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HIGHLIGHTED HANDOUT

MTH5101

Calculus I

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MTH5101

Calculus I

STUDY HANDOUT

A complete topic-wise study handout for MTH5101 Calculus I, prepared from the course video lectures (MTH5101 short-lecture series, Virtual University). Covers real numbers and functions, limits, continuity, differentiation and its applications, integration, and an introduction to sequences and series - with definitions, worked examples, key formulas and exam pointers. 

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Table of Contents

1. Real Numbers, Intervals and Inequalities
2. Functions: Domain, Range and Graphs
3. New Functions from Old: Transformations and Composition
4. Polynomial, Rational and Algebraic Functions
5. Exponential and Logarithmic Functions
6. Trigonometric Functions and Their Graphs
7. Intuitive Idea of Limits
8. Limit Laws and Computing Limits
9. Limits at Infinity and End Behaviour
10. Continuity
11. Tangent Lines and Rates of Change
12. The Derivative: Definition and Notation
13. Differentiation Rules: Power, Sum, Product, Quotient
14. Derivatives of Trigonometric Functions
15. The Chain Rule
16. Implicit Differentiation
17. Derivatives of Logarithmic and Exponential Functions
18. Related Rates
19. Linear Approximations and Differentials
20. Maximum and Minimum Values
21. The Mean Value Theorem and Its Consequences
22. First and Second Derivative Tests; Curve Sketching
23. Optimization Problems
24. Antiderivatives and the Indefinite Integral
25. The Definite Integral and Area
26. The Fundamental Theorem of Calculus
27. Integration by Substitution
28. Areas Between Curves
29. Volumes by Slicing: Disks and Washers
30. Volumes by Cylindrical Shells
31. Exponential Growth, Logarithms and L'Hopital's Rule
32. Sequences
33. Infinite Series: Convergence Tests I
34. Infinite Series: Convergence Tests II
35. Taylor and Maclaurin Series
36. Revision: Midterm Practice Set
37. Revision: Final-Term Practice Set

Lecture 1: Real Numbers, Intervals and Inequalities

Key terms: natural numbers, integers, rationals, irrationals, intervals, absolute value

Calculus begins with the real numbers: natural numbers (1, 2, 3, ...), integers (... , -2, -1, 0, 1, 2, ...), rational numbers (ratios p/q of integers) and irrational numbers (non-repeating decimals like π and $\sqrt{2}$). The reals form a complete ordered field - every bounded set has a least upper bound - which is why limits work.

Intervals describe sets of reals: open (a, b) , closed $[a, b]$, half-open, and infinite $(-\infty, a)$. Inequalities are solved by finding critical points and testing sign regions. The absolute value $|x|$ is the distance from x to 0; $|x| < a$ means $-a < x < a$.

- Properties of order: if $a < b$ then $a + c < b + c$; multiplying by a negative reverses the inequality.
- Triangle inequality: $|x + y| \leq |x| + |y|$ - very important for limit proofs. 
- Example: solve $|2x - 3| < 5$: $-5 < 2x - 3 < 5$, so $-1 < x < 4$.

EXAM POINTER: Inequality and absolute-value questions appear in nearly every midterm.
Practice sign-chart solving for rational inequalities like $(x-1)/(x+2) > 0$.

Lecture 2: Functions: Domain, Range and Graphs

Key terms: function, domain, range, vertical line test, even/odd functions

A function f from a set D (domain) to the reals assigns to each x in D exactly one value $f(x)$. The range is the set of all values $f(x)$. The vertical line test: a curve is a function's graph iff no vertical line meets it more than once. A function is even if $f(-x) = f(x)$ (symmetric about y -axis), odd if $f(-x) = -f(x)$ (symmetric about origin).

- Domain restrictions: denominators nonzero, even roots of nonnegatives, logarithms of positives.
- Piecewise functions are defined by different formulas on different intervals - check continuity at the joints.
- Example: $f(x) = \sqrt{x - 2}/(x - 5)$ has domain $[2, 5)$ union $(5, \infty)$.

EXAM POINTER: Finding domains of combined and composite functions is a favourite short question. Always list every restriction.

