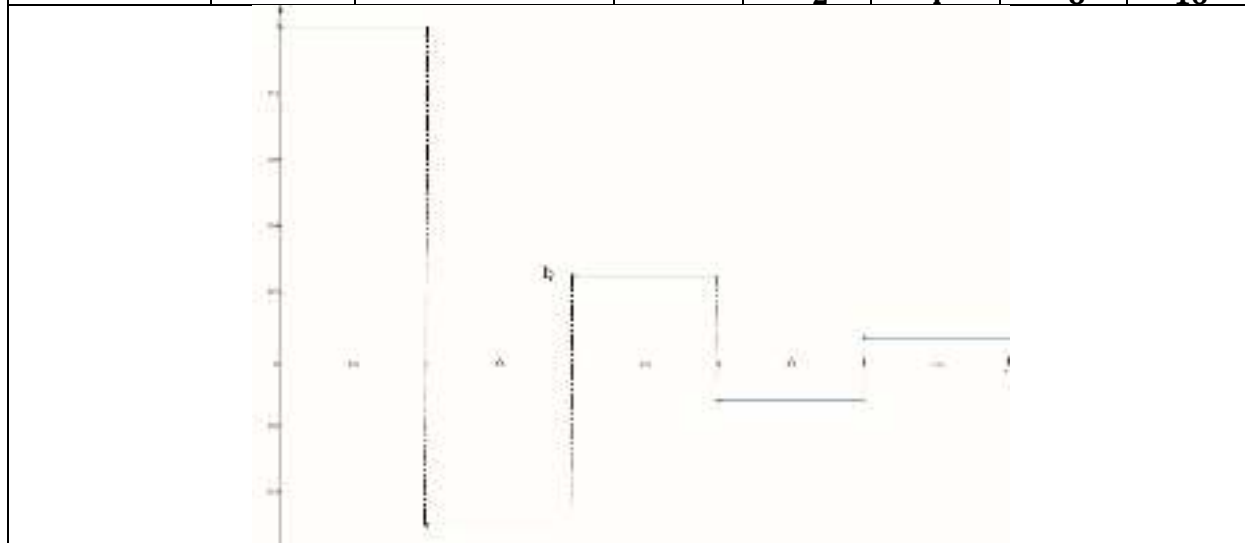
Region – I	Region – II	Region – III	Region – IV
$\left(\frac{b}{2} > 1\right) \vee \left(\frac{b}{2} \neq 0\right)$ non-oscillatory	$\left(\frac{b}{1} = 1\right) \vee \left(\frac{b}{1} \neq 0\right)$ non-oscillatory	$\left(0 < \frac{b}{\left(\frac{1}{2}\right)} < 1\right) \vee \left(\frac{b}{\left(\frac{1}{2}\right)} \neq 0\right)$ non-oscillatory	$\left(\frac{b}{0} = 0\right) \vee \left(\frac{b}{0} \neq 0\right)$ non-oscillatory
$\left(\frac{ b }{2} > 1\right) \vee \left(\frac{ b }{2} \neq 1\right)$ non-convergent (divergent)	$\left(\frac{ b }{1} = 1\right) \vee \left(\frac{ b }{1} \neq 1\right)$ non-convergent (neither)	$\left(0 < \frac{ b }{\left(\frac{1}{2}\right)} < 1\right) \vee \left(\frac{ b }{\left(\frac{1}{2}\right)} \neq 1\right)$ convergent	$\left(\frac{ b }{0} = 0\right) \vee \left(\frac{ b }{0} < 1\right)$ $y_c=0$ no deviation pre-converged

TOPIC 159: DYNAMIC STABILITY OF FIRST ORDER DIFFERENCE EQUATION: OSCILLATORY CASES

After explaining first four (I-IV) cases(regions), we explain the remaining three (V-VII) cases(regions). Out of seven cases (VII) cases, last three (V-VII) are **oscillatory** ($b < 0$), which are depicted below:

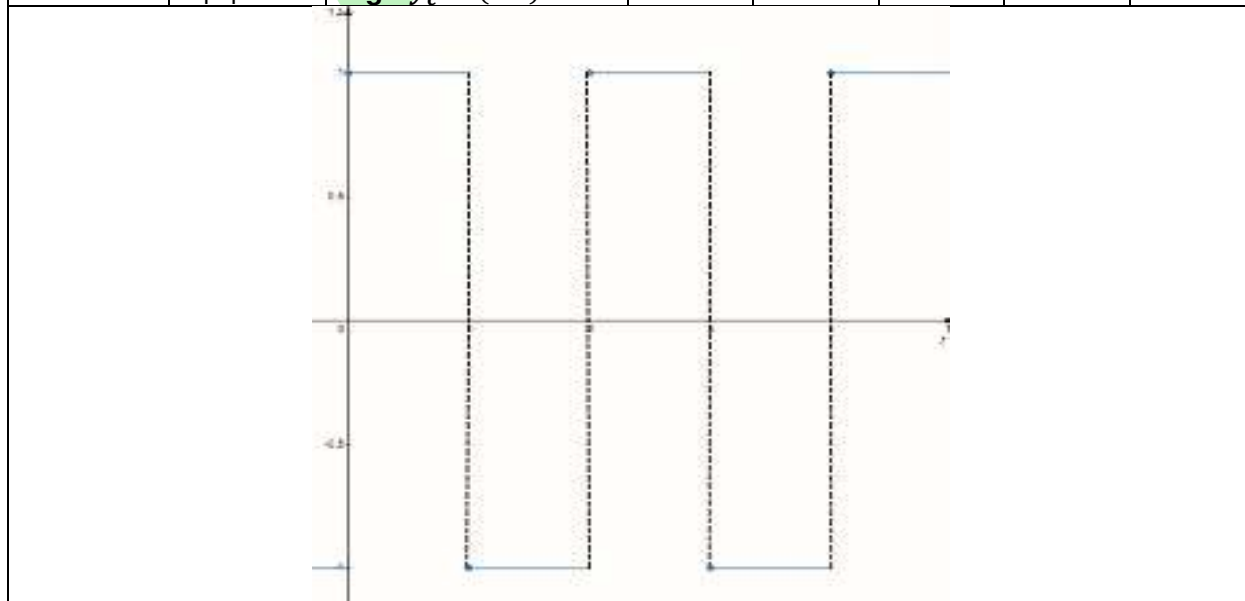
Region – V	$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
	$(-ve)^{(0 \text{ to } 4...)} \Rightarrow +ve/-ve$ (alternative signs): oscillatory.					

$-1 < b < 0$	$ b < 1$	e.g. $y_c = \left(-\frac{1}{2}\right)^t$	1	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{16}$
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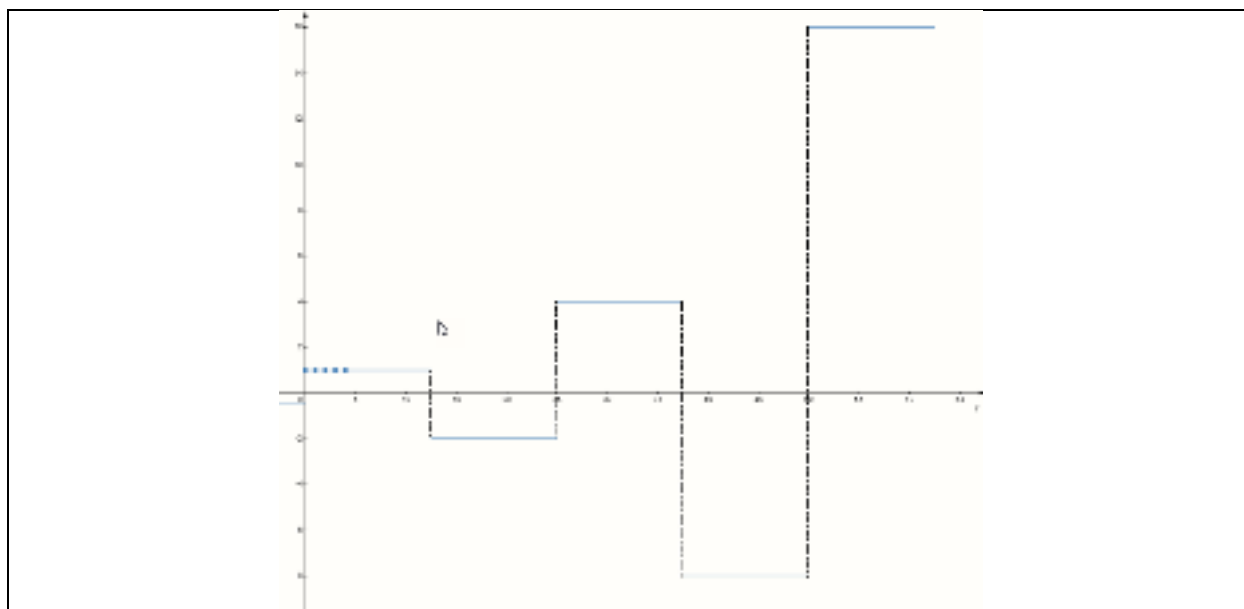
Dynamic Stability: Time-path is **convergent** & hence **dynamically stable**. It moves above and below (oscillates) the equilibrium with **decreasing vertical distance**.

Region – VI		$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
		$(-ve)^{(0 \text{ to } 4...)} \Rightarrow +ve/-ve$ (alternative signs): oscillatory.					
$b = -1$	$ b = 1$	e.g.: $y_c = (-1)^t$	1	-1	1	-1	1



Dynamic Stability: Time-path is **neither divergent nor convergent**. It moves above and below (oscillates) the equilibrium with **uniform vertical distance**.

Region – VII		$y_c = (b)^t$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
		$(-ve)^{(0 \text{ to } 4...)} \Rightarrow +ve/-ve$ (alternative signs): oscillatory.					
$b < -1$	$ b < 1$	e.g.: $y_c = (-2)^t$	1	-2	4	-8	16



Dynamic Stability: Time-path is **convergent** & hence **dynamically unstable**. It moves above and below (oscillates) the equilibrium with **increasing vertical distance**.

Summary: Cases (Regions) V-VII have negative b . A negative number when raised to any power generates numbers with **alternative signs** (+ve/-ve), which give rise to a graph (time-path, here) in 1st and 4th Quadrants – hence **oscillatory**.

$$b < 0 \Rightarrow \text{Oscillatory}$$

Only when $|b| < 1$, the time-path is convergent.

$$|b| < 1 \Rightarrow \text{Convergent}$$

Tabulating the observations:

Region – V	Region – VI	Region – VII
$\left(-1 < \underbrace{b}_{(-\frac{1}{2})} < 0 \right) \vee \underbrace{(b < 0)}_{\text{oscillatory}}$	$\left(\underbrace{b}_{-1} = -1 \right) \vee \underbrace{(b < 0)}_{\text{oscillatory}}$	$\left(\underbrace{b}_{(-2)} < -1 \right) \vee \underbrace{(b < 0)}_{\text{oscillatory}}$
$\left(\underbrace{ b }_{\frac{1}{2}} < 1 \right) \vee \underbrace{(b < 1)}_{\text{convergent}}$	$\left(\underbrace{ b }_1 = 1 \right) \vee \underbrace{(b \neq 1)}_{\text{non-convergent (neither)}}$	$\left(\underbrace{ b }_2 > 1 \right) \vee \underbrace{(b \neq 1)}_{\text{non-convergent (divergent)}}$

TOPIC 160: DYNAMIC STABILITY OF FIRST ORDER DIFFERENCE EQUATION: ROLE OF A

Recalling $y_c = \underbrace{A}_{\text{coefficient}} (b)^t$. Out of seven cases, we treat the **convergent ones**, as they are

the desired cases – **Case – III** & **Case – V**. In this selection, we get both of the patterns of time-path (**Non-oscillatory** and **Oscillatory**). However, remaining (undesired) 5 cases can also be treated in the same fashion.

(A) being a coefficient can affect y_c in either or both of the two ways:

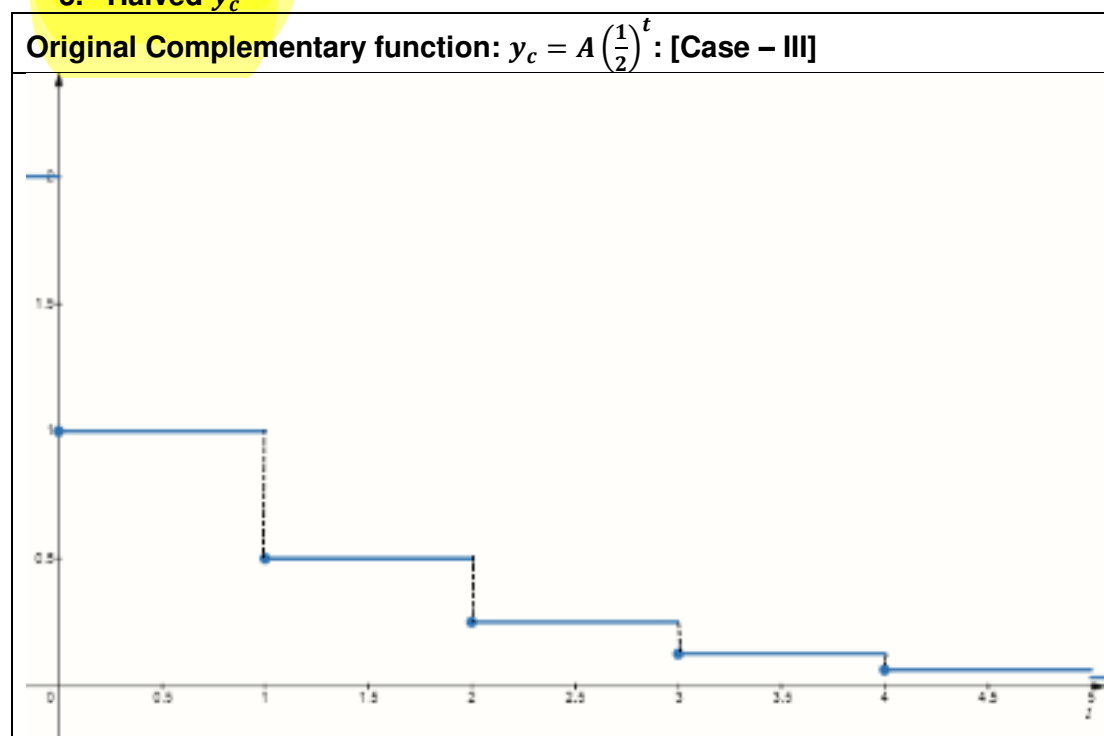
Dual Effect Value of A on y_c					
Magnitude				Sign	
$ A = 0$	$ A < 1$	$ A = 1$	$ A > 1$	$A > 0$	$A < 0$
Nullify	Minify	Neutral	Amplify	Retain	Reverse
$y_c = \underline{0}(b)^t$	$y_c = \underline{0.5}(b)^t$	$y_c = \underline{1}(b)^t$	$y_c = \underline{2}(b)^t$	$y_c = \underbrace{A}_{\geq 0} (b)^t$	$y_c = \underbrace{A}_{\leq 0} (b)^t$
$y_c = \underline{0}$	$y_c = \underline{0.5b^t}$	$y_c = \underline{b^t}$	$y_c = \underline{2b^t}$	$y_c > 0$	$y_c < 0$
-	Noteworthy	-	Noteworthy	-	Noteworthy
Scale Effect				Mirror Effect	

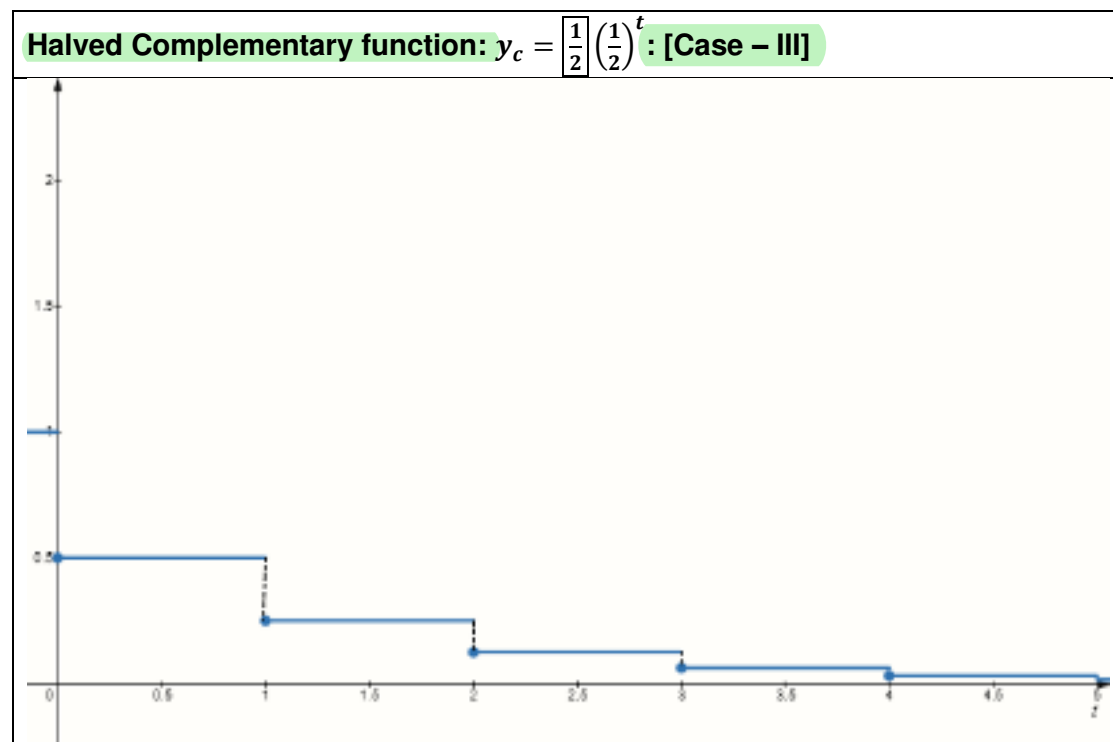
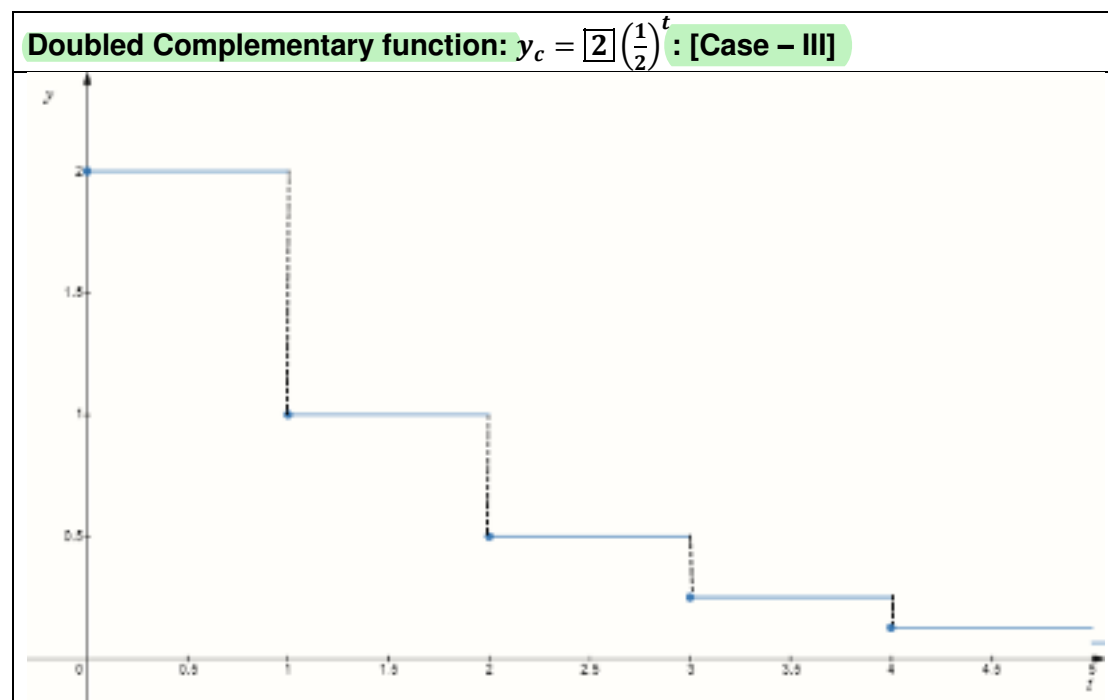
1. Scale Effect:

Case – III (Convergent & Non-oscillatory):

Three time-paths of complementary functions are plotted:

1. Original y_c
2. Doubled y_c
3. Halved y_c





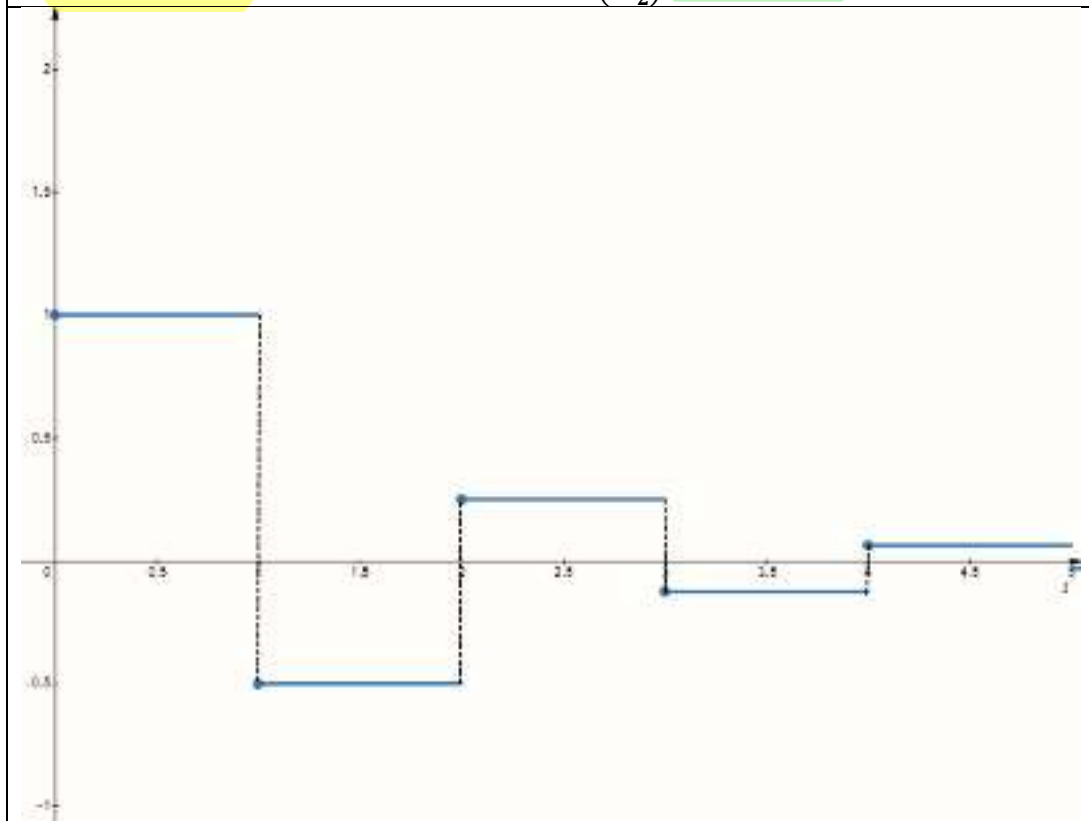
Interpretation: Original y_c starts from 1, however, Doubled y_c from 2 and Halved y_c from 0.5 respectively. It is evident that magnitude of $A (\geq 1)$ affects the scale of y_c by **upscaling or downscaling** the time-path. Its, therefore, called the **Scale Effect** of A .

Case – V (Convergent & Oscillatory):

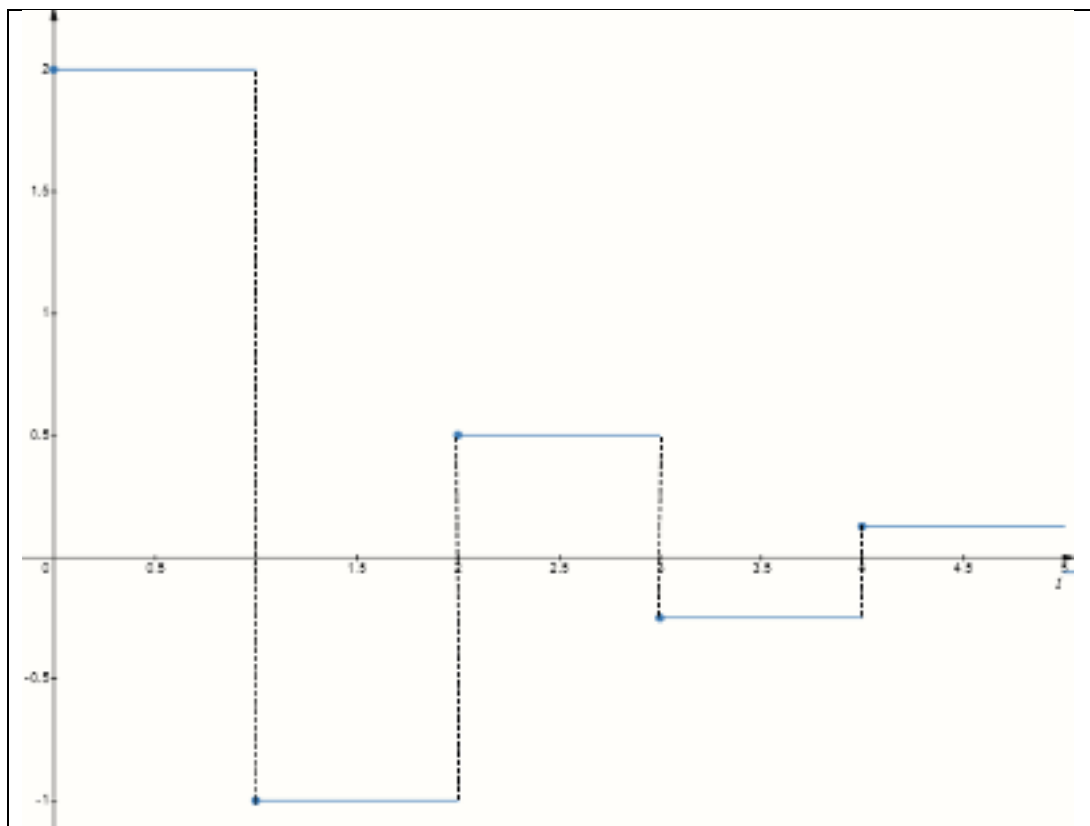
Three time-paths of complementary functions are plotted:

1. Original y_c
2. Doubled y_c
3. Halved y_c

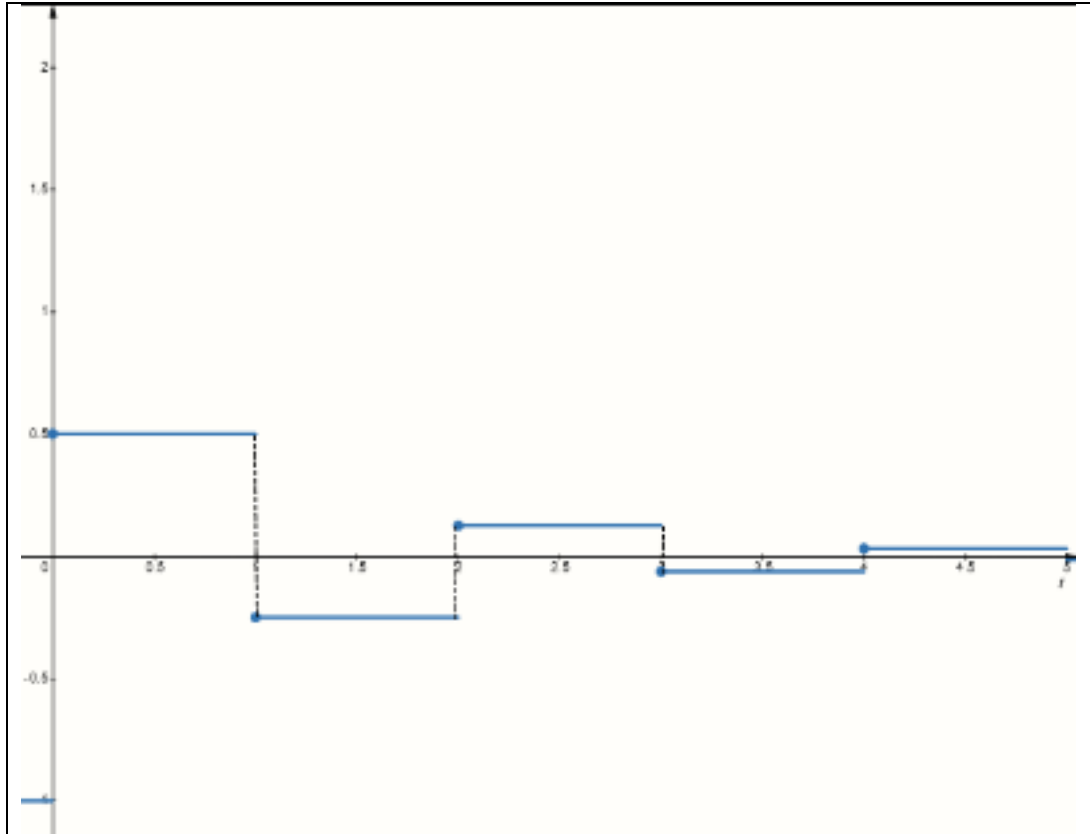
Original Complementary function: $y_c = A \left(-\frac{1}{2}\right)^t$: [Case – V]



Doubled Complementary function: $y_c = \boxed{2} \left(-\frac{1}{2}\right)^t$: [Case – V]



Halved Complementary function: $y_c = \frac{1}{2} \left(-\frac{1}{2}\right)^t$: [Case - V]



Interpretation: Original y_c starts from 1, however, Doubled y_c from 2 and Doubled y_c from 0.5 respectively. It is evident that magnitude of $A (\geq 1)$ affects the scale of y_c by **upscaling or downscaling** the time-path. It's, therefore, called the **Scale Effect** of A .

