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HIGHLIGHTED HANDOUT

ECO406

Mathematical Economics

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ECO406

Mathematical Economics

Study Handout

Complete lecture-wise notes for Virtual University students

About This Handout

This handout covers ECO406 Mathematical Economics -- the formal methods underlying economic theory. Each lesson introduces the mathematics, then immediately applies it to consumer theory, producer theory, equilibrium, and dynamics. The exam rewards both computational fluency (Lagrangians, Cramer's rule, elasticities) and economic interpretation (what does the math mean?). After each lesson, close the handout and re-derive the key results from memory -- mathematics is learned by doing. 

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Lesson 1: The Nature of Mathematical Economics

Learning Objectives

- Distinguish endogenous and exogenous variables
- Solve equilibrium in linear models
- Use function properties economically
- Explain why economics uses mathematics

Mathematical economics translates economic reasoning into formal mathematics: sets, functions, and equilibrium conditions. A market model has endogenous variables (determined inside the model: price, quantity) and exogenous variables/parameters (given: income, technology). An equilibrium is a state with no tendency to change -- mathematically, the solution to a system of equations. Example: $Q_d = a - bP$, $Q_s = c + dP$; equilibrium solves $a - bP = c + dP$, giving $P^* = (a-c)/(b+d)$.

Functions formalize behavior: demand $Q_d(P)$, cost $C(Q)$, production $Q = f(K,L)$. Key properties: continuity, differentiability, monotonicity, concavity/convexity -- each with economic meaning (diminishing marginal utility = concave utility; diminishing returns = concave production). Inverse functions matter: demand $P(Q)$ inverts $Q(P)$. Composition builds models: profit = revenue($Q(P)$) - cost($Q(P)$).

Why mathematize? Precision (assumptions explicit), generality (theorems cover cases), and refutability (predictions testable). The cost: abstraction from institutional detail -- mathematics is the language, not the substance, of economics.

Key Terms & Definitions

- Endogenous / exogenous — Determined in / given to the model.
- Equilibrium — Solution with no tendency to change.
- Concavity — Diminishing marginal returns/utility.
- Inverse function — $P(Q)$ from $Q(P)$; composition builds models.

Example / Illustration

- $Q_d = 100 - 2P$, $Q_s = 20 + 3P$. Equilibrium: $100 - 2P = 20 + 3P$ gives $P^* = 16$, $Q^* = 68$. A per-unit tax t shifts supply: $Q_s = 20 + 3(P - t)$.

Lesson Summary

- Equilibrium = solution of the model's equations.
- Function shape encodes economic behavior.
- Math gives precision; assumptions stay explicit.

Exam Pointers — Very Important

- ★ Solve linear supply-demand equilibrium and classify endogenous vs exogenous -- the standard opening numerical.

Lesson 2: Differentiation: Marginal Analysis

Learning Objectives

- Differentiate fluently (all rules)
- Translate marginal concepts
- Compute elasticities
- Use logs for growth models

One-variable differentiation: $f'(x) = dy/dx$ is the marginal concept -- marginal cost = $C'(Q)$, marginal revenue = $R'(Q)$, marginal utility = $U'(x)$. Rules: power, product, quotient, chain -- drill until automatic. Economic applications: profit max where $MC = MR$ (first-order condition); elasticity $\epsilon = (dQ/dP)(P/Q) = \text{marginal/average}$ -- the ratio form appears everywhere.

Higher derivatives: $f'' > 0$ convex (increasing marginal -- rare), $f'' < 0$ concave (diminishing marginal -- the normal case). Second-order conditions for optimization use these signs. Differentials: $dy = f'(x)dx$ approximates small changes -- used for comparative statics approximations and growth-rate decompositions ($d \ln y = dy/y$).

Exponential and log functions: continuous growth $y = Ae^{rt}$; logs linearize: $\ln y = \ln A + rt$. Rules of logs turn multiplicative models (Cobb-Douglas) into linear ones -- the workhorse transformation of applied economics.

Key Terms & Definitions

- Marginal — Derivative: MC, MR, MU.
- Elasticity — $(dQ/dP)(P/Q)$; marginal over average.
- Concave / convex — $f'' < 0$ diminishing / $f'' > 0$ increasing marginal.
- Log-linearization — $\ln$ turns multiplicative into additive.

Example / Illustration

- $C(Q) = 100 + 5Q + 0.1Q^2$. $MC = 5 + 0.2Q$. At $Q = 50$, $MC = 15$. Demand $P = 50 - 0.5Q$: elasticity at $Q=50$ ($P=25$): $\epsilon = (-2)(25/50) = -1$ (unit elastic).