Lecture 3: New Functions from Old: Transformations and Composition

Key terms: transformations, shifts, stretches, reflections, composition, inverse functions

From $y = f(x)$ we get: $y = f(x) + k$ (shift up k), $y = f(x - h)$ (shift right h), $y = cf(x)$ (vertical stretch if $c > 1$), $y = f(cx)$ (horizontal compression if $c > 1$), $y = -f(x)$ (reflect in x-axis), $y = f(-x)$ (reflect in y-axis). Composition $(f \circ g)(x) = f(g(x))$ applies g first, then f ; domain requires x in domain of g and $g(x)$ in domain of f .

A function has an inverse f^{-1} iff it is one-to-one (passes the horizontal line test). The graphs of f and f^{-1} are reflections in the line $y = x$. To find the inverse: write $y = f(x)$, solve for x , swap x and y .

- Order matters in composition: $f \circ g$ is generally different from $g \circ f$.
- Restricting the domain can make a function one-to-one (e.g. x^2 on $[0, \infty)$ has inverse $\sqrt{x}$).
- Example: $f(x) = 2x + 3$, $g(x) = x^2$: $(f \circ g)(x) = 2x^2 + 3$; $(g \circ f)(x) = (2x+3)^2 = 4x^2 + 12x + 9$.

Lecture 4: Polynomial, Rational and Algebraic Functions

Key terms: polynomials, degree, rational functions, asymptotes, algebraic functions

A polynomial $p(x) = a_n x^n + \dots + a_1 x + a_0$ has domain all reals; its end behaviour is governed by the leading term $a_n x^n$. Zeros are found by factoring; multiplicity affects whether the graph crosses (odd) or touches (even) the x-axis. Rational functions $f(x) = p(x)/q(x)$ have vertical asymptotes where $q(x) = 0$ (and $p(x) \neq 0$) and horizontal/slant asymptotes determined by comparing degrees.

- $\deg p < \deg q$: horizontal asymptote $y = 0$. $\deg p = \deg q$: $y = \text{ratio of leading coefficients}$.
- $\deg p = \deg q + 1$: slant asymptote (divide).
- Holes occur at common factors cancelled from p and q .
- Example: $f(x) = (x^2 - 1)/(x - 1) = x + 1$ for $x \neq 1$: a hole at $(1, 2)$, not an asymptote.

EXAM POINTER: Sketching rational functions with all asymptotes and intercepts labelled is a standard 10-mark question.

Lecture 5: Exponential and Logarithmic Functions

Key terms: exponential growth, e , natural logarithm, logarithm laws

The exponential function $f(x) = a^x$ ($a > 0$, $a \neq 1$) grows fast for $a > 1$ and decays for $0 < a < 1$. The natural exponential e^x ($e = 2.718\dots$) is special because its derivative equals itself. Its inverse is the natural logarithm $\ln x$, defined for $x > 0$. Logarithm laws: $\ln(ab) = \ln a + \ln b$, $\ln(a/b) = \ln a - \ln b$, $\ln(a^r) = r \ln a$.

- Change of base: $\log_a x = \ln x / \ln a$.

- Exponential equations: take logs of both sides; logarithmic equations: exponentiate, then check domain.
- Example: solve $3^{(2x)} = 5$: $2x \ln 3 = \ln 5$, so $x = \ln 5 / (2 \ln 3)$.

EXAM POINTER: You must know the logarithm laws cold - they are used in logarithmic differentiation and integration later. 

Lecture 6: Trigonometric Functions and Their Graphs

Key terms: radians, sine, cosine, tangent, amplitude, period, phase shift

Angles in calculus are measured in radians: π radians = 180 degrees. The sine and cosine functions have period 2π , range $[-1, 1]$; tangent has period π with vertical asymptotes at odd multiples of $\pi/2$. For $y = A \sin(Bx - C) + D$: amplitude $|A|$, period $2\pi/|B|$, phase shift C/B , vertical shift D .

- Key values: sin/cos of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ - memorise the unit circle.
- Pythagorean identities: $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$.
- Inverse trig functions: arcsin has domain $[-1, 1]$, range $[-\pi/2, \pi/2]$.