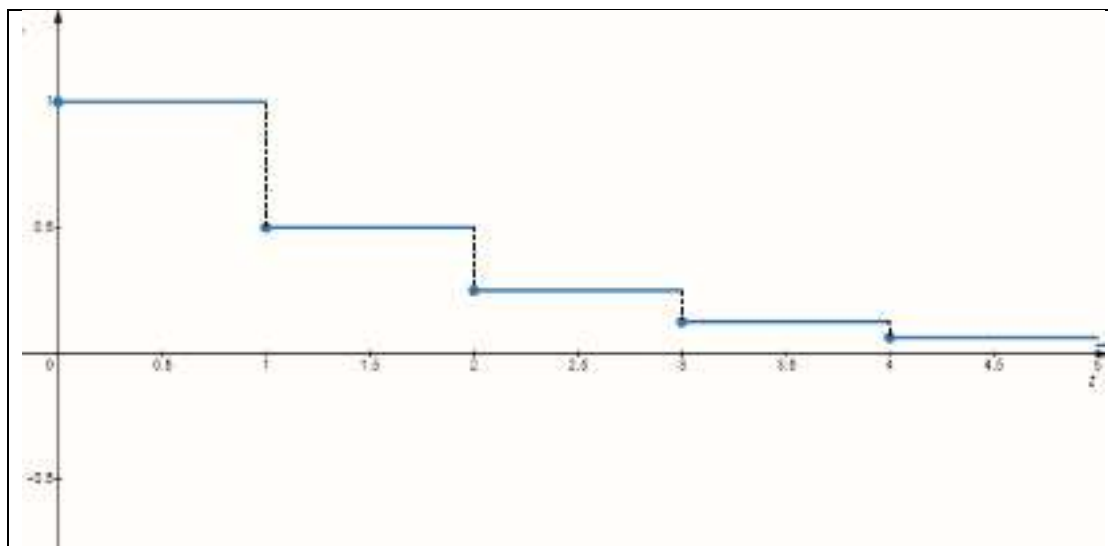
2. Mirror Effect:

Case – III (Convergent & Non-oscillatory):

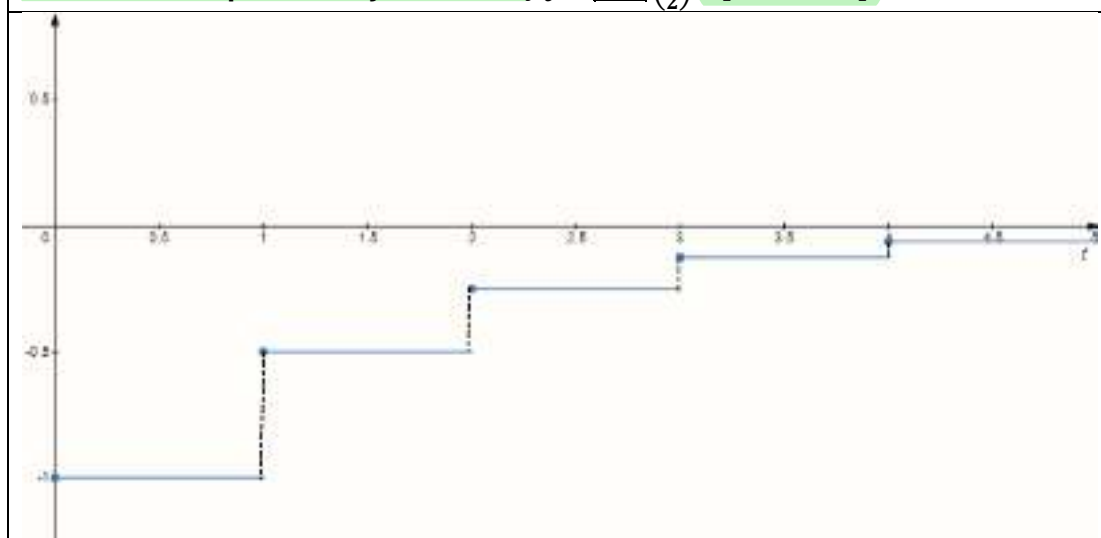
Two time-paths of complementary functions are plotted:

1. Original y_c
2. Reversed y_c

Original Complementary function: $y_c = A \left(\frac{1}{2}\right)^t$: [Case – III]



Reversed Complementary function: $y_c = \boxed{-1} \left(\frac{1}{2}\right)^t$: [Case – III]



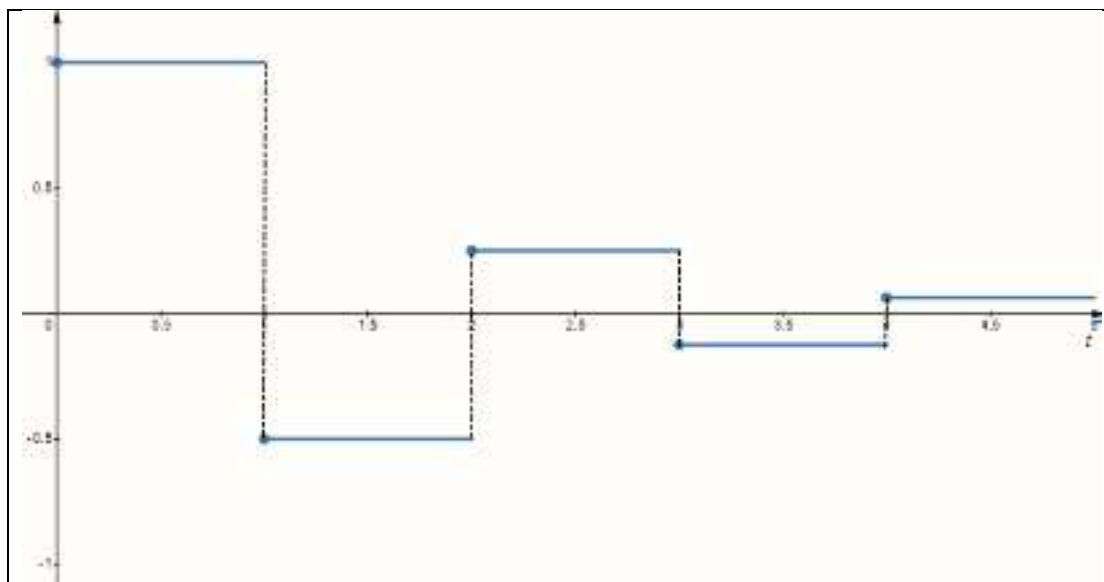
Interpretation: Both y_c start from 1, however, +1 and –1 respectively. It is evident that –ve sign of $A(< 0)$ reverses the shape of y_c by making a **mirror-image** of the original time-path. It's, therefore, called the **Mirror Effect** of A .

Case – V (Convergent & Oscillatory):

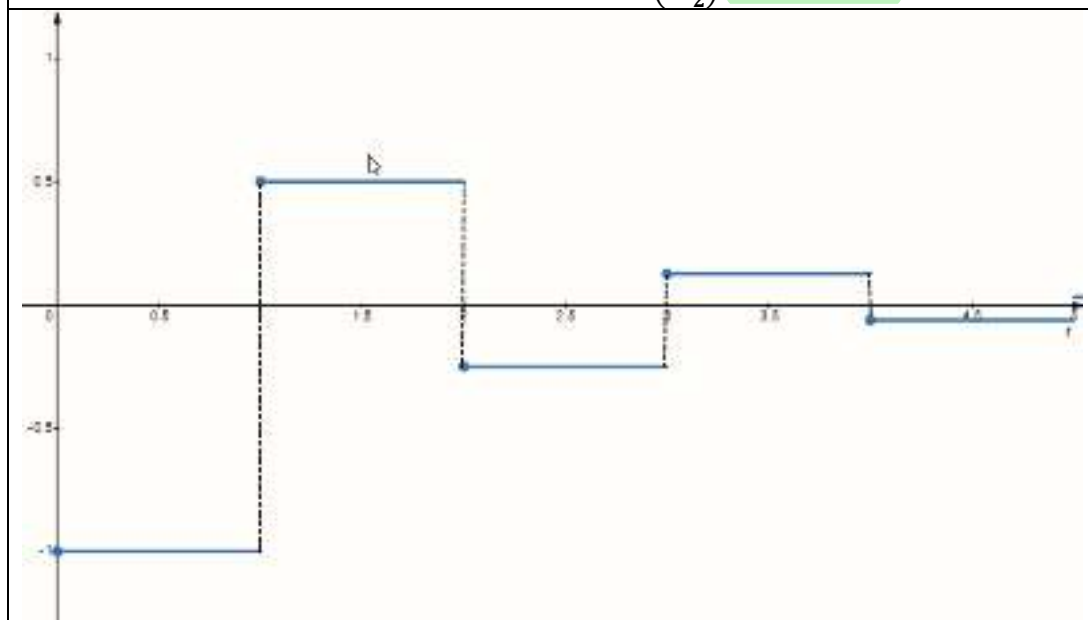
Two time-paths of complementary functions are plotted:

1. Original y_c
2. Reversed y_c

Original Complementary function: $y_c = A \left(-\frac{1}{2}\right)^t$: [Case – V]



Reversed Complementary function: $y_c = \boxed{-1} \left(-\frac{1}{2}\right)^t$: [Case - V]



Interpretation: Both y_c start from 1, however, +1 and -1 respectively. It is evident that $-ve$ sign of $A(< 0)$ reverses the shape of y_c by making a **mirror-image** of the original time-path. It's, therefore, called the **Mirror Effect** of A .

TOPIC 161: DYNAMIC STABILITY OF FIRST ORDER DIFFERENCE EQUATION: NUMERICALS

Example # 1: Check the dynamic stability the time-path $y_t = 2 \left(-\frac{4}{5}\right)^t + 9$?

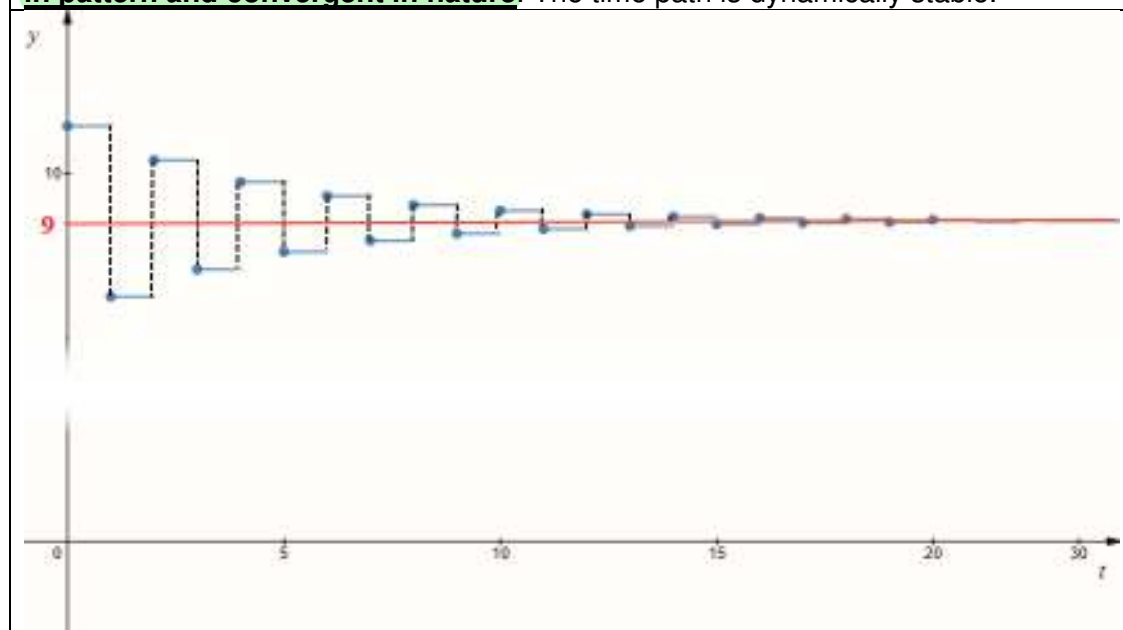
Since $y_t = 2 \left(-\frac{4}{5}\right)^t + \underbrace{9}_{\neq f(t)}$, the formula in the background is: $y_t = A(b)^t + \frac{c}{1+a}$.

Breaking-up the definite solution into complementary and particular solutions:

$$y_t = \overbrace{2 \left(-\frac{4}{5} \right)^t}^{y_c} + \overbrace{9}^{y_p}$$

Dynamic stability of time-path depends on original and absolute values of b .

Interpretation: Here, $b = -\frac{4}{5} \boxed{< 1}$ & $|b| = \frac{4}{5} \boxed{< 1}$. Therefore, time path is **oscillatory in pattern and convergent in nature**. The time path is dynamically stable.



Note: ☰

Time-path of $y_t = 2 \left(-\frac{4}{5} \right)^t + 9$ vs. Time-path of $y_t = -2 \left(\frac{4}{5} \right)^t + 9$

Example # 2: Check the dynamic stability the time-path $y_t = 3(2)^t + 4$?

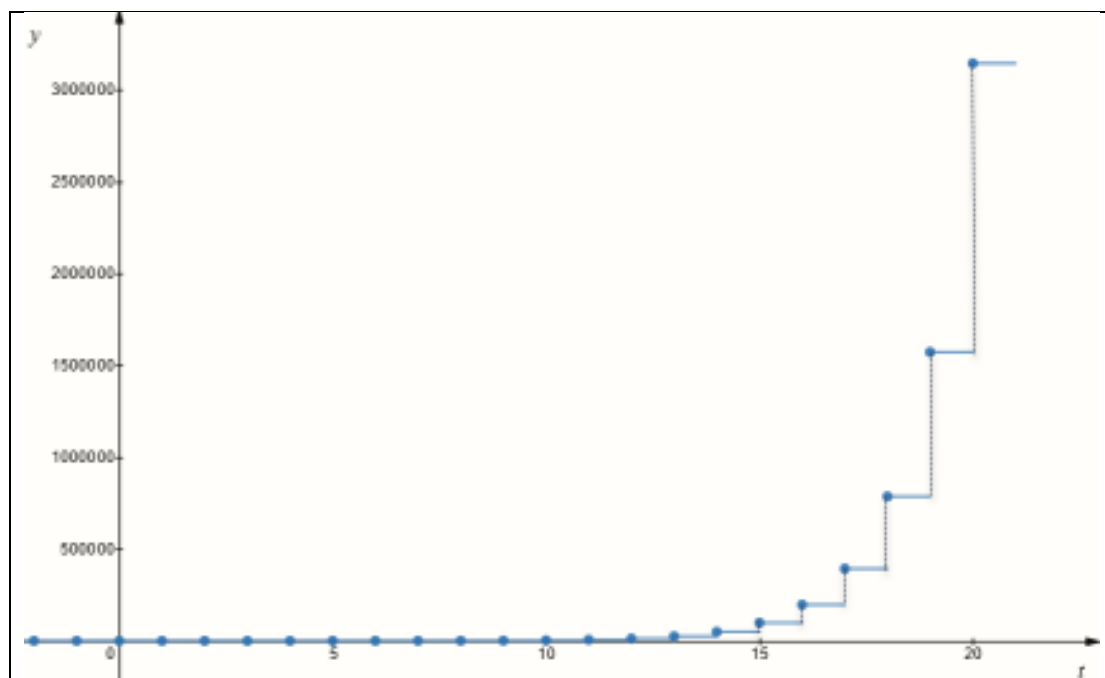
Since $y_t = 3(2)^t + \underbrace{4}_{\neq f(t)}$, the formula in the background is: $y_t = A(b)^t + \frac{c}{1+a}$.

Breaking-up the definite solution into complementary and particular solutions:

$$y_t = \overbrace{3(2)^t}^{y_c} + \overbrace{4}^{y_p}$$

Dynamic stability of time-path depends on original and absolute values of b .

Interpretation: Here, $b = 2 \boxed{\neq 1}$ & $|b| = 2 \boxed{\not< 1}$. Therefore, time path is **non-oscillatory in pattern and non convergent in nature**. The time path is dynamically unstable.



Lesson 36

INCOME DYNAMICS OF DIFFERENCE EQUATIONS

TOPIC 162: LAGGED INCOME DETERMINATION MODEL: FRAMEWORK

Unlike simple income determination model, we can introduce lags in consumption function:

$$\begin{aligned} C_t &= C_0 + cY_{t-1} \\ &\text{Income of last time-period} \\ &\text{determines current consumption} \\ Y_t &= C_t + I_t \end{aligned}$$

Here, $c = \frac{MPC}{(0 < c < 1)}$, $I_t = I_0$
autonomous investment

Substituting the value of C_t in $Y_t = C_t + I_t$

$$\begin{aligned} Y_t &= C_0 + cY_{t-1} + I_t \\ Y_t - cY_{t-1} &= C_0 + I_t \\ Y_t + \underbrace{(-c)}_{\substack{a \neq -1 \\ \text{as } MPC < 1}} Y_{t-1} &= \underbrace{C_0 + I_t}_c \end{aligned}$$

Here, $\underbrace{a}_{\substack{\text{coefficient} \\ \text{of FODE}}} = -c$, $\underbrace{C_0 + I_t}_{\substack{\text{constant} \\ \text{of FODE}}} = C_0 + I_t$

As coefficient of First Order Difference equation is $\neq -1$, we use the following formula for the definite time-path of National Income.

$$\begin{aligned} \hat{y}_t &= \left(\hat{y}_0 - \frac{(C_0 + I_t) \hat{c}}{1 + \underbrace{a}_{(-c)}} \right) \underbrace{(b)}_{(-a)}^t + \frac{(C_0 + I_t) \hat{c}}{1 + \underbrace{a}_{(-c)}} \\ \hat{y}_t &= \left(\hat{y}_0 - \frac{(C_0 + I_t) \hat{c}}{1 + \underbrace{a}_{(-c)}} \right) \underbrace{(-a)}_{a=-c}^t + \frac{(C_0 + I_t) \hat{c}}{1 + \underbrace{a}_{(-c)}} \\ Y_t &= \left\{ Y_0 - \frac{C_0 + I_t}{1 + (-c)} \right\} \{-(-c)\}^t + \frac{C_0 + I_t}{1 + (-c)} \\ Y_t &= \left\{ Y_0 - \frac{C_0 + I_t}{1 - c} \right\} (c)^t + \frac{C_0 + I_t}{1 - c} \end{aligned}$$

Interpretation: $Y_t = \left\{ Y_0 - \frac{C_0 + I_t}{1 - c} \right\} (c)^t + \frac{C_0 + I_t}{1 - c}$ is the definite time-path of the first order difference equation $Y_t - cY_{t-1} = C_0 + I_t$. Dynamic Stability of the time-path, mainly, depends on exponential expression (c^t) .

As per economic theory: $\underbrace{0 < MPC < 1}_{c \neq 0}$:

$$\left[\underbrace{\left(\frac{|c| \neq 0}{c < 1} \right)^t}_{\hat{c}} \right] : \frac{(|c| \neq 0) \Rightarrow \text{non-oscillatory}}{(c < 1) \Rightarrow \text{convergent}}$$

Therefore, the time-path is dynamically stable with a non-oscillatory pattern and a convergent nature.

$$Y_t = \underbrace{\left\{ Y_0 - \frac{C_0 + I_t}{1 - c} \right\}}_{\substack{\text{coefficient} \\ \text{of } (c)^t}} \underbrace{(c)^t}_{\substack{\rightarrow 0 \\ \text{as } t \rightarrow \infty}} + \underbrace{\frac{C_0 + I_t}{1 - c}}_{Y_p (= Y^*)}$$

$Y_c (\text{deviation from } Y^*)$
 $Y_c \rightarrow 0, \text{ as } t \rightarrow \infty$
 $Y_t \rightarrow Y^*, \text{ as } t \rightarrow \infty$

TOPIC 163: LAGGED INCOME DETERMINATION MODEL: NUMERICAL

In addition to symbolic determination of income determination model with time lags, we can solve it numerically, as well:

$$Y_t = \boxed{C_t} + \underline{I_t}$$

$$\boxed{C_t} = 200 + 0.9Y_{t-1}$$

*Income of last time-period
determines current consumption*

*autonomous
investment*

$$\underline{I_t} = 100$$

$$Y_0 = 4500$$

Here, $\underline{MPC = 0.9}$.
($0 < c < 1$)

Substituting the value of C_t and I_t in $Y_t = \boxed{C_t} + \underline{I_t}$

$$Y_t = \underbrace{\widetilde{200}}_{C_0} + \underbrace{\widetilde{0.9}}_{MPC} Y_{t-1} + \underline{100}$$

$$Y_t - \boxed{0.9Y_{t-1}} = \underline{200} + \underline{100}$$

$$Y_t + \underbrace{(-0.9)}_{a \neq -1} Y_{t-1} = \underbrace{300}_c$$

Comparing the standard form of First Order Difference equation:

$$y_t + ay_{t-1} = c$$

Since $a = -0.9 \neq -1$, the definite time-path will be:

$$\hat{y}_t = \left(\hat{y}_0 - \frac{\hat{c}}{1 + \hat{a}} \right) \left(\hat{b} \right)^t + \frac{\hat{c}}{1 + \hat{a}}$$

$$Y_t = \left(4500 - \frac{300}{1 - 0.9} \right) \{ -(-0.9) \}^t + \frac{300}{1 - 0.9}$$

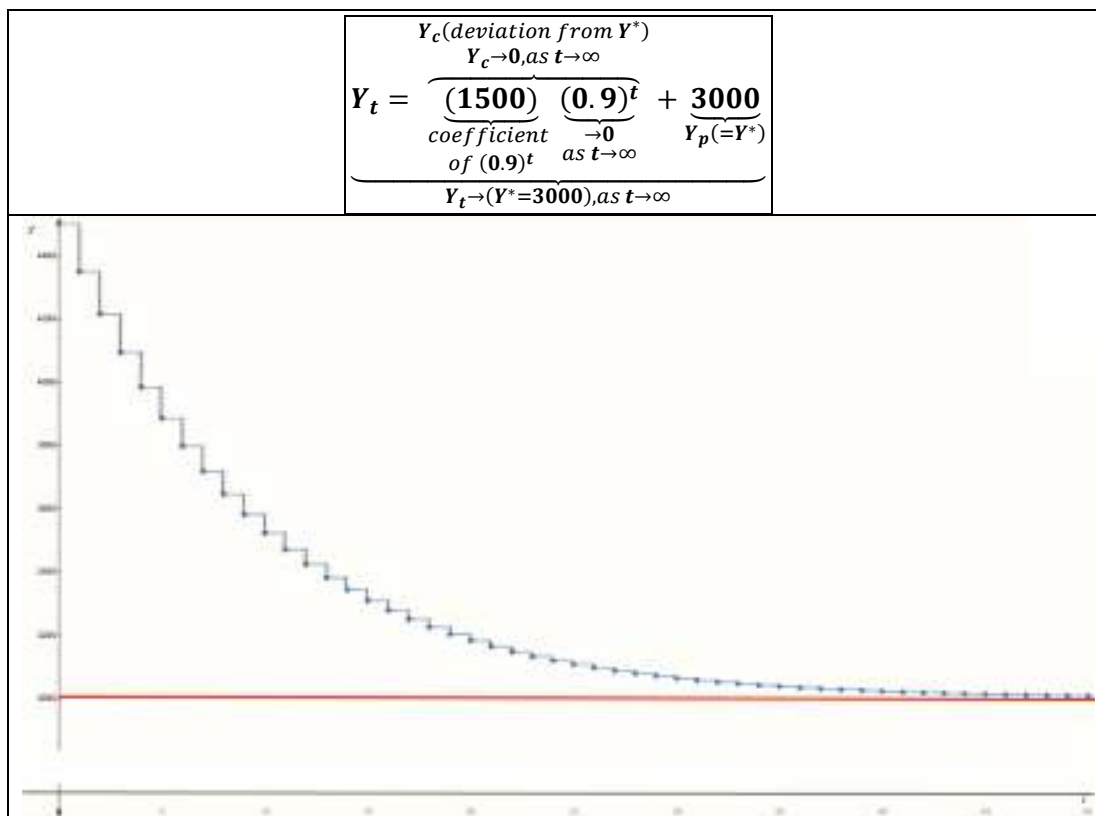
$$Y_t = (4500 - 3000)(0.9)^t + 3000$$

$$\boxed{Y_t = (1500)(0.9)^t + 3000}$$

Interpretation: $Y_t = (1500)(0.9)^t + 3000$ is the definite time-path of the first order difference equation $\boxed{Y_t - 0.9Y_{t-1} = 300}$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(0.9)^t$.

$$\left[\left\{ \begin{array}{l} |0.9| < 1 \\ (0.9) \\ 0.9 < 1 \end{array} \right\}^t \right] : \frac{(|0.9| < 1) \Rightarrow \text{non-oscillatory}}{(0.9 < 1) \Rightarrow \text{convergent}}$$

Therefore, the time-path is dynamically stable with a non-oscillatory pattern and a convergent nature.



TOPIC 164: INCOME DYNAMICS WITH INDUCED INVESTMENT USING DIFFERENCE EQUATIONS

After introducing induced investment alongside autonomous investment, the income determination model with time lags, can be done numerically, as follows:

$$Y_t = \underbrace{C_t}_{MPC} + I_t$$

$$\underbrace{C_t}_{\substack{\text{Income of last time-period} \\ \text{determines current consumption}}} = \underbrace{200}_{\text{autonomous}} + \underbrace{0.4 Y_{t-1}}_{\text{induced}}$$

$$I_t = \underbrace{100}_{\text{autonomous}} + \underbrace{0.2 Y_{t-1}}_{\text{induced}}$$

$$Y_0 = 950$$

Here, $\underbrace{MPC = 0.4}_{(0 < c < 1)}$.

Substituting the value of C_t and I_t in $Y_t = C_t + I_t$

$$Y_t = \underbrace{C_0}_{\text{autonomous}} + \underbrace{0.4 Y_{t-1}}_{MPC} + \underbrace{100}_{\text{autonomous}} + \underbrace{0.2 Y_{t-1}}_{\text{induced}}$$

$$Y_t - 0.4 Y_{t-1} - 0.2 Y_{t-1} = 200 + 100$$

$$Y_t - 0.6 Y_{t-1} = 300$$

$$Y_t + \underbrace{(-0.6)}_{a \neq -1} Y_{t-1} = \underbrace{300}_c$$

Comparing the standard form of First Order Difference equation:

$$y_t + ay_{t-1} = c$$

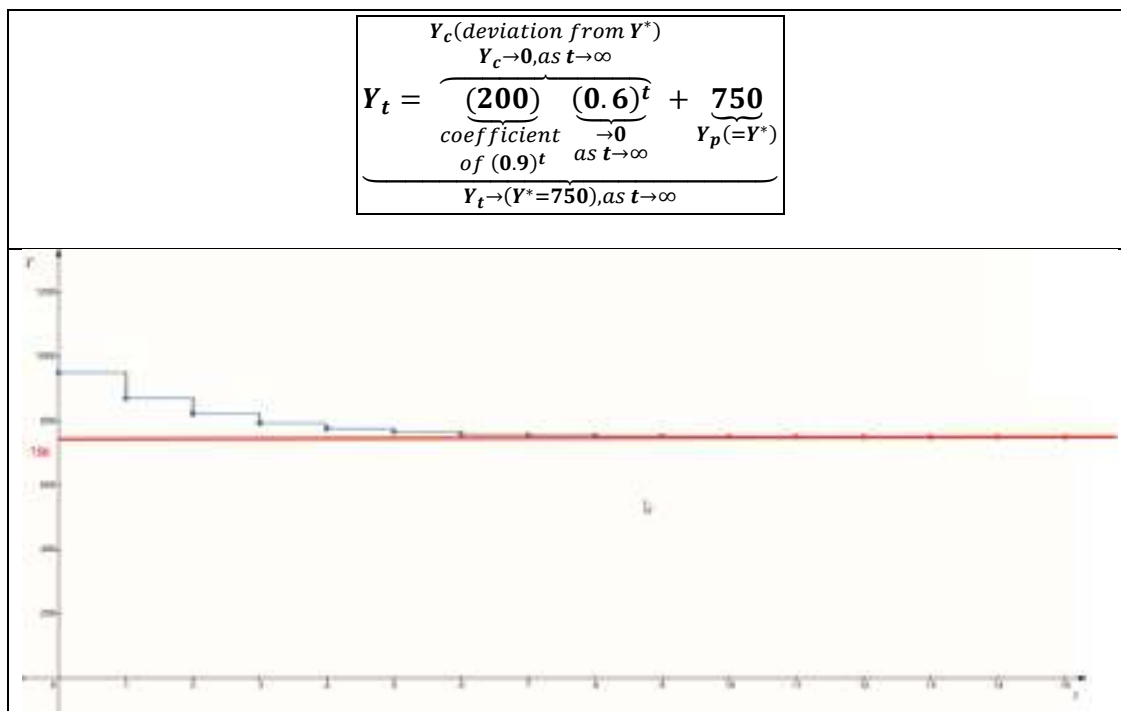
Since $a = -0.6 \neq -1$, the definite time-path will be:

$$\begin{aligned} \hat{y}_t &= \left(\hat{y}_0 - \frac{\hat{c}}{1 + \hat{a}} \right) (\hat{b})^t + \frac{\hat{c}}{1 + \hat{a}} \\ Y_t &= \left(950 - \frac{300}{1 - 0.6} \right) \{-(-0.6)\}^t + \frac{300}{1 - 0.6} \\ Y_t &= (950 - 750)(0.6)^t + 750 \\ Y_t &= (200)(0.6)^t + 750 \end{aligned}$$

Interpretation: $Y_t = (200)(0.6)^t + 750$ is the definite time-path of the first order difference equation $Y_t - 0.6Y_{t-1} = 300$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(0.6)^t$.

$$\left[\begin{array}{c} (|0.6| < 1) \\ (0.6) \\ (0.6 < 1) \end{array} \right]^t : \frac{(|0.6| < 1) \Rightarrow \text{non-oscillatory}}{(0.6 < 1) \Rightarrow \text{convergent}}$$

Therefore, the time-path is dynamically stable with a non-oscillatory pattern and a convergent nature.



TOPIC 165: INCOME DYNAMICS & CAPITAL MARKET USING DIFFERENCE EQUATIONS

Savings in an economy represent a share of current income given by the marginal propensity to save (s), whereas firms decide to invest a share of the income earned in the previous period. Thus:

$$S_t = sY_t \text{ \& } I_t = \beta Y_{t-1}$$

Elaborating the given equations:

$$S_t = \underbrace{\overbrace{\widehat{MPS}}^{(0 < s < 1)}}_{\substack{\text{proportion of last} \\ \text{year's income} \\ \text{that is invested}}} Y_t \quad s > 0$$

$$I_t = \underbrace{\widehat{\beta}}_{\substack{\text{Income of last time-period} \\ \text{determines current investment}}} Y_{t-1} \quad \beta < 1$$

For equilibrium, savings should be equal to investment: $S_t = I_t$

$$\begin{aligned} S_t &= I_t \\ sY_t &= \beta Y_{t-1} \\ sY_t - \beta Y_{t-1} &= 0 \end{aligned}$$

Normalizing the equation by dividing throughout by s :

$$\begin{aligned} Y_t - \left(\frac{\beta}{s}\right) Y_{t-1} &= 0 \\ Y_t + \underbrace{\left\{-\left(\frac{\beta}{s}\right)\right\}}_{a \neq -1} Y_{t-1} &= \underbrace{0}_c \end{aligned}$$

$$a = - \left(\frac{\overset{+ve}{\tilde{\beta}}}{\underset{+ve}{s}} \right) \underset{(\beta \neq s)}{\neq} -1$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $(a)_{(\beta \neq s)} = -\frac{\beta}{s} \neq -1$.

Here; $a = -\frac{\beta}{s}$, $c = 0$, $y_t = Y_t$, $y_0 = Y_0$, $b = -a$.

$$y_t = \left(y_0 - \frac{c}{1+a} \right) \left(\frac{(-a)}{\tilde{b}} \right)^t + \frac{c}{1+a}$$

$$Y_t = \left\{ Y_0 - \frac{0}{1 + \left(-\frac{\beta}{s} \right)} \right\} \{ (-a) \}^t + \frac{0}{1 + \left(-\frac{\beta}{s} \right)}$$

$$Y_t = \{ Y_0 - 0 \} \left[\left\{ - \left(-\frac{\beta}{s} \right) \right\} \right]^t + 0$$

$$\boxed{Y_t = Y_0 \left(\frac{\beta}{s} \right)^t}$$

Interpretation: $Y_t = Y_0 \left(\frac{\beta}{s} \right)^t$ is the definite time-path of the first order difference equation

$Y_t - \frac{\beta}{s} Y_{t-1} = 0$. Dynamic Stability of the time-path, mainly, depends on exponential expression $\left(\frac{\beta}{s} \right)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \tilde{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \tilde{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = \left \frac{\beta}{s} \right \leq 1 \xrightarrow{\text{if } (\beta < s)} \left \frac{\beta}{s} \right < 1$ <i>convergent</i>	$b = \left(\frac{\beta}{s} \right) \neq 0 \Rightarrow$ Non-oscillatory

Therefore, the time-path will be convergent in nature only if $(\beta < s)$ while it will move in a non-oscillatory pattern.

TOPIC 166: INCOME DYNAMICS & MONEY MARKET USING DIFFERENCE EQUATIONS

On the money market, the government sets money supply as a percentage of national income from the previous period. Furthermore, transaction demand for money depends on current income such that:

$$\begin{aligned} M_s &= sY_{t-1} \\ M_D &= \alpha + \beta Y_t \quad (0 < \beta < s < 1 \text{ \& } \alpha > 0) \end{aligned}$$

Find the aggregate national income that ensures equilibrium on the money market, and check whether the time path of national income is dynamically stable and oscillating.

Solution:

s : proportion of previous year's income that is converted into supply of money in current year.

Equilibrium in the money market requires $M_s = M_D$:

$$\begin{aligned} sY_{t-1} &= \alpha + \beta Y_t \\ -\beta Y_t + sY_{t-1} &= \alpha \\ \beta Y_t - sY_{t-1} &= -\alpha \end{aligned}$$

Normalizing the equation by dividing throughout by β :

$$\begin{aligned} Y_t + \left(-\frac{s}{\beta}\right)Y_{t-1} &= -\frac{\alpha}{\beta} \\ Y_t + \underbrace{\left\{\left(-\frac{s}{\beta}\right)\right\}}_{\substack{a \neq -1 \\ \text{+ve}}} Y_{t-1} &= \underbrace{-\frac{\alpha}{\beta}}_c \\ a &= -\underbrace{\left(\frac{\tilde{s}}{\beta}\right)}_{\substack{\neq \\ \text{+ve}}} \underbrace{-1}_{(\beta < s)} \end{aligned}$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $(a)_{(\beta < s)} = -\frac{\beta}{s} \neq -1$.