Lesson Summary

- Marginal = first derivative.
- f'' sign = diminishing vs increasing marginal.
- Elasticity = (marginal)/(average).

Exam Pointers — Very Important

- ★ Marginal = derivative translations and elasticity computations are guaranteed exam material.

Lesson 3: Unconstrained Optimization

Learning Objectives

- Solve one-variable optimization
- Apply FOC and SOC
- Distinguish profit vs revenue max
- Verify global optimality

Unconstrained optimization in one variable: $\max f(x)$ needs $f'(x) = 0$ (FOC) and $f''(x) < 0$ (SOC). Economic logic: $MB = MC$ at the optimum -- expand while marginal benefit exceeds marginal cost. Profit max: $\pi(Q) = R(Q) - C(Q)$; $\pi' = MR - MC = 0$ gives $MR = MC$; $\pi'' = MR' - MC' < 0$ needs MC rising faster than MR.

Revenue max vs profit max differ: $\max R(Q)$ sets $MR = 0$, while profit max sets $MR = MC > 0$ -- revenue maximizers produce more. Cost minimization for a target output uses the same machinery on $C(Q)$. Inventory, pricing, and input-choice problems all reduce to FOC/SOC.

Global vs local: FOC finds stationary points; check endpoints and SOC for global max. Concave f on an interval: any stationary point is the global max -- a theorem worth quoting. Sketching f' sign charts avoids sign errors.

Key Terms & Definitions

- FOC / SOC — First/second-order conditions.
- $MB = MC$ — Marginal benefit equals marginal cost at optimum.
- Stationary point — $f' = 0$; classify with f'' .
- Global max — Concave f : stationary point is global max.

Example / Illustration

- $P = 100 - 2Q$, $C = 20Q + Q^2$. $\pi = 100Q - 2Q^2 - 20Q - Q^2 = 80Q - 3Q^2$.
FOC: $80 - 6Q = 0$, $Q^* = 13.33$. SOC: $-6 < 0$, max. $\pi^* = 533.3$.

Lesson Summary

- FOC: $f' = 0$; SOC: $f'' < 0$ for max.
- Economics: $MB = MC$.
- Concavity guarantees globality.

Exam Pointers — Very Important

- ★ 'Maximize profit given demand and cost' -- the complete FOC/SOC workflow is a core numerical.

Lesson 4: Multivariable Calculus for Economics

Learning Objectives

- Compute partials and total differentials
- Test Hessian definiteness
- Apply Euler's theorem
- Interpret returns to scale

Partial derivatives give multivariable marginals: for $U(x,y)$, $MU_x = dU/dx$ (holding y fixed). Total differential: $dU = U_x dx + U_y dy$ -- the foundation of comparative statics. Young's theorem (Clairaut): $U_{xy} = U_{yx}$ with continuous second partials. Hessian matrix H of second partials; definiteness classifies: negative definite $\rightarrow$ local max, positive definite $\rightarrow$ local min.

Unconstrained max in n variables: gradient = 0 (n FOCs), Hessian negative definite (SOC). For 2 variables: $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Example: $\max U = x^a y^b$ subject to nothing? Unbounded -- constraints are what make economics interesting (next lesson). Production: $Q = F(K,L)$; $MPL = dQ/dL$, $MPK = dQ/dK$; diminishing returns = $F_{LL} < 0$.

Homogeneous functions: $F(tK,tL) = t^n F(K,L)$; degree n = returns to scale ($n > 1$ increasing). Euler's theorem: $K \cdot F_K + L \cdot F_L = n \cdot F$ -- for CRS ($n=1$), output exhausts factor payments (product exhaustion theorem) -- the mathematical backbone of marginal productivity theory.

Key Terms & Definitions

- Gradient — Vector of first partials; = 0 at optimum.
- Hessian — Matrix of second partials; definiteness classifies.
- Homogeneous — $F(tK,tL) = t^n F$; n = returns to scale.
- Euler's theorem — $K \cdot F_K + L \cdot F_L = n \cdot F$.

Example / Illustration

- $U = x^{0.5} y^{0.5}$: $U_x = 0.5x^{-0.5} y^{0.5}$, $U_{xx} = -0.25x^{-1.5} y^{0.5} < 0$ (diminishing MU). Homogeneous degree 1 (CRS): $x \cdot U_x + y \cdot U_y = U$.

Lesson Summary

- Gradient = 0; Hessian definiteness = SOC.
- Homogeneity degree = returns to scale.
- Euler: CRS implies product exhaustion.

Exam Pointers — Very Important

- ★ Hessian definiteness checks and Euler's theorem statements are favourite theory questions.

