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MTH634

Topology

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MTH634

TOPOLOGY

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Topic 001

Introduction to Topology

1. What is Topology?

Topology as a word is composition of two words.

1. Topos (Greek) means “place”.

2. Logy (Latin) means study/Analysis.

Topology means the study of something with respect to place/position.

2. What kind of Math it is?

Topology is a kind of Math in which we study shapes.

Topology is not Geometry. In Geometry we care about exact measurements such as length, angles and areas.

But in Topology, we don't care about measurements.

Geometrically, rectangle, triangle, circle and square are different shapes. But in Topology, they are same.

Instead of measurements, **in Topology we care about:**

1. Arrangements of shapes.

2. Relative positions of shapes.

3. Functions/deformations between shapes.

So, we can say that **topology is “Geometry of position”**. In Topology two shapes are considered same if one shape can be deformed to other shape.

As an example:

A circle can be deformed into a square. So, topologically circle and square are of same shapes. But a line and circle are of different shapes.

4. How useful this subject is if we can't tell the difference between shapes?

In-fact it is a matter of nature of the study.

Topological study is more qualitative in nature rather than quantitative.

There are many problems/mathematical models where qualitative study is more effective than quantitative.

So, topology is a very useful tool in such problems or one may say sometimes the only effective tool.

Topic 002

Topology on a set

Power set:

Let X be a set, power set of X denoted as $P(X)$ is the set of all subsets of X . Usually we use the word class for the set of sets.

Example:

Let $X = \{1,2,3\}$ then,

$$P(X) = \{ \Phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X \}$$

Remarks:

1. Φ and X are also members of $P(X)$.
2. If X is finite and has n elements, then

$$|P(X)| = 2^n$$

Let X be a non empty set and τ be a subclass of $P(X)$, then τ is called a topology on X if τ satisfies the following axioms:

1. $\emptyset, X \in \tau$
2. The union of any number of sets in τ belongs to τ .
3. The intersection of finite number of sets in τ belongs to τ .

To satisfy the 3rd axiom it is enough to show that intersection of two elements of τ belongs to τ .

Open set:

The elements of τ are called open sets or more specifically τ -open sets.

Topic 003

Topology on a set

Topology on a set can also be defined by stating only 2nd and 3rd axiom.

1st axiom will be followed by 2nd and 3rd .

$$\bigcup \{U \in \tau \mid U \in \emptyset\} = \emptyset$$

$$\bigcap \{U \in \tau \mid U \in \emptyset\} = X$$

Topic 004

Examples: Topology on a set

Example 1:

Let $X = \{a, b, c, d\}$ then,

$$P(X) =$$

$\{ \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, \{a,c,d\}, X \}$

Let us consider the sub-collection:

$$\tau = \{ \emptyset, \{a\}, \{c\}, \{a,c\}, X \}$$

Let us check all the three axioms one by one on τ

1. It is clear that $\Phi, X \in \tau$.

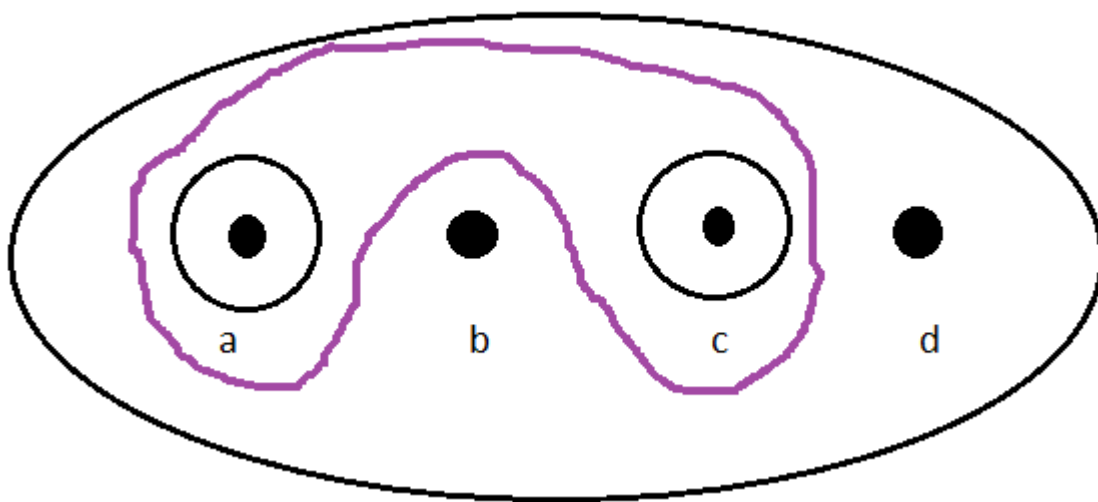
2. Now if we take union of any number of elements of τ , it belongs to τ .

3. The same is true for intersection.

So τ is topology on X .

Graphically,

$$\tau = \{ \Phi, \{a\}, \{c\}, \{a,c\}, X \}$$



Topic 005

Examples: Topology on a set

Let $X = \{a,b,c,d\}$

Let us consider:

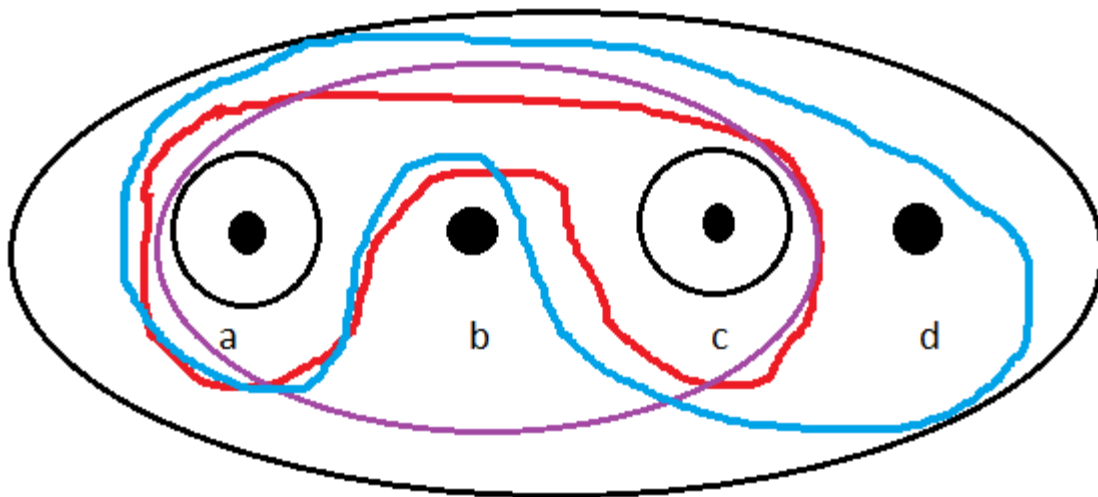
$$\tau' = \{ \Phi, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, X \}$$

1. Again in this example, it s clear that:

$$\Phi, X \in \tau'.$$

For 2nd and 3rd axiom:

$$\tau' = \{ \Phi, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}, X \}$$



This sub-collection is also closed under union and intersection.

Remark 1:

More than one topologies can be defined on a set.

Singelton set has only one topology.

Let:

$$X = \{a\}$$

$$Y = \{a,b\}$$

$$\tau = \{ \Phi, X \} \quad \tau_1 = \{ \Phi, \{a\}, Y \}$$

$$\tau_2 = \{ \Phi, \{b\}, Y \}$$

$$\tau_3 = \{ \Phi, \{a,b\}, Y \}$$

$$\tau_4 = \{ \Phi, \{a\}, \{b\}, Y \}$$

Let us list down all the topologies on a three point set.

$$Z = \{a, b, c\}$$

$$\tau_1 = \{ \Phi, Z \}$$

$$\tau_2 = \{ \Phi, \{a\}, Z \}$$

$$\tau_3 = \{ \Phi, \{b\}, Z \}$$

$$\tau_4 = \{ \Phi, \{c\}, Z \}$$

$$\tau_5 = \{ \Phi, \{a,b\}, Z \}$$

$$\tau_6 = \{ \Phi, \{a,c\}, Z \}$$

$$\tau_7 = \{ \Phi, \{b,c\}, Z \}$$

$$\tau_8 = \{ \Phi, \{a\}, \{a,b\}, Z \}$$

$$\tau_9 = \{ \Phi, \{a\}, \{a,c\}, Z \}$$

$$\tau_{16} = \{ \Phi, \{c\}, \{b,c\}, Z \}$$

$$\tau_{17} = \{ \Phi, \{a\}, \{b\}, \{a,b\}, Z \}$$

$$\tau_{18} = \{ \Phi, \{a\}, \{c\}, \{a,c\}, Z \}$$

$$\tau_{19} = \{ \Phi, \{b\}, \{c\}, \{b,c\}, Z \}$$

$$\tau_{20} = \{ \Phi, \{a\}, \{a,b\}, \{a,c\}, Z \}$$

$$\tau_{21} = \{ \Phi, \{b\}, \{a,b\}, \{b,c\}, Z \}$$

$$\tau_{22} = \{ \Phi, \{c\}, \{a,c\}, \{b,c\}, Z \}$$

$$\tau_{23} = \{ \Phi, \{a\}, \{b\}, \{a,b\}, \{b,c\}, Z \}$$

$$\tau_{24} = \{ \Phi, \{a\}, \{b\}, \{a,b\}, \{a,c\}, Z \}$$

$$\tau_{10} = \{ \Phi, \{a\}, \{b,c\}, Z \}$$

$$\tau_{25} = \{ \Phi, \{a\}, \{c\}, \{a,c\}, \{b,c\},$$

$$\tau_{11} = \{ \Phi, \{b\}, \{a,b\}, Z \}$$

$$\tau_{26} = \{ \Phi, \{a\}, \{c\}, \{a,c\}, \{a,b\},$$

$$\tau_{12} = \{ \Phi, \{b\}, \{a,c\}, Z \}$$

$$\tau_{27} = \{ \Phi, \{b\}, \{c\}, \{b,c\}, \{a,b\},$$

$$\tau_{13} = \{ \Phi, \{b\}, \{b,c\}, Z \}$$

$$\tau_{28} = \{ \Phi, \{b\}, \{c\}, \{b,c\}, \{a,c\},$$

$$\tau_{14} = \{ \Phi, \{c\}, \{a,b\}, Z \}$$

$$\tau_{29} = P(X)$$

$$\tau_{15} = \{ \Phi, \{c\}, \{a,c\}, Z \}$$

Hence, a finite set has finite many topologies.