Here; $a = -\frac{s}{\beta}$, $c = -\frac{\alpha}{\beta}$, $y_t = Y_t$, $y_0 = Y_0$, $b = -a$.

$$\begin{aligned} y_t &= \left(y_0 - \frac{c}{1+a}\right) \left(\frac{(-a)}{\tilde{b}}\right)^t + \frac{c}{1+a} \\ Y_t &= \left\{Y_0 - \frac{\left(-\frac{\alpha}{\beta}\right)}{1 + \left(-\frac{s}{\beta}\right)}\right\} (-a)^t + \frac{\left(-\frac{\alpha}{\beta}\right)}{1 + \left(-\frac{s}{\beta}\right)} \\ Y_t &= \left\{Y_0 + \frac{\frac{\alpha}{\beta}}{1 - \frac{s}{\beta}}\right\} \left\{-\left(-\frac{s}{\beta}\right)\right\}^t - \frac{\frac{\alpha}{\beta}}{1 - \frac{s}{\beta}} \\ Y_t &= \left\{Y_0 + \frac{\frac{\alpha}{\beta}}{\left(\frac{\beta-s}{\beta}\right)}\right\} \left(\frac{s}{\beta}\right)^t - \frac{\frac{\alpha}{\beta}}{\left(\frac{\beta-s}{\beta}\right)} \\ Y_t &= \left\{Y_0 + \frac{\alpha}{\beta} \times \left(\frac{\beta}{\beta-s}\right)\right\} \left(\frac{s}{\beta}\right)^t - \frac{\alpha}{\beta} \times \left(\frac{\beta}{\beta-s}\right) \\ \boxed{Y_t} &= \boxed{\left(Y_0 + \frac{\alpha}{\beta-s}\right) \left(\frac{s}{\beta}\right)^t - \frac{\alpha}{\beta-s}} \end{aligned}$$

$$Y_t = \underbrace{\left(Y_0 + \frac{\alpha}{\beta - s}\right)}_{Y_c} \underbrace{\left(\frac{s}{\beta}\right)^t}_{\text{Exponential expression}} - \underbrace{\frac{\alpha}{\beta - s}}_{Y_p}$$

Interpretation: $Y_t = \left(Y_0 + \frac{\alpha}{\beta - s}\right) \left(\frac{s}{\beta}\right)^t - \frac{\alpha}{\beta - s}$ is the definite time-path of the first order difference equation $Y_t - \frac{s}{\beta} Y_{t-1} = -\frac{\alpha}{\beta}$. Dynamic Stability of the time-path, mainly, depends on exponential expression $\left(\frac{s}{\beta}\right)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \underbrace{\tilde{b}}_{(<1): \text{convergent}} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \underbrace{\tilde{b}}_{(>0): \text{non-oscillatory}} \end{array} \right\}$
$ b = \left \frac{s}{\beta}\right \leq 1 \xrightarrow{\text{as given } (\beta < s)} \left \frac{s}{\beta}\right > 1$ <i>Time-path is Convergent</i>	$b = \left(\frac{s}{\beta}\right) \neq 0 \Rightarrow \text{Non-oscillatory}$

Therefore, the time-path is convergent in nature as $(\beta < s)$ while it will move in a non-oscillatory pattern.

TOPIC 167: INCOME DYNAMICS & BALANCED TRADE USING DIFFERENCE EQUATIONS

The imports of a country are a share of national income from the current period. Furthermore, exports depend both on current income and income from the previous period such that:

$$M_t = mY_t \quad 0 < m < 1$$

$$X_t = \alpha Y_t + \beta Y_{t-1} \quad (0 < \alpha, \beta < 1)$$

Find the time path of aggregate national income that ensures balanced foreign trade.

Solution: For balanced Foreign Trade; $M_t = X_t$:

$$mY_t = \alpha Y_t + \beta Y_{t-1}$$

$$mY_t - \alpha Y_t - \beta Y_{t-1} = 0$$

$$(m - \alpha)Y_t - \beta Y_{t-1} = 0$$

Normalizing the equation by dividing throughout by $(m - \alpha)$:

$$Y_t - \frac{\beta}{m - \alpha} Y_{t-1} = \frac{0}{m - \alpha}$$

$$Y_t + \underbrace{\left\{ \left(-\frac{\beta}{m - \alpha} \right) \right\}}_{a \neq -1} Y_{t-1} = \underbrace{0}_c$$

$$a = -\left(\frac{\beta}{m - \alpha} \right) \neq -1 \quad (m - \alpha \neq \beta)$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $(a)_{(m - \alpha \neq \beta)} = -\frac{\beta}{s} \neq -1$.

Here; $a = -\frac{\beta}{m - \alpha}$, $c = 0$, $y_t = Y_t$, $y_0 = Y_0$, $b = -a$.

$$y_t = \left(y_0 - \frac{0}{1 + a} \right) \left(\frac{(-a)}{\tilde{b}} \right)^t + \frac{c}{1 + a}$$

$$Y_t = \left\{ Y_0 - \frac{0}{1 + \left(-\frac{\beta}{m-\alpha}\right)} \right\} (-a)^t + \frac{0}{1 + \left(-\frac{\beta}{m-\alpha}\right)}$$

$$Y_t = \{Y_0 - 0\} \left\{ -\left(-\frac{\beta}{m-\alpha}\right) \right\}^t - 0$$

$$Y_t = Y_0 \left(\frac{\beta}{m-\alpha} \right)^t$$

$$\begin{array}{c} \text{Exponential} \\ \text{expression} \\ Y_t = Y_0 \underbrace{\left(\frac{\beta}{m-\alpha} \right)^t}_{Y_c} - \underbrace{0}_{Y_p} \end{array}$$

Interpretation: $Y_t = Y_0 \left(\frac{\beta}{m-\alpha} \right)^t$ is the definite time-path of the first order difference equation

$Y_t - \frac{\beta}{m-\alpha} Y_{t-1} = 0$. Dynamic Stability of the time-path, mainly, depends on exponential expression $\left(\frac{\beta}{m-\alpha} \right)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \underbrace{\tilde{b}} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \underbrace{\tilde{b}} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = \left \frac{\beta}{m-\alpha} \right \leq 1 \xrightarrow{\text{if } (m-\alpha > \beta)} \left \frac{\beta}{m-\alpha} \right < 1$ Time-path will be Convergent	$b = \left(\frac{\tilde{\beta}}{m-\alpha} \right)^{>0} \xrightarrow{\text{if convergent: } (m-\alpha > \beta)} \xrightarrow{\text{then: } (m-\alpha > 0)} > 0$ Non-Oscillatory

Therefore, the time-path will be convergent in nature if $(m - \alpha > \beta)$ while it will move in a non-oscillatory pattern as $(m - \alpha > 0)$.

Lesson 37

VARIOUS ECONOMIC APPLICATIONS OF DIFFERENCE EQUATIONS

TOPIC 168: INVESTMENT DYNAMICS USING DIFFERENCE EQUATIONS

I_t is the amount invested in period t :

$$I_t = 0.9I_{t-1} + 60$$

$$I_t + \underbrace{(-0.9)}_{a \neq -1} I_{t-1} = \underbrace{60}_c$$

And investment at $t = 0$ is £900m. Find the time-path of investment.

Comparing the standard form of First Order Difference equation:

$$y_t + ay_{t-1} = c$$

Since $a = -0.9 \neq -1$, the definite time-path will be:

$$\hat{y}_t = \left(\hat{y}_0 - \frac{\hat{c}}{1 + \underbrace{a}_{-0.9}} \right) \left(\underbrace{\hat{b}}^{-a} \right)^t + \frac{\hat{c}}{1 + \underbrace{a}_{-0.9}}$$

$$I_t = \left(900 - \frac{60}{1 - 0.9} \right) \{ -(-0.9) \}^t + \frac{60}{1 - 0.9}$$

$$I_t = (900 - 600)(0.9)^t + 600$$

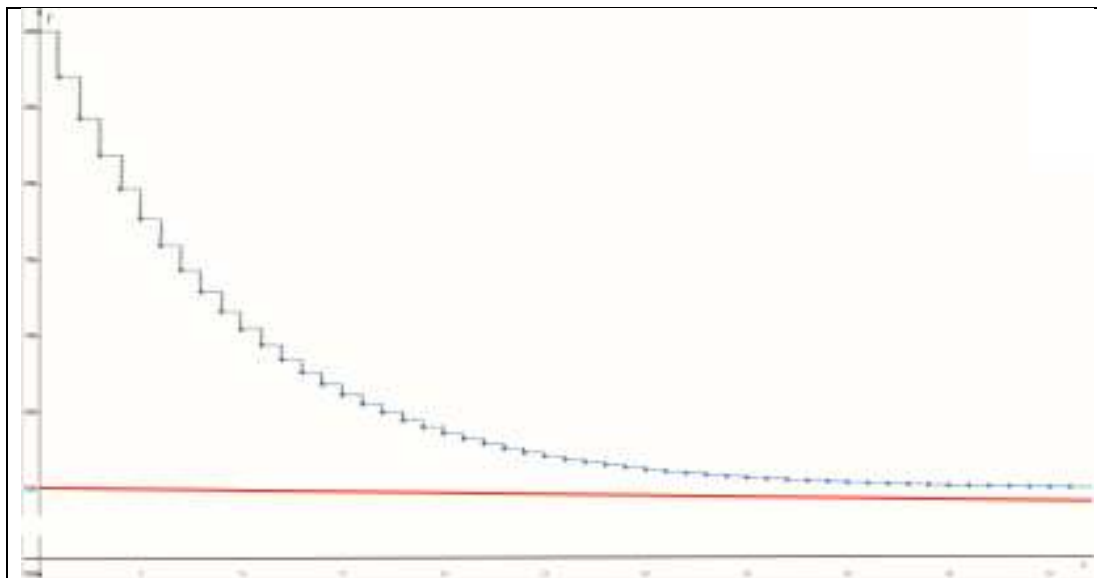
$$I_t = 300(0.9)^t + 600$$

Interpretation: $I_t = 300(0.9)^t + 600$ is the definite time-path of the first order difference equation $I_t - 0.9I_{t-1} = 60$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(0.9)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \underbrace{\hat{b}} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \underbrace{\hat{b}} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$\underbrace{ \hat{b} = -0.9 < 1}_{\text{Convergent}}$	$\underbrace{\hat{b} = (0.9) < 0}_{\text{Non-oscillatory}}$

Therefore, the time-path will be convergent in nature while it will move in a non-oscillatory pattern as.

$I_t = \underbrace{(300)}_{\substack{\text{coefficient} \\ \text{of } (0.9)^t}} \underbrace{(0.9)^t}_{\substack{\rightarrow 0 \\ \text{as } t \rightarrow \infty}} + \underbrace{600}_{I_p (= I^*)}$ $I_t \rightarrow (I^* = 600), \text{ as } t \rightarrow \infty$	$I_c(\text{deviation from } I^*)$ $I_c \rightarrow 0, \text{ as } t \rightarrow \infty$
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TOPIC 169: PRICE DYNAMICS & MONEY MARKET USING DIFFERENCE EQUATIONS

Money supply (M_s), is set by the government. Money demand (L_t) depends on the price level (P_t) as well as on real output (Y). Inflation can be measured as the difference between the price levels between consecutive time period ($P_{t+1} - P_t$), and depends on excess money supply ($M_s - L_t$) such that:

$$\begin{aligned} L_t &= \alpha P_t Y & (\alpha > 0) \\ P_{t+1} - P_t &= j(M_s - L_t) & (j > 0) \end{aligned}$$

Express the time path of the price level in the economy. Find its intertemporal equilibrium. How is this equilibrium related to money supply and aggregate output?

Solution:

Plugging in the value of L_t in $P_{t+1} - P_t = j(M_s - L_t)$:

$$P_{t+1} - P_t = j(M_s - \alpha P_t Y)$$

$$P_{t+1} - P_t = j(M_s - \alpha P_t Y)$$

$$P_{t+1} - P_t + \alpha j P_t Y = j M_s$$

$$P_{t+1} - (1 - \alpha j Y) P_t = j M_s$$

Recalling the Standard form of First Order Difference Equation:

$$\underbrace{P_{t+1}}_{y_{t+1}} + \underbrace{\{- (1 - \alpha j Y)\}}_a \underbrace{P_t}_{y_t} = \underbrace{j M_s}_c$$

$$\text{Here; } a = - \overbrace{\left(1 - \underbrace{\hat{\alpha} \hat{j} \hat{Y}}_{>0} \right)}^{=-1}, c = j M_s, y_t = P_t, y_0 = P_0, b = -a.$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a} \right) \left(\frac{-a}{\hat{b}} \right)^t + \frac{c}{1+a}$$

$$\begin{aligned}
 P_t &= \left\{ P_0 - \frac{(jM_s)}{1 + \{-(1 - \alpha jY)\}} \right\} [-\{-(1 - \alpha jY)\}]^t + \frac{(jM_s)}{1 + \{-(1 - \alpha jY)\}} \\
 P_t &= \left\{ P_0 - \frac{jM_s}{1 - (1 - \alpha jY)} \right\} (1 - \alpha jY)^t + \frac{jM_s}{1 - (1 - \alpha jY)} \\
 P_t &= \left(P_0 - \frac{jM_s}{1 - 1 + \alpha jY} \right) (1 - \alpha jY)^t + \frac{jM_s}{1 - 1 + \alpha jY} \\
 P_t &= \left(P_0 - \frac{jM_s}{\alpha jY} \right) (1 - \alpha jY)^t + \frac{jM_s}{\alpha jY} \\
 P_t &= \underbrace{\left(P_0 - \frac{M_s}{\alpha Y} \right)}_{P_c} \underbrace{(1 - \alpha jY)^t}_{\text{exponential expression}} + \underbrace{\frac{M_s}{\alpha Y}}_{P_p} \\
 P_t &= \left(P_0 - \frac{M_s}{\alpha Y} \right) (1 - \alpha jY)^t + \frac{M_s}{\alpha Y}
 \end{aligned}$$

Interpretation: $P_t = \left(P_0 - \frac{M_s}{\alpha Y} \right) (1 - \alpha jY)^t + \frac{M_s}{\alpha Y}$ is the definite time-path of the first order difference equation $P_{t+1} - (1 - \alpha jY)P_t = jM_s$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(1 - \alpha jY)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \underbrace{\quad \hat{b} \quad}_{(<1): \text{convergent}} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \underbrace{\quad \hat{b} \quad}_{(>0): \text{non-oscillatory}} \end{array} \right\}$
$ b = 1 - \alpha jY $ $\leq 1 \xrightarrow{\text{if } (\alpha jY < 1)} 1 - \alpha jY > 1$ <i>Time-path will be Convergent</i>	$b = (1 - \alpha jY)$ $\leq 0 \xrightarrow{\text{if } (\alpha jY < 1)} (1 - \alpha jY) \neq 0$ <i>Time-path will be Non-Oscillatory</i>

Therefore, the time-path is convergent in nature if $(\alpha jY < 1)$ while it will move in a non-oscillatory pattern.

TOPIC 170: FISH STOCK ANALYSIS USING DIFFERENCE EQUATION

A fish population increases by 5% each generation, but a catch of 8000 is lost from each generation. If P_t represents the size of the population for generation t :

(a) Write the difference equation which expresses P_t in terms of P_{t-1} .

(b) Solve the difference equation, given $P_0 = 100000$. Hence, describe the time path.

Solution: Mathematizing the given information:

$$\underbrace{F_t}_{\text{current fish stock}} = \underbrace{1.05}_{\substack{5\% \uparrow \\ \text{in } 100}} \underbrace{F_{t-1}}_{\substack{\text{fish stock} \\ \text{in last} \\ \text{generation}}} - \underbrace{8000}_{\substack{\text{fish catch in} \\ \text{every generation}}}$$

$$F_{[t]} = 1.05F_{[t-1]} - 8000$$

Introducing 1 generation lag:

Introducing 1 generation lag:

$$F_{t+1} = 1.05F_{t-1} - 8000$$

$$F_{[t+1]} = 1.05F_{[t]} - 8000$$

$$F_{t+1} - 1.05F_t = -8000$$

$$F_{t+1} + \underbrace{(-1.05)}_{a \neq -1} F_t = \underbrace{(-8000)}_c$$

Comparing the standard form of First Order Difference equation:

$$y_t + ay_{t-1} = c$$

Since $a = -1.05 \neq -1$, the definite time-path will be:

$$\hat{y}_t = \left(\hat{y}_0 - \frac{\hat{c}}{1 + \underbrace{a}_{-1.05}} \right) \left(\hat{b} \right)^t + \frac{\hat{c}}{1 + \underbrace{a}_{-1.05}}$$

$$F_t = (100000 - 160000)\{-(-1.05)\}^t + 160000$$

$$F_t = (-60000)(1.05)^t + 160000$$

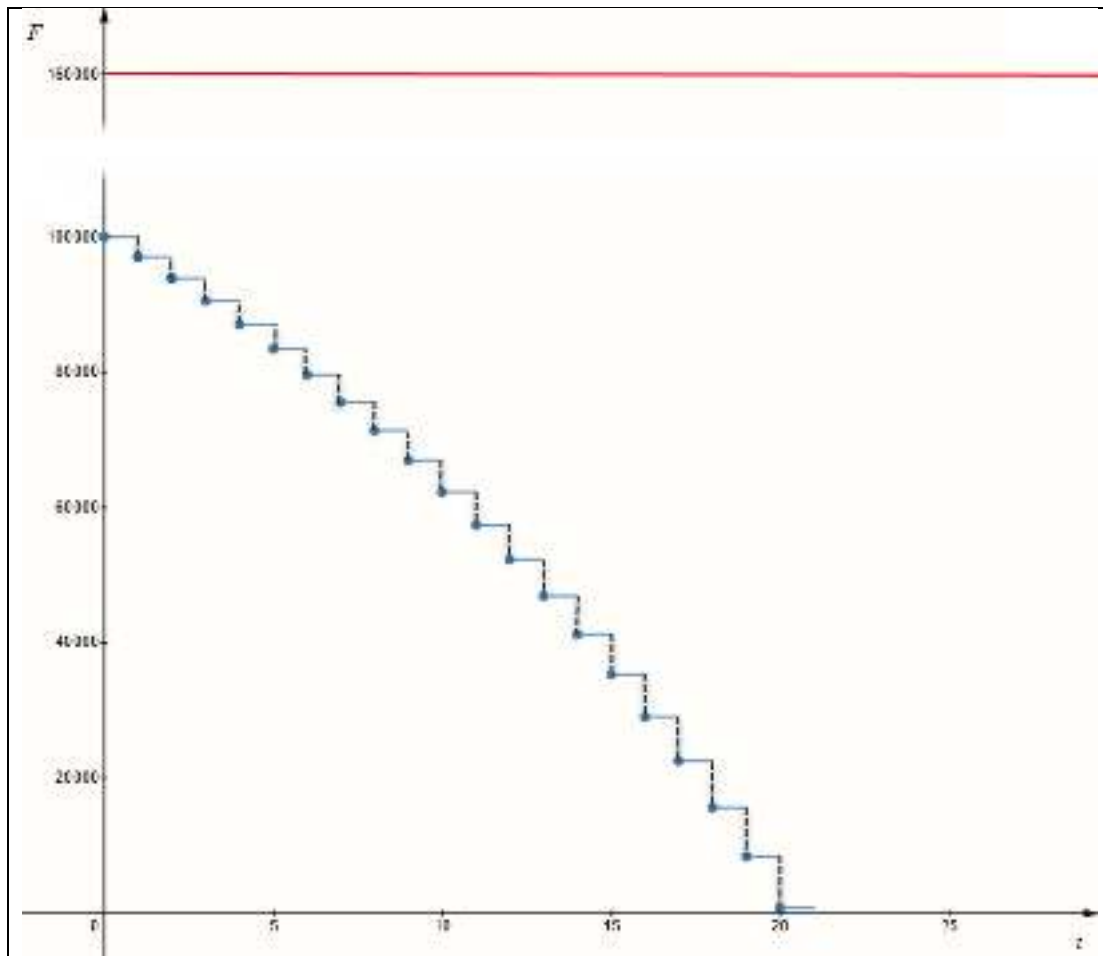
$$F_t = -60000(1.05)^t + 160000$$

Interpretation: $F_t = -60000(1.05)^t + 160000$ is the definite time-path of the first order difference equation $F_{t+1} - 1.05F_t = -8000$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(1.05)^t$.

Nature $\left\{ \begin{array}{c} (>1): \text{non-convergent} \\ \underbrace{\hat{b}} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{c} (<0): \text{oscillatory} \\ \underbrace{\hat{b}} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$\underbrace{ \hat{b} = 1.05 > 1}_{\text{Non-Convergent}}$	$\underbrace{\hat{b} = (1.05) > 0}_{\text{Non-oscillatory}}$

Therefore, the time-path will be non-convergent in nature while it will move in a non-oscillatory pattern as.

$F_t = \underbrace{(-60000)}_{\substack{\text{coefficient} \\ \text{of } (1.05)^t}} \underbrace{(1.05)^t}_{\substack{\rightarrow 0 \\ \text{as } t \rightarrow \infty}} + \underbrace{160000}_{F_p (= F^*)}$ $F_t \rightarrow (F^* = 160000), \text{ as } t \rightarrow \infty$	$F_c (\text{deviation from } F^*)$ $F_c \rightarrow 0, \text{ as } t \rightarrow \infty$
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TOPIC 171: INITIAL AMOUNT DEPOSITED IN COMPOUND INTEREST

Derive the formula for the value P_t of an initial amount of money P_0 deposited at i interest for t years when compounded annually.

When interest is compounded annually:

$$\underbrace{P_{t+1}}_{\substack{\text{principal} \\ + \\ \text{interest}}} = \underbrace{P_t}_{\substack{\text{initial} \\ \text{amount}}} + \underbrace{i}_{\substack{\text{interest} \\ \text{rate}}} \times \underbrace{P_t}_{\substack{\text{initial} \\ \text{amount}}} \\ \text{compound interest}$$

$$P_{t+1} = (1 + i)P_t$$

$$P_{t+1} - (1 + i)P_t = 0$$

Recalling the standard form of First Order Difference Equation:

$$y_{t+1} + ay_t = c$$

$$\underbrace{P_{t+1}}_{y_{t+1}} + \underbrace{\{-(1+i)\}}_{a \neq -1} \underbrace{P_t}_{y_t} = \underbrace{0}_c$$

$$\underbrace{\left\{ - \left(1 + \overset{\neq 0}{i} \right) \right\}}_{a \neq -1} : \{-(1+i)\} > -1 \therefore \{-(1+i)\} \neq -1.$$

Here; $a = \{-(1+i)\}$, $c = 0$, $y_t = P_t$, $y_0 = P_0$, $b = -a$.

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a}\right) \left(\frac{\tilde{b}}{1+a}\right)^t + \frac{c}{1+a}$$

$$P_t = \left\{P_0 - \frac{0}{1 + \{-(1+i)\}}\right\} (-a)^t + \frac{0}{1 + \{-(1+i)\}}$$

$$P_t = \{P_0\}[-\{-(1+i)\}]^t$$

$$\boxed{P_t = P_0(1+i)^t}$$

exponential
expression

$$P_t = \underbrace{P_0}_{Y_c} \underbrace{(1+i)^t}_{Y_p} + \underbrace{0}_{Y_p}$$

Interpretation: $P_t = P_0(1+i)^t$ is the definite time-path of the first order difference equation $P_{t+1} - (1+i)P_t = 0$. Dynamic Stability of the time-path, mainly, depends on exponential expression $(1+i)^t$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \tilde{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \tilde{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = 1+i \xrightarrow{\text{given } (i>0)} 1+i \nless 1$ <i>Time-path will be Non-convergent</i>	$b = (1+i) \xrightarrow{\text{given } (a>0)} (1+i) \nless 0$ <i>Time-path will be Non-Oscillatory</i>

Therefore, the time-path is non-convergent in nature as $(a > s)$ while it will move in a non-oscillatory pattern.

TOPIC 172: HARROD MODEL OF GROWTH

The Harrod model attempts to explain the dynamics of growth in the economy. It assumes

$$S_t = sY_t$$

where s is a constant equal to both the **MPS** and **APS**. It also assumes the *acceleration principle*, i.e., investment is proportional to the rate of change of national income over time.

$$I_t = a(Y_t - Y_{t-1})$$

where a is a constant equal to both the marginal and average capital-output ratios (**COR**). Develop the time path of national income and interpret.

In equilibrium, $I_t = S_t$. Therefore $I_t = a(Y_t - Y_{t-1})$ becomes:

$$S_t = a(Y_t - Y_{t-1})$$

$$sY_t = aY_t - aY_{t-1}$$

$$sY_t - aY_t + aY_{t-1} = 0$$

$$(s-a)Y_t + aY_{t-1} = 0$$

Normalizing by dividing throughout by $(s-a)$:

$$Y_t + \left\{ \frac{a}{(s-a)} \right\} Y_{t-1} = \frac{0}{(s-a)}$$

$$\boxed{Y_t + \left(\frac{a}{s-a} \right) Y_{t-1} = 0}$$

Recalling the standard form of First Order Difference Equation:

$$y_{t+1} + ay_t = c$$

Introducing a lag to make the given form comparable with the standard form:

$$Y_{t+1} + \left(\frac{a}{s-a} \right) Y_{t-1+1} = 0$$

$$\underbrace{Y_{t+1}}_{y_{t+1}} + \underbrace{\left(\frac{a}{s-a}\right)}_{a \neq -1} \underbrace{Y_t}_{y_t} = \underbrace{0}_c$$

$$\underbrace{\left(\frac{a}{s-a}\right)}_{a \neq -1} : \underbrace{\left(\frac{a}{s-a}\right)}_{\text{if } s=0, \text{ but } 0 < s} = -1 \therefore \underbrace{\left(\frac{a}{s-a}\right)}_{\text{as per economic theory}} \neq -1.$$

Here; $a = \left(\frac{a}{s-a}\right)$, $c = 0$, $y_t = Y_t$, $y_0 = Y_0$, $b = -a$.

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a}\right) \left(\frac{(-a)}{\hat{b}}\right)^t + \frac{c}{1+a}$$

$$Y_t = \left\{ Y_0 - \frac{0}{1 + \left(\frac{a}{s-a}\right)} \right\} \left\{ -\left(\frac{a}{s-a}\right) \right\}^t + \frac{0}{1 + \left(\frac{a}{s-a}\right)}$$

$$\boxed{Y_t = Y_0 \left(\frac{a}{a-s}\right)^t}$$

exponential
expression

$$Y_t = \underbrace{Y_0 \left(\frac{a}{a-s}\right)^t}_{Y_c} + \underbrace{0}_{Y_p}$$

Interpretation: $Y_t = Y_0 \left(\frac{a}{a-s}\right)^t$ is the definite time-path of the first order difference equation

$Y_{t+1} + \left(\frac{a}{s-a}\right) Y_{t-1} = 0$. Dynamic Stability of the time-path, mainly, depends on exponential expression $\left(\frac{a}{a-s}\right)^t$. a (the capital-output ratio) is usually greater than 1 while s (marginal propensity to save) is lesser than 1 (but also greater than zero):

Mathematically: $\underbrace{(a > 1) \wedge (s < 1)}_{a > s}$. The inequality $(a > s)$ shall be used in explaining the nature and pattern of time-path.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \hat{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \hat{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = \left \frac{a}{a-s} \right \leq 1 \xrightarrow{\text{given } (a>s)} \left \frac{a}{a-s} \right \not\leq 1$ Time-path will be Non-convergent	$b = \left(\frac{a}{a-s}\right) \leq 0 \xrightarrow{\text{as given } (a>s)} \left(\frac{a}{a-s}\right) \not\leq 0$ Time-path will be Non-Oscillatory

Therefore, the time-path is non-convergent in nature as $(a > s)$ while it will move in a non-oscillatory pattern.

TOPIC 173: WARRANTED RATE OF GROWTH

In the Harrod model the *warranted rate of growth* is of pivotal importance. Where, 'Warranted Rate of Growth' (i.e., the path the economy must follow to have equilibrium between saving and investment each year).

Recalling the time-path of national income $Y_t = \left(\frac{a}{a-s}\right) Y_0$. This time-path shows that current year's national income is $\left(\frac{a}{a-s}\right)$ times the national income of the previous period. e.g. for current period;

$$Y_1 = \left(\frac{a}{a-s}\right) Y_0$$

Intertemporal growth: $G = \frac{Y_1 - Y_0}{Y_0}$. Substituting the value of Y_1 becomes:

$$G = \frac{Y_1 - Y_0}{Y_0}$$

$$G = \frac{\left(\frac{a}{a-s}\right) Y_0 - Y_0}{Y_0}$$

$$G = \frac{\left\{\left(\frac{a}{a-s}\right) - 1\right\} Y_0}{Y_0}$$

$$G = \frac{a}{a-s} - 1$$

$$G = \frac{a - (a-s)}{a-s}$$

$$\begin{matrix} \text{Warranted} \\ \text{Growth rate} \end{matrix} \hat{G} = \frac{a - a + s}{a-s}$$

Warranted Growth rate:

$$G_w = \frac{s}{a-s}$$

Lesson 38

COBWEB MODEL AND DIFFERENCE EQUATIONS

TOPIC 174: COBWEB MODEL: FRAMEWORK

Using First Order Difference Equations, we can understand the price fluctuations of products with lags involved in between production and sale. For instance, agricultural products, in which planting and harvest occur with substantial gap.