Lesson 5: Constrained Optimization: Consumer Theory

Learning Objectives

- Set up the consumer Lagrangian
- Derive the equimarginal condition
- Solve Cobb-Douglas demands
- Interpret LAMBDA

The consumer's problem: $\max U(x,y)$ s.t. $p_x x + p_y y = M$. Lagrangian: $L = U(x,y) + \text{LAMBDA} \cdot (M - p_x x - p_y y)$. FOCs: $U_x = \text{LAMBDA} \cdot p_x$, $U_y = \text{LAMBDA} \cdot p_y$, budget binds. Key result: $MU_x/p_x = MU_y/p_y$ -- marginal utility per rupee equalized across goods (the equimarginal principle). LAMBDA = marginal utility of income.

Demand functions: solving FOCs + constraint gives $x^*(p_x, p_y, M)$, $y^*(p_x, p_y, M)$ -- Marshallian demands. Homogeneous of degree 0 in (p, M) (no money illusion); budget exhausted (Walras' law). Example: Cobb-Douglas $U = x^a y^{1-a}$ gives $x^* = aM/p_x$ -- constant expenditure shares, a famous result.

Duality preview: the expenditure-minimization problem gives Hicksian (compensated) demands; the Lagrangian machinery is identical. Second-order conditions use the bordered Hessian -- negative definiteness on the constraint set.

Key Terms & Definitions

- Lagrangian — $L = \text{objective} + \text{LAMBDA} \cdot (\text{constraint})$.
- Equimarginal principle — $MU_x/p_x = MU_y/p_y$.
- Marshallian demand — $x^*(p, M)$ from utility max.
- LAMBDA — Marginal utility of income.

Example / Illustration

- $U = xy$, $p_x = 2$, $p_y = 4$, $M = 120$. FOCs: $y = 2 \cdot \text{LAMBDA}$, $x = 4 \cdot \text{LAMBDA}$; budget: $2x + 4y = 120$. Solve: $x^* = 30$, $y^* = 15$. Check: $MU_x/p_x = 15/2 = MU_y/p_y = 30/4 = 7.5$.

Lesson Summary

- $L = U + \text{LAMBDA} \cdot (M - p_x x - p_y y)$.
- MU per rupee equalized.
- Cobb-Douglas: constant expenditure shares.

Exam Pointers — Very Important

- ★ Deriving Cobb-Douglas demands via Lagrangian is THE classic exam question of this course.

Lesson 6: Producer Theory: Profit and Cost

Learning Objectives

- Solve profit maximization
- Derive cost-minimization conditions
- Use Hotelling's and Shephard's lemmas
- Link returns to scale to cost curves

Firm's problem: $\max \pi = p \cdot F(K, L) - rK - wL$. FOCs: $p \cdot F_K = r$, $p \cdot F_L = w$ -- value of marginal product equals factor price for each input. These implicitly define factor demands $K^*(p, r, w)$, $L^*(p, r, w)$ and output supply. Profit function $\pi^*(p, r, w)$: Hotelling's lemma -- $d\pi^*/dp = y^*$ (supply), $d\pi^*/dw = -L^*$ (factor demand). Envelope theorem: only direct effects matter at the optimum.

Cost minimization dual: $\min rK + wL$ s.t. $F(K, L) = q$. Lagrangian gives $F_K/F_L = r/w$ -- MRTS equals input price ratio (tangency of isoquant and isocost). Conditional factor demands $K^c(r, w, q)$; cost function $C(r, w, q)$; Shephard's lemma: $dC/dw = L^c$. For Cobb-Douglas, cost function derivation is a standard exercise.

Returns to scale and cost: CRS implies constant $MC = AC$; IRS implies falling AC (natural monopoly); DRS implies rising MC . Long-run vs short-run: fix K in the short run, optimize only L -- the envelope of short-run costs is the long-run cost.

Key Terms & Definitions

- VMP = factor price — $p \cdot F_K = r$; $p \cdot F_L = w$.
- MRTS = r/w — Cost-min tangency.
- Hotelling's lemma — $d\pi^*/dp = \text{supply}$.
- Shephard's lemma — $dC/dw = \text{conditional labor demand}$.

Example / Illustration

- $Q = K^{0.5} L^{0.5}$, $r = 4$, $w = 1$, target $q = 10$. $MRTS = K/L$... FOCs: $0.5(KL)^{-0.5} \cdot L$... solve: $K^c = 10 \cdot (w/r)^{0.5} = 5$, $L^c = 10 \cdot (r/w)^{0.5} = 20$. $C = 4 \cdot 5 + 1 \cdot 20 = 40$.

Lesson Summary

- Profit max: VMP = factor price.
- Cost min: MRTS = price ratio.
- Duality: max profit and min cost mirror each other.