Topic 006

Examples: Topology on a set

Remark 2:

Not every collection of subsets of a set is eligible to be a topology on the set unless that collection does not satisfy all the stated conditions in the definition.

Example 3:

Let $X = \{a, b, c, d\}$ and consider the following collection of subsets of X .

$$\tau_1 = \{\{b\}, \{c\}, \{b, c\}, X\}$$

$$\tau_2 = \{\Phi, \{a\}, \{c\}, \{a, c\}\}$$

$$\tau_3 = \{\Phi, \{a\}, \{b, c\}, X\}$$

$$\tau_4 = \{\Phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$$

None of the above τ_i 's is topology on X .

Reasons:

$$\tau_1 = \{\{b\}, \{c\}, \{b, c\}, X\} \quad (\Phi \text{ does not belong to } \tau_1)$$

$$\tau_2 = \{\Phi, \{a\}, \{c\}, \{a, c\}\} \quad (X \text{ does not belong to } \tau_2)$$

$$\tau_3 = \{\Phi, \{a\}, \{b, c\}, X\}$$

$$\{a\} \cup \{b, c\} = \{a, b, c\} \notin \tau_3$$

$$\tau_4 = \{\Phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$$

$$\{a, b\} \cap \{b, c\} = \{b\} \notin \tau_4$$

Topic 007

More Examples: Topology on a set

Example 4:

Let $X = \mathbb{R}$ and τ be the class consisting Φ , $\mathbb{R}$ and all infinite open intervals of the kind of the following:

$I_a = (a, \infty)$ where a be any real number.

Then τ is the topology on $\mathbb{R}$.

Let us justify our claim.

It is given that: $\Phi, \mathbb{R} \in \tau$

For union:

Note that for any sub-collection $A = \{I_i \mid i \in J\}$ of τ , 

where $J \subset \mathbb{R}$.

$$\cup I_i = \begin{cases} (j, \infty) & \text{if } j = \inf(J) \\ (-\infty, \infty) & \text{if no inf exist} \end{cases}$$

In both cases union belongs to τ .

For intersection:

Let $A = (a, \infty)$ and $B = (b, \infty)$

Consider $a < b$.

$$A \cap B = (b, \infty) \in \tau$$

Example 5:

Let $X = \mathbb{R}$ and τ be the class consisting Φ , $\mathbb{R}$ and all infinite open intervals of the kind of the following:

$$A_q = (-\infty, q) \quad \text{where } q \in \mathbb{Q}$$

This collection is not a topology on real line.

Reason:

The condition of union is not satisfied by the given collection τ . Let us consider the following subclass of τ .

$$\tau = \{(-\infty, q) \mid q \in \mathbb{Q}, q < \pi\}$$

Since,

$$\cup (-\infty, q) = (-\infty, \pi) \notin \tau$$

Topic 008

Some special Topologies

For any set there are always two topologies. Those are very special.

1. Indiscrete Topology

2. Discrete Topology

Indiscrete Topology:

A topology on a set X consists of empty set and the set itself is called indiscrete Topology. It is also called Trivial Topology.

Remark: Indiscrete Topology is the smallest topology one can define on some set.

Task: Show that such collection is a topology on any set.

Let X be a set then according to the definition:

$\tau_{ind} = \{\emptyset, X\}$. It satisfies all the conditions. So, it is an indiscrete topology.

Discrete Topology:

A topology on a set X consists of all subsets of X is called discrete topology. i.e $\tau = P(X)$

Remark: Discrete topology is the largest topology one can define on some set X .

Task: Show that $\tau = P(X)$ is a topology on a set X .

Consider a set X . Let $P(X)$ is the set of all subsets of X .

$P(X)$ satisfies all the conditions of topology.

Topic 009

Co-finite Topology

Co-finite topology is also known as “Finite Complement Topology”. For a set X “Co-finite Topology” τ_{cof} be the collection of all subsets U of X such that U^c is either finite or complete X .

i.e $\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$

or

$$\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } \emptyset\}$$

Task: Show that $\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$

defines a topology on X .

Proof:

$$\emptyset \in \tau_{cof} \quad (\emptyset^c = X)$$

$$X \in \tau_{cof} \quad (X^c = \emptyset, \text{finite})$$

1st condition has satisfied.

Now for the 2nd condition:

$$\text{Let } A_i \in \tau_{cof} \quad A_i^c = \text{finite}, X \quad i \in I$$

Now we have to show that $\cup_{i \in I} A_i \in \tau_{cof}$

$$(\cup_{i \in I} A_i)^c = \cap_{i \in I} (A_i)^c = \text{finite} \quad \text{De Morgan's law}$$

Because $(A_i)^c$ is finite or X

Now 3rd condition:

Let $A_i \in \tau_{cof}$ $A_i^c = \text{finite}$, X $i \in I$

We have to check $\cap_{i \in I} (A_i) \in \tau_{cof}$?

$(\cap_{i \in I} (A_i))^c = \cup_{i \in I} (A_i)^c = \text{finite or } X$

Finite union of finite sets is finite.

Hence $\cap_{i \in I} (A_i) \in \tau_{cof}$. So $\tau_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$ is a topology on X .

Remark: If X is finite then the co-finite topology on X is discrete. i.e $\tau_{cof} = P(X)$ if X is finite.

Proof:

$P(X)$ = the set of all subsets of X .

$\tau_{cof} \subset P(X)$

Now let $u \in P(X)$. Since X is finite So u^c is finite.

This implies that $u \in \tau_{cof}$

Hence,

$\tau_{cof} = P(X)$ if X is finite.

Topic 010

Topological Space

Definition: Let τ be a topology on X then the pair (X, τ) is a topological space or sometime called τ -Topological space.

Example: Consider $X = \{a, b, c, d\}$

And $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$. Then (X, τ) is a topological space.

Example: Consider $X = \{a, b, c, d\}$

And $\tau' = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$

Then (X, τ') be another topological space.

Note that in both examples X is same but we define two different topological spaces. 

Remark: Correspond to single set one may define multiple topological spaces and all can be different from each other.

Correspond to a three point set one may define 29 different topological spaces.

Discrete space: A set with discrete topology is called discrete topological space or simply discrete space.

Indiscrete space: A set with indiscrete topology is called indiscrete topological space or simply indiscrete space.

Topic 011

Arbitrary Intersection

Let us recall the 3rd axiom of definition of topology on a set. “Finite Intersection”

Q: What about arbitrary intersection?

Specifically “**Infinite intersection**”. Before giving the answer to this question, let us look into the next two examples.

Example 1: Consider $\mathbb{R}$ “real line” and consider τ be the set of all open intervals and their unions. This collection is a topology on $\mathbb{R}$. Consider the following sub-collection:

$$A = \left\{ A_n = \left(-\frac{1}{n}, \frac{1}{n} \right) \mid n \in \mathbb{N}^* \right\}$$

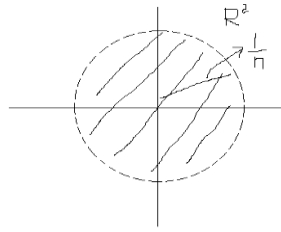
Now

$$\bigcap_{n \in \mathbb{N}^*} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\} \notin \tau$$

Example 2: Consider $\mathbb{R}^2$ “real plane” and consider τ the set of all open disks, their arbitrary unions and finite

intersection. This collection is topology on $\mathbb{R}^2$. Consider the following sub-collection of open disks:

$$A = \{A_n \mid n \in \mathbb{N}^*\} \text{ where } A_n = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < (\frac{1}{n})^2\}$$



$$\bigcap_{n \in \mathbb{N}^*} A_n = (0,0) \notin \tau$$

Remark: Arbitrary intersection of open sets may or may not be open.

Topic 012

Open sets, closed sets

Open set: Let τ be a topology on X . Then the elements of τ are called the open sets, more precisely open subsets of X .

Let us look into an example for better understanding of the openness.

Example 1: Let $X = \{a, b, c, d\}$. Consider

$$\tau = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}.$$
 Then,

$\emptyset, \{a\}, \{c\}, \{a,c\}, X$ are open subsets of X .

Example 2: Let $X = \{a,b,c,d\}$. Consider

$\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c,d\}, X\}$. Then

$\emptyset, \{c\}, \{a,c\}, \{b,c,d\}, X$ are open subsets of X .

Remark:

- Openness is a relative term.
- $\emptyset$ and X are always open.

Closed set: Let (X, τ) be the topological space. A subset A of X is closed iff the complement of A in X is open.

i.e $X - A \in \tau$. Let us discuss in example.

Example 3: Let $X = \{a,b,c,d\}$

$$\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,c,d\}, X\}$$

Now $\{a\}$ is closed. Since,

$$X - \{a\} = \{b,c,d\} \text{ is open. i.e } \in \tau$$

Similarly $\{b,d\}$ is closed. Since

$$\{b,d\}^c = X - \{b,d\} = \{a,c\} \in \tau$$

Example 4: Let $X = \{a,b,c,d\}$

$$\tau = \{\emptyset, \{c\}, \{a,c\}, \{b,d\}, \{b,c,d\}, X\}$$

Now, $\{a,c\}$ is open. i.e

$$\{a,c\}^c = X - \{a,c\} = \{b,d\} \in \tau$$

Remark:

- Closeness like openness is relative term.
- $\emptyset$ and X are always closed.
- A set can be open and closed at the same time.

Example: Let X be the set.

$\tau_{dis} = P(X)$. In discrete space all subsets of X are open and closed at the same time.

Topic 013

Properties of collection of closed sets

Let (X, τ) be the topological space. Consider the collection:

$$A = \{A \subset X \mid A^c \in \tau\}$$

A is the set of all those subsets of X whose compliment is open. Simply “Collection of all closed subsets”.