The Model: Cobweb model is based on demand and supply forces:

Demand Side $\{Q_d(t)\}$: The demand side of such products has no lags, since for a buyer:

P_t $Q_d(t)$
Price of crop in current time period $\Rightarrow$ **Demand of crop in current time period**

buyer is concerned with the current price of agricultural products

Mathematizing this basic-psychology:

$$\begin{array}{c} Q_d(t) \rightleftharpoons D(P_t) \\ \text{current} \quad \leftarrow \quad \text{current} \\ \text{demand} \quad \text{price} \\ \boxed{Q_d(t) = D(P_t)} \end{array}$$

Supply Side $\{Q_s(t)\}$:

P_t $Q_{s(t+1)}$
Price of crop in current time period $\Rightarrow$ **Supply (Output) of crop in next time period**

it takes a season (time period) for a crop to get ready

Mathematizing this agricultural phenomenon:

$$\begin{array}{c} Q_{s(t+1)} \rightleftharpoons S(P_t) \\ \text{future} \quad \leftarrow \quad \text{current} \\ \text{supply} \quad \text{price} \\ \boxed{Q_{s(t+1)}} = S(P_t) \end{array}$$

can't be equated to $Q_d(t)$

due to different time periods

Therefore, we convert $Q_{s(t+1)}$ into $Q_{s(t)}$ by introducing a lag (-1) :

$$\begin{array}{c} Q_{s(t+1[-1])} = S(P_{t[-1]}) \\ \boxed{Q_{s(t)} = S(P_{t-1})} \end{array}$$

Unlagged and Lagged Demand and Supply Functions:

Unlagged Demand Function	Lagged Supply Function
$Q_d(t) = D(P_t)$	$Q_{s(t)} = S(P_{t-1})$
Unlagged Demand Equation	Lagged Supply Equation
$Q_d(t) = \alpha - \beta(P_t)$	$Q_{s(t)} = -\gamma + \delta(P_{t-1})$

For equilibrium in (agricultural) market: $Q_d = Q_s$.

$$\begin{array}{l} Q_d(t) = Q_{s(t)} \\ \alpha - \beta(P_t) = -\gamma + \delta(P_{t-1}) \\ -\beta(P_t) - \delta(P_{t-1}) = -\alpha - \gamma \\ \beta(P_t) + \delta(P_{t-1}) = \alpha + \gamma \end{array}$$

Normalizing by dividing throughout by β :

$$(P_t) + \frac{\delta}{\beta}(P_{t-1}) = \frac{(\alpha + \gamma)}{\beta}$$

$$P_t + \frac{\delta}{\beta}(P_{t-1}) = \frac{\alpha + \gamma}{\beta}$$

Recalling the standard form of First Order Difference Equation:

$$y_{[t+1]} - ay_{[t]} = b$$

$$P_{[t]} + \frac{\delta}{\beta}(P_{[t-1]}) = \frac{\alpha + \gamma}{\beta}$$

Boxed subscripts show discrepancy of time, so we introduce time lead in price equation.

$$P_{t[+1]} + \frac{\delta}{\beta}(P_{t-1[+1]}) = \frac{\alpha + \gamma}{\beta}$$

$$\underbrace{P_{t+1}}_{y_{t+1}} + \underbrace{\left(\frac{\delta}{\beta}\right)}_a \underbrace{P_t}_{y_t} = \underbrace{\frac{\alpha + \gamma}{\beta}}_c$$

Comparing with the standard form: $a = \frac{\delta}{\beta}$, $c = \frac{\alpha + \gamma}{\beta}$, $y_{t+1} = P_{t+1}$, $y_t = P_t$.

$$\underbrace{\left(\frac{\delta}{\beta}\right)}_{\substack{>0 \\ <0 \\ a>0}} \Rightarrow \left(\frac{\delta}{\beta}\right) \neq -1.$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a}\right) \left(\frac{-a}{\hat{b}}\right)^t + \frac{c}{1+a}$$

$$P_t = \left\{ P_0 - \frac{\left(\frac{\alpha + \gamma}{\beta}\right)}{\left(1 + \frac{\delta}{\beta}\right)} \right\} (-a)^t + \frac{\left(\frac{\alpha + \gamma}{\beta}\right)}{\left(1 + \frac{\delta}{\beta}\right)}$$

$$P_t = \left\{ P_0 - \frac{\left(\frac{\alpha + \gamma}{\beta}\right)}{\left(\frac{\beta + \delta}{\beta}\right)} \right\} \left(-\frac{\delta}{\beta}\right)^t + \frac{\left(\frac{\alpha + \gamma}{\beta}\right)}{\left(\frac{\beta + \delta}{\beta}\right)}$$

stationary
equilibrium

$(\alpha, \beta, \gamma, \delta): \text{constants}$

$$P_t = \left\{ P_0 - \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta}\right)}_{P_p} \right\} \left(-\frac{\delta}{\beta}\right)^t + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta}\right)}_{P_p}$$

$$P_t = \left(P_0 - \underbrace{P_p}_{P^*} \right) \left(-\frac{\delta}{\beta}\right)^t + \underbrace{P_p}_{P^*}$$

$$P_t = (P_0 - P^*) \left(-\frac{\delta}{\beta}\right)^t + P^*$$

$$P_t = \underbrace{(P_0 - P^*)}_{A} \underbrace{\left(-\frac{\delta}{\beta}\right)^t}_b + \underbrace{P^*}_{P_p}$$

Role of A: It can be recalled adopting the following table:

$A = (P_0 - P^*)$ being a coefficient can affect P_p in either or both of the two ways:

Scale Effect: Absolute Value of $ P_0 - P^* $ on P_p			
Magnitude			
$ P_0 - P^* = 0$ Nullify	$ P_0 - P^* < 1$ Minify	$ P_0 - P^* = 1$ Neutral	$ P_0 - P^* > 1$ Amplify
$P_c = \boxed{0} \left(-\frac{\delta}{\beta}\right)^t$	$P_c = \boxed{0.5} \left(-\frac{\delta}{\beta}\right)^t$	$P_c = \boxed{1} \left(-\frac{\delta}{\beta}\right)^t$	$P_c = \boxed{2} \left(-\frac{\delta}{\beta}\right)^t$
$P_c = \boxed{0}$	$P_c = \boxed{0.5} \left(-\frac{\delta}{\beta}\right)^t$	$P_c = \boxed{\left(-\frac{\delta}{\beta}\right)^t}$	$y_c = \boxed{2} \left(-\frac{\delta}{\beta}\right)^t$
-	Noteworthy	-	Noteworthy

Mirror Effect: Original Value of $(P_0 - P^*)$ on P_p	
Sign	
$(P_0 - P^*) > 0$ Retain	$(P_0 - P^*) < 0$ Reverse
$P_c = \underbrace{(P_0 - P^*)}_{>0} \left(-\frac{\delta}{\beta}\right)^t$	$P_c = \underbrace{(P_0 - P^*)}_{<0} \left(-\frac{\delta}{\beta}\right)^t$
$P_c > 0$	$P_c < 0$
-	Noteworthy

TOPIC 175: COBWEB MODEL: DIAGRAMMATICALLY

Using First Order Difference Equation, we found the definite time-path

$$P_t = (P_0 - P^*) \left(-\frac{\delta}{\beta}\right)^t + P^* \text{ of price equation } P_{t+1} + \frac{\delta}{\beta}(P_t) = \frac{\alpha + \gamma}{\beta}$$

$$P_t = \underbrace{(P_0 - P^*)}_A \underbrace{\left(-\frac{\delta}{\beta}\right)^t}_b + \underbrace{P^*}_{P_p}$$

exponential expression

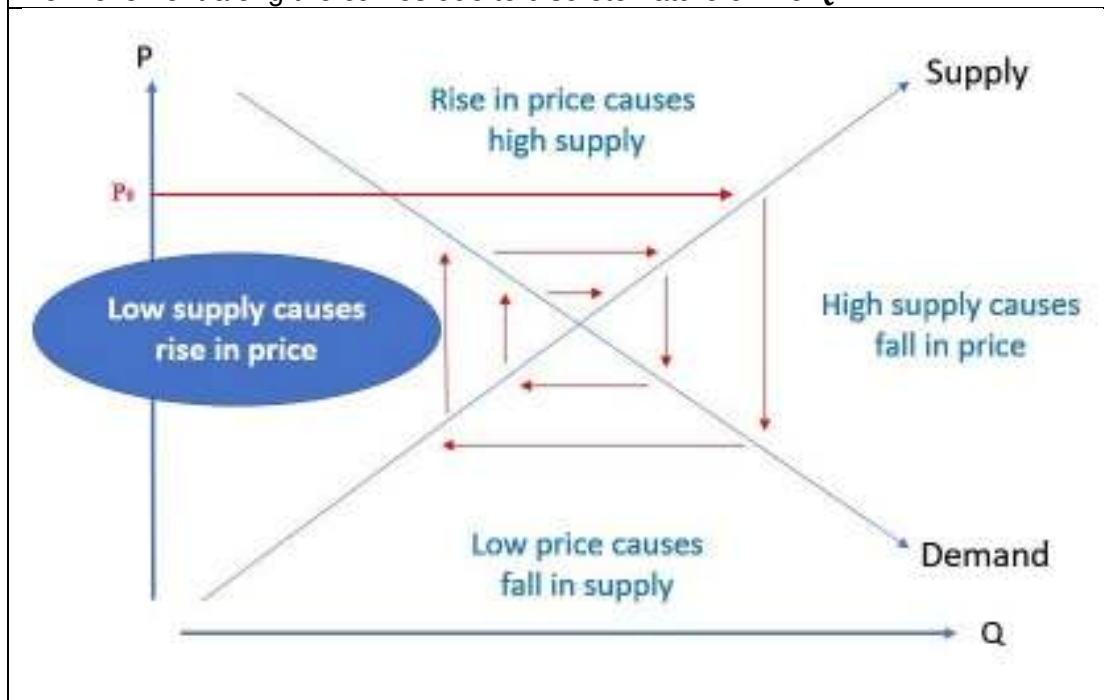
Within the boxed term, exponential expression $\left(-\frac{\delta}{\beta}\right)^t$ is critical for determining the nature and pattern of the time path of P_t . We borrow the base $\left(-\frac{\delta}{\beta}\right)$ of the expression for elaboration.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \boxed{b} \\ (<1): \text{convergent} \end{array} \right\}$		Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \boxed{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = \left -\frac{\delta}{\beta} \right \leq 1$		$b = \left(-\frac{\delta}{\beta}\right) \leq 0$
$\boxed{\delta < \beta}$	$\boxed{\delta > \beta}$	Both δ and β are positive. $(\delta, \beta) > 0$
$\underbrace{\left -\frac{\delta}{\beta} \right < 1}_{\text{Convergent}}$	$\underbrace{\left -\frac{\delta}{\beta} \right > 1}_{\text{Non-Convergent (Divergent)}}$	Regardless of $(\delta \leq \beta)$, $\underbrace{b < 0}_{\text{Oscillatory}}$

Therefore, the time-path can be either convergent or non-convergent in nature as $(\delta \leq \beta)$, while it will move in an oscillatory pattern.

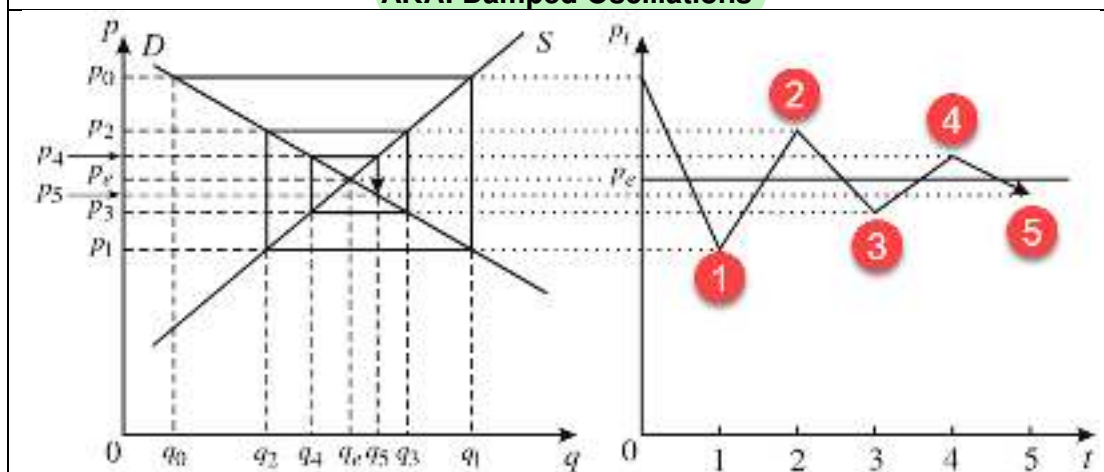
Underlying Mechanism of Cobweb Model:

No movement along the curves due to discrete nature of P & Q .



$\left\{ \left| -\frac{\delta}{\beta} \right| < 1 \right\}$ **Convergent and Oscillatory Time-path of Price**

AKA: Damped Oscillations

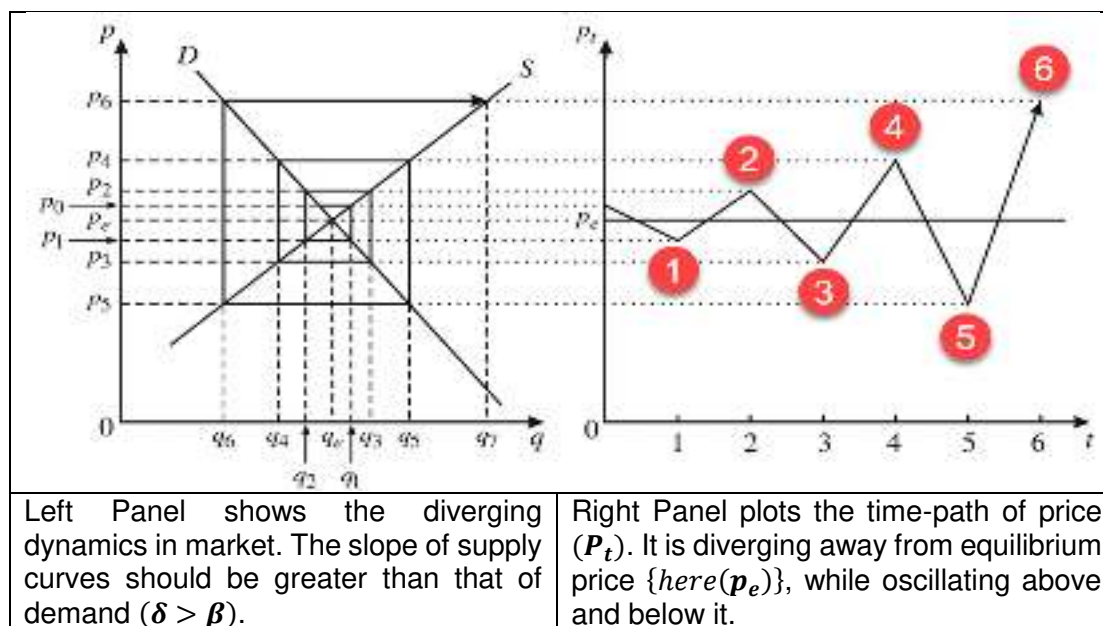


Left Panel shows the converging dynamics in market. The slope of supply curves should be lesser than that of demand ($\delta < \beta$).

Right Panel plots the time-path of price (P_t). It is converging towards equilibrium price {here(p_e)}, while oscillating above and below it.

$\left\{ \left| -\frac{\delta}{\beta} \right| > 1 \right\}$ **Non-Convergent (Divergent) and Oscillatory Time-path of Price**

AKA: Explosive Oscillations



Conclusion: The nature of time-path is primarily dependent on the relative slope of demand and supply curves ($\delta \leq \beta$).

TOPIC 176: COBWEB MODEL: RARE CASE

We have understood two **common** cases of convergence and divergence in cobweb model:

Common Cases: $\delta \leq \beta$	
$\delta < \beta$	$\delta > \beta$
$\left -\frac{\delta}{\beta} \right < 1$	$\left -\frac{\delta}{\beta} \right > 1$
Convergent	Non-Convergent (Divergent)

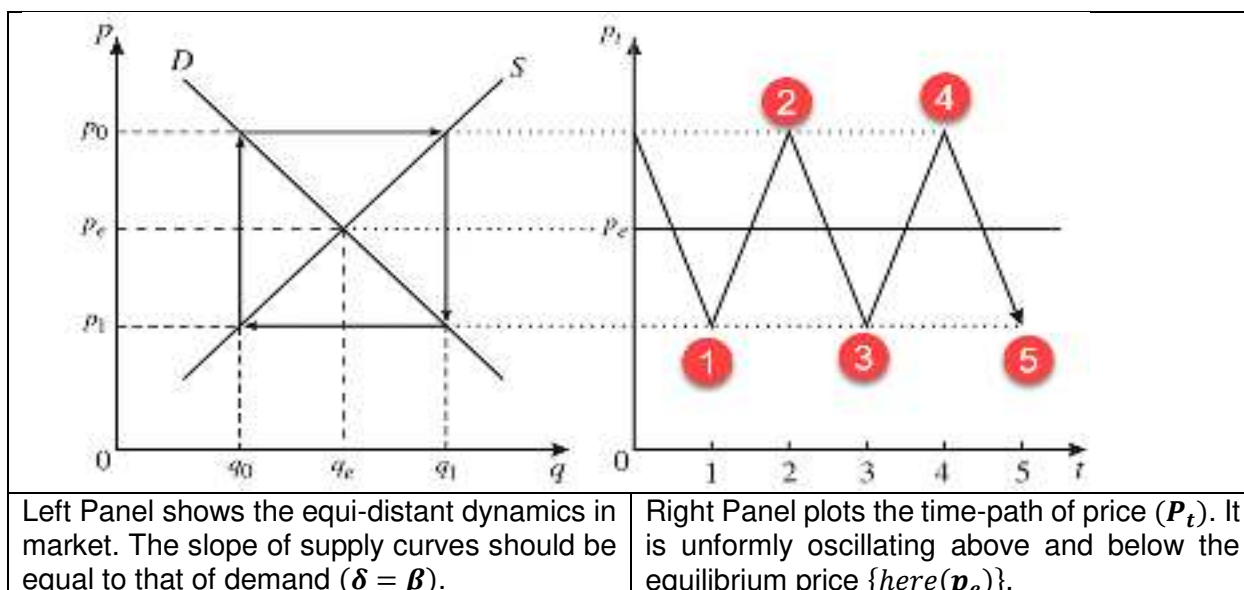
Now considering a **rare** case in cobweb model where ($\delta = \beta$):

Rare Case: $\delta = \beta$
$\left(-\frac{\delta}{\beta} \right) = -1$

Nature of time-path of Rare Case:

Nature $\left\{ \begin{array}{c} (>1): \text{non-convergent} \\ \hat{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{c} (<0): \text{oscillatory} \\ \hat{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$\delta = \beta$	$\delta = \beta$
$ b = \left -\frac{\delta}{\beta} \right = 1$	$b = \left(-\frac{\delta}{\beta} \right) = -1$
$\left -\frac{\delta}{\beta} \right = 1 \nless 1$ Non-Convergent (Neither)	$b = -1 < 0$ Oscillatory

$\left\{ \left -\frac{\delta}{\beta} \right = 1 \right\}$ Non-Convergent (Uniform) and Oscillatory Time-path of Price



Conclusion: The existence of uniform oscillation case is rare event as the equality of slope is a matter of chance ($\delta = \beta$). Any slight inequality can lead to convergence or divergence cases.

TOPIC 177: COBWEB MODEL: NUMERICAL

Given the demand function and Supply Functions:

$Q_{dt} = 18 - 3P_t$ and $Q_{st} = -3 + 4P_{t-1}$, find the time-path of price.

Solution:

Unlagged Demand Equation	Lagged Supply Equation
$Q_{d(t)} = \alpha - \beta(P_t)$	$Q_{s(t)} = -\gamma + \delta(P_{t-1})$
$Q_{dt} = 18 - 3(P_t)$	$Q_{st} = -3 + 4(P_{t-1})$

Here, $\alpha = 18$, $\beta = 3$, $\gamma = 3$ and $\delta = 4$

For equilibrium in market: $Q_d = Q_s$.

$$\begin{aligned}
 Q_{d(t)} &= Q_{s(t)} \\
 18 - 3P_t &= -3 + 4P_{t-1} \\
 -3P_t - 4P_{t-1} &= -3 - 18 \\
 -3P_t - 4P_{t-1} &= -21 \\
 3P_t + 4P_{t-1} &= 21
 \end{aligned}$$

Normalizing by dividing throughout by 3:

$$\begin{aligned}
 P_t + \frac{4}{3}(P_{t-1}) &= 7 \\
 P_t + \frac{4}{3}(P_{t-1}) &= 7
 \end{aligned}$$

Recalling the standard form of First Order Difference Equation:

$$\begin{aligned}
 y_{[t+1]} - ay_{[t]} &= b \\
 P_{[t]} + \frac{4}{3}(P_{[t-1]}) &= 7
 \end{aligned}$$

Boxed subscripts show discrepancy of time, so we introduce time lead in price equation.

$$P_{t[\boxed{+1}]} + \frac{4}{3}(P_{t-4[\boxed{+1}]}) = 7$$

$$\underbrace{P_{t+1}}_{y_{t+1}} + \underbrace{\left(\frac{4}{3}\right)}_a \underbrace{P_t}_{y_t} = \underbrace{7}_c$$

Comparing with the standard form: $\underbrace{a = \frac{4}{3}}_{a \neq -1}$, $c = 7$, $y_{t+1} = P_{t+1}$, $y_t = P_t$.

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a}\right) \left(\frac{(-a)}{\hat{b}}\right)^t + \frac{c}{1+a}$$

$$P_t = \left\{P_0 - \frac{7}{\left(1 + \frac{4}{3}\right)}\right\} (-a)^t + \frac{7}{\left(1 + \frac{4}{3}\right)}$$

$$P_t = \left\{P_0 - \frac{7}{\left(\frac{7}{3}\right)}\right\} (-a)^t + \frac{7}{\left(\frac{7}{3}\right)}$$

$$P_t = (P_0 - 3) \left(-\frac{4}{3}\right)^t + 3$$

$$P_t = \underbrace{\left(P_0 - \underbrace{3}_{P_p}\right)}_{P_c} \left(\underbrace{-\frac{4}{3}}_{\hat{b}}\right)^t + \underbrace{3}_{P_p}$$

Interpretation: The time-path of in case of cobweb model is $P_t = (P_0 - 3) \left(-\frac{4}{3}\right)^t + 3$, if $Q_d = 18 - 3P_t$ and $Q_s = -3 + 4P_{t-1}$.

Nature $\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \hat{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern $\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \hat{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
$ b = \left -\frac{\delta}{\beta} \right = \left -\frac{4}{3} \right > 1$ <i>Non-Convergent (Divergent)</i>	$b = \left(-\frac{\delta}{\beta} \right) = \left(-\frac{4}{3} \right) < 0$ <i>Oscillatory</i>

Hence the time-path is divergent in nature and oscillatory in pattern – dynamically unstable.

TOPIC 178: DIFFERENCE EQUATIONS AND ADAPTIVE EXPECTATIONS

Consider the following market model:

$$\begin{aligned} Q_{dt} &= Q_{st} \\ Q_{dt} &= -\alpha + \beta P_t \\ Q_{st} &= \gamma - \delta P_t^* \end{aligned}$$

Here, P_t^* is the expected price in current time period t .

Moreover, the type of expectations is Adaptive Expectations.

Mathematically; $P_t^* = P_{t-1}^* + \eta(P_{t-1} - P_{t-1}^*)$ ($0 < \eta \leq 1$)

		expectation- adjustment coefficient	
$\underbrace{P_t^*}_{\substack{\text{expected price} \\ \text{in current} \\ \text{time period}}}$	=	$\underbrace{P_{t-1}^*}_{\substack{\text{expected price} \\ \text{in previous} \\ \text{time period}}}$	+ $\underbrace{\eta}_{\substack{\text{expectation-} \\ \text{adjustment} \\ \text{coefficient}}} \underbrace{(P_{t-1} - P_{t-1}^*)}_{\substack{\text{discrepancy between} \\ \text{actual and expected prices} \\ \text{in previous time period}}}$

Effect of the Adaptive Price Expectations:

If $(P_{t-1} > P_{t-1}^*)$	If $(P_{t-1} < P_{t-1}^*)$
Price in Previous time period is greater than that in Current	Price in Previous time period is lesser than that in Current
$\underbrace{P_{t-1} = 20, P_{t-1}^* = 10}_{\text{under-expectation}}$	$\underbrace{P_{t-1} = 20, P_{t-1}^* = 30}_{\text{over-expectation}}$
Also assume that $\eta = 0.5$	
$P_t^* = P_{t-1}^* + \eta(P_{t-1} - P_{t-1}^*)$	
$P_t^* = 20 + 0.5(20 - 10)$	$P_t^* = 20 + 0.5(20 - 30)$
$P_t^* = 25$	$P_t^* = 15$
Due to under-expectations, price expectations are revised upwards (from 20 to 25).	Due to over-expectations, price expectations are revised downwards (from 20 to 15).

Role of η : Let $\eta = 1$

Assuming $\eta = 1$, we rewrite the price expectations equation as follows:

$$\begin{aligned} P_t^* &= P_{t-1}^* + \underbrace{\eta}_{=1} (P_{t-1} - P_{t-1}^*) \\ P_t^* &= \underbrace{P_{t-1}^*}_{\substack{\text{expected price in} \\ \text{current time period}}} + \underbrace{P_{t-1} - P_{t-1}^*}_{\substack{\text{actual price in} \\ \text{previous time period}}} \\ P_t^* &= P_{t-1} \end{aligned}$$

Conclusion: It implies that seller is expecting the price this year to be same as that in last year. Such situation is observed in Cobweb model. Therefore, we find that adaptive expectations have a significant role in the working of Cobweb model.

Adaptive Expectations in Differential vs. Difference Equations:

	Differential Equations	Difference Equations
	$\frac{d\pi}{dt} = j(p - \pi)$	$P_t^* = P_{t-1}^* + \eta(P_{t-1} - P_{t-1}^*)$ $P_t^* - P_{t-1}^* = \eta(P_{t-1} - P_{t-1}^*)$
Change	$\frac{d\pi}{dt}$	$P_t^* - P_{t-1}^*$
Adjustment coefficient	j	η

Discrepancy term	$(p - \pi)$	$(P_{t-1} - P_{t-1}^*)$
Observation: Both, Differential & Difference equations have the same structure while including the Adaptive Expectations.		

Lesson 39

A MARKET MODEL WITH INVENTORY USING DIFFERENCE EQUATIONS

TOPIC 179: A MARKET MODEL WITH INVENTORY: FRAMEWORK

In Cobweb model, market was assumed to be cleared at every time period. It is possible either if product is perishable which can't be stored or are not stocked.

However, in real life there are many products that can be stored. For such products we develop a model with possibility of inventory.

Premises of The Model

Let us assume the following:

- I. Both the quantity demanded, (Q_d) and the quantity (Q_s) are currently produced, are unlagged linear functions of price (P_t).

$$\overbrace{Q_{dt} = D(P_t)}^{\text{linear function}} \text{ and } \overbrace{Q_{st} = S(P_t)}^{\text{linear function}}$$

unlagged dependence unlagged dependence

- II. Adjustment of price is affected not through market clearance in every period, but through a process of price-setting by the sellers (seller-determined price). At the beginning of each period, the sellers set a price for that period after taking into consideration the inventory situation.

$$\overbrace{(P_t - P_{t-1})}^{\text{price adjustment}} = f \left(\underbrace{Q_s - Q_d}_{\text{inventory}} \right)$$

$\neq f(Q_d = Q_s)$

Accumulated inventory in preceding-period drops the price in current-period, so the merchandise could be 'moved'.

Conversely, a decumulated inventory raises the current price higher than before.

- III. The price adjustment made from period to period is inversely proportional to the observed change in the inventory (stock).

Mathematizing the premises:

Premise – I	$Q_{dt} = \alpha - \beta P_t$ ($\alpha, \beta > 0$)
	$Q_{st} = -\gamma + \delta P_t$ ($\gamma, \delta > 0$)
Premise – II & III	$P_{t+1} = P_t - \sigma(Q_{st} - Q_{dt})$ ($\sigma > 0$)

σ : stock-induced-price-adjustment coefficient.

Note: ($Q_{st} - Q_{dt}$) is considered here as the inventory (stock), whereas ($Q_{dt} - Q_{st}$) is considered as excess demand.

TOPIC 180: A MARKET MODEL WITH INVENTORY: SOLUTION

Given the demand, supply and price-adjustment functions:

$$\begin{aligned} Q_{dt} &= \alpha - \beta P_t & (\alpha, \beta > 0) \\ Q_{st} &= -\gamma + \delta P_t & (\gamma, \delta > 0) \\ P_{t+1} &= P_t - \sigma(Q_{st} - Q_{dt}) & (\sigma > 0) \end{aligned}$$

Substituting the values of Q_{dt} and Q_{st} in price-adjustment equation:

$$P_{t+1} = P_t - \sigma(Q_{st} - Q_{dt})$$

$$\begin{aligned}
P_{t+1} &= P_t - \sigma(-\gamma + \delta P_t - \alpha + \beta P_t) \\
P_{t+1} &= P_t - \sigma(-\gamma + \delta P_t - \alpha + \beta P_t) \\
P_{t+1} &= P_t + \sigma\gamma - \sigma\delta P_t + \sigma\alpha - \sigma\beta P_t \\
P_{t+1} &= P_t - \sigma\beta P_t - \sigma\delta P_t + \sigma\alpha + \sigma\gamma \\
P_{t+1} - P_t + \sigma\beta P_t + \sigma\delta P_t &= \sigma\alpha + \sigma\gamma \\
P_{t+1} - (1 - \sigma\beta - \sigma\delta)P_t &= \sigma\alpha + \sigma\gamma \\
P_{t+1} - \{1 - \sigma(\beta + \delta)\}P_t &= \sigma(\alpha + \gamma)
\end{aligned}$$

First Order Difference Equation representing a market model with inventory. Elaborating the equation by comparing with the standard form:

$$\begin{array}{ccccccc}
& & y_{t+1} + ay_t = c & & & & \\
P_{t+1} + [-\{1 - \sigma(\beta + \delta)\}]P_t &= & \sigma(\alpha + \gamma) & & & & \\
\text{y}_{t+1} & & a & & \text{y}_t & & c
\end{array}$$

Here; $a = [-\{1 - \sigma(\beta + \delta)\}]$, $c = \sigma(\alpha + \gamma)$, $y_{t+1} = P_{t+1}$, $y_t = P_t$

$$a = \left[- \left\{ 1 - \underbrace{\sigma}_{>0} \left(\underbrace{\tilde{\beta}}_{>0} + \underbrace{\tilde{\delta}}_{>0} \right) \right\} \right] \Rightarrow \left(\frac{\delta}{\beta} \right) \neq -1.$$

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a} \right) \left(\frac{(-a)}{\tilde{b}} \right)^t + \frac{c}{1+a}$$

$$P_t = \left[P_0 - \frac{\sigma(\alpha + \gamma)}{1 + [-\{1 - \sigma(\beta + \delta)\}]} \right] \left(\frac{(-a)}{\tilde{b}} \right)^t + \frac{\sigma(\alpha + \gamma)}{1 + [-\{1 - \sigma(\beta + \delta)\}]}$$

$$P_t = \left[P_0 - \frac{\sigma(\alpha + \gamma)}{1 - 1 + \sigma(\beta + \delta)} \right] [-\{1 - \sigma(\beta + \delta)\}]^t + \frac{\sigma(\alpha + \gamma)}{1 - 1 + \sigma(\beta + \delta)}$$

$$P_t = \left[P_0 - \frac{\sigma(\alpha + \gamma)}{\sigma(\beta + \delta)} \right] \{1 - \sigma(\beta + \delta)\}^t + \frac{\sigma(\alpha + \gamma)}{\sigma(\beta + \delta)}$$

$$P_t = \left[P_0 - \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{P^*} \right] \{1 - \sigma(\beta + \delta)\}^t + \underbrace{\left(\frac{\alpha + \gamma}{\beta + \delta} \right)}_{P^*}$$

$$P_t = (P_0 - P^*)\{1 - \sigma(\beta + \delta)\}^t + P^*$$

stationary
equilibrium (P_p)

$$\begin{array}{c}
P_c \\
P_t = \overbrace{(P_0 - P^*)\{1 - \sigma(\beta + \delta)\}^t}^{P_c} + \underbrace{P^*}_{P_p} \\
\boxed{P_t = \underbrace{(P_0 - P^*)}_A \underbrace{\{1 - \sigma(\beta + \delta)\}^t}_b + \underbrace{P^*}_{P_p}}
\end{array}$$

Interpretation: The definite time-path of market model with inventory is obtained.

TOPIC 181: A MARKET MODEL WITH INVENTORY: TYPES OF TIME PATH

$$P_t = \underbrace{(P_0 - P^*)}_A \underbrace{\{1 - \sigma(\beta + \delta)\}^t}_b + \underbrace{\bar{P}^*}_{P_p}$$

The time-path of price in market with inventory is a First Order Difference Equation. $b = \{1 - \sigma(\beta + \delta)\}$ which can make seven cases(regions) depending on value of b : ($-1 > b > 1$).

Region – I: ($b > 1$) $\Rightarrow$ Divergent & Non-oscillatory

$$\begin{aligned} \{1 - \sigma(\beta + \delta)\} &> 1 \\ \{1 - 1 - \sigma(\beta + \delta)\} &> 1 - 1 \\ \{-\sigma(\beta + \delta)\} &\geq 0 \end{aligned}$$

Dividing throughout by $-(\beta + \delta)$ & hence reversing the inequality.

$$\left\{ -\frac{\sigma(\beta + \delta)}{-(\beta + \delta)} \right\} \leq \frac{0}{-(\beta + \delta)}$$

$$\sigma < 0$$

Already given that
($\sigma > 0$), so this case is not
acceptable as per given
parametric restriction

Therefore, we rule-out Region – I

Region – II: ($b = 1$) $\Rightarrow$ Non-convergent & Non-oscillatory

$$\begin{aligned} \{1 - \sigma(\beta + \delta)\} &= 1 \\ -\sigma(\beta + \delta) &= 0 \\ 0 \\ \sigma &= \frac{0}{-(\beta + \delta)} \end{aligned}$$

$$\sigma = 0$$

Already given that
($\sigma > 0$), so this case is not
acceptable as per given
parametric restriction

Therefore, we rule-out Region – II

Region – III: ($0 < b < 1$) $\Rightarrow$ Convergent & Non-oscillatory

$$\begin{aligned} 0 &< \{1 - \sigma(\beta + \delta)\} < 1 \\ 0 - 1 &< \{1 - 1 - \sigma(\beta + \delta)\} < 1 - 1 \\ -1 &< \{-\sigma(\beta + \delta)\} < 0 \\ -1 &\leq \{-\sigma(\beta + \delta)\} \leq 0 \end{aligned}$$

Dividing throughout by $-(\beta + \delta)$ & hence reversing the inequality.

$$\frac{-1}{-(\beta + \delta)} \geq \left\{ -\frac{\sigma(\beta + \delta)}{-(\beta + \delta)} \right\} \geq \frac{0}{-(\beta + \delta)}$$

$$\frac{1}{\beta + \delta} > \{\sigma\} > 0$$

$$0 < \sigma < \frac{1}{\beta + \delta}$$

Region – IV: ($b = 0$) $\Rightarrow$ Pre-convergent & Non-oscillatory

$$\{1 - \sigma(\beta + \delta)\} = 0$$

$$-\sigma = -\frac{1}{(\beta + \delta)}$$

$$\sigma = \frac{1}{\beta + \delta}$$

Region – V: $(-1 < b < 0) \Rightarrow$ Convergent & Oscillatory

$$-1 < \{1 - \sigma(\beta + \delta)\} < 0$$

$$-1 \boxed{-1} < \{1 \boxed{-1} - \sigma(\beta + \delta)\} < 1 \boxed{-1}$$

$$-2 \leq \{-\sigma(\beta + \delta)\} \leq 0$$

Dividing throughout by $-(\beta + \delta)$ & hence reversing the inequality.

$$-\frac{2}{-(\beta + \delta)} \geq \left\{ -\frac{\sigma(\beta + \delta)}{-(\beta + \delta)} \right\} \geq \frac{0}{-(\beta + \delta)}$$

$$\frac{2}{(\beta + \delta)} > \{\sigma\} > 0$$

$$0 < \sigma < \frac{2}{\beta + \delta}$$

Region – VI: $(b = -1) \Rightarrow$ Non-Convergent & Oscillatory

$$\{1 - \sigma(\beta + \delta)\} = -1$$

$$-\sigma(\beta + \delta) = -1 - 1$$

$$-\sigma(\beta + \delta) = -2$$

Dividing throughout by $-(\beta + \delta)$.

$$-\frac{\sigma(\beta + \delta)}{-(\beta + \delta)} = -\frac{2}{-(\beta + \delta)}$$

$$\sigma = \frac{2}{\beta + \delta}$$

Region – VII: $(b < -1) \Rightarrow$ Divergent & Oscillatory

$$\{1 - \sigma(\beta + \delta)\} < -1$$

$$\{1 \boxed{-1} - \sigma(\beta + \delta)\} < -1 \boxed{-1}$$

$$\{-\sigma(\beta + \delta)\} \leq -2$$

Dividing throughout by $-(\beta + \delta)$ & hence reversing the inequality.

$$\left\{ -\frac{\sigma(\beta + \delta)}{-(\beta + \delta)} \right\} \geq -\frac{2}{-(\beta + \delta)}$$

$$\sigma > \frac{2}{\beta + \delta}$$

Conclusion: For market model with inventory, we experimented with seven cases(regions), out of which last five (5) are feasible. While, first two (2) are ruled out. All cases relate stock-induced-price-adjustment coefficient (σ) with slopes of demand and supply curves (β & δ), respectively.