Exam Pointers — Very Important

- ★ Cost-minimization tangency $F_K/F_L = r/w$ with Cobb-Douglas is a standard numerical.

Lesson 7: Comparative Statics

Learning Objectives

- Totally differentiate equilibrium conditions
- Apply Cramer's rule
- Sign comparative-static effects
- State the correspondence principle

Comparative statics: how does equilibrium respond to parameter changes? Method: totally differentiate the equilibrium conditions (FOCs + constraints), solve the linear system for $dx^*/d\alpha$. The implicit function theorem guarantees the solution exists when the Jacobian is nonzero. Sign results often need no functional forms -- only curvature assumptions (concavity) -- which is the power of the method.

Example: demand $x^*(p_x, p_y, M)$. Totally differentiate FOCs + budget: get dx^*/dM (income effect sign = normality), dx^*/dp_x (own-price: substitution + income via Slutsky -- next lesson). Cramer's rule on the 3×3 system; the bordered Hessian's sign pattern signs the effects.

General equilibrium preview: excess demand $Z(p) = 0$ defines equilibrium prices; differentiate for policy effects (tax incidence: dP^*/dt from differentiated market-clearing). Stability: tatonnement $dp/dt = Z(p)$; stable if $Z'(p^*) < 0$ -- the correspondence principle (Samuelson): meaningful comparative statics need stability.

Key Terms & Definitions

- Comparative statics — $dx^*/d\alpha$ from differentiated equilibrium.
- Implicit function theorem — Guarantees $x^*(\alpha)$ exists.
- Cramer's rule — Solves the differentiated linear system.
- Correspondence principle — Stability underlies meaningful statics.

Example / Illustration

- From FOCs $U_x = \lambda p_x$ etc., $d(\text{totally})/dM$ gives $dx^*/dM = (\text{bordered-Hessian cofactor})/|H| > 0$ iff x normal. Signs from curvature, not functional form.

Lesson Summary

- Differentiate FOCs; solve for $dx^*/d\alpha$.
- Curvature signs effects without functional forms.
- Stability first, statics second.

Exam Pointers — Very Important

- ★ Totally-differentiate-then-Cramer is the method question; practice the 2-good consumer case.

Lesson 8: The Slutsky Equation and Duality

Learning Objectives

- Decompose price effects (Slutsky)
- Distinguish Hicksian and Marshallian
- Identify Giffen conditions
- Apply to tax and labor supply

Slutsky equation: $\frac{dx^*}{dp_x} = \frac{dx^c}{dp_x} - x^*(\frac{dx^*}{dM})$ -- total price effect = substitution effect (Hicksian, always ≤ 0) + income effect ($-x^*$ times income effect). The substitution effect is the pure relative-price response (utility held constant); the income effect is the purchasing-power change. Giffen goods: strongly inferior, income effect dominates -- upward-sloping demand, a theoretical curiosity.

Derivation via duality: differentiate the identity $x^c(p_x, p_y, u) = x^*(p_x, p_y, E(p_x, p_y, u))$ (Hicksian = Marshallian at minimizing expenditure). Matrix form: the Slutsky matrix is symmetric negative semidefinite -- testable restrictions of consumer theory! Symmetry of cross-substitution effects: $\frac{dx^c_i}{dp_j} = \frac{dx^c_j}{dp_i}$.

Elasticity form: $\epsilon_{xx} = \epsilon^c_{xx} - s_x \cdot \eta_{xM}$ (s_x = budget share, η = income elasticity). Uses: tax incidence decomposition, welfare measurement (EV/CV via Hicksian demands), labor supply (substitution vs income effects of wage changes -- the backward-bending supply curve).

Key Terms & Definitions

- Slutsky equation — Total = substitution + income effect.
- Hicksian demand — Utility-constant (compensated) demand.
- Giffen good — Upward-sloping; income effect dominates.
- Slutsky symmetry — Cross-substitution effects symmetric.

Example / Illustration

- p_x rises 10%, x^* falls 15%: total -15%. If income effect is -5% (normal good, real income fell), substitution = -10%. Both negative: ordinary (non-Giffen) demand.

Lesson Summary

- $\frac{dx^*}{dp_x} = \text{substitution } (\leq 0) + \text{income effect.}$
- Symmetry gives testable restrictions.
- Giffen needs strong inferiority.

Exam Pointers — Very Important

★ State and interpret the Slutsky equation; decompose a price change -- core theory question.