Example: Let $X = \{a,b,c\}$ and

$$\tau = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\}$$

Then, $A = \{X, \{b,c\}, \{a,b\}, \{c\}, \{b\}, \emptyset\}$ (compliments of τ)

is the collection of all closed subsets of X .

Example: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, d\}, \{a, c, d\}, X\}$ then,

$A = \{X, \{b, c, d\}, \{b, c\}, \{b\}, \emptyset\}$ is the collection of all closed subsets of X .

Example: Let $X = \mathbb{R}$ and $\tau = \{(a, \infty) \mid a \in \mathbb{R}\}$. Then,

$A = \{(-\infty, a) \mid a \in \mathbb{R}\}$ is the collection of all closed subsets of $\mathbb{R}$ with respect to τ .

Theorem: Let (X, τ) be a topological space. Then A , the collection of all closed subsets of X , satisfies the following properties.

1) $\emptyset, X \in \tau$.

2) A is closed under finite set union.

i.e $A_i \in A (i \in \{1, 2, \dots, n\}) \Rightarrow \cup_{i=1}^n A_i \in A$

3) A is closed under arbitrary set intersection.

i.e $A_i \in A (i \in I) \Rightarrow \cap_{i \in I} A_i \in A$

Proof: Let $A = \{A \subset X \mid A^c \in \tau\}$

1) $\emptyset \in A \quad \therefore \emptyset^c = X \in \tau$

$X \in A \quad \therefore X^c = \emptyset \in \tau$

2) Let $A_i \in A$ i.e $(A_i)^c \in A$ $i = 1, 2, \dots, n$

We have to show $\bigcup_{i=1}^n A_i \in A$??

$$(\bigcup_{i=1}^n A_i)^c = \bigcap_{i=1}^n (A_i)^c \in T$$

$\therefore$ The finite intersection of open sets is always open.

$$\Rightarrow \bigcup_{i=1}^n A_i \in A$$

3) Let $A_i \in A$ $i \in I$

$$\Rightarrow (A_i)^c \in T \quad \forall i \in I$$

We have to show that $\bigcap_{i \in I} A_i \in A$??

$$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} (A_i)^c \quad A_i \text{ are open.}$$

So, the union of open sets is always open. i.e $\bigcup_{i \in I} A_i \in T$

$$\Rightarrow \bigcap_{i \in I} A_i \in A \quad \text{proved!}$$

Topic 014

Intersection of Topologies

Let T_1 and T_2 be two topologies on X . Now what can we say about the intersection of T_1 and T_2 ?

Does $T_1 \cap T_2$ define a topology on X ?

Before answering to this question let us look in to an example.

Example: $X = \{a, b, c\}$ and

$$T_1 = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\}$$

$$T_2 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$$

$$\text{Then } T_1 \cap T_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$$

It is clear that $T_1 \cap T_2$ is a topology on X .

Theorem: Let $\{T_i \mid i \in I\}$ be any collection of topologies on set X . Then $\bigcap_{i \in I} T_i$ is also a topology on X .

Proof: Let $\{T_i \mid i \in I\}$. Now we have to show that $\bigcap_{i \in I} T_i$ is also a topology on X .

$$1) \emptyset \in \bigcap_{i \in I} T_i \quad \because T_i \text{ is a topology on } X \quad \forall i$$

$$\text{So } \emptyset \in T_i \quad \forall i$$

$$X \in \bigcap_{i \in I} T_i \quad \because T_i \text{ is a topology on } X \quad \forall i$$

$$\text{So } X \in T_i \quad \forall i$$

$$2) \text{Let } u_j \in \bigcap_{i \in I} T_i$$

$$\Rightarrow u_j \in T_i \quad \forall i$$

$$\text{But every } T_i \text{ is a topology. So, } \bigcup_{j \in J} u_j \in T_i \quad \forall i \in I$$

$$\Rightarrow \bigcup_{j \in J} u_j \in \bigcap_{i \in I} T_i \quad \text{Proved!}$$

$$3) \text{Let } u_j \in \bigcap_{i \in I} T_i \quad j=1,2,3,\dots,n$$

$$\Rightarrow u_j \in T_i \quad \forall i \in I \quad j \in J$$

But every T_i is a topology. $\forall i \in I$

So, $\bigcap_{j=1}^n u_j \in T_i \quad \forall i \in I$

$\Rightarrow \bigcap_{j=1}^n u_j \in \bigcap_{i \in I} T_i$ **Proved!**

So, $\bigcap_{i \in I} T_i$ is also a topology on X .

Topic 015

Union of Topologies

Let T_1 and T_2 be two topologies on X . Now what can we say about the union of T_1 and T_2 ?

As like intersection, can we say that $T_1 \cup T_2$ defines a topology on X ?

Let us see some examples.

Example: Let $X = \{a, b, c\}$ and

$$T_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$$

$$T_2 = \{\emptyset, \{b\}, \{a, b\}, X\}$$

$$T_1 \cup T_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

In this example $T_1 \cup T_2$ is a topology on X .

Does it mean that it is true in general?

No, unless we prove it.

Example: Let $X = \{a, b, c\}$ and

$$T_1 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$$

$$T_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

$$T_1 \cup T_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$$

Here, $T_1 \cup T_2$ is not a topology on X .

Since $\{b\} \cup \{c\} = \{b, c\} \notin T_1 \cup T_2$

Remark: Union of topologies on a set X may or may not be a topology on X .

Topic 016

Usual topology on $\mathbb{R}$

Real line: The real number set $\mathbb{R}$, one of the most studied/dominated object in Mathematics. Most of the concepts in Mathematics are the generalization/abstraction of the properties of the real numbers. To study $\mathbb{R}$ someone needs a usual structure so the rest of the terminologies can be defined according to that.

This usual structure is provided by usual topology.

Here we define usual topology on $\mathbb{R}$ by using order relation.

Open interval: Let $a, b \in \mathbb{R}$ with usual order relation “ $a < b$ ” then the open interval from a to b is;

$$(a, b) = \{x \mid a < x < b\} \subset \mathbb{R}$$

Open set in $\mathbb{R}$: A subset A of $\mathbb{R}$ is open iff for every $a \in A$ there exist an open interval I_a , containing a , which contained in A . i.e. $a \in I_a \subset A$.

Usual topology on $\mathbb{R}$: The set of all open sets of $\mathbb{R}$, T_u (as defined in above def.) is a topology on $\mathbb{R}$ called usual topology on $\mathbb{R}$.

Task: Prove that T_u is a topology on $\mathbb{R}$.

Proof: 1) $\emptyset \in T_u$	$\therefore \nexists x \in \emptyset$	Criteria
$X = \mathbb{R} \in T_u$	$\forall x \in \mathbb{R}$	
	$I_x \subset \mathbb{R}$	
		$A \subset \mathbb{R}, A \in T_u$ iff
		$\forall a \in A \quad \exists I_a$
2) Let $u_i \in T_u$	$i \in I$	s.t $a \in I_a \subset A$

$$\text{Let } x \in \bigcup_{i \in I} u_i$$

$$\Rightarrow x \in u_i \quad \text{for some } i \in I$$

But, u_i is open in $\mathbb{R}$. So, $\exists I_x \subset u_i \subset \bigcup_{i \in I} u_i$

$$\Rightarrow x \in I_x \subset \bigcup_{i \in I} u_i$$

$$\Rightarrow \bigcup_{i \in I} u_i \in T_u$$

3) Let $u_1, u_2 \in T_u$

$$x \in u_1 \cap u_2 \Rightarrow x \in u_1 \text{ and } u_2$$

But u_1 and u_2 both are open. So,

$\exists I_{1x}, I_{2x}$ such that $I_{1x} \subset u_1$ and $I_{2x} \subset u_2$

Now, $I_{1x} \cap I_{2x} = I_{3x} \subset u_1, u_2$

$$x \in \underbrace{I_{3x}}_{\text{open interval}} \subset u_1 \cap u_2 \Rightarrow u_1 \cap u_2 \in T_u$$

open interval

Hence, T_u is a topology on $\mathbb{R}$.

Topic 017

Usual Topology on $\mathbb{R}^2$

Real plane $\mathbb{R}^2$: The real plane is the Cartesian product of two real lines. i.e $\mathbb{R} \times \mathbb{R}$. The real plane is usually written as $\mathbb{R}^2$.

Open disk: An open disk $D_{(x,y)}$ in $\mathbb{R}^2$, centered at $(x,y) \in \mathbb{R}^2$ is the set of all those points (x',y') in $\mathbb{R}^2$ whose distance from (x,y) is less than r , where r is the positive real number.

$$D_{(x,y)} = \{(x',y') | \sqrt{(x-x')^2 + (y-y')^2} < r\} \subset \mathbb{R}^2$$

Open set in $\mathbb{R}^2$: A subset U of $\mathbb{R}^2$ is open in $\mathbb{R}^2$ iff for every $a=(x,y) \in U$ there exists an open disk D_a , which contained in U . i.e $a \in D_a \subset U$

Usual topology on $\mathbb{R}^2$: The set of all open sets of $\mathbb{R}^2$, T_u (as defined in above def.) is a topology on $\mathbb{R}^2$, called usual topology on $\mathbb{R}^2$.