TOPIC 182: A MARKET MODEL WITH INVENTORY: GRAPHICALLY

In order to graphically depict the all possible cases of market model with inventory, we put them in one table:

			Nature
--	--	--	---------------

	$b = \{1 - \sigma(\beta + \delta)\}$	σ	Pattern
I	$b > 1$	$\sigma < 0$ (violating $\sigma > 0$)	Divergent
			Non-oscillatory
II	$b = 1$	$\sigma = 0$ (violating $\sigma > 0$)	Non-convergent
			Non-oscillatory
III	$0 < b < 1$	$0 < \sigma < \frac{1}{\beta + \delta}$	Convergent
			Non-oscillatory
IV	$b = 0$	$\sigma = \frac{1}{\beta + \delta}$ (Rectangular Hyperbola)	Pre-convergent
			Non-oscillatory
V	$-1 < b < 0$	$\frac{1}{\beta + \delta} < \sigma < \frac{2}{\beta + \delta}$	Convergent
			Oscillatory
VI	$b = -1$	$\sigma = \frac{2}{\beta + \delta}$ (Rectangular Hyperbola)	Non-Convergent
			Oscillatory
VII	$b < -1$	$\sigma > \frac{2}{\beta + \delta}$	Divergent
			Oscillatory

First two of the cases are ruled-out as they violate the parametric restriction ($\sigma > 0$).

We can try to plot cases (III-VII).

Both **Case – IV** and **Case – VI**, are equalities and can be plotted with ease.

Both of them are ‘rectangular hyperbola’ functions.

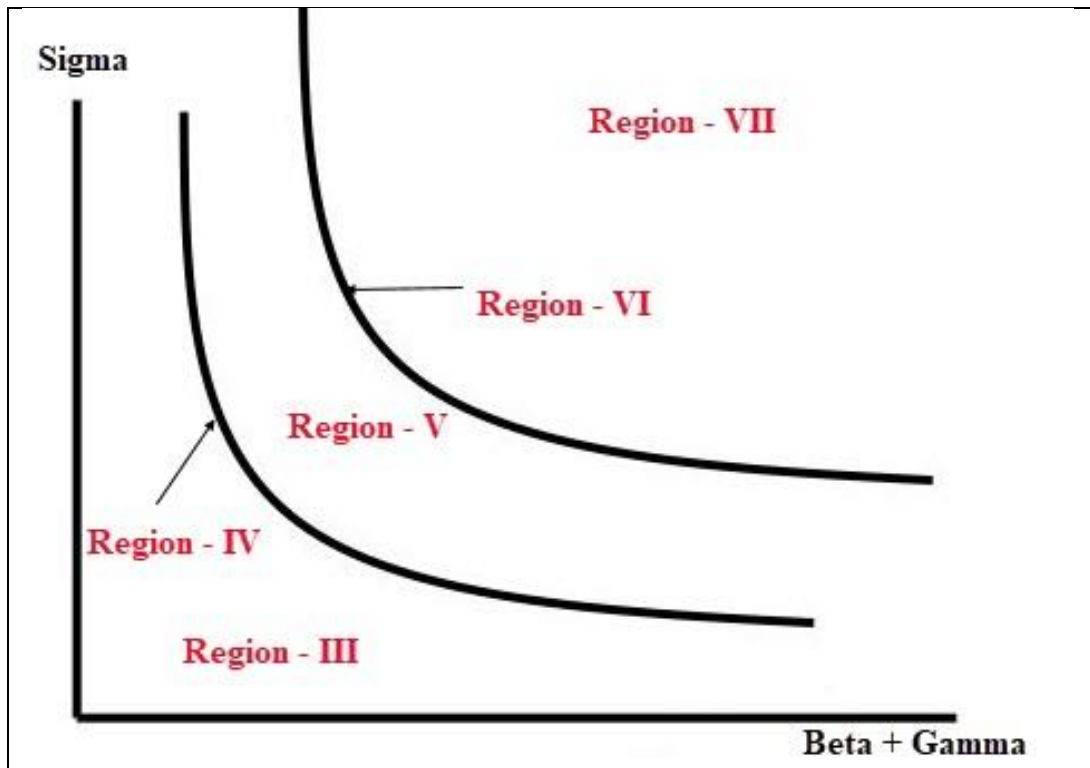
Moving from **Case – III** to **Case – VII**, we observe that σ is increasing.

$0 < \sigma < \frac{1}{\beta + \delta}$	$\sigma = \frac{1}{\beta + \delta}$	$\frac{1}{\beta + \delta} < \sigma < \frac{2}{\beta + \delta}$	$\sigma = \frac{2}{\beta + \delta}$	$\sigma > \frac{2}{\beta + \delta}$
σ				
between 0 & $\frac{1}{\beta + \delta}$	equals $\frac{1}{\beta + \delta}$	between $\frac{1}{\beta + \delta}$ & $\frac{2}{\beta + \delta}$	equals $\frac{2}{\beta + \delta}$	Greater than $\frac{2}{\beta + \delta}$

Accordingly, the regions should be in an ascending order in the graph.

Moreover, 1st quadrant shall be used as both $\sigma > 0$ and $(\beta + \delta) > 0$.

Graph: Cases (Regions): III – VII



Interpretation: Using Cases III – VII, we plotted a graph. It shows all feasible regions of Market Model with Inventory in one diagram.

TOPIC 183: A MARKET MODEL WITH INVENTORY: NUMERICAL

Example # 1: Given the demand, supply and price-adjustment functions:

$$Q_{dt} = 21 - 2P_t$$

$$Q_{st} = -3 + 6P_t$$

$$P_{t+1} = P_t - 0.3(Q_{st} - Q_{dt})$$

Find the time-path P_t and determine its dynamic stability.

Solution: The stock-induced-price-adjustment coefficient (σ) is the critical parameter which is $\sigma = 0.3 > 0$.

as per theory

Substituting the values of Q_{dt} and Q_{st} in price-adjustment equation:

$$P_{t+1} = P_t - 0.3(Q_{st} - Q_{dt})$$

$$P_{t+1} = P_t - 0.3(-3 + 6P_t - (21 - 2P_t))$$

$$P_{t+1} = P_t + 0.9 - 1.8P_t + 6.3 - 0.6P_t$$

$$P_{t+1} = 7.2 - 1.4P_t$$

$$P_{t+1} + 1.4P_t = 7.2$$

First Order Difference Equation representing a market model with inventory. Elaborating the equation by comparing with the standard form:

$$y_{t+1} + ay_t = c$$

$$\underbrace{P_{t+1}}_{y_{t+1}} + \underbrace{1.4}_{a} \underbrace{P_t}_{y_t} = \underbrace{7.2}_c$$

Here; $\underbrace{a = 1.4}_{a \neq 1}$, $c = 7.2$, $y_{t+1} = P_{t+1}$, $y_t = P_t$

Recalling the formula for definite solution of First Order Difference equation with assumption that $a \neq -1$.

$$y_t = \left(y_0 - \frac{c}{1+a} \right) \left(\frac{-a}{\hat{b}} \right)^t + \frac{c}{1+a}$$

$$P_t = \left(P_0 - \frac{7.2}{1+1.4} \right) (-1.4)^t + \frac{7.2}{1+1.4}$$

$$P_t = (P_0 - 3)(-1.4)^t + 3$$

stationary equilibrium (P_p)

$$P_t = \overbrace{(P_0 - 3)}^{P_c} (-1.4)^t + \underbrace{3}_{P_p}$$

$$P_t = \underbrace{(P_0 - 3)}_A \underbrace{(-1.4)^t}_b + \underbrace{3}_{P_p}$$

Interpretation: The definite time-path of market model with inventory is obtained.

Dynamic Stability using b :

Dynamic stability can be checked using the standard procedure used for First Order Difference Equations.

Nature	$\left\{ \begin{array}{l} (>1): \text{non-convergent} \\ \hat{b} \\ (<1): \text{convergent} \end{array} \right\}$	Pattern	$\left\{ \begin{array}{l} (<0): \text{oscillatory} \\ \hat{b} \\ (>0): \text{non-oscillatory} \end{array} \right\}$
	$ b = -1.4 > 1$ Non-Convergent (Divergent)		$b = -1.4 < 0$ Oscillatory

Hence the time-path is divergent in nature and oscillatory in pattern – dynamically unstable.

Dynamic Stability using σ , β and δ : Dynamic stability can also be verified using values of σ , β and δ :

	$b = \{1 - \sigma(\beta + \delta)\}$	σ	Nature Pattern
I	$b > 1$	$\sigma < 0$ (violating $\sigma > 0$)	Divergent Non-oscillatory
II	$b = 1$	$\sigma = 0$ (violating $\sigma > 0$)	Non-convergent Non-oscillatory
III	$0 < b < 1$	$0 < \sigma < \frac{1}{\beta + \delta}$	Convergent Non-oscillatory
IV	$b = 0$	$\sigma = \frac{1}{\beta + \delta}$	Pre-convergent Non-oscillatory
V	$-1 < b < 0$	$\frac{1}{\beta + \delta} < \sigma < \frac{2}{\beta + \delta}$	Convergent Oscillatory
VI	$b = -1$	$\sigma = \frac{2}{\beta + \delta}$	Non-Convergent Oscillatory
VII	$b < -1$	$\sigma > \frac{2}{\beta + \delta}$	Divergent Oscillatory

Choosing Case – VII through $b < -1$, and converting the condition to σ , β , δ .

$\sigma = 0.3, \beta = 2, \delta = 6.$	$\underbrace{\sigma}_{0.3} > \underbrace{\frac{2}{\beta + \delta}}_{0.25}$	Divergent & Oscillatory
Therefore, we get same result of time-path either by using		

b or σ, β, δ.
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Lesson 40

SECOND ORDER DIFFERENCE EQUATIONS

TOPIC 184: SECOND-ORDER DIFFERENCE EQUATIONS: PARTICULAR INTEGRAL

Similar to Second Order Differential Equations, Difference Equations can also acquire second order of difference (recurrence).

Understanding Second Difference:

$\Delta^2 y_t$ shall be called second difference of y_t .

$$\begin{aligned}\Delta^2 y_t &= \Delta(\Delta y_t) \\ \Delta^2 y_t &= \Delta(y_{t+1} - y_t) \\ \Delta^2 y_t &= \Delta y_{t+1} - \Delta y_t \\ &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\ &= y_{t+2} - y_{t+1} - y_{t+1} + y_t \\ \Delta^2 y_t &= y_{t+2} - 2y_{t+1} + y_t\end{aligned}$$

Second difference of y_t has an elaborated expression involving lead/lag of 2, lead/lag of 1 and a term with no lead/lag.

Standard Form of Second Order Difference Equation: A linear, non-homogeneous version with constant coefficients (a_1, a_2) and constant term c .

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

Order: 2nd (due to 2nd difference)

Degree: Linear/First degree (due to highest power '1' of the 2nd difference) Constant Coefficients (a 's) and Constant term (c).

Delimitations:

- For simplicity variable coefficients and variable term are not considered.
- This course deals with linear cases only.
- Cross terms – product of function ' y_t ' and its differences (leads/lags) y_{t+1} – are not included.

Possible Cases: Depending on the values of c , we get 2 possible broad cases:

Main Cases based on value of (c)						
Identification	c = 0			c ≠ 0		
Terminology	Homogenous Case			Non-Homogenous Case		
Components of Solution	y _p = 0			y _p ≠ 0		
Sub-Cases based on values of (a ₁) and/or (a ₂)						
Identification	a ₂ ≠ 0, (a ₁ ≠ 0) ∨ (a ₁ = 0)	a ₂ = 0, a ₁ ≠ 0	a ₂ = 0, a ₁ = 0	a ₂ ≠ 0, (a ₁ ≠ 0) ∨ (a ₁ = 0)	a ₂ = 0, a ₁ ≠ 0	a ₂ = 0, a ₁ = 0
Particular Solution (y _p)	y _p = $\frac{(0)}{a_2}$	y _p = $\frac{(0)}{a_1}t$	y _p = $\frac{(0)}{2}t^2$	y _p = $\frac{b}{a_2}$	y _p = $\frac{b}{a_1}t$	y _p = $\frac{b}{2}t^2$

Particular Integral (y_p): Any particular solution (containing no arbitrary constants) of complete equation. y_p provides equilibrium value of y_t , in intertemporal sense:

y_t <i>Time Path</i>	$=$	y_c <i>Deviation from equilibrium value of $y(t)$</i>	$+$	y_p <i>Equilibrium value of y_t</i>
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Subcase – I: Since the particular integral (y_p) can be any solution, i.e., any value of y that satisfies this nonhomogeneous equation, we should always try the simplest possible type; namely, $y = a \text{ constant}$. If $y = k$, it follows that:

$$y = y_t = k$$

$$y_{t+1} = k$$

$$y_{t+2} = k$$

Substituting the values of y_t , y_{t+1} and y_{t+2} in the general form $y_{t+2} + a_1 \cdot y_{t+1} + a_2 \cdot y_t = c$:

$$k + a_1 \cdot k + a_2 \cdot k = c$$

$$k(1 + a_1 + a_2) = c$$

$$k = \frac{c}{(1 + a_1 + a_2)}$$

Since k is a particular value of y , that satisfies the difference equation, we can call it Particular Solution (y_p):

$$y_p = \frac{c}{1 + a_1 + a_2}$$

$y_p = \frac{c}{1 + \underbrace{a_1 + a_2}_{\neq -1}}$
--

Interpretation: Particular Integral has actually a constant value as c and a_1 , a_2 are constants and so will be their ratio.

Caveat: For a defined Particular Solution (y_p), $a_1 + a_2 \neq -1$.

Subcase – II: If $a_1 + a_2 = -1$, the formula of particular solution ($y_p = \frac{c}{1 + a_1 + a_2}$) shall give undefined answer. i.e. ($y_p = \frac{c}{0} = \infty$). Therefore, we need some other formula for (y_p). For another formula, we need another assumption regarding solution value of y_t . Instead of considering constant a solution y , we try a nonconstant (variable); i.e. $y = kt$.

$$y_t = kt$$

$$y_{t+1} = k(t+1)$$

$$y_{t+2} = k(t+2)$$

Substituting the values of y_t , y_{t+1} , y_{t+2} and in the general form $y''(t) + a_1 \cdot y'(t) + a_2 \cdot y(t) = c$:

$$\begin{aligned} y_{t+2} + a_1 \cdot y_{t+1} + a_2 \cdot y_t &= c \\ k(t+2) + a_1 \cdot k(t+1) + a_2 \cdot kt &= c \\ k\{(t+2) + a_1 \cdot (t+1) + a_2 \cdot t\} &= c \\ k\{(t+2) + (a_1 t + a_1) + a_2 t\} &= c \\ k(t + a_1 t + a_2 t + a_1 + 2) &= c \\ k\left\{\left(1 + \overbrace{a_1 + a_2}^{-1}\right)t + a_1 + 2\right\} &= c \end{aligned}$$

$$\begin{aligned}\underline{k} \left[\{1 + (-1)\}t + a_1 + 2 \right] &= c \\ \underline{k} \left[\{0\}t + a_1 + 2 \right] &= c \\ \underline{k} [a_1 + 2] &= c \\ \underline{k} &= \frac{c}{a_1 + 2}\end{aligned}$$

Since the assumed solution value of y was $y = \underline{k}t$, we substitute the value of k to get the solution value of y :

$$y = \left[\frac{c}{a_1 + 2} \right] t$$

Since solution value is actually the Particular Solution (y_p):

$$y_p = \left(\frac{c}{\underbrace{a_1}_{\neq -2} + 2} \right) t$$

Interpretation: Particular Solution has actually a nonconstant value regardless of c , a_1 and a_2 being constants. It is due to t that gets multiplied with the ratio $\left(\frac{c}{a_1+2}\right)$. In this subcase; $a_1 + a_2 = -1$ is permitted.

Caveat: For a defined Particular Solution (y_p), $a_1 \neq -2$.

Subcase – III: If $a_2 = -2$ and already $a_1 + a_2 = -1$, we check if **Subcase – I** and **Subcase – II** formula of particular solution are applicable:

Subcase – I	Subcase – II
$y_p = \frac{c}{1 + a_1 + a_2}$	$y_p = \frac{c}{a_1 + 2} t$
Substituting values of Subcase – III ($a_1 = -2, a_1 + a_2 = -1$)	
$y_p = \frac{c}{0} = \infty$	$y_p = \frac{c}{0} t = \infty$
Therefore, the particular integral using both Subcase – III give an undefined answer. We need another formula to find the value of y_p .	

Building on the assumptions $y = k$ and $y = kt$, we use $y = kt^2$ in **Subcase – III**.

$$\begin{aligned}y_t &= kt^2 \\ y_{t+1} &= k(t+1)^2 \\ y_{t+2} &= k(t+2)^2\end{aligned}$$

Substituting the values of y_t , y_{t+1} , y_{t+2} , and a_1 in the standard form: $y_{t+2} + \underline{a_1} \cdot y_{t+1} + \underline{a_2} \cdot y_t = c$:

$$\begin{aligned}\frac{k(t+2)^2}{k(t^2 + 4t + 4)} + a_1 \cdot \frac{k(t+1)^2}{k(t^2 + 2t + 1)} + a_2 \cdot \frac{kt^2}{kt^2} &= c \\ k \left\{ \frac{(t^2 + 4t + 4)}{(t^2 + 2t + 1)} + a_1 \cdot \frac{(t^2 + 2t + 1)}{(t^2 + 2t + 1)} + a_2 \cdot \frac{t^2}{t^2} \right\} &= c \\ k(t^2 + a_1 t^2 + a_2 t^2 + 4t + 2a_1 t + 4 + a_1) &= c \\ k \left\{ t^2 \left(1 + \underbrace{a_1 + a_2}_{=-1} \right) + t \left(4 + 2 \underbrace{a_1}_{=-2} \right) + 4 + \underbrace{a_1}_{=-2} \right\} &= c\end{aligned}$$

$$y_p = \left(\frac{12}{1+2} \right) t$$

$$\boxed{y_p = 4t}$$

Interpretation: $y_p = 4t$ is particular solution of the **Non-Homogenous Case – II** of Second Order Differential Equation $y_{t+2} + y_{t+1} - 2y_t = 12$.

Non-Homogenous Case – III: $c \neq 0$, $a_1 + a_2 = -1$, $a_1 = -2$.

Example: Consider a Second Order Differential Equation:

$$y_{t+2} - 2y_{t+1} + y_t = 12$$

Comparing with the given specific equation with the standard form:

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = b$$

$$y_{t+2} + \underbrace{(-2)}_{a_1} y_{t+1} + \underbrace{(1)}_{a_2} y_t = \underbrace{12}_c$$

Subcase-III *Non-Homogenous Case*

Here; $(a_1 = -2)$, $(a_2 = 1)$, $b = 12 \neq 0$.

Suitable Formula for Particular Solution: $y_p = \frac{c}{2} t^2$

$$y_p = \frac{12}{2} \times t^2$$

$$\boxed{y_p = 6t^2}$$

Interpretation: $y_p = 6t^2$ is particular solution of the **Non-Homogenous Case – III** of Second Order Differential Equation $y_{t+2} - 2y_{t+1} + y_t = 12$.

Summary: Non-Homogenous Case		
Subcase – I	Subcase – II	Subcase – III
$y_{t+2} - 3y_{t+1} + 4y_t = \boxed{12}$	$y_{t+2} + y_{t+1} - 2y_t = \boxed{12}$	$y_{t+2} - 2y_{t+1} + y_t = \boxed{12}$
$y_p = \frac{\boxed{c}}{1 + a_1 + a_2}$	$y_p = \frac{\boxed{c}}{a_1 + 2} t$	$y_p = \frac{\boxed{c}}{2} t^2$
In Non-Homogenous case, $b = 12$		
$y_p = \frac{\boxed{12}}{1 + (-3) + 4}$	$y_p = \frac{\boxed{12}}{1 + 2} t$	$y_p = \frac{\boxed{12}}{2} t^2$
$y_p = 6$	$y_p = 4t$	$y_p = 6t^2$

HOMOGENOUS SUBCASES

Since formulae of all subcases of y_p have c in numerator:

Summary: Non-Homogenous Case		
Subcase – I	Subcase – II	Subcase – III
$y_{t+2} - 3y_{t+1} + 4y_t = \boxed{0}$	$y_{t+2} + y_{t+1} - 2y_t = \boxed{0}$	$y_{t+2} - 2y_{t+1} + y_t = \boxed{0}$
$y_p = \frac{\boxed{c}}{1 + a_1 + a_2}$	$y_p = \frac{\boxed{c}}{a_1 + 2} t$	$y_p = \frac{\boxed{c}}{2} t^2$
In Non-Homogenous case, $b = 0$		
$y_p = \frac{\boxed{0}}{1 + (-3) + 4}$	$y_p = \frac{\boxed{0}}{1 + 2} t$	$y_p = \frac{\boxed{0}}{2} t^2$
$y_p = 0$	$y_p = 0$	$y_p = 0$

All subcases have a zero value of particular solution; i.e. $(y_p = 0)$.

TOPIC 186: SECOND-ORDER DIFFERENCE EQUATIONS: y_c DISTINCT REAL ROOTS

Complementary Function (y_c): General solution (containing arbitrary constants) of reduced equation. y_c provides deviation of time path $y(t)$ from equilibrium value, at every point in time.

$y(t)$ <i>Time Path</i>	=	y_c <i>Deviation from equilibrium value of $y(t)$</i>	+	y_p <i>Equilibrium value of $y(t)$</i>
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Adaptation of the Complementary Function (y_c):

For the complementary function (y_c), we can adopt from the second order differential equations. The natural exponential expression is replaced with exponential function for this purpose:

Second Order Differential Equation	Second Order Difference Equation
$y_c = A_1 \underbrace{e^{r_1 t}}_{\text{natural exponential function}} + A_2 \underbrace{e^{r_2 t}}_{\text{natural exponential function}}$	$y_c = A_1 \underbrace{(b_1)^t}_{\text{exponential function}} + A_2 \underbrace{(b_2)^t}_{\text{exponential function}}$

Numerical Example: Find the solution of $y_{t+2} + y_{t+1} - 2y_t = 12$.

Solution: Elaborating the given equation:

$$y_{t+2} + \underbrace{(1)}_{a_1} y_{t+1} + \underbrace{(-2)}_{a_2} y_t = \underbrace{12}_c$$

$$y_p = \left(\frac{c}{a_1 + 2} \right) t$$

$a_1 + a_2 = -1, a_1 = -2$

$$y_p = \left(\frac{12}{1 + 2} \right) t$$

$$\boxed{y_p = 4t}$$

Nature of Roots: $\boxed{(a_1^2 - 4a_2) \geq 0}$

$$\underbrace{9 > 0}_{\substack{\text{Distinct} \\ \text{Real Roots}}}$$

$$\{(1)^2 - 4(-2)\} \geq 0$$

$$\underbrace{9 > 0}_{\substack{\text{Distinct} \\ \text{Real Roots}}}$$

Calculation of Roots:

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$b_1, b_2 = \frac{-(1) \pm \sqrt{(1)^2 - 4(-2)}}{2}$$

$$b_1, b_2 = \frac{-1 \pm \sqrt{9}}{2}$$

$$b_1 = \frac{-1 + 3}{2}, b_2 = \frac{-1 - 3}{2}$$

$$\boxed{b_1 = 1} \text{ or } \boxed{b_2 = -2}$$

These roots can be substituted in the formula of Complementary Function (y_c):

Complementary Function: $y_c = A_1(b_1)^t + A_2(b_2)^t$

$$y_c = A_1(1)^t + A_2(-2)^t$$

$$\boxed{y_c = A_1 + A_2(-2)^t}$$

General Solution: $y(t) = y_c + y_p$

$$\boxed{y(t) = \underbrace{A_1 + A_2(-2)^t}_{y_c} + \underbrace{4t}_{y_p}}$$

Definite Solution: For definite solution, we substitute $t = 0$ in the General Solution; $y(t) = A_1 + A_2(-2)^t + 4t$:

$$\underbrace{y(0)}_{\text{Given}} = A_1 + A_2(-2)^{(0)} + 4(0)$$

$$\underbrace{y(0)}_4 = A_1 + A_2(1) + 0$$

$$4 = A_1 + A_2$$

$$4 = A_1 + A_2$$

As there are two unknowns A_1 and A_2 , we need two (simultaneous) equations to evaluate them. So, develop Second equation putting next value of t i.e. ($t = 1$) in y_t , as y_1 is given.

$$y_t = A_1 + A_2(-2)^t + 4t$$

$$\underbrace{y_1}_{\text{given}} = A_1 + A_2(-2)^{(1)} + 4(1)$$

$$\underbrace{y_t}_5 = A_1 - 2A_2 + 4$$

$$5 = A_1 - 2A_2 + 4$$

$$1 = A_1 - 2A_2$$

Simultaneously solving the two equations with A_1 and A_2 :

$$4 = A_1 + A_2$$

$$\boxed{A_1 = 4 - A_2}$$

$$1 = \boxed{A_1} - 2A_2$$

$$1 = \boxed{4 - A_2} - 2A_2$$

$$1 = 4 - 3A_2$$

$$A_2 = 1$$

For evaluating A_1 , put $A_2 = 1$ in $A_1 = 4 - A_2$

$$A_1 = 4 - A_2$$

$$A_1 = 4 - 1$$

$$A_1 = 3$$

Returning value of A_1 and A_2 in General Solution $y(t) = 3 + 1(-2)^t + 4t$ to get the Definite Solution:

$$y(t) = 3 + (-2)^t + 4t$$

Interpretation: Definite Solution of Second Order Difference Equation $y_{t+2} + y_{t+1} - 2y_t = 12$ with case of Distinct Real Roots is $y(t) = 3 + (-2)^t + 4t$.

TOPIC 187: SECOND-ORDER DIFFERENCE EQUATIONS: y_c REPEATED REAL ROOTS

Adaptation of the Complementary Function (y_c):

For the complementary function (y_c), we can adopt from the second order differential equations. The natural exponential expression is replaced with exponential function for this purpose:

Second Order Differential Equation	Second Order Difference Equation
$y_c = A_3 \underbrace{e^{r_1 t}}_{\text{natural exponential function}} + A_4 \times t \times \underbrace{e^{r_2 t}}_{\text{natural exponential function}}$	$y_c = A_3 \underbrace{(b_1)^t}_{\text{exponential function}} + A_4 \times t \times \underbrace{(b_2)^t}_{\text{exponential function}}$

Numerical Example: Find the solution of $y_{t+2} + 6y_{t+1} + 9y_t = 4$.

Solution: Elaborating the given equation:

$$y_{t+2} + \underbrace{(6)}_{a_1} y_{t+1} + \underbrace{(9)}_{a_2} y_t = \underbrace{4}_c$$

$$y_p = \left(\frac{c}{1 + a_1 + a_2} \right)$$

$$y_p = \left(\frac{4}{1 + 6 + 9} \right)$$

$$y_p = \frac{1}{4}$$

Nature of Roots: $(a_1^2 - 4a_2) \geq 0$

$$\underline{0 = 0}$$

Repeated
Real Roots

$$\{(6)^2 - 4(9)\} \geq 0$$

$$\underline{0 = 0}$$

Repeated
Real Roots

Calculation of Roots:

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$b_1, b_2 = \frac{-(6) \pm \sqrt{(6)^2 - 4(9)}}{2}$$

$$b_1, b_2 = \frac{-6 \pm \sqrt{0}}{2}$$

$$b_1 = \frac{-6}{2}$$

$$\boxed{b_1 = 3} \text{ or } \boxed{b_2 = -3} \Rightarrow \boxed{b = -3}$$

These roots can be substituted in the formula of Complementary Function (y_c):

Complementary Function: $y_c = A_3(b)^t + A_4t(b)^t$

$$y_c = A_3(-3)^t + A_4t(-3)^t$$

$$\boxed{y_c = A_3(-3)^t + A_4t(-3)^t}$$

General Solution: $y(t) = y_c + y_p$

$$y(t) = \underbrace{A_3(-3)^t + A_4t(-3)^t}_{y_c} + \underbrace{\frac{1}{4}}_{y_p}$$

Interpretation: General Solution of Second Order Difference Equation $y_{t+2} + 6y_{t+1} + 9y_t = 4$

with case of Repeated Real Roots is $y(t) = A_3(-3)^t + A_4t(-3)^t + \frac{1}{4}$.

Note: Definite solution can also be found, if initial conditions y_0 and y_1 are given.

TOPIC 188: SECOND-ORDER DIFFERENCE EQUATIONS: y_c COMPLEX ROOTS

For the complementary function (y_c), we can adopt from the second order differential equations. The natural exponential expression is replaced with exponential function for this purpose:

Second Order Differential Equation	Second Order Difference Equation
$y_c = \underbrace{e^{ht}}_{\text{natural exponential function}} (A_5 \cos vt + A_6 \sin vt)$	$y_c = \underbrace{R^t}_{\text{exponential function}} (A_5 \cos \theta t + A_6 \sin \theta t)$

Numerical Example: Find the solution of $y_{t+2} + \frac{1}{4}y_t = 5$.

Solution: Elaborating the given equation:

$$y_{t+2} + \overbrace{\underbrace{(0)}_{a_1}}^{\text{self generated}} y_{t+1} + \underbrace{\left(\frac{1}{4}\right)}_{a_2} y_t = \underbrace{5}_c$$

$$y_p = \left(\frac{c}{1 + a_1 + a_2} \right)$$

$$y_p = \left(\frac{5}{1 + 0 + \frac{1}{4}} \right)$$

$$\boxed{y_p = 4}$$

Nature of Roots: $\boxed{(a_1^2 - 4a_2) \geq 0}$

$$\left\{ (0)^2 - 4 \left(\frac{1}{4} \right) \right\} \geq 0$$

$$\underbrace{-1 < 0}_{\substack{\text{Complex} \\ \text{Roots}}}$$

$$\underbrace{-1 < 0}_{\substack{\text{Complex} \\ \text{Roots}}}$$

Calculation of Roots:

$$b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$b_1, b_2 = \frac{-(0) \pm \sqrt{(0)^2 - 4 \left(\frac{1}{4} \right)}}{2}$$

$$b_1, b_2 = \frac{0 \pm \sqrt{-1}}{2}$$

$$b_1, b_2 = \overset{h}{0} \pm \overset{v}{\left(\frac{1}{2} \right)} i$$

$$\boxed{h = 0} \text{ and } \boxed{v = \frac{1}{2}}$$

Using h and v , we find R and θ .

$R = \sqrt{h^2 + v^2}$ $R = \sqrt{(0)^2 + \left(\frac{1}{2} \right)^2}$ $\boxed{R = \frac{1}{2}}$	$\sin \theta = \frac{v}{R} \Rightarrow \theta = \sin^{-1} \left(\frac{v}{R} \right)$ $\theta = \sin^{-1} \left(\frac{\frac{1}{2}}{\frac{1}{2}} \right)$ $\theta = \sin^{-1}(1)$ $\boxed{\theta = 90^\circ}$
--	--

R and θ can be substituted in the formula of Complementary Function (y_c):

Complementary Function:

$$y_c = R^t (A_5 \cos \theta t + A_6 \sin \theta t)$$

$$y_c = \left(\frac{1}{2} \right)^t \left\{ A_5 \cos \left(\frac{\pi}{2} \right) t + A_6 \sin \left(\frac{\pi}{2} \right) t \right\}$$

General Solution: $y_t = y_c + y_p$

$$y_t = \underbrace{\left(\frac{1}{2} \right)^t \left\{ A_5 \cos \left(\frac{\pi}{2} \right) t + A_6 \sin \left(\frac{\pi}{2} \right) t \right\}}_{y_c} + \underbrace{4}_{y_p}$$

Interpretation: General Solution of Second Order Difference Equation $y_{t+2} + \frac{1}{4}y_t = 5$ with case

of Complex Roots is $y_t = \left(\frac{1}{2} \right)^t \left\{ A_5 \cos \left(\frac{\pi}{2} \right) t + A_6 \sin \left(\frac{\pi}{2} \right) t \right\} + 4$.