Lesson 9: Market Equilibrium and Welfare

Learning Objectives

- Analyze market stability
- Compute tax incidence
- Describe general equilibrium
- Solve Cournot duopoly

Market equilibrium: $D(p) = S(p)$. Stability: Walrasian (price adjusts to excess demand: $dp/dt = k \cdot Z(p)$; stable iff $Z'(p^*) < 0$) vs Marshallian (quantity adjusts). Tax incidence: per-unit tax t on sellers: $D(p) = S(p - t)$; $dp^*/dt = S'/(S' - D') = \eta_s/(\eta_s - \eta_d)$... in elasticity form, consumer burden share = $|\eta_s|/(|\eta_s| + |\eta_d|)$ -- the more inelastic side bears the tax. Mathematics makes incidence exact.

General equilibrium (2x2x2): Edgeworth box, contract curve (MRS equalized), production possibility frontier; equilibrium: $MRS = MRT = \text{price ratio}$. Existence (fixed-point theorems), welfare theorems: (1) competitive equilibrium is Pareto optimal; (2) any Pareto optimum is an equilibrium with transfers -- the mathematical core of welfare economics.

Game theory preview: Nash equilibrium as mutual best responses -- $\max_{s_i} u_i(s_i, s_{-i})$ for each i ; the FOC machinery applies per player. Oligopoly models (Cournot: quantity competition; Bertrand: price) are equilibrium computations -- Cournot-Nash via reaction functions $R_1(q_2)$, $R_2(q_1)$, intersection = equilibrium.

Key Terms & Definitions

- Tax incidence — Burden split by relative elasticities.
- Edgeworth box — 2-agent exchange; contract curve.
- Welfare theorems — Equilibrium = Pareto optimal (1st).
- Cournot-Nash — Reaction-function intersection.

Example / Illustration

- $\eta_d = -0.5$, $\eta_s = 1.0$. Consumer share of tax = $1.0/(1.0+0.5) = 2/3$. Demand inelastic relative to supply, so consumers bear more -- compute, don't guess.

Lesson Summary

- Incidence follows inelasticity.
- GE: $MRS = MRT = p$ ratio.
- Nash = mutual best responses.

Exam Pointers — Very Important

- ★ Tax-incidence formula in elasticity form is a high-yield numerical.

Lesson 10: Dynamics, Uncertainty, and Growth

Learning Objectives

- Solve intertemporal consumption
- Apply NPV/IRR
- Measure risk aversion
- Find Solow steady states

Intertemporal choice: max sum of discounted utilities subject to lifetime budget -- Euler equation: $U'(c_t) = \beta(1+r)U'(c_{t+1})$ -- marginal utility smoothed across time (consumption smoothing). Present value: $PV = \sum x_t/(1+r)^t$; NPV rule for investment: invest iff $NPV > 0$; $IRR = r$ with $NPV = 0$.

Uncertainty: expected utility max; risk aversion = concave U (Arrow-Pratt: $-U''/U'$). Insurance: full insurance optimal at actuarially fair premiums. Portfolio: two-asset allocation, CAPM preview -- $E[R_i] = R_f + \beta_i(E[R_m] - R_f)$, β from covariance -- the math of diversification.

Growth: Solow model -- $k' = s \cdot f(k) - (n+\delta) \cdot k$; steady state $s \cdot f(k^*) = (n+\delta) \cdot k^*$; golden rule max c^* . Dynamics: phase diagrams, stability of steady states -- differential equations meet economics. These extensions show the same optimization + equilibrium machinery conquering dynamics and uncertainty.

Key Terms & Definitions

- Euler equation — $U'(c_t) = \beta(1+r)U'(c_{t+1})$; smoothing.
- NPV / IRR — Investment criteria.
- Arrow-Pratt — $-U''/U'$; risk aversion measure.
- Solow steady state — $s \cdot f(k^*) = (n+\delta) \cdot k^*$.

Example / Illustration

- $U = \ln c$, $\beta(1+r) = 1$: Euler gives $1/c_t = 1/c_{t+1}$ -- perfectly smooth consumption. Solow: $s = 0.2$, $f = k^{0.5}$, $n+\delta = 0.1$: $0.2 \cdot \sqrt{k^*} = 0.1 \cdot k^*$ gives $k^* = 4$.

Lesson Summary

- Euler: smooth marginal utility over time.
- Risk aversion = concave utility.
- Solow: savings = break-even investment at k^* .

Exam Pointers — Very Important

★ Euler equation interpretation and Solow steady-state algebra are the standard dynamic questions.