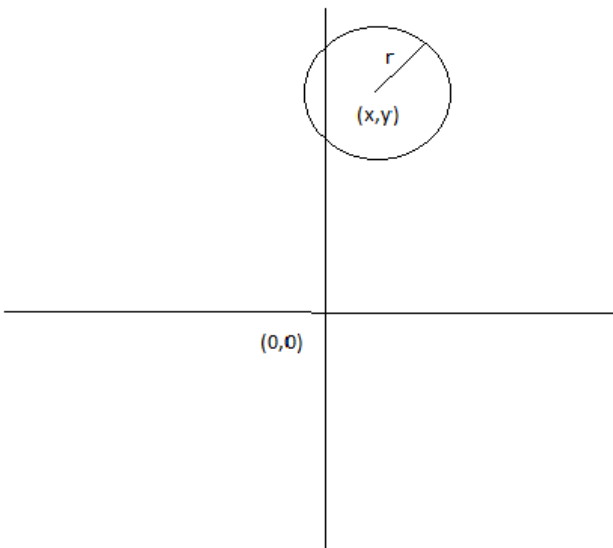


Figure: Open Disk

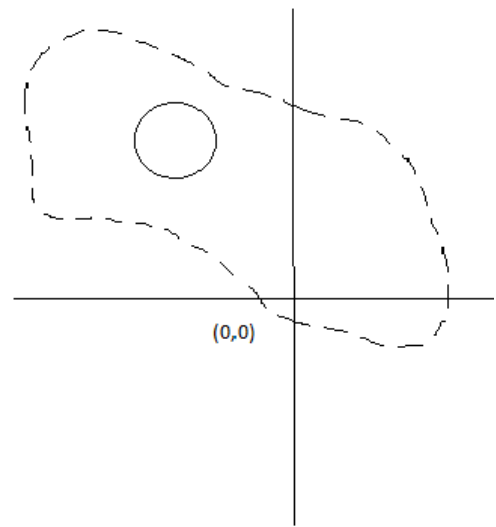


Figure: Open subset in $\mathbb{R}^2$

Task: Prove that T_u is the topology on $\mathbb{R}^2$.

Proof: Do by yourself.

Topic 018

Usual Topology in $\mathbb{R}^n$

Open n-Ball: An open ball B_x in $\mathbb{R}^n$ centered at

$x=(x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$, is the set of all those points $y=(y_1, y_2, y_3, \dots, y_n) \in \mathbb{R}^n$ whose distance from x is less than r , where r is the positive real number.

$$B_x = \{y: \sqrt{\sum_{i=1}^n (x_i - y_i)^2} < r\}$$

Open set in $\mathbb{R}^n$: A subset U of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ iff for every $a \in U$ there exist an open n -Ball B_a contained in U . i.e for every $a \in U \exists B_a$ such that $a \in B_a \subset U$.

Usual topology in $\mathbb{R}^n$: The set of all open sets of $\mathbb{R}^n$, T_u (as defined in above def.) is a topology on $\mathbb{R}^n$, called usual topology on $\mathbb{R}^n$.

Task: Prove that T_u is the topology on $\mathbb{R}^n$.

Proof: Do by yourself.

Topic 019

Comparison of Topologies

As we have already seen that more than one topologies can be defined on a set X with more than one element. So it is quite natural to ask the following question.

Question: Is there any relation between two different topologies?

As the topologies consist of collection of subset of X . So one may put a question on the size of two topologies.

But this question may not be answered properly unless we will not consider the relation between the elements of two topologies and the elements are the subsets of a set.

Naturally we have the relation that is, the set inclusion. And we have the positive answer of our question i.e “yes”. There can be class inclusion relation between two topologies.

Let T_1 and T_2 be two topologies on a non empty set X . Then T_1 is finer than T_2 if and only if $T_2 \subseteq T_1$.

i.e $\forall U \in T_2 \Rightarrow U \in T_1$

If T_1 is finer than T_2 then we also say that T_2 is weaker than T_1 .

Example: Let $X = \{a, b, c\}$

$$T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

$$T_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$$

It is clear that $T_2 \subset T_1$

Example: Let X be any non empty set. Let T be any topology on X and T_{inD} be the indiscrete topology on X . Then it is not very hard to show that $T_{\text{inD}} \subseteq T$.

$\therefore$ Every topology contains $\emptyset$ and X and $T_{\text{inD}} = \{\emptyset, X\}$.

Remark: Indiscrete topology is the weakest topology on a set X .

Example: Let X be any non empty set. Let T be any topology on X and T_{dis} be the discrete topology on X . Then it is not very hard to show that $T \subseteq T_{\text{dis}}$.

$\therefore$ Every $T \subseteq P(X) = T_{\text{dis}}$

Remark: Discrete topology is the finest topology on X .

Comparable topologies: Two topologies on a set X are comparable iff one is finer(weaker) than other.

Remark: Not all topologies on a set are comparable.

Example: Let $X = \{a, b, c, d\}$

$$T_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$

$$T_2 = \{\emptyset, \{c\}, \{c, d\}, X\}$$

T_1 and T_2 are not comparable since $T_1 \not\subseteq T_2$ and $T_2 \not\subseteq T_1$.
i.e no one is finer no one is weaker in comparison of each other.

Theorem: The collection $T = \{T_i\}$ of all topologies on X is partially ordered by class inclusion.

Hint:

Show that the following properties hold for all elements of T .

1. Reflexivity: $T \subseteq T$.

2. Antisymmetry: If $T \subseteq T'$ and $T' \subseteq T$, then $T = T'$.

3. Transitivity: If $T \subseteq T'$ and $T' \subseteq T''$, then $T \subseteq T''$.

Topic 020

Limit Point of a set

Limit Point: Let (X, T) be a topological space and A be a subset of X . A point $x \in X$ is a limit point of A iff every open set U_x containing x contains a point of A different from x . i.e $\forall U_x \in T, x \in U_x$

$$A \cap (U_x \setminus \{x\}) \neq \emptyset$$

Limit point is also called accumulation point or derived point.

Example:

Let $X = \{a, b, c, d, e\}$ and

$$T = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X . Consider $A = \{a, b, c\} \subset X$. Let us check all the elements of X one by one for limit point of A .

For a: a is not a limit point of A . Since there exist an open set $\{a\}$ containing a such that $A \cap (\{a\} \setminus \{a\}) = \emptyset$

For b: b is the limit point of $A = \{a, b, c\}$. Since all the open sets containing b intersect A other than b .

$$A \cap (\{b, c, d, e\} \setminus \{b\}) = \{c\} \neq \emptyset$$

$$A \cap (X \setminus \{b\}) = \{a, c\} \neq \emptyset$$

For c: c is not a limit point of $A = \{a, b, c\}$. Since there exist an open set $\{c, d\}$ containing c such that

$$A \cap (\{c, d\} \setminus \{c\}) = \emptyset$$

For d: d is a limit point of $A = \{a, b, c\}$. Since all the open sets containing d intersect A other than d .

$$A \cap (\{c, d\} \setminus \{d\}) = \{c\} \neq \emptyset$$

$$A \cap (\{a, c, d\} \setminus \{d\}) = \{a, c\} \neq \emptyset$$

$$A \cap (\{b, c, d, e\} \setminus \{d\}) = \{b, c\} \neq \emptyset$$

$$A \cap (X \setminus \{d\}) = A \neq \emptyset$$

For e: e is the limit point of $A = \{a, b, c\}$. Since all the open sets containing e intersect A other than e.

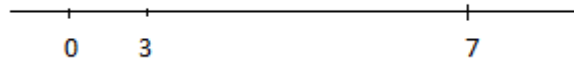
$$A \cap (\{b, c, d, e\} \setminus \{e\}) = \{b, c\} \neq \emptyset$$

$$A \cap (X \setminus \{e\}) = A \neq \emptyset$$

Example: Let $X = \mathbb{R}$ with usual topology.

$A = (0, 3)$, Is 7 the limit point of this open interval?

The answer is No.



Is 1.5 a limit point of A?? Answer is Yes.

Is 3 a limit point of A?? Answer is Yes.

Example:

Let $X = \mathbb{R}$ with usual topology.

$$B = \{1, 1/2, 1/3, 1/4, \dots\} = \{1/n \mid n \in \mathbb{N}^*\}$$

In this 0 is a limit point of B.

Topic 021

Derived Set

Derived Set: Let (X, T) be a topological space and A be a subset of X . Then the derived set of A , denoted as A' , is the set of all limit points of A .

$$\text{i.e } A' = \{x \in X \mid U_x \in T, A \cap (U_x \setminus \{x\}) \neq \emptyset\}$$

Remarks:

- Derived set of A may or may not be the subset of A .
- Derived set of empty set is empty.

Example 1: Let $X = \{a, b, c, d, e\}$ and

$$T = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X .

Consider $A = \{a, b, c\} \subset X$. then,

$$A' = \{b, d, e\}$$

Consider $B = \{b, e\} \subset X$. then,

$$B' = \{b, e\}$$

Example 2: Let $X = \mathbb{R}$ with usual topology and

$$A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}. \text{ Then } A' = \{0\}$$

Example 3: Let $X = \mathbb{R}$ with usual topology and

$$A = (-1, 2). \text{ Then } A' = [-1, 2]$$

Example 4: Let $X=\mathbb{R}$ with usual topology and

$A=\mathbb{Q}$. Then $A' = \mathbb{R}$.

Topic 022

Derived set: Special Topologies

As we have studied two special topologies on a set X named as “Indiscrete Topology” and “Discrete Topology”

Let us figure out the derived set of a subset of X with respect to these special topologies.

Consider an indiscrete space X , i.e X with indiscrete topology $T_{\text{inD}} = \{\emptyset, X\}$

Note that for every point x in X there is one and only one open set containing x and that is X itself. 

So a point $x \in X$ will be limit point of some subset A of X iff $A \cap (X \setminus \{x\}) \neq \emptyset$

Now if $A=\{y\}$ then any point $x \neq y$ of X satisfies the above said condition. i.e $\{y\} \cap (X \setminus \{x\}) = \{y\} \neq \emptyset$

So, in this case: $A' = A^c$

Now, if A contains two or more points then any point x of X satisfies $A \cap (X \setminus \{x\}) = A \setminus \{x\} \neq \emptyset$.

So in this case, $A' = X$.

Let us summarize it as our concluding remark.

Remark:

In an indiscrete space X , the derived set A' of any subset A of X is as follows:

$$A' = \begin{cases} \emptyset & \text{if } A = \emptyset \\ A^c = X \setminus \{y\} & \text{if } A = \{y\} \\ X & \text{if otherwise} \end{cases}$$

Now Let us discuss for discrete topology.

Consider a discrete space X . i.e X with discrete topology

$$\tau_{\text{dis}} = \mathcal{P}(X)$$

Note that for any point x in X every subset of X containing x is open. 

So a point $x \in X$ will be a limit point of some subset A of X iff every subset U of X containing x must intersect with A other than x .

But for every x in a discrete space there is a singleton set containing x i.e $\{x\}$ which does not intersect with A other than x .

So no point of X can be limit point of any subset A of X .