Note: Definite solution can also be found, if initial conditions y_0 and y_1 are given.

Lesson 41

DYNAMIC STABILITY: DISTINCT REAL ROOTS

TOPIC 189: DYNAMIC STABILITY: DISTINCT REAL ROOTS

Recalling the under Distinct Real Roots case:

$$y_t = \underbrace{A_1(b_1)^t + A_2(b_2)^t}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{Try to mitigate it}}} + \underbrace{y_p}_{\text{equilibrium}}$$

$$\underbrace{\underbrace{\underbrace{\widehat{A}_1}_{\text{arbitrary constant}} \underbrace{(b_1)}_{\text{independent variable}}}_{\text{deviation from equilibrium}} + \underbrace{\underbrace{\widehat{A}_2}_{\text{arbitrary constant}} \underbrace{(b_2)}_{\text{independent variable}}}_{\text{deviation from equilibrium}}}_{\text{deviation from equilibrium}}$$

A_1 and A_2 are constant and their values depend on initial conditions. t is independent variable spanning $(0 - \infty)$. Deviation from equilibrium will depend on roots (b_1, b_2) .

Hair Splitting:

$$A_1 e^{\widehat{r}_1(t)} + A_2 e^{\widehat{r}_2(t)}$$

b_1 & b_2 , determine the dynamic stability.

If both b_1 & b_2 are $-ve$, then the exponential terms $\{A_1(b_1)^t \& A_2(b_2)^t\}$ will fade away due to exponential decay. Automatically their sum also fades away:

$$\left\{ A_1 \times \underbrace{\left(\widehat{b}_1 \right)^t}_{\text{exponential decay}} + A_2 \times \underbrace{\left(\widehat{b}_2 \right)^t}_{\text{exponential decay}} \right\} \rightarrow 0 \text{ when } t \rightarrow \infty, \therefore y_c \rightarrow 0.$$

Caveat: Both b_1 and b_2 should be negative, else convergence may not take place.

Numerical Example:

	$y(t) = 3(1)^t + (2)^t + 4t$			$y(t) = 3(0.5)^t + (-2)^t + 4t$			$y(t) = 3(-0.8)^t + (0.1)^t + 4t$		
	$3 \underbrace{(1)^t}_{ b_1 < 1} + \underbrace{(2)^t}_{ b_2 < 1}$	y_c		$3 \underbrace{(0.5)^t}_{ b_1 < 1} + \underbrace{(-2)^t}_{ b_2 < 1}$	y_c		$3 \underbrace{(-0.8)^t}_{ b_1 < 1} + \underbrace{(0.1)^t}_{ b_2 < 1}$	y_c	
	$A_1(b_1)^t$	$A_2(b_2)^t$		$A_1(b_1)^t$	$A_2(b_2)^t$		$A_1(b_1)^t$	$A_2(b_2)^t$	
t	$3 \underbrace{(1)^t}_{+ve}$	$\underbrace{(2)^t}_{+ve}$		$3 \underbrace{(0.5)^t}_{+ve}$	$\underbrace{(-2)^t}_{-ve}$		$3 \underbrace{(-0.8)^t}_{-ve}$	$\underbrace{(0.1)^t}_{+ve}$	
0	3	1	4	3	1	4	3	1	4
1	3	2	5	1.5	-2	-0.5	-2.4	0.1	-2.3

2	3	4	7	0.75	4	4.8	1.9	0.01	1.9
3	3	8	11	0.375	-8	-7.6	-1.5	0.001	-1.5
4	3	16	19	0.188	16	16.2	1.2	0.0001	1.2
5	3	32	35	0.094	-32	-31.9	-0.98	0.00001	-0.98
6	3	64	67	0.046	64	64.1	0.8	0.000001	0.79
↑	$\overline{A_1(b_1)^t}$	$A_2(b_2)^t$	(y_c)	$A_1(b_1)^t$	$ A_2(b_2)^t $	$ y_c $	$ A_1(b_1)^t $	$A_2(b_2)^t$	$ y_c $
	1 st exponential term remains constant, 2 nd one increases		Deviation increases	1 st exponential term increases, 2 nd one decreases		Deviation increases	Both exponential terms decrease		Deviation decreases
	Divergence			Divergence			Convergence		
	Dynamically Unstable			Dynamically Unstable			Dynamically Stable		
Hence it is numerically confirmed that for Dynamic Stability, Absolute value of BOTH the roots must be: $(b_1 \wedge b_2) < 1$.									

TOPIC 190: DYNAMIC STABILITY: REPEATED REAL ROOTS

Recalling the under Repeated Real Roots case:

$$y_t = \underbrace{A_3(b)^t + A_4 t(b)^t}_{y_c} + y_p$$

$$y_t = \underbrace{A_3(b)^t + A_4 t(b)^t}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{Try to mitigate it}}} + \underbrace{y_p}_{\text{equilibrium}}$$

$$\underbrace{A_3(b)^t + A_4 t(b)^t}_{\substack{\text{deviation from} \\ \text{equilibrium}}}$$

$$\underbrace{\underbrace{\widehat{A}_3}_{\substack{\text{arbitrary} \\ \text{constant}}} (b)^t}_{\substack{\text{independent} \\ \text{variable}}} + \underbrace{\underbrace{\widehat{A}_4}_{\substack{\text{arbitrary} \\ \text{constant}}} \underbrace{t}_{\substack{\text{independent} \\ \text{variable}}} (b)^t}_{\substack{\text{independent} \\ \text{variable}}}$$

deviation from equilibrium

A_3 and A_4 are constant and their values depend on initial conditions. t is independent variable spanning $(0 - \infty)$. Deviation from equilibrium will depend on root (b) .

Hair Splitting:

$$A_3 \underbrace{(b)^t}_{-ve} + A_4 t \underbrace{(b)^t}_{-ve}$$

b , determine the dynamic stability.

If b is $-ve$, then the exponential terms $\{A_1(b)^t \& A_2 t(b)^t\}$ will fade away due to exponential decay. Though, the second term $(A_2 t(b)^t)$ has an additional term t being multiplied, yet it doesn't reverse the dynamics of this exponential term. Eventually, sum of the two exponential terms also fades away:

$$\left\{ A_3 \times \underbrace{(b)^t}_{\substack{\text{exponential} \\ \text{decay}}} + A_4 \times \underbrace{t}_{\substack{\text{Neutral} \\ \text{multiplier}}} \underbrace{(b)^t}_{\substack{\text{exponential} \\ \text{decay}}} \right\} \rightarrow 0 \text{ when } t \rightarrow \infty, \therefore y_c \rightarrow 0.$$

Numerical Example:

Numerical Example						
	$y_t = 2(3)^t + t(3)^t + \frac{1}{4}$			$y_t = 2(0.3)^t + t(0.3)^t + \frac{1}{4}$		
	$2(3)^t + t(3)^t$		y_c	$2(0.3)^t + t(0.3)^t$		y_c
	$A_3 b^t$	$A_4 t b^t$		$A_3 b^t$	$A_4 t b^t$	
t	$2 \underbrace{(3)^t}_{+ve}$	$t \underbrace{(3)^t}_{+ve}$		$2 \underbrace{(0.3)^t}_{+ve}$	$t \underbrace{(0.3)^t}_{+ve}$	
0	2	0	2	2	0	2
1	6	3	9	0.6	0.3	0.9
2	18	18	36	0.18	0.18	0.36
3	54	81	135	0.054	0.081	0.135
4	162	324	486	0.016	0.032	0.049
5	486	1215	1701	0.005	0.012	0.017
↑↓	$(A_3 b^t)↑$	$(A_4 t b^t)↑$	$(y_c)↑$	$(A_3 b^t)↓$	$(A_4 t b^t)↓$	$(y_c)↓$
	Both exponential terms increase		Deviation increases	Both exponential terms decrease		Deviation decreases
	Divergence			Convergence		
	Dynamically Unstable			Dynamically Stable		
Hence it is numerically confirmed that for Dynamic Stability, the absolute value of repeated root (b) must be less than 1: $ b < 1$						

TOPIC 191: DYNAMIC STABILITY: COMPLEX ROOTS

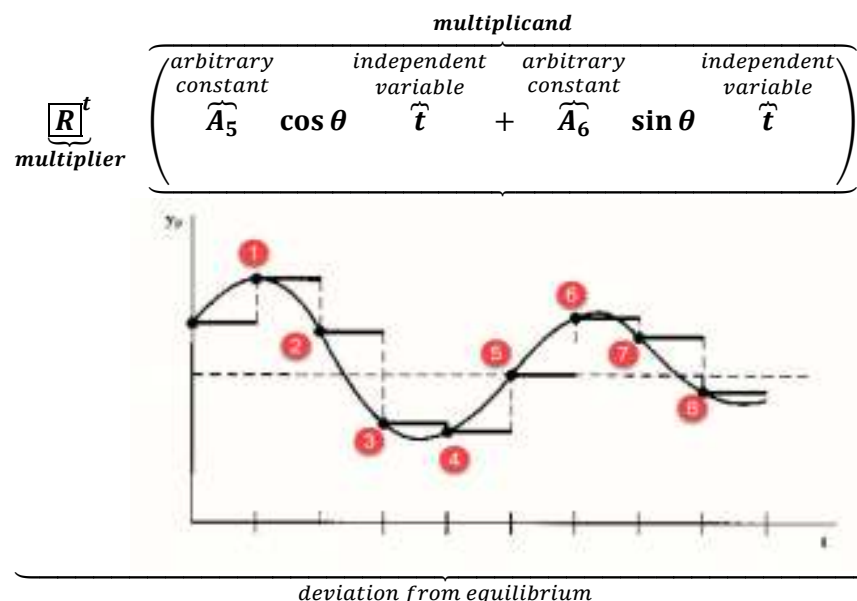
Recalling the under Complex Roots case:

$$y_t = \underbrace{R^t(A_5 \cos \theta t + A_6 \sin \theta t)}_{y_c} + y_p$$

Hair Splitting:

$$y_t = \underbrace{R^t(A_5 \cos \theta t + A_6 \sin \theta t)}_{\substack{\text{deviation from} \\ \text{equilibrium} \\ \text{Try to mitigate it}}} + \underbrace{y_p}_{\text{equilibrium}}$$

$$\underbrace{R^t(A_5 \cos \theta t + A_6 \sin \theta t)}_{\substack{\text{deviation from} \\ \text{equilibrium}}}$$



Stepped Fluctuation: The multiplicand ($A_5 \cos \theta t + A_6 \sin \theta t$) has the knack of having a wave-like graph. But due to discrete nature of t , the **smoothness** of the graph is '**stepped**' as shown above. This shall have uniform amplitude above and below the equilibrium. So, convergence shall not take place. However, multiplier R^t can mitigate the amplitude of the graph if it's value declines with time (t): i.e. $\underbrace{R^t}_{R < 1} \rightarrow 0$ as $t \rightarrow \infty$.

Numerical Example:

Numerical Example									
	$(2)^t\{\cos(1)t + 3 \sin(1)t\} + 4$			y_t $(1)^t\{\cos(1)t + 3 \sin(1)t\} + 4$			$\left(\frac{1}{2}\right)^t \{\cos(1)t + 3 \sin(1)t\} + 4$		
	$(2)^t\{\cos(1)t + 3 \sin(1)t\}$		y_c	$(1)^t\{\cos(1)t + 3 \sin(1)t\}$		y_c	$\left(\frac{1}{2}\right)^t \{\cos(1)t + 3 \sin(1)t\}$		y_c
	R^t	$(A_5 \cos \theta t + A_6 \sin \theta t)$		R^t	$(A_5 \cos \theta t + A_6 \sin \theta t)$		R^t	$(A_5 \cos \theta t + A_6 \sin \theta t)$	
t	$\underbrace{(2)^t}_{\neq 1}$	$\{\cos(1)t + 3 \sin(1)t\}$		$\underbrace{(1)^t}_{\neq 1}$	$\{\cos(1)t + 3 \sin(1)t\}$		$\underbrace{\left(\frac{1}{2}\right)^t}_{< 1}$	$\{\cos(1)t + 3 \sin(1)t\}$	
0	1	0	0	1	0	0	1	0	0
1	2	3.06	6.13	1	3.06	3.06	0.5	3.06	1.53
2	4	2.31	9.25	1	2.31	2.31	0.25	2.31	0.58
3	8	-0.57	-4.53	1	-0.57	-0.57	0.13	-0.57	-0.07
4	16	-2.92	-46.78	1	-2.92	-2.92	0.06	-2.92	-0.18
5	32	-2.59	-82.98	1	-2.59	-2.59	0.03	-2.59	-0.08
6	64	0.12	7.80	1	0.12	0.12	0.02	0.12	0.002
7	128	2.72	348.78	1	2.72	2.72	0.01	2.72	0.021
8	256	2.82	722.58	1	2.82	2.82	0.003	2.82	0.011
9	512	0.33	166.52	1	0.33	0.33	0.002	0.33	0.001
10	1024	-2.47	-2530.44	1	-2.47	-2.47	0.001	-2.47	-0.002
$\downarrow \uparrow$	$(R^t) \uparrow$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \downarrow \uparrow$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \uparrow$	$(\overline{R^t})$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \downarrow \uparrow$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \downarrow \uparrow$	$(R^t) \downarrow$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \downarrow \uparrow$	$\begin{pmatrix} +ve \\ -ve \end{pmatrix} \downarrow$
	Increasing	Fluctuation	Explosive	constant	Fluctuation	Uniform	Decreasing	Fluctuation	Damped
	Explosive Fluctuation & Divergent			Uniform Fluctuation & Neither Divergent nor Convergent			Damped Fluctuation & Convergent		
	Dynamically Unstable			Neither			Dynamically Stable		
Hence it is numerically confirmed that for Dynamic Stability, in Complex root happens if; $(R < 1)$.									

Lesson 42

SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL

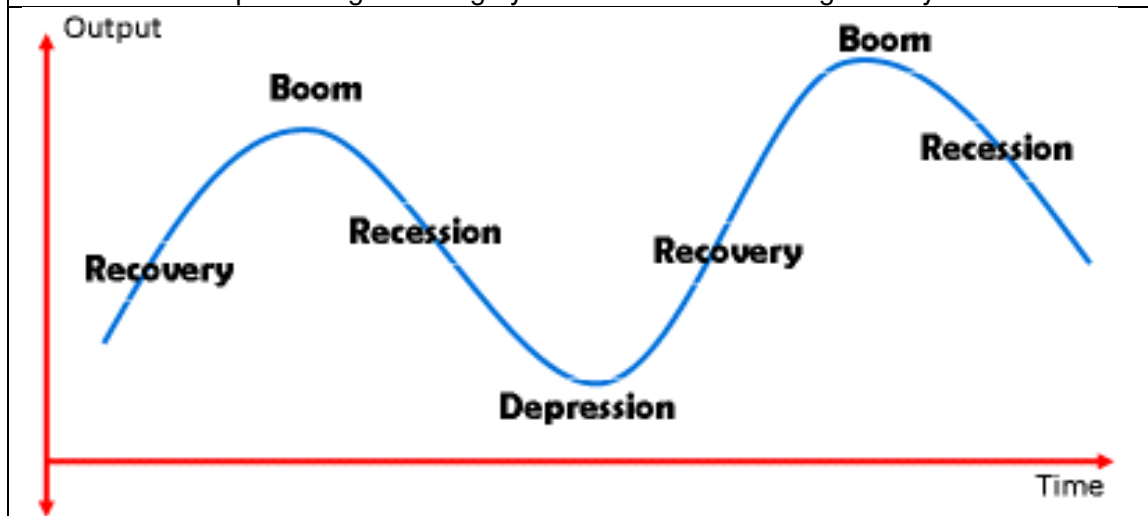
TOPIC 192: SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL: FRAMEWORK

- Attributed to Professor Paul A. Samuelson (the first Economist to win the Nobel Prize).



- An application of Second Order Difference Equations in Economics.
- It seeks to explore the 'dynamic' process of income determination when the acceleration principle is in operation along with the Keynesian multiplier.

Among other things, that model shows that mere interaction of the multiplier and the accelerator is capable of generating cyclical fluctuations endogenously.



$$\overbrace{(\uparrow \Delta I_o) \xrightarrow[\text{MPC}]{\text{Multiplier}} (\uparrow \Delta Y) \xrightarrow[\text{(Investors' expectations regarding consumption)}]{\text{Accelerator}} (\uparrow \Delta I_i) \xrightarrow[\text{MPC}]{\text{Multiplier}} (\uparrow \Delta Y) \dots}^{\text{Recovery or Boom}} \Rightarrow$$

- And Vice Versa i.e. Recession or depression.

$$\overbrace{(\downarrow \Delta I_o) \xrightarrow[\text{MPC}]{\text{Multiplier}} (\downarrow \Delta Y) \xrightarrow[\text{(Investors' expectations regarding consumption)}]{\text{Accelerator}} (\downarrow \Delta I_i) \xrightarrow[\text{MPC}]{\text{Multiplier}} (\downarrow \Delta Y) \dots}^{\text{Recession or Depression}} \Rightarrow$$

National Income (Y_t) has three component expenditure streams: consumption (C_t), investment (I_t), and government expenditure (G_t).

Consumption (C_t):

Consumption is envisaged as a function not of current income (Y_t) but of the income of the prior period (Y_{t-1}).

For simplicity, assume that (C_t) is strictly proportional to (Y_{t-1}):

$$C_t = \underbrace{\gamma Y_{t-1}}_{\text{no intercept}} \text{ where } (0 < \gamma < 1) \text{ \& is actually MPC.}$$

Investment (I_t):

(Induced) investment (I_t), is a function of the prevailing trend of consumer spending (C_t):
 $(\uparrow \Delta I_t) \xrightarrow[\text{MPC}]{\text{Multiplier}} (\uparrow \Delta Y)$.

This induced investment triggers the acceleration mechanism in the model.

Role of consumption:

Investment (I_t): It is assumed that investment is induced on the basis on consumption patterns.

$$I_t = \alpha \underbrace{(\Delta C_{t-1})}_{\substack{\text{Consumption} \\ \text{increment}}} \\ I_t = \alpha \underbrace{(\Delta C_{t-1})}_{C_t - C_{t-1}} \\ I_t = \alpha(C_t - C_{t-1})$$

$\alpha > 0$, as the relationship between investment and consumption pattern is positive.

Economic Significance of α : It is actually the 'Acceleration Coefficient'.

Moreover, it is considered a Fixed Ratio

$$I_t = \alpha(C_t - C_{t-1}) \\ \frac{I_t}{C_t - C_{t-1}} = \alpha \\ \alpha = \frac{I_t}{C_t - C_{t-1}} \\ \alpha = \frac{I_t}{\Delta C_t} (\text{fixed})$$

Government Expenditure (G_t):

Third component is Government expenditure (G_t), which is considered exogenous (G_0).

Therefore, the framework of Samuelson Multiplier-Acceleration Interaction Model is composed of following system of equations:

$$C_t = \gamma Y_{t-1} \\ I_t = \alpha(C_t - C_{t-1}) \\ G_t = G_0$$

TOPIC 193: SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL: SOLUTION

Borrowing the framework of Samuelson Multiplier-Acceleration Interaction Model which is composed of following system of equations:

$$C_t = \gamma Y_{t-1} \\ I_t = \alpha(C_t - C_{t-1})$$

$$G_t = G_0$$

Combining all three components in Expenditure approach Equation:¹

$$Y_t = C_t + \bar{I}_t + G_t$$

$$Y_t = \gamma Y_{t-1} + \alpha(C_t - \overline{C_{t-1}}) + G_0$$

Introducing a lag in $C_t = \gamma Y_{t-1}$, we get, $\overline{C_{t-1}} = \gamma Y_{t-1[-1]}$

$$\overline{C_{t-1}} = \gamma Y_{t-2}$$

Y_t equation becomes:

$$Y_t = \gamma Y_{t-1} + \alpha(\gamma Y_{t-1} - \overline{\gamma Y_{t-2}}) + G_0$$

$$Y_t = \gamma Y_{t-1} + \alpha\gamma Y_{t-1} - \alpha\gamma Y_{t-2} + G_0$$

$$Y_t = \gamma(1 + \alpha)Y_{t-1} - \alpha\gamma Y_{t-2} + G_0$$

$$Y_t - \gamma(1 + \alpha)Y_{t-1} + \alpha\gamma Y_{t-2} = G_0$$

Introducing a lead of 2 on both sides:

$$Y_{t+2} - \gamma(1 + \alpha)Y_{t+1} + \alpha\gamma Y_t = G_0$$

$$Y_{t+2} - \gamma(1 + \alpha)Y_{t+1} + \alpha\gamma Y_t = G_0$$

comparable with second order difference equation ($y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$):

$$\underbrace{Y_{t+2}}_{y_{t+2}} + \underbrace{\{-\gamma(1 + \alpha)\}}_{a_1} \underbrace{Y_{t+1}}_{y_{t+1}} + \underbrace{\alpha\gamma}_{a_2} \underbrace{Y_t}_{y_t} = \underbrace{G_0}_c$$

Here; $y = Y$, $a_1 = -\gamma(1 + \alpha)$, $a_2 = \alpha\gamma$ & $c = G_0$

Particular Integral (Y_p): y_p depends on a_1 & a_2 .

$$a_1 + a_2 = -\gamma(1 + \alpha) + \alpha\gamma$$

$$a_1 + a_2 = -\gamma - \alpha\gamma + \alpha\gamma$$

$$a_1 + a_2 = -\gamma \neq -1$$

Recalling the formula of y_p with assumption $a_1 + a_2 \neq -1$:

$$y_p = \frac{c}{1 + a_1 + a_2}$$

$$Y_p = \frac{G_0}{1 - \gamma(1 + \alpha) + \alpha\gamma}$$

$$Y_p = \frac{G_0}{1 - \gamma - \alpha\gamma + \alpha\gamma}$$

$$\underbrace{Y_p}_{Y^*} = \frac{G_0}{1 - \gamma}$$

Hence, we have obtained the intertemporal equilibrium of national income (Y).

Complementary Solution (Y_c): y_c depends on $a_1^2 \gtrless 4a_2$.

Case – I: Distinct Real Roots ($a_1^2 > 4a_2$)

$$\{-\gamma(1 + \alpha)\}^2 > 4(\alpha\gamma)$$

$$\gamma^2(1 + \alpha)^2 > 4\alpha\gamma$$

$$\gamma(1 + \alpha)^2 > 4\alpha$$

$$\gamma > \frac{4\alpha}{(1 + \alpha)^2}$$

Case – II: Repeated Real Roots ($a_1^2 = 4a_2$)

$$\{-\gamma(1 + \alpha)\}^2 = 4(\alpha\gamma)$$

¹ Letting $I_t = 0$, shall lead to p-547, Example no. 2 situation.

$$\begin{aligned}\gamma^2(1+\alpha)^2 &= 4\alpha\gamma \\ \gamma(1+\alpha)^2 &= 4\alpha \\ \gamma &= \frac{4\alpha}{(1+\alpha)^2}\end{aligned}$$

Case – III: Complex Roots ($a_1^2 < 4a_2$)

$$\begin{aligned}\{-\gamma(1+\alpha)\}^2 &< 4(\alpha\gamma) \\ \gamma^2(1+\alpha)^2 &< 4\alpha\gamma \\ \gamma(1+\alpha)^2 &< 4\alpha \\ \gamma &< \frac{4\alpha}{(1+\alpha)^2}\end{aligned}$$

Interpretation: In terms of α and γ , we have conditions of all three cases of complementary functions (Y_c). Remaining calculations can be done using their values in a Second Order Difference Equation.

TOPIC 194: SAMUELSON MULTIPLIER-ACCELERATION INTERACTION MODEL: NUMERICALS

Under Samuelson Multiplier-Acceleration Interaction Model, the characteristic equation is similar to standard form of SODE.

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c$$

Reduced form:

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Trial Solutions: ($y_t = Ab^t$) $\Rightarrow$ ($y_{t+1} = Ab^{t+1}$) & ($y_{t+2} = Ab^{t+2}$).

$$\begin{aligned}Ab^{t+2} + a_1 Ab^{t+1} + a_2 Ab^t &= 0 \\ Ab^t(b^2 + a_1 b + a_2) &= 0\end{aligned}$$

$$\underbrace{b^2 + a_1 b + a_2 = 0}_{\substack{\text{characteristic equation} \\ \text{for SODE}}} \Rightarrow b_1, b_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

Recalling a_1 and a_2 from Samuelson Multiplier-Acceleration Interaction Model:

$$a_1 = -\gamma(1+\alpha), \quad a_2 = \alpha\gamma.$$

$$b_1, b_2 = \frac{-\{-\gamma(1+\alpha)\} \pm \sqrt{\{-\gamma(1+\alpha)\}^2 - 4(\alpha\gamma)}}{2}$$

$$b_1, b_2 = \frac{\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

Multiplier-Accelerator Model-specific characteristic roots

Numerical # 1: For Samuelson Multiplier-Acceleration Interaction Model, if $\alpha = 3.5$ and $\gamma = 0.8$, then find characteristic roots and judge the dynamic stability of the time-path.

$$b_1, b_2 = \frac{\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}}{2}$$

Multiplier-Accelerator Model-specific characteristic roots

$$\begin{aligned}b_1, b_2 &= \frac{(0.8)(1+3.5) \pm \sqrt{(0.8)^2(1+3.5)^2 - 4(3.5)(0.8)}}{2} \\ b_1, b_2 &= \frac{3.6 \pm \sqrt{1.76}}{2}\end{aligned}$$

$$b_1, b_2 = \frac{3.6 \pm 1.3266}{2}$$

$$b_1, b_2 = \frac{3.6 + 1.3266}{2} \text{ and } \frac{3.6 - 1.3266}{2}$$

$$\boxed{b_1 = 2.46} \text{ and } \boxed{b_2 = 1.14}$$

Interpretation: Both $|b_1| > 1$ and $|b_2| > 1$, therefore, the Interaction time-path shall be divergent & hence dynamically unstable.

Numerical # 2: For Samuelson Multiplier-Acceleration Interaction Model, if $\alpha = 0.2$ and $\gamma = 0.9$, then find characteristic roots and judge the dynamic stability of the time-path.

$$b_1, b_2 = \frac{\gamma(1 + \alpha) \pm \sqrt{\gamma^2(1 + \alpha)^2 - 4\alpha\gamma}}{2}$$

Multiplier-Accelerator Model-specific characteristic roots

$$b_1, b_2 = \frac{(0.9)(1 + 0.2) \pm \sqrt{(0.9)^2(1 + 0.2)^2 - 4(0.2)(0.9)}}{2}$$

$$b_1, b_2 = \frac{1.08 \pm \sqrt{0.4464}}{2}$$

$$b_1, b_2 = \frac{1.08 \pm 0.6681}{2}$$

$$b_1, b_2 = \frac{1.08 + 0.6681}{2} \text{ and } \frac{1.08 - 0.6681}{2}$$

$$\boxed{b_1 = 0.87} \text{ and } \boxed{b_2 = 0.21}$$

Interpretation: Both $|b_1| < 1$ and $|b_2| < 1$, therefore, the Interaction time-path shall be convergent & hence dynamically stable.

Lesson 43

SECOND ORDER DIFFERENCE EQUATION WITH VARIABLE TERM

TOPIC 195: SECOND-ORDER DIFFERENCE EQUATION WITH VARIABLE TERM

Second Order Difference Equation with constant coefficient and constant term:

$$y_{t+2} + \underbrace{a_1}_{\text{constant coefficient}} y_{t+1} + \underbrace{a_2}_{\text{constant coefficient}} y_t = \underbrace{c}_{\text{constant term}}$$

What if: instead of c , there is a variable term $\{f(t)\}$ in terms of t .

constant-coefficient variable-term difference equation			
$y_{t+2} +$	$\underbrace{a_1}_{\text{constant coefficient}}$	$y_{t+1} +$	$\underbrace{a_2}_{\text{constant coefficient}} y_t = \underbrace{f(t)}_{\substack{\text{cm}^t: c \wedge m \neq 0 \\ (y_p) \text{ shall change}}}$
no change on L.H.S $\Rightarrow (y_c)$ remains unchanged, as it always depends on $b^2 + a_1 b + a_2 = 0$.			

Solution: Such situation calls for a certain method, known as **Method of Undetermined Coefficients**.

Pre-condition: Method of Undermined Coefficients requires that the variable term and its successive differences together contain only distinct types of expressions:

Function & its Differences	Distinct Type of Expressions
y_t	7^t
Δy_t	$\Delta 7^t = 7^{t+1} - 7^t$ $= (7 - 1)7^t$ $\Delta 7^t = 6(7^t)$
$\Delta^2 y_t$	$\Delta^2 (7^t) = \Delta(\Delta 7^t)$ $\Delta^2 (7^t) = \Delta\{6(7^t)\}$ $\Delta^2 (7^t) = 6\{\Delta(7^t)\}$ $\Delta^2 (7^t) = 6(7^{t+1} - 7^t)$ $\Delta^2 (7^t) = 6(7 - 1)7^t$ $\Delta^2 (7^t) = 36(7^t)$

Numerical Example: Find the Particular Integral of:

$$y_{t+2} + \underbrace{(1)}_{a_1} y_{t+1} + \underbrace{(-3)}_{a_2} y_t = \underbrace{7^t}_{\substack{\text{exponential term} \\ f(t)}}$$

Here, $a_1 = 1$, $a_2 = -3$ and $f(t) = 7^t$.

Here, particular integral (y_p) is calculated differently due to change in L.H.S.

$$\overbrace{y_{t+2} + y_{t+1} - 3y_t}^{\text{L.H.S}} \stackrel{\text{must be equal}}{=} \overbrace{7^t}^{\text{R.H.S}}$$

$y_{t+2}, y_{t+1} \& y_t$ in terms of (t) exponential term $f(t)$

Both sides are in terms of (t) , however, the polynomial on L.H.S is not explicit i.e. functions; $y_{t+2}, y_{t+1} \& y_t$.

Equality (=) requires that if **R.H.S** is exponential function in terms of (t), then **L.H.S** should also be exponential in terms of (t). So, we assume a (trial) general exponential function that could satisfy y_t, y_{t+1} & y_{t+2} on **L.H.S** to satisfy equality.

As found in table above:

$y_t = 7^t$
$\Delta y_t = 6(7^t)$
$\Delta^2 y_t = 36(7^t)$
A constant grows with 7^t
Observed pattern: $B(7^t)$
Generally speaking, $y_t = B(7^t)$ can serve as exponential type of solution, which produces distinct type of expressions. Therefore, we use it as a trial (general) solution of the given equation.

$$y_t = B(7^t)$$

Generating y_{t+1} and y_{t+2} for the given equation ($y_{t+2} + y_{t+1} - 3y_t = 7^t$).

$$y_{t+1} = B(7^{t+1})$$

$$y_{t+2} = B(7^{t+2})$$

Substituting, y_t, y_{t+1} and y_{t+2} in the given equation ($y_{t+2} + y_{t+1} - 3y_t = 7^t$).

$$B(7^{t+2}) + B(7^{t+1}) - 3B(7^t) = 7^t$$

$$B(7^{t+2}) + B(7^{t+1}) - 3B(7^t) = 7^t$$

$$B\{(7^2) + (7) - 3\}(7^t) = 7^t$$

$$B\{58\}(7^t) = 7^t$$

$$58B(7^t) = 7^t$$

$$58B7^t = \{1\}7^t$$

comparing the coefficients of 7^t on both sides

$$58B = \{1\}$$

$$58B = 1$$

$$B = \frac{1}{58}$$

The trial (general) solution was: $y_t = B(7^t)$, so substitute value of B :

$$y_t = \left(\frac{1}{58}\right)(7^t)$$

Interpretation: $y_t = \left(\frac{1}{58}\right)(7^t)$ is the particular integral (y_p) of given equation

$y_{t+2} + y_{t+1} - 3y_t = 7^t$ which is found by deploying the Method of Undetermined Coefficients. Whereas, (y_c) can be found using methods same as 'constant-coefficient and constant-term second order difference equation'.

TOPIC 196: SECOND-ORDER DIFFERENCE EQUATION WITH VARIABLE TERM (CT^n)

We solved the Second Order Difference Equation with constant coefficient and variable term (cm^t):

What if: instead of (cm^t), there is a variable term (ct^n).

$$\overbrace{y_{t+2} + \underbrace{a_1}_{\substack{\text{constant} \\ \text{coefficient}}} y_{t+1} + \underbrace{a_2}_{\substack{\text{constant} \\ \text{coefficient}}} y_t}_{\substack{\text{no change on L.H.S} \Rightarrow (y_c) \text{ remains unchanged,} \\ \text{as it always depends on } b^2 + a_1 b + a_2 = 0}} = \underbrace{f(t)}_{\substack{ct^n: c \wedge n \neq 0 \\ (y_p) \text{ shall} \\ \text{change}}}$$

Comparison of (cm^t) and (ct^n) : (c) is a common coefficient in both, however:

(m^t)	(t^n)
Exponential Function	Power Function

Solution: Such situation calls for a certain method, known as **Method of Undetermined Coefficients**.