Lesson 11: Linear Algebra for Economics

Learning Objectives

- Do matrix algebra fluently
- Apply Cramer's rule
- Use determinants for existence
- Find eigenvalues

Matrices organize multi-variable linear systems: $Ax = b$. Matrix multiplication, transpose, inverse ($A^{-1}A = I$). Determinant $|A|$: nonzero iff A invertible (unique solution). Cramer's rule: $x_i = |A_i|/|A|$ -- each endogenous variable as a ratio of determinants -- the engine of comparative statics. 2x2 inverse: $(1/|A|)*[[d,-b],[-c,a]]$ -- memorize it.

Rank, linear dependence: n equations, n unknowns, $|A| \neq 0$ gives unique equilibrium. Homogeneous systems $Ax = 0$: nontrivial solutions iff $|A| = 0$ -- used in stability analysis (characteristic roots). Eigenvalues: $|A - L*I| = 0$; stability of dynamic systems needs $|\text{eigenvalues}| < 1$ (discrete) or negative real parts (continuous).

Economic uses: IS-LM as 2-equation system (solve for Y, r via Cramer's rule); input-output inverses; Markowitz portfolio (covariance matrix). Practice: 2x2 systems by hand until instant; 3x3 via expansion or row reduction.

Key Terms & Definitions

- Determinant — Nonzero iff unique solution.
- Cramer's rule — $x_i = |A_i|/|A|$.
- Eigenvalues — $|A - L*I| = 0$; stability analysis.
- Inverse — $A^{-1} = \text{adj}(A)/|A|$.

Example / Illustration

- $Y = C + I$, $C = 100 + 0.8Y$, $I = 50$: $Y - 0.8Y = 150$ gives $Y^* = 750$. Matrix form $[[1,-1],[-0.8,1]]$; Cramer's rule: $|A| = 1-0.8 = 0.2$, $Y = 150/0.2 = 750$.

Lesson Summary

- $|A| \neq 0$: unique equilibrium.
- Cramer gives each variable explicitly.
- Eigenvalues govern dynamics stability.

Exam Pointers — Very Important

- ★ Solve a 2-equation macro model with Cramer's rule -- a staple numerical.

Lesson 12: Integral Calculus: Surplus and Aggregation

Learning Objectives

- Integrate economic functions
- Compute CS/PS/DWL as integrals
- Recover totals from marginals
- Value perpetuities

The definite integral sums continuous flows: consumer surplus = integral of (demand - price) from 0 to Q^* ; producer surplus = integral of (price - supply). These are the welfare measures of applied economics -- computed, not hand-waved. Example: $P = 100 - 2Q$, $P^* = 16$, $Q^* = 42$: $CS = \int_0^{42} (84 - 2Q)dQ = [84Q - Q^2]_0^{42} = 3528 - 1764 = 1764$.

Integration rules: substitution, parts, partial fractions -- economic integrands are usually polynomials/exponentials, so the basics suffice. Improper integrals: present values are integrals to infinity: $PV = \int_0^{\infty} x \cdot e^{-rt} dt = x/r$ (perpetuity) -- finance as calculus.

Applications: total from marginal ($C(Q) = \int MC + FC$ -- recover levels from margins); Gini coefficient from the Lorenz curve (area ratios); deadweight loss triangles as integrals (exact for nonlinear curves). The fundamental theorem: differentiation and integration invert -- marginal/level duality.

Key Terms & Definitions

- Consumer surplus — Integral of (demand - price).
- Perpetuity — $PV = x/r$ via improper integral.
- Total from marginal — $C(Q) = FC + \int MC$.
- DWL — Deadweight loss; Harberger triangle.

Example / Illustration

- $MC = 5 + 0.2Q$, $FC = 100$: $C(Q) = 100 + 5Q + 0.1Q^2$. DWL of tax t : triangle $0.5 \cdot t \cdot \Delta Q$ for the linear case.

Lesson Summary

- Surplus and DWL are integrals.
- Integrate marginals to get levels.
- PV of perpetuity = x/r .

Exam Pointers — Very Important

- ★ CS/PS as integrals and total-from-marginal recoveries are standard numericals.

Lesson 13: Differential Equations and Dynamics

Learning Objectives

- Solve first-order linear ODEs
- Draw phase diagrams
- Analyze price-adjustment stability
- Use Solow dynamics

Differential equations model continuous-time dynamics: $dy/dt = f(y,t)$. First-order linear: $dy/dt + a*y = b$ -- solution $y(t) = b/a + (y_0 - b/a)e^{-at}$; converges to b/a iff $a > 0$ (stability). Economic example: price adjustment $dp/dt = k*(D(p) - S(p))$ -- Walrasian tatonnement; stability needs excess demand decreasing in p .

Phase diagrams: plot dy/dt vs y ; arrows show motion; steady states where $dy/dt = 0$; stable if arrows point inward. Solow model: $dk/dt = s*f(k) - (n+\delta)*k$ -- phase diagram shows convergence to k^* . Qualitative analysis without closed-form solutions -- the working economist's skill.