Following is our concluding remark.

Remark: In a discrete space X , the derived set A' of any subset A of X is empty.

Topic 023

Derived set: Some Results

In this topic we prove some results related to derived set. Before going towards the theorem. Let us see when does a point $x \in X$ not belong to derived set of some set A i.e. not be a limit point of A .

So it means that $x \notin A'$ iff there exist an open set U_x containing x such that $(U_x \setminus \{x\}) \cap A = \emptyset$.

or equivalently:

Iff there exist an open set U_x containing x such that either $U_x \cap A = \emptyset$ or $U_x \cap A = \{x\}$.

or equivalently:

Iff there exist an open set U_x containing x such that $U_x \cap A \subset \{x\}$.

Theorem: Let (X, T) be a topological space. Let $A, B \subset X$. If $A \subset B$ then $A' \subset B'$.

Proof:

Let $x \in A' \Rightarrow \forall U_x$ (open set containing x)

$$(U_x \setminus \{x\}) \cap A \neq \emptyset$$

Now given $A \subset B$ so,

$$\emptyset \neq (U_x \setminus \{x\}) \cap A \supset (U_x \setminus \{x\}) \cap B$$

$$\Rightarrow x \in B'$$

Proved!

Theorem: Let (X, T) be a topological space. Let $A, B \subset X$.

Then $(A \cup B)' = A' \cup B'$.

Proof: Let us first show that $A' \cup B' \subset (A \cup B)'$

$$A \subset A \cup B \Rightarrow A' \subset (A \cup B)'$$

$$B \subset A \cup B \Rightarrow B' \subset (A \cup B)'$$

$$\text{So, } A' \cup B' \subset (A \cup B)'$$

Now we show that $(A \cup B)' \subset A' \cup B'$. For this we take an element from $(A' \cup B')^c$ and show that it belongs to $((A \cup B)')^c$.

Let $x \notin (A' \cup B')$ this implies that $x \notin A'$ and $x \notin B'$

$$\Rightarrow \exists U_x \text{ and } V_x \text{ (open sets containing } x)$$

Such that $U_x \cap A \subset \{x\}$ and $V_x \cap B \subset \{x\}$

Now $U_x \cap V_x$ is an open set containing x and

$$(U_x \cap V_x) \cap (A \cup B) = ((U_x \cap V_x) \cap A) \cup ((U_x \cap V_x) \cap B) \subset (U_x \cap A) \cup (V_x \cap B) \subset \{x\} \cup \{x\} = \{x\}$$

$$\Rightarrow x \notin (A \cup B)' \Rightarrow (A \cup B)' \subset A' \cup B'$$

So $(A \cup B)' = A' \cup B'$ is proved.

Topic 024

Closed set and Derived set

As we have already studied that a subset A of a topological space X is closed iff A^c is open.

In this topic we characterize the closed set in terms of its limit points. It provides us a criterion to check closeness of a set.

Theorem: A subset A of a topological space X is closed iff $A' \subset A$.

Example: Consider $X = \{a, b, c, d\}$ with

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, c\}, \{a, b, d\}, X\}$$

Consider $A = \{b, d\}$, $B = \{d\}$, $C = \{b, c, d\}$

Now A and C are closed since $A' = \{b, d\} = A$ and $C' = \{b, d\} \subset C$ but B is not closed because $B' = \{b\} \not\subset B$.

Example: Consider $\mathbb{R}$ with usual topology. Consider following subsets of $\mathbb{R}$.

- $A = [0, 1)$ not closed
- $B = \{0, 1\}$ closed
- $C = \{1, 1/2, 1/3, \dots\}$ not closed

Topic 025

Closed set and Derived set

Proof: Let A is closed. Now we show that $A' \subset A$.

Let $x \notin A \Rightarrow x \in A^c$

$\therefore A$ is closed. So A^c is open and we have an open set A^c containing x that does not intersect A at all. i.e

$$A^c \cap A = \emptyset \Rightarrow (A^c \setminus \{x\}) \cap A = \emptyset$$

So, $x \notin A' \Rightarrow A' \subset A$

Now, let us prove the converse.

Let $A' \subset A$. Now we show that A is closed or equivalently A^c is open.

Let $x \in A^c \Rightarrow x \notin A \Rightarrow x \notin A'$ as given $A' \subset A$

$\Rightarrow \exists U_x(\text{open set containing } x) \text{ s.t. } (U_x \setminus \{x\}) \cap A = \emptyset$

But $x \notin A$ so, $U_x \cap A = \emptyset \Rightarrow U_x \subset A^c$.

So for every x in A^c we have U_x (open set containing x) s.t. $U_x \subset A^c$.

And we can write A^c as union of all these open subsets i.e.

$$A^c = \bigcup_{x \in A^c} U_x.$$

This shows that A^c is open and this implies that A is closed. **Proved!**

Topic 026

Criterion for openness

Let (X, T) be a topological space and we have already studied that the elements of T are called the open sets.

Now the question is “Does there exist any other criterion to check the openness”? The answer is Yes. Let us state it in following theorem.

Theorem: A subset A of topological space X is open iff

$$\forall x \in A \exists U_x (\text{open set containing } x) \text{ s.t. } U_x \subset A.$$

Example: Consider $X = \{a, b, c, d\}$ and

$$T = \{\emptyset, \{a\}, \{a,c\}, \{b,d\}, \{a,b,d\}, X\}$$

Consider

- $A = \{a, b, d\}$
- $B = \{a, b\}$
- $C = \{d\}$

Now A is open but B and C are not open.

Example: Consider $\mathbb{R}$ with usual topology.

- $A = (0, 1]$ not open set
- $B = \{1, 4\}$ not open set

Example: Consider $\mathbb{R}^2$ with usual topology.

- $A = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ not open set
- $B = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$ open set

Proof: One way is very obvious. i.e

Let A is open. It is obvious that $\forall x \in A \exists U_x$ (open set containing x) s.t. $U_x \subset A$.

$\therefore$ For every $x \in A$ there exists always the best candidate, that satisfies the condition, is A itself. So,

$$\forall x \in A \exists U_x = A \text{ (open set containing } x \text{) s.t. } A \subset A.$$

Let us prove the converse.

Let $\forall x \in A \exists U_x$ (open set containing x) s.t. $U_x \subset A$. So we can write A as union of all U_x i.e.

$$A = \bigcup_{x \in A} U_x$$

This shows that A is open.

Proved!

Topic 027

Closure of a Set

Closure of a set: Let (X, T) be a topological space and A be a subset of X . The closure of A , denoted as $\bar{A}$, is the intersection of all closed subsets of X containing A . i.e

$\bar{A} = \bigcap_{C \in C_A} C$ where C_A is the class of all closed subsets of X containing A .

Example 1: Let $X = \{a, b, c, d, e\}$ and

$T = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ be a topology of X .

So, here we have the closed collection

$$C = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \emptyset\}$$

Consider $A = \{c\} \subset X$. Then $C_A = \{X, \{c, d, e\}\}$ is the class of all closed subsets of X containing A . So,

$$\bar{A} = \bigcap_{C \in C_A} C = \{c, d, e\} \cap X = \{c, d, e\}$$

Example 2: Let $X = \{a, b, c, d, e\}$ and

$T = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$ be a topology of X .

So, here we have the closed collection

$$C = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \emptyset\}$$

Consider $B = \{a, b\} \subset X$. Then $C_B = \{X, \{a, b, e\}\}$ is the class of all closed subsets of X containing B . So,

$$\bar{B} = \bigcap_{C \in C_B} C = \{a, b, e\} \cap X = \{a, b, e\}$$

Remarks:

- $A \subset \bar{A}$
- $\bar{A}$ is the smallest closed superset of A .
- $\overline{\emptyset} = \emptyset$
- $\bar{X} = X$

Topic 028

Computation of Closure

For a given topological space X and its subset A , one way to compute the closure of A is to apply the definition in the following three steps.

1. List all the closed subsets of X .

2. Write C_A the set of all closed subsets of X containing A .

3. Take intersection of all the elements of C_A .

The second way to compute closure of A is to figure out the smallest closed superset of A .

Example: Let $X=\mathbb{R}$ with usual topology and $A=(0,2]$. Then the smallest closed superset of A is $[0,2]$.

So, $\bar{A} = [0,2]$.

Example: Let $X=\mathbb{R}^2$ with usual topology and

$$A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}.$$

Then the smallest closed superset of A is

$$\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

$$\text{So, } \bar{A} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

The third way to compute $\bar{A}$ is to use the relation between the $\bar{A}$, A and A' given in the following theorem.

Theorem 1: Let A be a set of topological space X . Then

$$\bar{A} = A \cup A'$$

Proof: Let us first prove that $A \cup A' \subset \bar{A}$.

Now $A \subset \bar{A}$. Now let $x \in A' \Rightarrow \forall U_x$ (open set containing x)

$(U_x \setminus \{x\}) \cap A \neq \emptyset \Rightarrow U_x \cap A \neq \emptyset \Rightarrow x \in \bar{A}$ (By th. 2)

$$\Rightarrow A' \subset \bar{A} \Rightarrow A \cup A' \subset \bar{A}$$

Now let us prove the converse.

$$\bar{A} \subset A \cup A'$$

Let $x \in \bar{A}$ if $x \in A$ we have done.

$$\Rightarrow \bar{A} \subset A \subset A \cup A'$$

If $x \notin A$. But $x \in \bar{A}$ (By theorem 2)

$$\forall U_x, U_x \cap A \neq \emptyset$$

$$(U_x \cap A \setminus \{x\}) \neq \emptyset \Rightarrow x \in A'$$

$$\Rightarrow x \in A \cup A' \Rightarrow \bar{A} \subset A \cup A'$$

$$\Rightarrow \bar{A} = A \cup A'$$

Proved!

Theorem 2: Let A be a subset of topological space X . Then $x \in \bar{A}$ iff for every open set U_x containing x intersects A .

Topic 029

Properties of Closure of a set

Let us prove some results related to closure of a set.

Proposition:

Let A be a subset of topological space X . Then

1. $\bar{A}$ is closed subset of X .