Pre-condition: Method of Undetermined Coefficients requires that the variable term and its successive differences together contain only distinct types of expressions:

Function & its Differences	Distinct Type of Expressions
y_t	t^2
Δy_t	$\Delta t^2 = (t+1)^2 - t^2$ $\Delta t^2 = \cancel{t^2} + 2t + 1 - \cancel{t^2}$ $\Delta t^2 = 2t + 1$
$\Delta^2 y_t$	$\Delta^2(t^2) = \Delta(\Delta t^2)$ $\Delta^2(t^2) = \Delta(2t + 1)$ $= \{(2(t+1) + 1) - (2t + 1)\}$ $= \{2t + 2 + 1 - 2t - 1\}$ $\Delta^2(t^2) = 2$

Numerical Example: Find the Particular Integral of: $y_{t+2} + 5y_{t+1} + 2y_t = t^2$

$$y_{t+2} + \underbrace{5}_{a_1} y_{t+1} + \underbrace{2}_{a_2} y_t = \underbrace{t^2}_{\substack{\text{variable term} \\ f(t)=ct^n}}$$

Here, $a_1 = 5$, $a_2 = 2$ and $f(t) = t^2$.

Particular integral (y_p) is calculated differently due to change in **L.H.S**.

$$\overbrace{y_{t+2} + 5y_{t+1} + 2y_t}^{\substack{\text{L.H.S} \\ \text{must be equal} \\ \text{R.H.S}}} \cong \overbrace{t^2}^{\substack{\text{quadratic} \\ \text{in terms of } (t)}}$$

Both sides are in terms of (t) , however, the degree of polynomial on **L.H.S** is not explicit.

Equality ($=$) requires that if **R.H.S** is quadratic in terms of (t) , then **L.H.S** should also be quadratic in terms of (t) . So, we assume a (trial) general quadratic function that could satisfy y_t, y_{t+1} & y_{t+2} on **L.H.S** to satisfy equality:

$$y_t = B_1 t^2 + B_2 t + B_3$$

Developing y_{t+1} and y_{t+2} for the given $(y_{t+2} + 5y_{t+1} + 2y_t = t^2)$.

$$y_{t+1} = B_1(t+1)^2 + B_2(t+1) + B_3$$

$$y_{t+1} = B_1(t^2 + 2t + 1) + B_2 t + B_2 + B_3$$

$$y_{t+1} = B_1 t^2 + 2B_1 t + B_2 t + B_1 + B_2 + B_3$$

$$y_{t+1} = B_1 t^2 + (2B_1 + B_2)t + B_1 + B_2 + B_3$$

$$y_{t+2} = B_1(t+2)^2 + B_2(t+2) + B_3$$

$$\begin{aligned}
 y_{t+2} &= B_1(t^2 + 4t + 4) + B_2t + 2B_2 + B_3 \\
 y_{t+2} &= B_1t^2 + 4B_1t + 4B_1 + B_2t + 2B_2 + B_3 \\
 y_{t+2} &= B_1t^2 + 4B_1t + B_2t + 4B_1 + 2B_2 + B_3 \\
 y_{t+2} &= B_1t^2 + (4B_1 + B_2)t + 4B_1 + 2B_2 + B_3
 \end{aligned}$$

Since y_t , y_{t+1} and y_{t+2} are now symbolized, we substitute them on L.H.S:

$$\begin{aligned}
 \overbrace{y_{t+2}}^{L.H.S} + 5\overbrace{y_{t+1}}^{L.H.S} + 2\overbrace{y_t}^{L.H.S} &= \overbrace{t^2}^{R.H.S} \\
 B_1t^2 + (4B_1 + B_2)t + 4B_1 + 2B_2 + B_3 + 5[B_1t^2 + (2B_1 + B_2)t + B_1 + B_2 + B_3] \\
 + 2[B_1t^2 + B_2t + B_3] &= t^2 \\
 B_1t^2 + (4B_1 + B_2)t + 4B_1 + 2B_2 + B_3 + 5B_1t^2 + 5(2B_1 + B_2)t + 5B_1 + 5B_2 + 5B_3 \\
 + 2B_1t^2 + 2B_2t + 2B_3 &= t^2 \\
 B_1t^2 + 5B_1t^2 + 2B_1t^2 + (4B_1 + B_2)t + 5(2B_1 + B_2)t + 2B_2t + 4B_1 + 2B_2 + B_3 \\
 + 5B_1 + 5B_2 + 5B_3 + 2B_3 &= t^2 \\
 B_1 + 5B_1 + 2B_1 t^2 + \{4B_1 + B_2 + 10B_1 + 5B_2 + 2B_2\}t + 4B_1 + 5B_1 + 2B_2 + 5B_2 + B_3 \\
 + 5B_3 + 2B_3 &= t^2 \\
 \overbrace{8B_1t^2 + (14B_1 + 8B_2)t + 9B_1 + 7B_2 + 8B_3}^{L.H.S} &= \overbrace{1t^2 + (0)t + (0)}^{R.H.S} \\
 &\text{self-generated for comparison}
 \end{aligned}$$

Comparing the coefficients of t^2 & t and the constants on both sides of equation:

t^2	t	constants
$8B_1 = 1$	$(14B_1 + 8B_2) = (0)$	$9B_1 + 7B_2 + 8B_3 = (0)$
$B_1 = \frac{1}{8}$	$\frac{7}{4} + 8B_2 = 0$	$9\left[\frac{1}{8}\right] + 7B_2 + 8B_3 = \bar{0}$
	$B_2 = -\frac{7}{32}$	$\frac{9}{8} + 7\left(-\frac{7}{32}\right) + 8B_3 = \bar{0}$
		$B_3 = \frac{13}{256}$

So, the **Undetermined** coefficients are now **determined**.

Substituting B_1 , B_2 and B_3 in $y(t) = B_1t^2 + B_2t + B_3$ assumed above as the (particular) solution.

$$y_p = \left(\frac{1}{8}\right)t^2 + \left(-\frac{7}{32}\right)t + \left(\frac{13}{256}\right)$$

$$y_p = \frac{1}{8}t^2 - \frac{7}{32}t + \frac{13}{256}$$

Interpretation: So, the particular integral of the 'constant-coefficient and variable-term second order difference equation' $y_{t+2} + 5y_{t+1} + 2y_t = t^2$ is

$$y_p = \frac{1}{8}t^2 - \frac{7}{32}t + \frac{13}{256}$$

Method used to find (y_p) is Method of Undetermined Coefficients.

Whereas, (y_c) can be found using methods same as 'constant-coefficient and constant-term second order difference equation'.

TOPIC 197: HIGHER-ORDER LINEAR DIFFERENCE EQUATION

After comprehending the solution of second order difference equations, we extend our analysis to Higher ($n > 2$) Order Difference Equations.

Standard Form of Higher Order Difference Equation: A linear, non-homogeneous version with ' n ' constant coefficients ($a_1, a_2, \dots, a_n$) and constant term c .

$$\begin{aligned} \widetilde{a_0}^{(=1)} y_{t+n} + a_1 y_{t+(n-1)} + a_2 y_{t+(n-2)} + \dots + a_{n-1} y_{t+1} + a_n y_t &= c \\ \text{order of difference increases } \{y_t, y_{t+1} \text{ to } y_{t+n}\} \\ \overbrace{a_0 y_{t+n} + a_1 y_{t+(n-1)} + a_2 y_{t+(n-2)} + \dots + a_{n-1} y_{t+1} + a_n y_t}^{\text{coefficient-subscript increases } (a_0 \text{ to } a_n)} &= c \end{aligned}$$

Order: n^{th} (due to n^{th} difference)

Degree: Linear/First degree (due to highest power '1' of the n^{th} difference) Constant Coefficients (a 's) and Constant term (c).

Delimitations:

1. For simplicity variable coefficients and variable term are not considered.
2. This course deals with linear cases only.
3. Cross terms – product of function ' y_t ' and its differences (leads/lags) $y_{t+1}, \dots, y_{t+n}$ – are not included.

Solution: Solution of Higher Order Difference equations is merely the extension of solution of Second Order Difference equations.

Guidelines:

1. Particular Solution has same buildup of cases. e.g.:
Case – I has same condition; ($a_1 + a_2 + \dots + a_n \neq -1$)
2. n^{th} order implies n number of characteristic roots.
3. A hybrid complementary solution can exist, e.g.: out of n roots, if first 3 are repeated real and last two are complex roots, then

$$y_t = \underbrace{A_1 b_1^t + A_1 t b_1^t + A_3 t^2 b_1^t}_{\substack{\text{1st 3 are repeated real roots} \\ \text{Number of:}}} + \dots + \underbrace{R^t (A_{n-1} \cos \theta t + A_n \sin \theta t)}_{\substack{\text{last 2 are complex roots}}}$$

4. Characteristic Roots (b_n) = arbitrary constants (A_n) = $\underbrace{n \text{ initial conditions}}_{\text{for definite solution}}$

Numerical Example: Find the General Solution of third-order difference equation:

$$\begin{aligned} y_{t+3} - \frac{7}{8} y_{t+2} + \frac{1}{8} y_{t+1} + \frac{1}{32} y_t &= 9 \\ y_{t+3} + \underbrace{\left(-\frac{7}{8}\right)}_{a_1} y_{t+2} + \underbrace{\left(\frac{1}{8}\right)}_{a_2} y_{t+1} + \underbrace{\left(\frac{1}{32}\right)}_{a_3} y_t &= \underbrace{9}_c \end{aligned}$$

Here; $a_1 = -\frac{7}{8}$, $a_2 = \frac{1}{8}$, $a_3 = a_n = \frac{1}{32}$ and $\underbrace{c = 9 \neq 0}_{\text{non-homogenous case}}$.

Particular Integral (y_p): Testing for **Case – I**:

$$\begin{aligned} a_1 + a_2 + a_3 &\neq -1 \\ -\frac{7}{8} + \frac{1}{8} + \frac{1}{32} &= -\frac{23}{32} \neq -1 \end{aligned}$$

Recalling the formula of y_p with assumption $a_1 + a_2 + a_3 \neq -1$:

$$y_p = \frac{c}{1 + a_1 + a_2 + \boxed{a_3}}$$

similar formula; extended to $n=3$

$$y_p = \left[\frac{9}{\left\{ 1 + \left(-\frac{7}{8}\right) + \frac{1}{8} + \frac{1}{32} \right\}} \right]$$

$$\boxed{y_p = y^* = 32}$$

static equilibrium

Complementary Function (y_c): For Second Order Difference equation, the characteristic equation was:

$$b^2 + a_1b + a_2 = 0$$

Accordingly, for Third Order Difference equation, the characteristic equation will be:

$$b^3 + a_1b^2 + a_2b + a_3 = 0$$

To find the roots (b), we need to solve the characteristic equation, therefore, factorizing

$$b^3 - \frac{7}{8}b^2 + \frac{1}{8}b + \frac{1}{32} = 0, \text{ we get:}$$

$$\left(b - \frac{1}{2}\right) \times \left(b - \frac{1}{2}\right) \times \left(b + \frac{1}{8}\right) = 0$$

Either;

$\left(b_1 - \frac{1}{2}\right) = 0$	$\left(b_1 - \frac{1}{2}\right) = 0$	$\left(b_3 + \frac{1}{8}\right) = 0$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$b_1 = \frac{1}{2}$	$b_2 = \frac{1}{2}$	$b_3 = -\frac{1}{8}$

Summarizing the (hybrid) group of roots:

$$\overbrace{b_1 = \frac{1}{2}, b_2 = \frac{1}{2}}^{\text{repeated real roots}}, \underbrace{b_3 = -\frac{1}{8}}_{\text{distinct real root}}$$

Accordingly, the complementary function will also be hybrid:

<i>Hybrid nature of complementary function</i>	
$y_c = \underbrace{A_1(b_1)^t + A_2t(b_2)^t}_{\text{Repeated Real roots segment where } b_2=b_1=b}$	$+ \underbrace{A_3(b_3)^t}_{\text{Distinct real root segment}}$
<i>Hybrid nature of complementary function</i>	
$y_c = \underbrace{A_1\left(\frac{1}{2}\right)^t + A_2t\left(\frac{1}{2}\right)^t}_{\text{Repeated Real roots segment where } b_2=b_1=b}$	$+ \underbrace{A_3\left(-\frac{1}{8}\right)^t}_{\text{Distinct real root segment}}$

General Solution (y_t): Like any other difference equation:

$$y_t = y_c + y_p$$

Substituting the values of y_c , y_p we get:

$$y_t = \underbrace{A_1\left(\frac{1}{2}\right)^t + A_2t\left(\frac{1}{2}\right)^t + A_3\left(-\frac{1}{8}\right)^t}_{y_c} + \underbrace{32}_{y_p}$$

Interpretation: $y_t = A_1 \left(\frac{1}{2}\right)^t + A_2 t \left(\frac{1}{2}\right)^t + A_3 \left(-\frac{1}{8}\right)^t + 32$ is the general solution of Higher (Third)

Order Difference Equation $y_{t+3} - \frac{7}{8}y_{t+2} + \frac{1}{8}y_{t+1} + \frac{1}{32}y_t = 9$. Complementary Function is hybrid with a combination of Repeated Real roots and Distinct Real root.

Note: If $n (= 3)$ number of initial conditions are given then, we can also find the definite solution from general solution which has $n (= 3)$ number arbitrary constants A_n .

Dynamic Stability: Similar to any other Difference Equation, the roots should be less than 1 in terms of their magnitude. Here,

$$\left\{ \left(|b_1| = |b_2| = \left| \frac{1}{2} \right| \right) \wedge \left(|b_3| = \left| -\frac{1}{8} \right| \right) \right\} < 1$$

Convergent time-path of the 3rd Order Difference Equation

TOPIC 198: CONVERGENCE AND THE SCHUR THEOREM

After comprehending the solution of Higher order difference equations, we realize that it can be a cumbersome **quantitative** exercise to find n number of roots.

$$\stackrel{(<1)}{\widetilde{a_0}} y_{t+n} + a_1 y_{t+(n-1)} + a_2 y_{t+(n-2)} + \cdots + a_{n-1} y_{t+1} + a_n y_t = c$$

n^{th} order implies n number of characteristic roots.

An easier **qualitative** method is available – **Schur Theorem**.

Without having to perform quantitative solution of higher order difference equations, one can find the dynamic stability of the time path.

Determinants are developed and evaluated using the coefficients in given difference equations.

Number of determinants = Order of Equation

For second-order difference equation (SODE) to be dynamically stable:

$$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} \leq 0$$

$$\Delta_2 = \begin{vmatrix} a_0 & 0 & a_2 & a_1 \\ a_1 & a_0 & 0 & a_2 \\ a_2 & 0 & a_0 & a_1 \\ a_1 & a_2 & 0 & a_0 \end{vmatrix} \leq 0$$

Interpretation		
$(\Delta_1 \wedge \Delta_2) < 0$	$\left\{ \begin{array}{c} (\Delta_1 < 0) \wedge (\Delta_2 > 0) \\ \vee \\ (\Delta_1 > 0) \wedge (\Delta_2 < 0) \end{array} \right\}$	$(\Delta_1 \wedge \Delta_2) > 0$
Divergent	Neither	Convergent

Numerical Example # 1: Find the Dynamic stability of the higher (second) order difference equation without solving it quantitatively:

$$y_{t+2} + 3y_{t+1} + 2y_t = 12$$

$$\underbrace{1}_{a_0} y_{t+2} + \underbrace{(3)}_{a_1} y_{t+1} + \underbrace{(2)}_{a_2} y_t = \underbrace{12}_c$$

Here; $a_0 = 1$, $a_1 = 3$, $a_2 = 2$, and $\underbrace{c = 12 \neq 0}_{\text{non-homogenous case}}$.

Recalling two Schur Theorem determinants for second-order difference equation:

$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} \leq 0$	However, we evaluate the second determinant (Δ_2) for the sake of rehearsal.
--	---

$\Delta_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$ $\Delta_1 < 0$ First Determinant (Δ_1) is already found to be negative. It violates the requirement Convergence in Schur Theorem which requires all of the determinants to be positive.	$\Delta_2 = \begin{vmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 3 & 2 & 0 & 1 \end{vmatrix} = 0$ $\Delta_2 = 0$ Second determinant (Δ_2) also defies the requirement of Convergence in Schur Theorem.
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Interpretation		
$(\Delta_1 \wedge \Delta_2) < 0$	$\begin{cases} (\Delta_1 < 0) \wedge (\Delta_2 > 0) \\ \vee \\ (\Delta_1 > 0) \wedge (\Delta_2 < 0) \end{cases}$	$(\Delta_1 \wedge \Delta_2) \neq 0$
DIVERGENT	Not applicable	Not applicable

Verification: It can be done by finding the roots via (quantitative) solution and comparing their absolute values with < 1 . The roots of the given Higher Order Difference equation are -1 , -2 , absolute value of both of which are not lesser than 1 . Hence it is verified that Schur Theorem gave us correct interpretation.

Numerical Example # 2: Test the convergence of the path of $y_{t+2} + \frac{1}{6}y_{t+1} - \frac{1}{6}y_t = 2$

$$\underbrace{1}_{a_0} y_{t+2} + \underbrace{\left(\frac{1}{6}\right)}_{a_1} y_{t+1} + \underbrace{\left(-\frac{1}{6}\right)}_{a_2} y_t = \underbrace{2}_c$$

Here; $a_0 = 1$, $a_1 = \frac{1}{6}$, $a_2 = -\frac{1}{6}$, and $c = 2 \neq 0$.
non-homogenous case

Recalling two Schur Theorem determinants for second-order difference equation:

$\Delta_1 = \begin{vmatrix} a_0 & a_2 \\ a_2 & a_0 \end{vmatrix} \leq 0$ $\Delta_1 = \begin{vmatrix} 1 & -\frac{1}{6} \\ -\frac{1}{6} & 1 \end{vmatrix} = \frac{35}{36}$ $\Delta_1 > 0$ First Determinant (Δ_1) is found to be positive. It follows the requirement Convergence in Schur Theorem which requires all of the determinants to be positive.	Therefore, we evaluate the second determinant (Δ_2) in hope of another positive value. $\Delta_2 = \begin{vmatrix} 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 1 & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & 1 & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & 0 & 1 \end{vmatrix}$ $\Delta_2 = \frac{1176}{1296} = \frac{49}{54} > 0$ Second determinant (Δ_2) also follows the requirement of Convergence in Schur Theorem.
--	--

Interpretation		
$(\Delta_1 \wedge \Delta_2) \neq 0$	$\left\{ \begin{array}{c} (\Delta_1 < 0) \wedge (\Delta_2 > 0) \\ \vee \\ (\Delta_1 > 0) \wedge (\Delta_2 < 0) \end{array} \right\}$	$(\Delta_1 \wedge \Delta_2) > 0$
Not applicable	Not applicable	CONVERGENT

Limitations:

1. Dynamic stability is investigated but the characteristic roots remain unknown.
2. For $n > 2$, the determinants become huge – contrary to our expectation of ease.

Lesson 44

LINEAR AND NON-LINEAR PROGRAMMING

TOPIC 199: LINEAR PROGRAMMING: GLASSWARE MAKING

“A problem well stated is a problem half solved.”

Charles Franklin Kettering

Major Contributors were Leonid Kantorovich, George Dantzig, John von Neumann during 1939 to 1947.



In addition to calculus-based optimization, algebra-based & graphical optimization can also be done – Linear Programming. Linear programming addresses real life problems of allocating finite resources such as labor, machine time or raw materials to maximize profits or minimize costs.

Process: Mathematize the given information into Decision Variables, Objective Function, and (multiple) Constraints.

Limitation: It only deals with linear objective and constraint functions.

Generally, done in three steps.

Numerical Example: Glassware making involve processes like; glassblowing, annealing and addition of ingredients like silica sand. Profit of the glassware firm depends on decision regarding production of bowls (x) and plates (y):

x = number of bowls

y = number of plates

STEP – I: Mathematization of Given Information

$$\pi = f(x, y)$$

Profit per bowl = \$150 $\Rightarrow$ Total Profit from bowls = \$150 $\times x$ = 150 x

Profit per plate = \$100 $\Rightarrow$ Total Profit from bowls = \$100 $\times y$ = 100 y

Total profit from both:

Objective Function to be maximized.	
$\pi = 150x + 100y$	
π_{bowls}	π_{plates}

Limited resources put constraints:

1st Constraint (Glassblowing Time): Total time available for glassblowing is 70 hours per week. A bowl takes 2 hours in making while a plate takes 1 hour:

(time consumed for glassblowing of bowls and plates)		(should be less or equal)	
$2x$	+	y	$\leq$
(time needed for bowls)		(time needed for plates)	
			70 (available hours per week)

2nd Constraint (Annealing Time): Total time available for Annealing is 130 hours per week. A bowl takes 4 hours in making while a plate takes 1 hour:

$$\frac{\begin{array}{c} \text{(time consumed for annealing)} \\ \text{of bowls and plates} \end{array}}{\begin{array}{c} \underbrace{4x}_{\text{(time needed)}} + \underbrace{y}_{\text{(time needed)}} \\ \text{for bowls} \quad \text{for plates} \end{array}} \begin{array}{c} \text{(should be)} \\ \text{less or} \\ \text{equal} \end{array} \leq \underbrace{130}_{\begin{array}{c} \text{(available)} \\ \text{hours per} \\ \text{week} \end{array}}$$

3rd Constraint (mass of Silica Sand): Total amount available of Silica Sand is 45 kg:

$$\frac{\begin{array}{c} \text{(mass consumed of Silica)} \\ \text{sand for bowls and plates} \end{array}}{\begin{array}{c} \underbrace{x}_{\text{(mass needed)}} + \underbrace{y}_{\text{(mass needed)}} \\ \text{for bowls} \quad \text{for plates} \end{array}} \begin{array}{c} \text{(should be)} \\ \text{less or} \\ \text{equal} \end{array} \leq \underbrace{45}_{\begin{array}{c} \text{(available)} \\ \text{hours per} \\ \text{week} \end{array}}$$

Since production cannot be negative, we develop 2 additional constraints:

Non-negativity Constraints:

4th Constraint (minimum one bowl):

$$\frac{x}{\begin{array}{c} \text{(production)} \\ \text{of bowls} \end{array}} \geq \underbrace{0}_{\begin{array}{c} \text{(can't be)} \\ \text{negative} \end{array}}$$

5th Constraint (minimum one plate):

$$\frac{y}{\begin{array}{c} \text{(production)} \\ \text{of plates} \end{array}} \geq \underbrace{0}_{\begin{array}{c} \text{(can't be)} \\ \text{negative} \end{array}}$$

Tabulating all the extracted information:

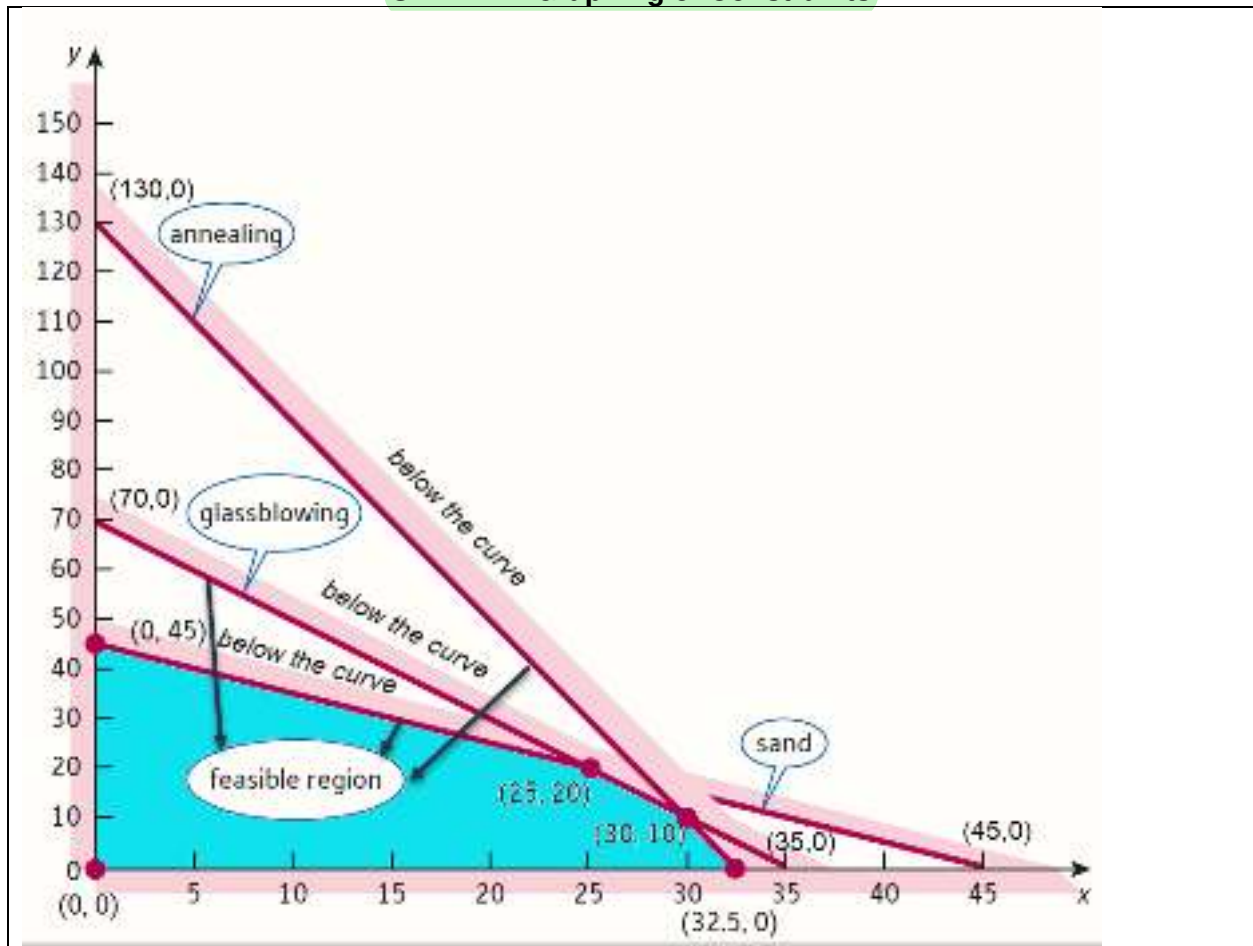
Objective (Profit) Function (to be maximized)	$\pi_{max} = 150x + 100y$
Subject to:	
1 st Constraint (Glassblowing Time)	$2x + y \leq 70$
2 nd Constraint (Annealing Time)	$4x + y \leq 130$
3 rd Constraint (mass of Silica Sand)	$x + y \leq 45$
4 th Constraint (minimum one bowl)	$x \geq 0$
5 th Constraint (minimum one plate)	$y \geq 0$

Plotting the Equalities from Inequalities Constraints: Reduce the inequalities to equalities that are more meaningful to plot: $\leq \Rightarrow \equiv$

We use the intercepts to plot these linear functions:

Constraints	Equalities/ Inequalities	x-intercept	y-intercept
1 st (Glassblowing Time) $\leq \Rightarrow \text{below the curve}$	$2x + y \equiv 70$	$2x + (0) = 70$ $x = 35 \Rightarrow (35, 0)$	$2(0) + y = 70$ $y = 70 \Rightarrow (0, 70)$
2 nd (Annealing Time) $\leq \Rightarrow \text{below the curve}$	$4x + y \equiv 130$	$4x + (0) = 130$ $x = 32.5 \Rightarrow (32.5, 0)$	$4(0) + y = 130$ $y = 130 \Rightarrow (0, 130)$
3 rd (mass of Silica Sand) $\leq \Rightarrow \text{below the curve}$	$x + y \equiv 45$	$x + (0) = 45$ $x = 45 \Rightarrow (45, 0)$	$(0) + y = 45$ $y = 45 \Rightarrow (0, 45)$
4 th (minimum one bowl)	$x \geq 0$	1 st Quadrant	

5 th (minimum one plate)	$y \geq 0$	
-------------------------------------	------------	--

STEP – II: Graphing of Constraints**STEP – III: Selection of Optimum**

The highlighted feasible region satisfies all five given constraints. So, we choose the points from boundary to find maximum profit.

Points (x, y)	Profit $(\pi = 150x + 100y)$
$(0, 0)$	$150(0) + 100(0) = 0$
$(0, 45)$	$150(0) + 100(45) = 4500$
$(25, 20)$	$150(25) + 100(20) = 5750$
$(30, 10)$	$150(30) + 100(10) = 5500$
$(32.5, 0)$	$150(32.5) + 100(0) = 4875$

Observation: While following all constraints, \$5750 is the maximum amount of profit earned by producing $(x^*, y^*) = (25, 20)$ of bowls and plates, respectively.

TOPIC 200: LINEAR PROGRAMMING: EMPLOYMENT COST MINIMIZATION

After understanding maximization problem, we execute a minimization problem:

Numerical Example: An insurance company employs full- and part-time staff, who work **40** and **20** hours per week, respectively. Full-time staff are paid **\$800** per week and part-time staff **\$320**. In addition, it is company policy that the number of part-time staff should not exceed **one-third** of the number of full-time staff.

If the number of worker-hours per week required to deal with the company's work is **900**, how many workers of each type should be employed in order to complete the workload at minimum cost?