Systems: $dY/dt, dr/dt$ in dynamic IS-LM; stability via Jacobian eigenvalues (negative real parts). Applications: capital accumulation, resource depletion (Hotelling), inflation dynamics. Separable and linear equations cover most exam problems.

Key Terms & Definitions

- Steady state — $dy/dt = 0$.
- Phase diagram — dy/dt vs y ; arrows = motion.
- Tatonnement — $dp/dt = k*\text{excess demand}$.
- Stability — Converges iff $a > 0$ (linear case).

Example / Illustration

- $dp/dt = 2*(100 - 2p - (20 + 3p)) = 2*(80 - 5p)$. Steady state $p^* = 16$; solution $p(t) = 16 + (p_0 - 16)e^{-10t}$ -- converges (coefficient $-10 < 0$).

Lesson Summary

- $dy/dt + ay = b$: exponential convergence iff $a > 0$.
- Phase diagrams give qualitative dynamics.
- Tatonnement stability = downward excess demand.

Exam Pointers — Very Important

- ★ Solve $dy/dt + ay = b$ and draw the phase diagram -- the standard dynamics question.

Lesson 14: Difference Equations and the Cobweb

Learning Objectives

- Solve linear difference equations
- Analyze cobweb stability
- Explain the multiplier-accelerator
- Distinguish damped/explosive cycles

Difference equations model discrete-time dynamics: $y_{t+1} = f(y_t)$. Linear: $y_{t+1} = a*y_t + b$ -- solution $y_t = b/(1-a) + (y_0 - b/(1-a))*a^t$; converges iff $|a| < 1$ (oscillating if $a < 0$). The cobweb model: $D(p_t) = S(p_{t-1})$ (supply responds to last period's price) -- $p_t = (a-c)/b - (d/b)*p_{t-1}$; converges iff $d < b$.

Cobweb economics: agricultural cycles (hog cycle) -- farmers base planting on current prices, harvest gluts crash prices. Stability condition in elasticity terms. Nonlinear maps (logistic) can produce chaos -- bifurcations as parameters change; economics mostly stays in the stable linear region.

Higher-order: $y_{t+1} = a*y_t + c*y_{t-1} + b$ -- characteristic equation; Samuelson multiplier-accelerator: $Y_t = C_t + I_t$ with $C_t = a*Y_{t-1}$, $I_t = b*(C_t - C_{t-1})$ -- generates business cycles (damped, explosive, or uniform oscillation by parameters) -- the classic cycle model.

Key Terms & Definitions

- Cobweb model — $D(p_t) = S(p_{t-1})$; hog cycle.
- Stability — $|a| < 1$ for $y_{t+1} = a*y_t + b$.
- Multiplier-accelerator — Samuelson cycle model.
- Oscillation — $a < 0$ gives alternating paths.

Example / Illustration

- D: $Q = 100 - 2p_t$, S: $Q = 20 + 3p_{t-1}$. $p_t = 40 - 1.5p_{t-1}$: $|a| = 1.5 > 1$ -- explosive oscillation (supply more price-sensitive than demand).

Lesson Summary

- $y_{t+1} = ay_t + b$ converges iff $|a| < 1$.
- Cobweb: stability iff demand steeper than supply.
- Multiplier-accelerator generates cycles.

Exam Pointers — Very Important

- ★ Cobweb stability condition (d vs b) with a diagram is a favourite question.

Lesson 15: Game Theory: Strategic Equilibrium

Learning Objectives

- Find Nash equilibria
- Solve 2x2 games (pure/mixed)
- Apply backward induction
- Model oligopoly as games

Game theory: players, strategies, payoffs; Nash equilibrium -- each player's strategy is a best response to others' (no unilateral profitable deviation). Dominant strategy: best regardless of others (prisoners' dilemma: defect dominates, Nash = (defect, defect) though (cooperate, cooperate) Pareto-dominates -- the tension). Best-response functions; Nash = intersection (Cournot duopoly revisited).

Mixed strategies: randomize to keep opponents indifferent -- matching pennies; indifference conditions solve for mixing probabilities. Sequential games: game trees, backward induction (subgame perfect equilibrium) -- Stackelberg duopoly (leader commits quantity; follower reacts) vs Cournot (simultaneous).

Applications: oligopoly (Bertrand paradox -- price competition gives marginal-cost pricing with 2 firms!), auctions (Vickrey: bid true value), public goods (free-riding as PD), bargaining. Equilibrium refinements select among multiple Nash equilibria. The math: fixed-point theorems guarantee Nash existence (mixed).