2. A is closed iff $\bar{A} = A$.

3. $\overline{(\bar{A})} = \bar{A}$

Proof:

1. $\bar{A}$ is closed subset of X .

By definition:

$$\bar{A} = \bigcap_{C \in \mathcal{C}_A} C \quad C \text{ is the closed superset of } A$$

We know that the intersection of closed sets is closed.

$\Rightarrow \bar{A}$ is closed.

Proved!

2. A is closed iff $\bar{A} = A$.

Let A is closed.

$\therefore A \subset \bar{A}$ and $\bar{A}$ is the smallest closed superset of A .

$\Rightarrow \bar{A} \subset A$

Hence, $\bar{A} = A$.

Now let $\bar{A} = A$. Since $\bar{A}$ is closed so A is also closed.

Proved!

3. $\overline{(\bar{A})} = \bar{A}$

Since $\bar{A}$ is closed. Then by (2) i.e A is closed iff $\bar{A} = A$.

Hence, $\overline{(\bar{A})} = \bar{A}$

Proved!

Theorem: Let X be a topological space and $A, B \subset X$. If $A \subset B$ then $\bar{A} \subset \bar{B}$.

Proof: $A \subset B \Rightarrow A' \subset B'$

$\Rightarrow A \cup A' \subset B \cup B' \Rightarrow \bar{A} \subset \bar{B}$

Proved!

Theorem: Let X be a topological space and $A, B \subset X$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof: First we will prove that $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

By previous theorem:

$A \subset A \cup B \Rightarrow \bar{A} \subset \overline{A \cup B}$

Also $B \subset A \cup B \Rightarrow \bar{B} \subset \overline{A \cup B}$

$\Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$

Now let us prove that $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$

We know that $A \subset \bar{A}$ and $B \subset \bar{B}$

$\Rightarrow A \cup B \subset \bar{A} \cup \bar{B}$

$\Rightarrow \overline{A \cup B} \subset \overline{\bar{A} \cup \bar{B}}$

But $\bar{A}$ and $\bar{B}$ both are closed so $\bar{A} \cup \bar{B}$ is closed.

We also know that a set is closed iff set = closure of itself.

$$\Rightarrow \overline{A \cup B} \subset \overline{A} \cup \overline{B}$$

$$\text{Hence, } \overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proved!

Topic 030

Dense Set

Dense set: Let A be a subset of a topological space X . Then A is said to be dense in X iff every point $p \in X$ is either point of A or limit point of A .

Or in other words:

$$A \text{ is dense in } X \text{ iff } \overline{A} = X$$

Example 1: Let $X = \{a, b, c, d, e\}$ and

$$\tau = \{\emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X .

Consider $A = \{a, b, c\}$, $B = \{b, e\} \subset X$.

Then A is dense in X . Since

$$\overline{A} = \{a, b, c, d, e\}$$

But B is not dense in X . Since

$$\overline{B} = \{b, e\} \neq X$$

Example 2: Let $X = \mathbb{R}$ with usual topology and consider $\mathbb{Q} \subset \mathbb{R}$. Then

$$\overline{\mathbb{Q}} = \mathbb{R}$$

So $\mathbb{Q}$ is dense in $\mathbb{R}$.

Now consider $\mathbb{N} \subset \mathbb{R}$. Then

$$\overline{\mathbb{N}} = \mathbb{N} \neq \mathbb{R}$$

So $\mathbb{N}$ is not dense in $\mathbb{R}$.

Remarks:

- X is dense in itself.

Proof: Since $\overline{X} = X$

- A proper closed subset of X can never be dense in X .

Proof: Let A be a closed proper subset of X . Then

$$\overline{A} = A \neq X$$

Topic 031

Convergent Sequence

We will define convergence of a sequence in the language of topology.

Definition:

Let (X, τ) be a topological space and $\{x_1, x_2, \dots\} \subset X$ be a sequence of points in X . The sequence $\{x_n\}$ converges to a point $x \in X$ if for every U_x (open set containing x) in X , there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the points $x_n \in U_x$.

If $\{x_n\}$ converges to x then we say that x is the limit of the sequence $\{x_n\}$ and it is written as:

$$\lim_n x_n = x \quad \text{or} \quad x_n \rightarrow x$$

Example 1: Consider the real line $\mathbb{R}$ with usual topology.

Let $\{1, 1, \dots\}$ be the constant sequence. We know that the constant sequence always converge to the point from where it is defined i.e 1.

Because according to the definition of convergence in topology, if we take any open set containing 1 i.e open interval. We will get complete sequence in that interval containing 1. Here N is 1.

Example 2: Consider the real line $\mathbb{R}$ with usual topology.

Let $\{1, 2, 3, \dots\}$ be the sequence. This sequence does not converge to a point because we can't get any real number containing the infinite part of the sequence.

Topic 032

Examples: Convergent Sequence

Example 1: Consider the real line $\mathbb{R}$ with usual topology.

Consider the sequence $\{\frac{1}{n}\}$. We have studied in calculus that this sequence converges to 0.

According to the definition in topology, we have to show that there is a point 0 on the real line $\mathbb{R}$ such that for that point 0 for every open set containing 0, N there exist a n such that for every $n \geq N$ all the terms of the sequence should contain in that open set.

Example 2: Let X be an indiscrete space and $\{x_n\} \subset X$ be a sequence of points in X . Let y be a point in X . Now note that:

- There is only one open set containing y that is X itself.
- X contains all the terms of $\{x_n\}$.

So $x_n \rightarrow y$

Remark: In an indiscrete space X any sequence in X converges to any point in X .

Example 3: Let X be a discrete space and $\{x_1, x_2, x_3, \dots\} \subset X$ be a sequence of points in X . Now note that: 

- There is a singleton open set containing y that is $\{y\}$.
So x_n can not converge to y unless there exists an $N \in \mathbb{N}$ such that $x_n = y$ for all $n \geq N$.

Example 4: Consider $\mathbb{R}$ with usual topology and consider the sequence $\{(-1)^n\}$.

This sequence does not converge to a single point. Hence, it diverges.

Example 5: Consider $\mathbb{R}$ with partition topology

$$\tau_p = \{[4a - 2, 4a + 2) \mid a \in \mathbb{Z}\}$$

$$\tau_p = \{\dots, [-6, -2), [-2, 2), [2, 6), [6, 10), \dots\}$$

and consider the sequence $\{(-1)^n\}$.

This sequence converges to x and $x \in [-2, 2)$.

Topic 033

Interior Point

Interior point: Let (X, τ) be a topological space and A be a subset of X . A point $a \in A$ is an interior point of A iff

there exists U_a (open set containing a) such that $U_a \subset A$.

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X .

Let $A = \{a, b, c\}$.

Here, a and b are interior points of A .

Example 2: Let $X = \mathbb{R}$ with usual topology and

$$A = [0, 1).$$

$\frac{1}{2}$ is an interior point of A . Because we find an open interval $(0.3, 0.7)$ containing $\frac{1}{2}$ that is a subset of A .

Example 3: Let $X = \mathbb{R}^2$ with usual topology and $A = \{(x, y) | x^2 + y^2 < 3\}$.

Topic 034

Interior of a set

Definition: Let (X, τ) be a topological space and A be

a subset of X . Interior of A is the set of all interior points of A . i.e $\{a \in A | \exists U_a \in \tau, a \in U_a \subset A\}$.

Interior of A is denoted by $\text{int}(A)$ or A° .

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X .

Let $A = \{a, b, c\}$

$A^o = \{a, b\}$ because a and b are the interior points of A .

$B = \{c, d\} \quad B^o = \emptyset$

$C = \{a, b, c, d\} \quad C^o = X$

$X^o = X$

Example 2: Let $X = \mathbb{R}$ with usual topology.

$A = (0, 1] \quad A^o = (0, 1)$

$B = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \quad B^o = \emptyset$

$C = \mathbb{Q} \quad \mathbb{Q}^o = \emptyset$

Example 3: Let $X = \mathbb{R}^2$ with usual topology.

$A = \{(x, y) | x^2 + y^2 \leq 1\}$

$A^o = \{(x, y) | x^2 + y^2 < 1\}$

$B = \{(x, y) | x = y\} \quad B^o = \emptyset$

Remarks:

$$\diamond A^o \subset A$$

$$\diamond \emptyset^o = \emptyset$$

$$\diamond X^o = X$$

Topic 035

Properties of Interior

Proposition: Let A be a subset of topological space X .
Then

1. A^o is the union of all open subsets of X contained in A .
2. A^o is open in X .
3. A^o is the largest open subset of X contained in A i.e if there exists an open subset B contained in A then $B \subset A^o \subset A$.
4. A is open iff $A = A^o$.

Proof:

1.

First we have to show that interior is the union of all open subsets of X contained in A . i.e

Let $\mathcal{u}$ is the class of all open subsets of X contained in A .
Then $A^o = \bigcup_i u_i$ $u_i \in \mathcal{u}$

Let us first show that interior of A is the subset of this union.

Let $a \in A^o$ i.e $\exists U_a$ such that $a \in U_a \subset A$

$$\Rightarrow U_a \in \mathcal{u}$$

$$\Rightarrow A^o \subset \bigcup_i u_i$$

Now let $x \in \bigcup_i u_i \Rightarrow \exists j$ such that $x \in u_j$.

But u_j is open subset of A . Because $u_j \in \mathcal{u}$.

It means $x \in A^o$. $\Rightarrow \bigcup_i u_i \subset A^o$

Hence, $A^o = \bigcup_i u_i$

Proved!

2.

We have to show that A^o is open.

By 1. i.e $A^o = \bigcup_i u_i$

As we know union of open sets is open.

Hence, A^o is open.

Proved!

3.

A^o is the largest open subset of X contained in A i.e if there exists an open subset B contained in A then

$$B \subset A^o \subset A.$$

Since B is open i.e $B \in \mathcal{u}$ and we know $A^o = \bigcup_i u_i$

$\exists j$ such that $u_j = B$.

Hence, $B \subset A^o \subset A$.

Proved!

4.

A is open iff $A = A^o$.

If A is open then by 3. $A \subset A^o \subset A$.

$\Rightarrow A^o = A$

If $A = A^o$ then, as we know A^o is open, hence, A is open.