Solution: Weekly Employment Cost of the insurance company depends full- and part-time staff:

x = full-time staff

y = part-time staff

STEP – I: Mathematization of Given Information

$$C = f(x, y)$$

Wage per full-time worker = **\$800** $\Rightarrow$ Total Wage of full-time worker = **\$800** $\times x = 800x$

Wage per part-time worker = **\$320** $\Rightarrow$ Total Wage of part-time worker = **\$320** $\times y = 320y$

Total Wage Bill (Cost) from both types of workers:

<i>Objective Function to be minimized.</i> $\pi = \underbrace{800x}_{C_{full}} + \underbrace{320y}_{C_{part}}$

Limited resources put constraints:

1st Constraint (Total Worker Hours): Total time available for work is at least **70** hours per week. Full-time workers work 40 hours and Part-time workers work 20 hours:

$\underbrace{40x}_{\substack{\text{(time consumed working)} \\ \text{by full-time workers}}} + \underbrace{20y}_{\substack{\text{(time consumed working)} \\ \text{by part-time workers}}}$	$\stackrel{\substack{\text{(should be)} \\ \text{more or} \\ \text{equal}}}{\geq}$	$\underbrace{70}_{\substack{\text{(required)} \\ \text{hours per} \\ \text{week}}}$
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2nd Constraint (Proportion of Part-time workers): Number of Part-time workers (y) should be exceed $\frac{1}{3}$ rd Number of Full-time workers (x):

y $\underbrace{\text{(Number of)}}_{\text{Part-time workers}}$	$\stackrel{\substack{\text{(should be)} \\ \text{less or} \\ \text{equal}}}{\geq}$	$\frac{1}{3} \times x$ $\underbrace{\text{(one-third of)}}_{\text{(Number of)}} \text{ Full-time workers}$
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Since worker cannot be negative, we develop 2 additional constraints:

Non-negativity Constraints:

3rd Constraint (minimum one Full-time worker):

x $\underbrace{\text{(Full-time)}}_{\text{workers}}$	$\stackrel{\substack{\text{(can't be)} \\ \text{negative}}}{\geq}$	0
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4th Constraint (minimum one Part-time worker):

$$\underbrace{y}_{\substack{\text{(Part-time)} \\ \text{workers}}} \underbrace{\geq 0}_{\substack{\text{(can't be} \\ \text{negative)}}$$

Tabulating all the extracted information:

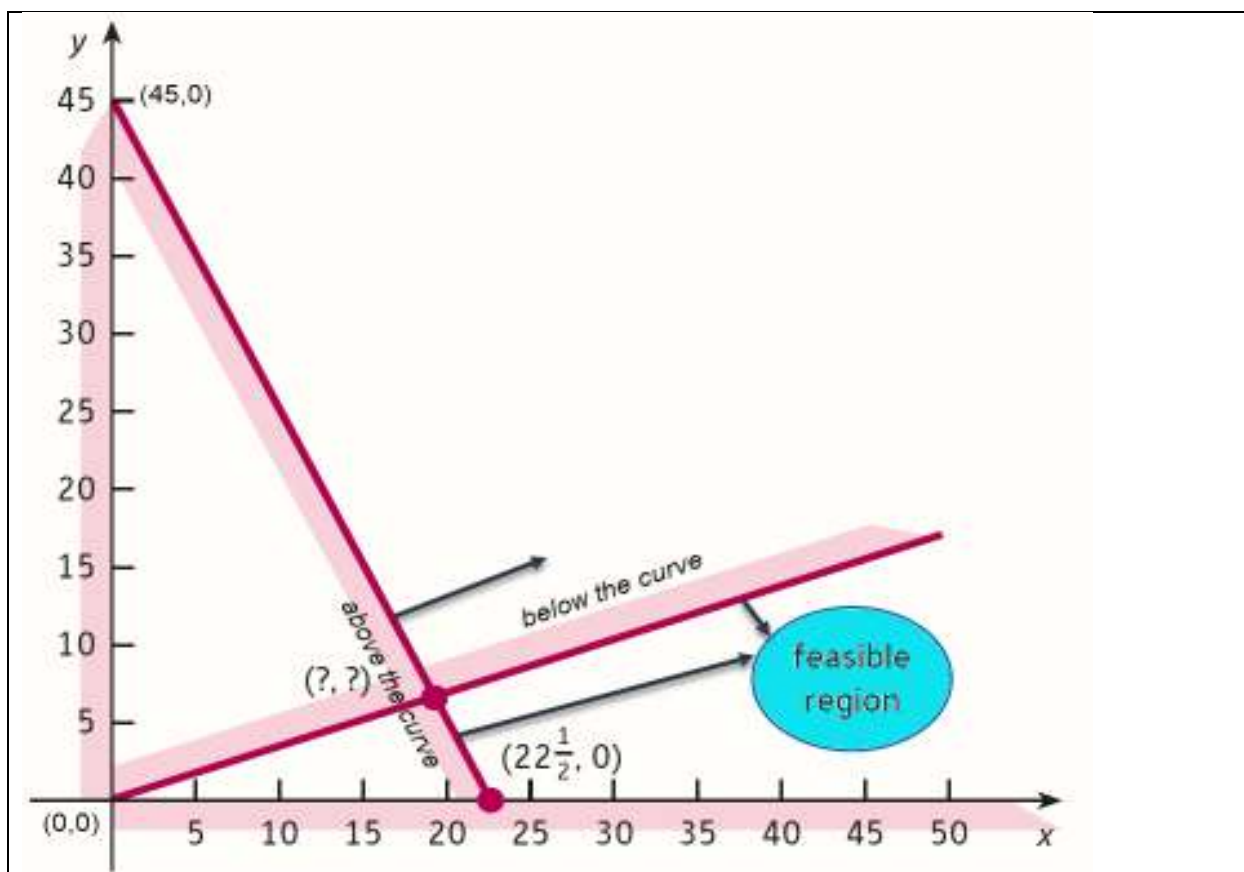
Objective (Cost) Function (to be minimized)	$C_{min} = 800x + 320y$
Subject to	
1st Constraint (Total Worker Hours)	$40x + 20y \geq 900$
2nd Constraint (Proportion of Part-time workers)	$y \leq \frac{1}{3}x$
3rd Constraint (minimum one Full-time worker)	$x \geq 0$
4th Constraint (minimum one Part-time worker)	$y \geq 0$

Plotting the Equalities from Inequalities Constraints: Reduce the inequalities to equalities that are more meaningful to plot: $\geq$ & $\leq \Rightarrow \equiv$

We use the intercepts to plot these linear functions:

Constraints	Equalities/ Inequalities	x-intercept	y-intercept
1st Constraint (Total Worker Hours) $\geq \Rightarrow$ above the curve	$40x + 20y \equiv 900$	$40x + 20(0) = 900$ $x = 22.5$ $\Rightarrow (22.5, 0)$	$40(0) + 20y = 900$ $y = 45$ $\Rightarrow (0, 45)$
2nd Constraint (Proportion of Part-time workers) $\leq \Rightarrow$ below the curve	linear function $y \equiv \frac{1}{3}x$ $y=mx$	Positively sloped linear function: $(0, 0), (1, \frac{1}{3}), (2, \frac{2}{3}), (3, 1), (4, \frac{4}{3})$ as shown in figure.	
3rd Constraint (minimum one Full-time worker)	$x \geq 0$	1st Quadrant	
4th Constraint (minimum one Part-time worker)	$y \geq 0$		

STEP – II: Graphing of Constraints



$(?, ?)$ are the coordinates of feasible area which can be found by simultaneously solving the two equations, as $(?, ?)$ they intersect each other.

$$y = \frac{1}{3}x \text{ and } 40x + 20y = 900 \Rightarrow 40x + 20\left(\frac{1}{3}\right)x = 900$$

$$x = \frac{135}{7} \text{ \& } y = \left(\frac{1}{3}\right) \underbrace{x}_{=\frac{135}{7}} \Rightarrow y = \left(\frac{1}{3}\right) \frac{135}{7} = \frac{45}{7}$$

$$(x, y) = (19.3, 6.4)$$

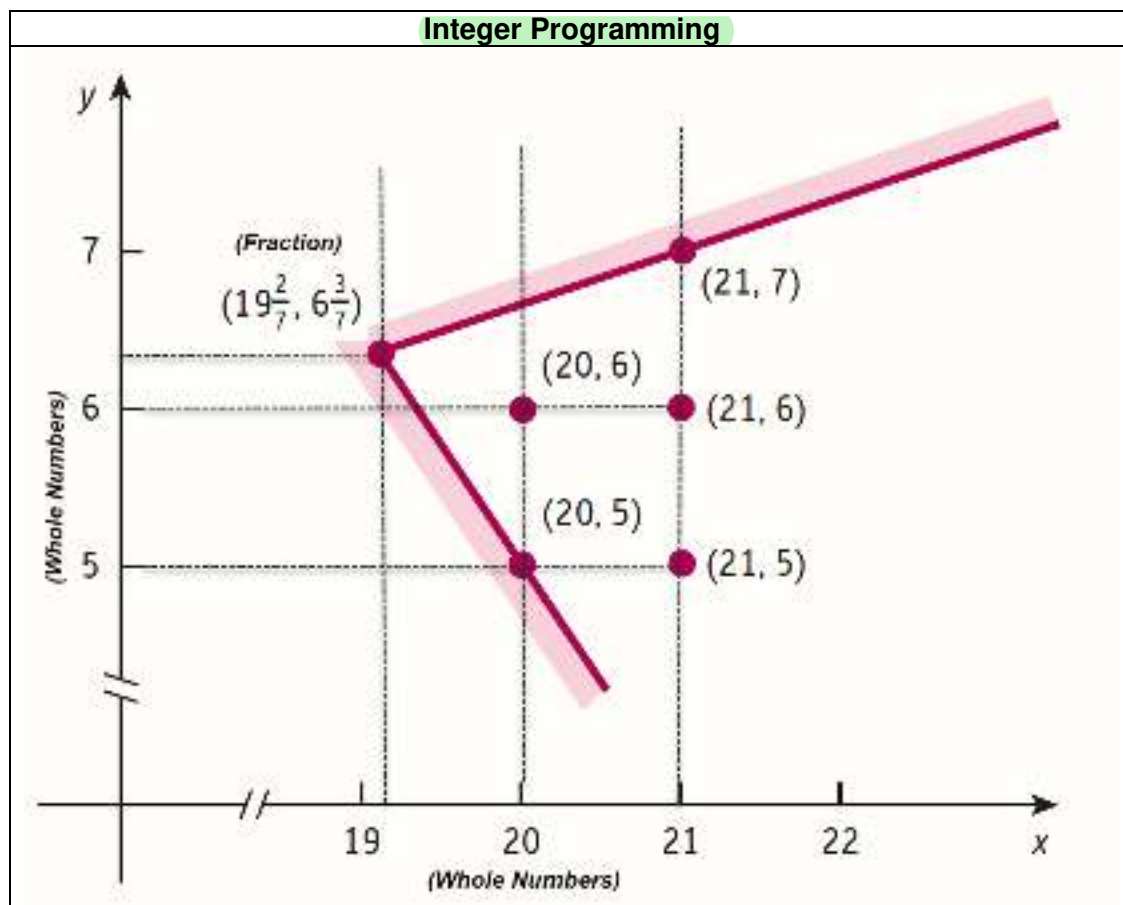
STEP – III: Selection of Optimum

The feasible region satisfies all five given constraints. So, we choose the points from boundary to find minimum cost.

Points (x, y)	Employment Cost $(C = 800x + 320y)$
$(0, 0)$	$800(0) + 320(0) = 0$
$(19.3, 6.4)$	$800(19.3) + 320(6.4) = 17485.7$
$(22.5, 0)$	$800(22.5) + 320(0) = 18000$
Observation: While following all constraints, 17485.7 is the minimum amount of employment cost incurred by insurance company $(19.3, 6.4)$ of Full- and Part-time workers, respectively.	

Need for Integer Programming:

Workers cannot be in fractions and $(19.3, 6.4)$ seem somewhat awkward. For this, one can experiment **WHOLE** numbers like $(20, 5)$, $(20, 6)$, $(21, 5)$, $(21, 6)$ and $(21, 7)$ among others.



Points (x, y)	Employment Cost ($C = 800x + 320y$)
(20, 5)	$800(20) + 320(5) = 17600$
(20, 6)	$800(20) + 320(6) = 17920$
(21, 5)	$800(21) + 320(5) = 18400$
(21, 6)	$800(21) + 320(6) = 18720$
(21, 7)	$800(21) + 320(7) = 19040$

Interpretation: In this case (20, 5) shall give the minimum value of cost, making (20, 5) even more meaningful than (19.3, 6.4). This additional step is known as **Integer Programming**.

Though $\underbrace{(C = 17600)}_{\text{integer programming at (20,5)}} > \underbrace{(C = 17485.7)}_{\text{linear programming at (19.3,6.4)}}$, yet this compromise was necessary as workers cannot be in fractions.

TOPIC 201: NONLINEAR PROGRAMMING: UTILITY MAXIMIZATION

For nonlinear objective and constraint functions, linear programming cannot be used – need for Nonlinear Programming. It can handle Nonlinear Objective Functions as well as nonlinear inequality constraints.

Process: Nonlinear programming has a similar procedure. However, it differs in terms of conditions known as Kuhn-Tucker conditions (1951).

Kuhn-Tucker conditions are similar to first order conditions.

Kuhn-Tucker Conditions for Maximization: For j number of variables, there are j number of conditions. For n number of variables, there are n number of conditions:

Variable-Based Conditions				
$\left(\frac{\partial Z}{\partial x_j}\right) \leq 0$	$x_j \geq 0$	$\Rightarrow$ either or both (=0)	$x_j \left(\frac{\partial Z}{\partial x_j}\right) = 0$ complementary slackness	Where $j = 1, 2, \dots, n$
Multiplier-Based Conditions				
$\left(\frac{\partial Z}{\partial \lambda_i}\right) \geq 0$	$\lambda_i \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_i \left(\frac{\partial Z}{\partial \lambda_i}\right) = 0$ complementary slackness	Where $i = 1, 2, \dots, m$

Numerical Example:

Objective Function	<i>product term–nonlinear</i> $\widehat{U = xy}$
General Constraints	$x + y \leq 100$
	$x \leq 40$
Nonnegativity Constraints	$x, y \geq 0$

STEP – I: Lagrangian Function:

Introduce the two constraints and hence two multipliers.

$$Z(x, y, \lambda_1, \lambda_2) = \underbrace{xy}_{\text{objective function}} + \underbrace{\widehat{\lambda}_1}_{\text{1st multiplier}} \underbrace{(100 - x - y)}_{\text{1st constraint}} + \underbrace{\widehat{\lambda}_2}_{\text{2nd multiplier}} \underbrace{(40 - x)}_{\text{2nd constraint}}$$

STEP – II: Kuhn-Tucker Conditions – Minimization:

For two variables (x, y) there are two conditions and for two multipliers (λ_1, λ_2) there are also two conditions.

Variable-Based Conditions			
$\left(\frac{\partial Z}{\partial x}\right) \leq 0$	$x \geq 0$	$\Rightarrow$ either or both (=0)	$x \left(\frac{\partial Z}{\partial x}\right) = 0$
$\left(\frac{\partial Z}{\partial y}\right) \leq 0$	$y \geq 0$	$\Rightarrow$ either or both (=0)	$y \left(\frac{\partial Z}{\partial y}\right) = 0$
Multiplier-Based Conditions			
$\left(\frac{\partial Z}{\partial \lambda_1}\right) \geq 0$	$\lambda_1 \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_1 \left(\frac{\partial Z}{\partial \lambda_1}\right) = 0$
$\left(\frac{\partial Z}{\partial \lambda_2}\right) \geq 0$	$\lambda_2 \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_2 \left(\frac{\partial Z}{\partial \lambda_2}\right) = 0$

Developing First Order Conditions:

$$\frac{\partial Z}{\partial x} = \frac{\delta}{\delta x} \{xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)\} \leq 0 \Rightarrow \boxed{\frac{\partial Z}{\partial x} = y - \lambda_1 - \lambda_2 \leq 0}$$

$$\frac{\partial Z}{\partial y} = \frac{\delta}{\delta y} \{xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)\} \leq 0 \Rightarrow \boxed{\frac{\partial Z}{\partial y} = x - \lambda_1 \leq 0}$$

$$\frac{\partial Z}{\partial \lambda_1} = \frac{\delta}{\delta \lambda_1} \{xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)\} \geq 0 \Rightarrow \boxed{\frac{\partial Z}{\partial \lambda_1} = 100 - x - y \geq 0}$$

$$\frac{\partial Z}{\partial \lambda_2} = \frac{\delta}{\delta \lambda_2} \{xy + \lambda_1(100 - x - y) + \lambda_2(40 - x)\} \geq 0 \Rightarrow \boxed{\frac{\partial Z}{\partial \lambda_2} = 40 - x \geq 0}$$

Converting the inequalities into equalities:

$$\frac{\partial Z}{\partial x} = y - \lambda_1 - \lambda_2 \equiv 0$$

$$\frac{\partial Z}{\partial y} = x - \lambda_1 \equiv 0$$

$$\frac{\partial Z}{\partial \lambda_1} = 100 - x - y \equiv 0$$

$$\frac{\partial Z}{\partial \lambda_2} = 40 - x \equiv 0$$

STEP – III: Trial Solution Cases:

We try two trial solutions:

Trial Solution Case – I: $\lambda_1, \lambda_2 > 0$	Trial Solution, Case – II: $x, y > 0$
$(\lambda_1 > 0) \wedge \left\{ \lambda_1 \underbrace{\left(\frac{\delta Z}{\delta \lambda_1} \right)}_{\text{then } \left(\frac{\delta Z}{\delta \lambda_1} \right) = 0} = 0 \right\}$	$(x > 0) \wedge \left\{ x \underbrace{\left(\frac{\delta Z}{\delta x} \right)}_{\text{then } \left(\frac{\delta Z}{\delta x} \right) = 0} = 0 \right\}$
$(\lambda_2 > 0) \wedge \left\{ \lambda_2 \underbrace{\left(\frac{\delta Z}{\delta \lambda_2} \right)}_{\text{then } \left(\frac{\delta Z}{\delta \lambda_2} \right) = 0} = 0 \right\}$	$(y > 0) \wedge \left\{ y \underbrace{\left(\frac{\delta Z}{\delta y} \right)}_{\text{then } \left(\frac{\delta Z}{\delta y} \right) = 0} = 0 \right\}$
$\boxed{\lambda_1, \lambda_2 > 0} \Rightarrow \left\{ \left(\frac{\delta Z}{\delta \lambda_1} \right), \left(\frac{\delta Z}{\delta \lambda_2} \right) \right\} = 0$	$\boxed{x, y > 0} \Rightarrow \left\{ \left(\frac{\delta Z}{\delta x} \right), \left(\frac{\delta Z}{\delta y} \right) \right\} = 0$

Trial Solution Case – I: $\boxed{\lambda_1, \lambda_2 > 0} \Rightarrow \left\{ \left(\frac{\delta Z}{\delta \lambda_1} \right), \left(\frac{\delta Z}{\delta \lambda_2} \right) \right\} = 0$

$\left\{ \left(\frac{\delta Z}{\delta \lambda_1} \right), \left(\frac{\delta Z}{\delta \lambda_2} \right) \right\} = 0$	
$\boxed{\left(\frac{\delta Z}{\delta \lambda_1} \right) = 0}$	$\boxed{\left(\frac{\delta Z}{\delta \lambda_2} \right) = 0}$
$\boxed{\underbrace{(100 - x - y)}_{(y=100-x)} = 0}$	$\boxed{\underbrace{(40 - x)}_{x=40} = 0}$
$y = 100 - \boxed{x}$ $y = 100 - \boxed{40}$ $\boxed{y = 60}$	
Trial Solution Case – I gives us values of both of the variables $(x^*, y^*) = (40, 60)$. Multipliers can also be found.	

For values of λ_1, λ_2 substitute x^* and y^* in $\left(\frac{\partial Z}{\partial x}\right)$ and $\left(\frac{\partial Z}{\partial y}\right)$.	
$\left(\frac{\partial Z}{\partial y}\right) = 0$ as $y > 0, \left(\frac{\partial Z}{\partial y}\right) > 0$	$\left(\frac{\partial Z}{\partial x}\right) = 0$ as $x > 0, \left(\frac{\partial Z}{\partial x}\right) > 0$
$\underbrace{x - \lambda_1}_{\substack{(x=40) \\ \left(\frac{\partial Z}{\partial \lambda_2}=0\right)}} = 0 \Rightarrow \lambda_1 = 40$	$\underbrace{y - \lambda_1 - \lambda_2}_{\substack{60 \\ 40}} = 0 \Rightarrow \lambda_2 = 20$
$(\lambda_1, \lambda_2) = (40, 20)$	

Verification: We found $(x, y) = (40, 60)$ and $(\lambda_1, \lambda_2) = (40, 20)$. Now we check if the values satisfy all of the functions:

Objective Function	$U = xy$	$U = (40)(60) = 2400$
General Constraints	$x + y \leq 100$	$40 + 60 \leq 100$
	$x \leq 40$	$40 \leq 40$
Nonnegativity Constraints	$x, y \geq 0$	$40, 60 \geq 0$
All constraints are satisfied. $(x, y) = (40, 60)$ are the undisputed values at equilibrium that maximize the utility (2400).		

TOPIC 202: NONLINEAR PROGRAMMING: COST MINIMIZATION

Kuhn-Tucker Conditions for Minimization: For j number of variables, there are j number of conditions. For n number of variables, there are n number of conditions:

Variable-Based Conditions				
$\left(\frac{\partial Z}{\partial x_j}\right) \geq 0$	$x_j \geq 0$	$\Rightarrow$ either or both (=0)	$x_j \left(\frac{\partial Z}{\partial x_j}\right) = 0$ complementary slackness	Where $j = 1, 2, \dots, n$
Multiplier-Based Conditions				
$\left(\frac{\partial Z}{\partial \lambda_i}\right) \leq 0$	$\lambda_i \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_i \left(\frac{\partial Z}{\partial \lambda_i}\right) = 0$ complementary slackness	Where $i = 1, 2, \dots, m$

Note: Complementary Slackness is used for the product as the two terms λ_i and $\left(\frac{\partial Z}{\partial \lambda_i}\right)$ complement each other in fading to slackness (0).

Numerical Example:

Objective Function	Quadratic (nonlinear) function $C = (x_1 - 4)^2 + (x_2 - 4)^2$
General Constraints	$2x_1 + 3x_2 \geq 6$
	$-3x_1 - 2x_2 \geq -12$
Nonnegativity Constraints	$x_1, x_2 \geq 0$

STEP – I: Lagrangian Function:

Introduce the two constraints and hence two multipliers.

$$Z(x_1, x_2, \lambda_1, \lambda_2) = \underbrace{(x_1 - 4)^2 + (x_2 - 4)^2}_{\substack{\text{objective function} \\ \text{2nd} \\ \text{multiplier}}} + \underbrace{\widehat{\lambda}_1}_{\substack{\text{1st} \\ \text{multiplier}}} \underbrace{(6 - 2x_1 - 3x_2)}_{\substack{\text{1st constraint}}} + \underbrace{\widehat{\lambda}_2}_{\substack{\text{2nd} \\ \text{multiplier}}} \underbrace{(-12 + 3x_1 + 2x_2)}_{\substack{\text{2nd constraint}}}$$

STEP – II: Kuhn-Tucker Conditions – Minimization:

For two variables (x_1, x_2) there are two conditions and for two multipliers (λ_1, λ_2) there are also two conditions.

Variable-Based Conditions			
$\left(\frac{\partial Z}{\partial x_1}\right) \geq 0$	$x_1 \geq 0$	$\Rightarrow$ either or both (=0)	$x_1 \left(\frac{\partial Z}{\partial x_1}\right) = 0$
$\left(\frac{\partial Z}{\partial x_2}\right) \geq 0$	$x_2 \geq 0$	$\Rightarrow$ either or both (=0)	$x_2 \left(\frac{\partial Z}{\partial x_2}\right) = 0$
Multiplier-Based Conditions			
$\left(\frac{\partial Z}{\partial \lambda_1}\right) \leq 0$	$\lambda_1 \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_1 \left(\frac{\partial Z}{\partial \lambda_1}\right) = 0$
$\left(\frac{\partial Z}{\partial \lambda_2}\right) \leq 0$	$\lambda_2 \geq 0$	$\Rightarrow$ either or both (=0)	$\lambda_2 \left(\frac{\partial Z}{\partial \lambda_2}\right) = 0$

Developing First Order Conditions:

$$\frac{\partial Z}{\partial x_1} = \frac{\delta}{\delta x_1} \{(x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1(6 - 2x_1 - 3x_2) + \lambda_2(-12 + 3x_1 + 2x_2)\} \geq 0$$

$$\frac{\partial Z}{\partial x_1} = 2(x_1 - 4) - 2\lambda_1 + 3\lambda_2 \geq 0$$

$$\frac{\partial Z}{\partial x_2} = \frac{\delta}{\delta x_2} \{(x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1(6 - 2x_1 - 3x_2) + \lambda_2(-12 + 3x_1 + 2x_2)\} \geq 0$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2 - 4) - 3\lambda_1 + 2\lambda_2 \geq 0$$

$$\frac{\partial Z}{\partial \lambda_1} = \frac{\delta}{\delta \lambda_1} \{(x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1(6 - 2x_1 - 3x_2) + \lambda_2(-12 + 3x_1 + 2x_2)\}$$

$$\frac{\partial Z}{\partial \lambda_1} = 6 - 2x_1 - 3x_2 \leq 0$$

$$\frac{\partial Z}{\partial \lambda_2} = \frac{\delta}{\delta \lambda_2} \{(x_1 - 4)^2 + (x_2 - 4)^2 + \lambda_1(6 - 2x_1 - 3x_2) + \lambda_2(-12 + 3x_1 + 2x_2)\}$$

$$\frac{\partial Z}{\partial \lambda_2} = -12 + 3x_1 + 2x_2 \leq 0$$

Converting the inequalities into equalities:

$$\frac{\partial Z}{\partial x_1} = 2(x_1 - 4) - 2\lambda_1 + 3\lambda_2 \equiv 0$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2 - 4) - 3\lambda_1 + 2\lambda_2 \equiv 0$$

$$\frac{\partial Z}{\partial \lambda_1} = 6 - 2x_1 - 3x_2 \equiv 0$$

$$\frac{\partial Z}{\partial \lambda_2} = -12 + 3x_1 + 2x_2 \equiv 0$$

STEP – III: Trial Solution Cases:

We try two trial solutions:

Trial Solution Case – I: $\lambda_1, \lambda_2 > 0$	Trial Solution, Case – II: $x_1, x_2 > 0$
$(\lambda_1 > 0) \wedge \left\{ \lambda_1 \left(\frac{\partial Z}{\partial \lambda_1} \right) = 0 \right\}$ then $\left(\frac{\partial Z}{\partial \lambda_1} \right) = 0$	$(x_1 > 0) \wedge \left\{ x_1 \left(\frac{\partial Z}{\partial x_1} \right) = 0 \right\}$ then $\left(\frac{\partial Z}{\partial x_1} \right) = 0$
$(\lambda_2 > 0) \wedge \left\{ \lambda_2 \left(\frac{\partial Z}{\partial \lambda_2} \right) = 0 \right\}$ then $\left(\frac{\partial Z}{\partial \lambda_2} \right) = 0$	$(x_2 > 0) \wedge \left\{ x_2 \left(\frac{\partial Z}{\partial x_2} \right) = 0 \right\}$ then $\left(\frac{\partial Z}{\partial x_2} \right) = 0$
$\lambda_1, \lambda_2 > 0 \Rightarrow \left\{ \left(\frac{\partial Z}{\partial \lambda_1} \right), \left(\frac{\partial Z}{\partial \lambda_2} \right) \right\} = 0$	$x_1, x_2 > 0 \Rightarrow \left\{ \left(\frac{\partial Z}{\partial x_1} \right), \left(\frac{\partial Z}{\partial x_2} \right) \right\} = 0$

Trial Solution Case – I: $\lambda_1, \lambda_2 > 0 \Rightarrow \left\{ \left(\frac{\partial Z}{\partial \lambda_1} \right), \left(\frac{\partial Z}{\partial \lambda_2} \right) \right\} = 0$

$\left\{ \left(\frac{\partial Z}{\partial \lambda_1} \right), \left(\frac{\partial Z}{\partial \lambda_2} \right) \right\} = 0$	
$\left(\frac{\partial Z}{\partial \lambda_1} \right) = 0$	$\left(\frac{\partial Z}{\partial \lambda_2} \right) = 0$
$\left(\frac{6 - 2x_1 - 3x_2}{x_1 = \frac{6-3x_2}{2}} \right) = 0$	$(-12 + 3x_1 + 2x_2) = 0$
$-12 + 3 \frac{6 - 3x_2}{2} + 2x_2 = 0$ $\frac{-24 + 3(6 - 3x_2) + 4x_2}{2} = 0 \Rightarrow -24 + 3(6 - 3x_2) + 4x_2 = 0$ $-24 + 18 - 9x_2 + 4x_2 = 0$ $-6 = 5x_2 \Rightarrow x_2 = -\frac{6}{5}$	
$x_1 = \frac{6 - 3x_2}{2} \Rightarrow \frac{6 - 3(-\frac{6}{5})}{2} \Rightarrow \frac{6 + \frac{18}{5}}{2} = \frac{\frac{30 + 18}{5}}{2} \Rightarrow x_1 = \frac{24}{8}$	
$(x_1, x_2) = \left(\frac{24}{8}, -\frac{6}{5} \right); \text{ Nonnegativity constraint is being violated for } x_2.$ $(x_1 < 0)$	
Trial Solution Case – I violates the nonnegativity conditions, so we try Trial Solution Case – II.	

Trial Solution Case – II: $x_1, x_2 > 0 \Rightarrow \left\{ \left(\frac{\partial Z}{\partial x_1} \right), \left(\frac{\partial Z}{\partial x_2} \right) \right\} = 0$

$\left\{ \left(\frac{\delta Z}{\delta x_1} \right), \left(\frac{\delta Z}{\delta x_2} \right) \right\} = 0$	
$\left(\frac{\delta Z}{\delta x_1} \right) = 0$	$\left(\frac{\delta Z}{\delta x_2} \right) = 0$
$\frac{2(x_1 - 4) - 2\lambda_1 + 3\lambda_2}{\left\{ \lambda_2 = \frac{2\lambda_1 - 2(x_1 - 4)}{3} \right\}} = 0$	$\frac{2(x_2 - 4) - 3\lambda_1 + 2\lambda_2}{\left\{ \lambda_2 = \frac{3\lambda_1 - 2(x_2 - 4)}{2} \right\}} = 0$
Equating (λ_2), we get: $\frac{2\lambda_1 - 2(x_1 - 4)}{3} = \frac{3\lambda_1 - 2(x_2 - 4)}{2}$ $\frac{2\lambda_1 - 2x_1 + 8}{3} = \frac{3\lambda_1 - 2x_2 + 8}{2}$ $\frac{3}{2}(2\lambda_1 - 2x_1 + 8) = \frac{2}{3}(3\lambda_1 - 2x_2 + 8)$ $4\lambda_1 - 4x_1 + 16 = 9\lambda_1 - 6x_2 + 24$ $0 = -4\lambda_1 + 4x_1 - 16 + 9\lambda_1 - 6x_2 + 24$ $4x_1 - 6x_2 + 5\lambda_1 + 8 = 0$ <i>3 unknowns $\Rightarrow$ let $\lambda_1 = 0$</i> $4x_1 - 6x_2 + 5(0) + 8 = 0$ $4x_1 - 6x_2 + 8 = 0$ $2x_1 - 3x_2 = -4$ $\underbrace{2x_1 - 3x_2 = -4}_{f(x_1, x_2)} \Rightarrow$ need for another equation in terms of (x_1, x_2), to find values of x_1 and x_2.	
Using condition $\lambda_2 > 0 \Rightarrow \left\{ \left(\frac{\delta Z}{\delta \lambda_2} \right) \right\} = 0$	
$\left(\frac{\delta Z}{\delta \lambda_2} \right) = 0$ $\underbrace{-12 + 3x_1 + 2x_2 = 0}_{g(x_1, x_2)}$ $3x_1 + 2x_2 = 12$	
$\underbrace{2x_1 - 3x_2 = -4}_{f(x_1, x_2)}$	$\underbrace{3x_1 + 2x_2 = 12}_{g(x_1, x_2)}$
Solving simultaneously for (x_1, x_2)	
$\underbrace{2x_1 - 3x_2 = -4}_{x_1 = \frac{-4 + 3x_2}{2}}$	$3\left[\frac{-4 + 3x_2}{2} \right] + 2x_2 = 12$
$3 \left[\frac{-4 + 3x_2}{2} \right] + 2x_2 = 12 \Rightarrow -6 + \frac{9x_2}{2} + 2x_2 = 12$ $\frac{13x_2}{2} = 18 \Rightarrow \boxed{x_2 = \frac{36}{13}}$	
For x_1 , we substitute x_2 in $x_1 = \frac{-4 + 3x_2}{2}$.	
$x_1 = \frac{-4 + 3\left(\frac{36}{13}\right)}{2} \Rightarrow \boxed{x_1 = \frac{28}{13}}$	
$(x_1^*, x_2^*) = \left(\frac{28}{13}, \frac{36}{13} \right)$	
For values of λ_1, λ_2 substitute x_1^* and x_2^* in $\left(\frac{\delta Z}{\delta x_1} \right)$ and $\left(\frac{\delta Z}{\delta x_2} \right)$.	
$2(x_1^* - 4) - 2\lambda_1 + 3\lambda_2 = 0$	$2(x_2^* - 4) - 3\lambda_1 + 2\lambda_2 = 0$

$2\left(\frac{28}{13} - 4\right) - 2\lambda_1 + 3\lambda_2 = 0$	$2\left(\frac{36}{13} - 4\right) - 3\lambda_1 + 2\lambda_2 = 0$
$-2\lambda_1 + 3\lambda_2 = \frac{48}{13}$ $\lambda_1 = \frac{1}{2}\left(3\lambda_2 - \frac{48}{13}\right)$	$-3\lambda_1 + 2\lambda_2 = \frac{32}{13}$
$-3\left[\frac{1}{2}\left(3\lambda_2 - \frac{48}{13}\right)\right] + 2\lambda_2 = \frac{32}{13} \Rightarrow -\frac{9}{2}\lambda_2 + \frac{144}{26} + 2\lambda_2 = \frac{32}{13} \Rightarrow \lambda_2 = \frac{16}{13}$	
$\lambda_1 = \frac{1}{2}\left(3\left[\frac{16}{13}\right] - \frac{48}{13}\right) \Rightarrow \lambda_1 = \frac{1}{2}\left(3\left[\frac{16}{13}\right] - \frac{48}{13}\right) \Rightarrow \lambda_1 = 0$	
$(\lambda_1, \lambda_2) = \left(0, \frac{16}{13}\right)$	

Verification: We found $(x_1^*, x_2^*) = \left(\frac{28}{13}, \frac{36}{13}\right)$ and $(\lambda_1, \lambda_2) = \left(0, \frac{16}{13}\right)$. Now we check if the values satisfy all of the functions:

Objective Function	$C = (x_1 - 4)^2 + (x_2 - 4)^2$	$C = \left(\frac{28}{13} - 4\right)^2 + \left(\frac{36}{13} - 4\right)^2 = \frac{68}{13}$
General Constraints	$2x_1 + 3x_2 \geq 6$	$2\left(\frac{28}{13}\right) + 3\left(\frac{36}{13}\right) = 12.6 \geq 6$
	$-3x_1 - 2x_2 \geq -12$	$-3\left(\frac{28}{13}\right) - 2\left(\frac{36}{13}\right) = -12 \geq -12$
Nonnegativity Constraints	$x_1, x_2 \geq 0$	$\frac{28}{13}, \frac{36}{13} \geq 0$
All constraints are satisfied. $(x_1^*, x_2^*) = \left(\frac{28}{13}, \frac{36}{13}\right)$ are the undisputed values at equilibrium that minimize the cost $\left(\frac{68}{13}\right)$.		