Key Terms & Definitions

- Nash equilibrium — Mutual best responses.
- Dominant strategy — Best no matter what others do.
- Backward induction — Solve sequential games from the end.
- Mixed strategy — Randomize; indifference conditions.

Example / Illustration

- PD: payoffs (C,C)=(3,3), (C,D)=(0,5), (D,C)=(5,0), (D,D)=(1,1). D dominates for both; Nash (D,D) with $(1,1) < (3,3)$ -- equilibrium vs optimum tension.

Lesson Summary

- Nash: no unilateral deviation pays.
- PD: dominant strategies vs Pareto optimum.
- Sequential: backward induction (Stackelberg).

Exam Pointers — Very Important

- ★ Find Nash equilibria (pure + mixed) of a 2x2 game -- guaranteed game-theory question.

Lesson 16: Linear Programming and Input-Output

Learning Objectives

- Solve LPs graphically
- State and interpret the dual
- Use shadow prices
- Apply the Leontief inverse

Linear programming: $\max c'x$ s.t. $Ax \leq b, x \geq 0$ -- the workhorse of constrained linear optimization. Graphical solution (2 vars); simplex method (corner-point search); duality: $\min b'y$ s.t. $A'y \geq c$ -- dual variables = shadow prices (marginal values of resources) -- the economic meaning of Lagrange multipliers in linear form. Complementary slackness links primal and dual optima.

Input-output (Leontief): $x = Ax + d$ -- $(I - A)^{-1}d$ = total (direct + indirect) output for final demand d ; the Leontief inverse. Hawkins-Simon conditions for viability. Applications: planning, impact analysis (what if exports rise?).

Activity analysis generalizes LP to production theory: efficient production = vertices of the technology set. Von Neumann growth model preview. The mathematics: linear algebra (matrices, inverses, eigenvalues for stability) serving economics -- this lesson is the bridge to computational economics.

Key Terms & Definitions

- Simplex — Corner-point search algorithm.
- Duality — $\min b'y$ s.t. $A'y \geq c$; dual vars = shadow prices.
- Shadow price — Marginal value of a resource/constraint.
- Leontief inverse — $(I - A)^{-1}$; direct + indirect requirements.

Example / Illustration

- Max $3x + 5y$ s.t. $x + 2y \leq 12, 2x + y \leq 12, x, y \geq 0$. Corners: $(0,0)=0, (0,6)=30, (4,4)=32, (6,0)=18$. Optimum $(4,4) = 32$. Dual: $\min 12a + 12b$ s.t. $a + 2b \geq 3, 2a + b \geq 5$.

Lesson Summary

- LP: optimize linear over a polyhedron.
- Dual variables = shadow prices.
- Leontief: $(I-A)^{-1}d$ for impact analysis.

Exam Pointers — Very Important

- ★ Solve a 2-variable LP graphically and state the dual -- a standard numerical.

Lesson 17: Revision: Mathematical Economics Drill

Learning Objectives

- Run a timed mixed-topic drill
- Translate between math and economics
- Recall all results cold
- Self-assess readiness

Drill (two per lesson): (1) linear supply-demand equilibrium; (2) elasticity computation; (3) profit max FOC/SOC; (4) Hessian definiteness; (5) Euler theorem check; (6) Cobb-Douglas demands via Lagrangian; (7) cost-min tangency; (8) Hotelling/Shephard application; (9) comparative statics via Cramer; (10) Slutsky decomposition; (11) tax incidence by elasticities; (12) Cournot-Nash; (13) Euler equation; (14) Solow steady state; (15) LP graphical solution + dual; (16) Leontief inverse impact; (17) Cramer 2-equation macro; (18) CS/PS integrals; (19) ODE phase diagram; (20) cobweb stability; (21) 2x2 game Nash (pure+mixed). Formula sheet: FOC/SOC; Lagrangian; Cramer; Slutsky; incidence shares; Euler equation; Solow; LP duality; Leontief; cobweb condition.

Method: translate economics to math, solve, translate back -- interpretation marks equal computation marks.

Final checklist: marginal=derivative; MB=MC; equimarginal; VMP=price; MRTS=ratio; Slutsky; incidence; Nash; Euler; Solow; duality; Cramer. State each cold.

Key Terms & Definitions

- Formula sheet — One page: all boxed results.
- Translate-solve-translate — The exam method.
- Checklist — Twelve must-know results.

Lesson Summary

- Twenty-one drills across all lessons.
- Interpretation = computation in marks.
- Translate-solve-translate.

Exam Pointers — Very Important

- ★ Do the full 21-drill set closed-book before the paper.