Proved!

Topic 036

Exterior of a set

Exterior point: Let (X, τ) be a topological space and A be a subset of X . A point $b \in A^c$ is an exterior point of A iff there exists U_b (open set containing b) such that $U_b \subset A^c$.

i.e Exterior point of A is interior point of A^c .

Exterior of a set: Let (X, τ) be a topological space and A be a subset of X . Then exterior of A denoted as $Ext(A)$,

is the interior of A^c . i.e $Ext(A) = (A^c)^o$

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X .

Let $A = \{a, c\}$

$$Ext(A) = (A^c)^o$$

$$A^c = \{b, d\} \quad (A^c)^o = \{b\} = Ext(A)$$

Let $B = \{d\}$

$$B^c = \{a, b, c\} \quad (B^c)^o = \{a, b, c\} = B^c = Ext(B)$$

Example 2: Let $X = \mathbb{R}$ with usual topology and

$$A = (-1, 2)$$

$$A^c = (-\infty, 1] \cup [2, \infty)$$

$$(A^c)^o = (-\infty, 1) \cup (2, \infty) = Ext(A)$$

Now let $\mathbb{Q}$

$$\mathbb{Q}^c = \mathbb{Q}'$$

$$(\mathbb{Q}^c)^o = (\mathbb{Q}')^o = \emptyset = Ext(\mathbb{Q})$$

Topic 037

Boundary of a set

Boundary point: Let (X, τ) be a topological space and A be a subset of X . A point $x \in X$ is called a boundary point of A iff for all U_x (open set containing x)

$$U_x \cap A \neq \emptyset$$

and

$$U_x \cap A^c \neq \emptyset$$

Boundary point is also called **Frontier point.**

Boundary of a set: Let (X, τ) be a topological space and A be a subset of X . Boundary of A is the set of all boundary points of A .

Boundary of A is denoted as $fr(A)$ or $bd(A)$.

$$bd(A) = (A^o \cup Ext(A))^c$$

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X .

Let $A = \{a, b, c\}$. We have to find boundary points of A .

$$A^c = \{d\}$$

$$a: \{a\} \cap A = \{a\} \neq \emptyset \quad \{a\} \cap A^c = \emptyset$$

a is not a boundary point of A .

$$b: \{a, b\} \cap A = \{a, b\} \neq \emptyset \quad \{a, b\} \cap A^c = \emptyset$$

b is not a boundary point of A .

$$c: \{a, c, d\} \cap A = \{a, c\} \neq \emptyset \quad \{a, c, d\} \cap A^c = \{d\} \neq \emptyset$$

$$X \cap A = \{a, b, c\} \neq \emptyset \quad X \cap A^c = \{d\} \neq \emptyset$$

c is a boundary point of A .

$$d: \{a, c, d\} \cap A = \{a, c\} \neq \emptyset \quad \{a, c, d\} \cap A^c = \{d\} \neq \emptyset$$

$$X \cap A = \{a, b, c\} \neq \emptyset \quad X \cap A^c = \{d\} \neq \emptyset$$

d is a boundary point of A .

Example 2: Consider $X = \mathbb{R}$ with usual topology.

$$\text{Let } A = [0, 2) \quad bd(A) = \{0, 2\}$$

$$B = \mathbb{Q} \quad bd(\mathbb{Q}) = \mathbb{R}$$

Example 3: Consider $X = \mathbb{R}^2$ with usual topology.

$$\text{Let } A = \{(x, y) | x^2 + y^2 < 4\}$$

$$bd(A) = \{(x, y) | x^2 + y^2 = 4\}$$

Remarks:

- $bd(A)$ is a closed set.
- Boundary of a set A is empty iff A is both open and closed.

Topic 038

Partition of a set

For a subset A of a topological space X we discussed following three types of sets:

1. Interior of A

2. Exterior of A

3. Boundary of A

The set X can be partitioned by these three sets. i.e

$$X = A^o \cup Ext(A) \cup bd(A)$$

Example: Let $X = \mathbb{R}$ with usual topology.

Let $A = (0,1]$

$$A^o = (0,1)$$

$$Ext(A) = (A^c)^o = (-\infty, 0) \cup (1, \infty)$$

$$bd(A) = \{0,1\}$$

$$X = \mathbb{R} = (0,1) \cup (-\infty, 0) \cup (1, \infty) \cup \{0,1\}$$

Now we will prove this fact. i.e

$$X = A^o \cup Ext(A) \cup bd(A)$$

Let $A \subset X$

As we know:

$$A^o \subset X, \text{Ext}(A) \subset X, \text{bd}(A) \subset X$$

$$\Rightarrow A^o \cup \text{Ext}(A) \cup \text{bd}(A) \subset X$$

Now let $x \in X$

Either $x \in A$ or $x \in A^c$

If $x \in A$ then we have two options. i.e

$$x \in A^o \quad \text{or} \quad x \notin A^o$$

$$\text{If } x \in A^o \text{ then } X \subset A^o \cup \text{Ext}(A) \cup \text{bd}(A)$$

$$\text{But if } x \notin A^o, \nexists U_x \text{ s.t. } x \in U_x \subset A$$

$$\Rightarrow \forall U_x, U_x \cap A \neq \emptyset \text{ and } U_x \cap A^c \neq \emptyset$$

$$\Rightarrow x \in \text{bd}(A) \Rightarrow X \subset A^o \cup \text{Ext}(A) \cup \text{bd}(A)$$

Now if $x \in A^c$

$$\text{Then } x \in \text{Ext}(A) = (A^c)^o$$

$$\Rightarrow X \subset \text{Ext}(A) \subset A^o \cup \text{Ext}(A) \cup \text{bd}(A)$$

$$\text{But if } x \notin (A^c)^o$$

$$\Rightarrow \forall U_x, U_x \not\subset A^c$$

$$\Rightarrow U_x \cap A^c \neq \emptyset \text{ and } U_x \cap A \neq \emptyset$$

$$\Rightarrow x \in \text{bd}(A)$$

$$\Rightarrow X \subset bd(A) \subset A^o \cup Ext(A) \cup bd(A)$$

Hence, $X = A^o \cup Ext(A) \cup bd(A)$

Proved!

Topic 039

Neighborhood System

Neighborhood of a point: Let p be point in a topological space (X, τ) . A subset N of X is a neighborhood of p iff there exists an open set U_p such that

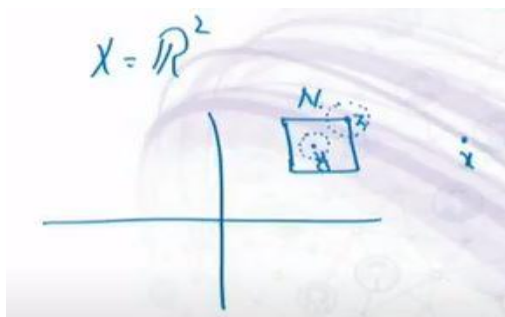
$$p \in U_p \subset N$$

One may say that N is a neighborhood of p iff p is an interior point of N .

Open Neighborhood: A neighborhood N of a point p is called an open neighborhood iff N is a neighborhood of all of its points. i.e N is an open set.

Neighborhood System: The class of all neighborhoods of $p \in X$ is called the neighborhood system of p . It is denoted by N_p .

Graphical Explanantion:



Example: Let $X = \{a, b, c\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ be a topology of X .

Here, $\{a, b\}$ is a neighborhood of point b .

$\{b, c\}$ is not a neighborhood of point b .

Example: Let $X = \mathbb{R}$ with usual topology.

Let $A = [0, 1)$

A is not a neighborhood of 0 and 1.

$B = (-\frac{1}{2}, \frac{1}{2})$

B is a neighborhood of its each point.

$C = \{1, 2, 3\}$

C is not a neighborhood of its any point.

Example: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X .

All the neighborhoods of point b are:

$\{a, b\}, \{a, b, c\}, \{a, b, d\}, X$

$N_b = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$

Topic 040

Properties: Neighborhood System

Proposition: Let N_p is the neighborhood system of p .
Then:

1. N_p is non empty.
2. $p \in \cap_{N_p \in N_p} N_p$.
3. N_p is closed under finite set intersection.
4. Every superset of a member of N_p belongs to N_p .

Proof:

1. N_p is non empty.

Note that X always is the neighborhood of p . i.e $X \in N_p$ 

Hence, N_p is non empty.

Proved!

2. $p \in \cap_{N_p \in N_p} N_p$

We have to show that $p \in N \quad \forall N \in N_p$

By definition:

N is neighborhood of p iff there exists an open set U_p such that

$$p \in U_p \subset N$$

$$\Rightarrow p \in \bigcap_{N_p \in \mathcal{N}_p} N_p \quad \textbf{Proved!}$$

3. N_p is closed under finite set intersection.

Let $M, N \in \mathcal{N}_p$

$\Rightarrow \exists U_p, V_p$ such that

$$p \in U_p \subset N \quad \text{and} \quad p \in V_p \subset M$$

$$\Rightarrow p \in U_p \cap V_p \subset N \cap M$$

(Intersection of two open sets is open)

$N \cap M$ is neighborhood of p .

$$\Rightarrow N \cap M \in \mathcal{N}_p \quad \textbf{Proved!}$$

4. Every superset of a member of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

Let $N \in \mathcal{N}_p$

$$\Rightarrow \exists U_p \text{ such that } p \in U_p \subset N$$

$$\text{Let } N \subset M \Rightarrow p \in U_p \subset N \subset M$$

$$\Rightarrow p \in U_p \subset M$$

Hence, M is the neighborhood of point p .

Proved!

Topic 041

Subspace Topology

In this topic, we just define another topology on the subset of set X that will be known as subspace topology.

Subspace Topology: Let A be a non empty subset of a topological space (X, τ) . Then the collection

$$\tau_A = \{V = U \cap A \mid U \in \tau\}$$

of subsets of A defines a topology on A .

τ_A is called subspace topology or relative topology on A relative to (X, τ) .

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X .

Let $A = \{a, b, c\}$

$$\tau_A = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, A\}$$

Example 2: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X .

Let $A = \{a, c\}$

$$\tau_A = \{\emptyset, \{a\}, A\}$$

Example 3: Let $X = \mathbb{R}$ with usual topology.

Let $A = \mathbb{Z}$

Let $\left(-\frac{1}{2}, \frac{1}{2}\right) \cap \mathbb{Z} = \{0\}$. In this way:

$\forall x \in \mathbb{Z} \quad \exists U \in \tau_{\mathbb{R}}$ such that $\{x\} = U \cap \mathbb{Z}$

Hence, $\tau_{\mathbb{Z}}$ is discrete topology on $\mathbb{Z}$ relative to $\tau_{\mathbb{R}}$.

Topic 042

Proof: Subspace Topology

Let A be a non empty subset of a topological space (X, τ) .
Then show that the collection

$$\tau_A = \{V = U \cap A \mid U \in \tau\}$$

of subsets of A defines a topology on A .

Proof:

$$\tau_A = \{V = U \cap A \mid U \in \tau\}$$

We have to show that τ_A satisfies all the axioms of a topology.

$\emptyset \in \tau_A \quad \because \emptyset \in \tau$ and we can write $\emptyset = \emptyset \cap A$

$A \in \tau_A \quad \because X \in \tau$ and we can write $A = X \cap A$

Now let $V_i \in \tau_A$ then $\exists U_i \in \tau$ such that $V_i = U_i \cap A$

Now $\cup_i V_i = \cup_i (U_i \cap A) = (\cup U_i) \cap A$

$\because \cup U_i \in \tau \Rightarrow \cup_i V_i \in \tau_A$

Now let $V_1, V_2 \in \tau_A$ then $\exists U_1, U_2 \in \tau$ such that

$V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$

$\Rightarrow V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A$

$\because U_1 \cap U_2 \in \tau \Rightarrow V_1 \cap V_2 \in \tau_A$

Hence, τ_A is a topology on A .

Proved!

Topic 043

Relative Open Sets

Definition: Let (A, τ_A) be subspace of a topological space (X, τ) . Then a subset H of A is open in A or relative open to A iff there exists an open subset G of X such that $H = G \cap A$.

One may say that the elements of τ_A are the open subsets of A or called relative open subsets of A .

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X . Let $A = \{a, b, c\}$ with $\tau_A = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, A\}$.

All the subsets of A are:

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A$$

Now:

$$\{a\} = \{a\} \cap A \quad \text{open in } A$$

$\emptyset, \{a\}, \{a, b\}, \{a, c\}, A$ are open in A .

Example 2: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X .

Let $A = \{a, c\}$. All the subsets of A are:

$$\emptyset, \{a\}, \{c\}, A$$

Open sets of A are: $\emptyset, \{a\}, A$

Example 3: Let $X = \mathbb{R}$ with usual topology.

Consider $A = (0, 1]$.

$(0, \frac{1}{2}) \subset A$ is open in A . Because it can be written as an

intersection of A . i.e Let $(-\frac{1}{2}, \frac{1}{2}) \cap A = (0, \frac{1}{2})$

$(0.7, 1]$ is open in A . Because it can be written as an intersection of A . i.e Let $(0.7, 3) \cap A = (0.7, 1]$.

The open sets of A may or may not open in $\mathbb{R}$. e.g $(0.7,1]$

Example 4: Let $X = \mathbb{R}$ with usual topology.

Consider $A = \mathbb{Z}$.

$\tau_{\mathbb{Z}}$ = discrete topology.

Every singleton subset of $\mathbb{Z}$ is open in $\mathbb{Z}$ relative to the usual topology on $\mathbb{R}$.

Topic 044

Basis for a Topology

Definition: Let (X, τ) be a topological space. A subcollection $\mathcal{B} \subset \tau$ is called a basis for τ if and only if $\forall U \in \tau$ and $\forall x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark: If $\mathcal{B}$ is a basis for a topology τ on X then every element of τ can be written as union of the elements of $\mathcal{B}$.

Proof: By definition of basis $\forall U \in \tau$ and $\forall x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

We can see that for every x there exists a basis element. It means that if we collect all B_x we can write:

$$U = \bigcup_{x \in U} B_x$$

Proved!

Remark: Let $\mathcal{B}$ be a basis for a topology τ on X . Then every superclass $\mathcal{B}^*$ of open subsets of X of $\mathcal{B}$ is also a basis for τ . i.e $\forall \mathcal{B} \subset \mathcal{B}^* \subset \tau$.

Proof: Since $\mathcal{B}$ is a basis so $\forall U \in \tau$ and $\forall x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. But we know that $\mathcal{B} \subset \mathcal{B}^*$ and thus $B \subset \mathcal{B}^*$.

$\Rightarrow \forall U \in \tau$ and $\forall x \in U$

$B \in \mathcal{B}^*$ such that $x \in B \subset U$

Proved!

Topic 045

Examples: Basis for a Topology

Example 1: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$ be a topology of X .
Consider $\mathcal{B} = \{\{a\}, \{b\}, \{a, c, d\}\}$

$\mathcal{B}$ is a basis because every element of τ can be written as a union of elements of $\mathcal{B}$.

As: $\emptyset = \cup \{\text{empty subclass of } \mathcal{B}\}$

$\{a\} = \{a\}$

$\{b\} = \{b\}$

$\{a, b\} = \{a\} \cup \{b\}$

$$\{a, c, d\} = \{a, c, d\}$$

$$X = \{b\} \cup \{a, c, d\}$$

Example 2: Let $X = \{a, b, c, d\}$ and

$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$ be a topology of X .

Consider $\mathcal{B} = \{\{a\}, \{b\}, \{a, b, c\}\}$

$\mathcal{B}$ is not a basis because $X \in \tau$ cannot be written as a union of elements of $\mathcal{B}$.

Example 3: Consider $\mathbb{R}$ with usual topology.

Consider $\mathcal{B} = \{(a, b) | a, b \in \mathbb{R}\}$

Usual topology can be generated by using set of all open intervals on $\mathbb{R}$.

Topic 046

Criteria for a collection to be basis

As we discussed for a given topology one can write a collection of open subsets of X that satisfies the condition of basis.

Now the question is "Can a topology be formed by a given collection $\mathcal{B}$ of subsets of X by using set union operation?"

If "Yes" then what should be the conditions on that collection?

One condition is very much clear that is

$$X = \cup \{B : B \in \mathcal{B}\}$$

Because X is always open in every topology on X .

Is this condition enough for any collection to be a basis?

To understand this let us look an example.

Example: Let $X = \{a, b, c\}$ and $\mathcal{B} = \{\{a, b\}, \{b, c\}\}$. Let us generate the elements from this collection.

$$\tau = \{\emptyset, \{a, b\}, \{b, c\}, X\}$$

This does not define a topology on X because intersection is not closed. i.e $\{a, b\} \cap \{b, c\} = \{b\} \notin \tau$

Theorem: Let $\mathcal{B}$ be the class of subsets of non empty set X . Then $\mathcal{B}$ forms a basis for some topology on X if and only if $\mathcal{B}$ satisfies the following conditions:

$$1. \forall x \in X \exists B \in \mathcal{B} \text{ s.t } x \in B$$

$$2. \text{ For } B_1, B_2 \in \mathcal{B} \text{ and } \forall x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B} \text{ s.t } x \in B_3 \subset B_1 \cap B_2$$

Example: Let $X = \{a, b, c\}$ and $\mathcal{B} = \{\{c\}, \{a, c\}, \{b, c\}\}$. It is clear that $\mathcal{B}$ satisfies both conditions stated in the theorem. Let us generate the elements from this collection.

$$\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} = X$$

Example: Let $X = \{a, b, c, d\}$ and $\mathcal{B} = \{\{a, b\}, \{c, d\}, \{d\}\}$. Let us generate the elements from this collection.

$$\tau = \left\{ \begin{array}{l} \emptyset, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \\ \{a, b, d\}, \{a, b, c, d\} \end{array} \right\} = X$$

Topic 047

Basis for some topologies

In this topic we will see basis for discrete and indiscrete topology.

Basis for Indiscrete topology:

Consider indiscrete space $\{\emptyset, X\}$. It is clear that there is only one non empty open set in this collection that is X . So basis of indiscrete space is $\{X\}$.

Basis of Discrete topology:

Example: Let $X = \{a, b, c\}$, consider the set of all singletons i.e

$$\mathcal{B} = \{\{a\}, \{b\}, \{c\}\}$$

It is clear that this collection satisfies both conditions of basis. Let us generate topology from $\mathcal{B}$.

$$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$$

It is discrete topology.

Remark: Consider a discrete space. That is a non empty set X with $\tau = P(X)$. Then the set of all singletons forms a basis for discrete topology. i.e $\mathcal{B} = \{\{x\} | x \in X\}$

Proof: First condition: $\forall x \in X \exists B = \{x\} \in \mathcal{B}$

Second condition: For $B_1, B_2 \in \mathcal{B}$ and $\forall x \in B_1 \cap B_2 \exists B_3 \in \mathcal{B}$ s.t $x \in B_3 \subset B_1 \cap B_2$

In this case $B_1 \cap B_2 = \emptyset$ Hence Proved!

Topic 048

Basis for usual topology on R^n

In this topic we will see the basis for usual topology on R^n . Basically, we will look for the basis those are usually used for the usual topology on R^n .

Basis for usual topology on R :

Consider R_u i.e R with usual topology. The usual basis for R_u is the set of all open intervals.

Basis for usual topology on R^2 :

Consider R_u^2 i.e R^2 with usual topology. The usual basis for R_u^2 is the set of all open disks.

We will show that it satisfies both conditions of basis.

1. For every element of space there is a basis element.

Such that $B = \{(x', y') | \sqrt{(x - x')^2 + (y - y')^2} < r\}$

2. If two open disks intersect then for every point of intersection there exists an open disk that is subset of intersection of both basis.

We also have other basis like set of all open triangles(interior part of triangle), set of all open squares, set of all open rectangles.

The usual basis for R_u^3 is the set of all open spheres.

Basis for usual topology on R^n :

Consider R_u^n i.e R^n with usual topology. The usual basis for R_u^n is the set of all open $n - balls$.

Topic 049

Topologies on Real Line