Lecture 7: Intuitive Idea of Limits

Key terms: limit, left-hand limit, right-hand limit, infinite limits

The limit $\lim_{x \rightarrow a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to (but not equal to) a . The function's value at a is irrelevant - the function need not even be defined at a . One-sided limits consider approach from the left ($x \rightarrow a^-$) or right ($x \rightarrow a^+$); the two-sided limit exists iff both one-sided limits exist and are equal.

- Numerical/graphical estimation: build a table of values approaching a from both sides.
- Infinite limits: $\lim_{x \rightarrow 0} 1/x^2 = \infty$ describes vertical asymptotes.
- Example: $f(x) = (x^2 - 4)/(x - 2)$ has limit 4 as $x \rightarrow 2$ even though $f(2)$ is undefined - factor and cancel.

EXAM POINTER: State the precise relationship between one-sided and two-sided limits - examiners ask this as a theory short question.

Lecture 8: Limit Laws and Computing Limits

Key terms: limit laws, direct substitution, factoring, rationalising, squeeze theorem

If $\lim f(x) = L$ and $\lim g(x) = M$, then limits of sums, differences, products, quotients ($M \neq 0$), powers and roots behave as expected. Strategy: try direct substitution first; if you get $0/0$,

factor and cancel, rationalise, or use known special limits. The squeeze theorem: if $g(x) \leq f(x) \leq h(x)$ near a and g, h have the same limit L at a , then f has limit L too.

- $\lim_{x \rightarrow 0} \sin x / x = 1$ and $\lim_{x \rightarrow 0} (1 - \cos x)/x = 0$ - the two fundamental trig limits.
- $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$ - the definition of e as a limit.
- Example: $\lim_{x \rightarrow 0} \sin(3x)/x = 3 * \lim_{x \rightarrow 0} \sin(3x)/(3x) = 3$.

EXAM POINTER: The 0/0 indeterminate form with trig functions almost always uses $\sin x / x =$ 
1. Learn to spot the substitution.

Lecture 9: Limits at Infinity and End Behaviour

Key terms: limits at infinity, horizontal asymptotes, end behaviour

$\lim_{x \rightarrow \infty} f(x) = L$ means $f(x)$ approaches L as x grows without bound; the line $y = L$ is a horizontal asymptote. For rational functions, divide numerator and denominator by the highest power of x . For polynomials, the leading term dominates. Infinite limits at infinity (e.g. $x^3 \rightarrow \infty$) describe end behaviour.

- $\lim_{x \rightarrow \infty} 1/x^r = 0$ for $r > 0$ - the workhorse for rational limits at infinity.
- Exponential dominance: e^x grows faster than any power x^n ; $\ln x$ grows slower than any positive power.
- Example: $\lim_{x \rightarrow \infty} (3x^2 + 2x)/(5x^2 - 1) = 3/5$ (ratio of leading coefficients).

Lecture 10: Continuity

Key terms: continuity, removable/jump/infinite discontinuity, Intermediate Value Theorem

f is continuous at a if: (1) $f(a)$ is defined, (2) $\lim_{x \rightarrow a} f(x)$ exists, (3) the limit equals $f(a)$. Polynomials, rationals (on their domain), roots, trig, exponential and log functions are continuous where defined. Discontinuities: removable (hole - limit exists), jump (one-sided limits differ), infinite (vertical asymptote).

The Intermediate Value Theorem: if f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then $f(c) = N$ for some c in (a, b) . It guarantees roots: if $f(a)$ and $f(b)$ have opposite signs, f has a zero between a and b - the basis of bisection root-finding.

- Continuity on an interval plus differentiability are the hypotheses of the big theorems (Rolle, MVT).
- Example: $f(x) = (x^2 - 9)/(x - 3)$ for $x \neq 3$, $f(3) = 6$ is continuous everywhere (removable discontinuity filled).

EXAM POINTER: Be able to classify a discontinuity from a graph or formula and to apply IVT to prove a root exists in an interval.

Lecture 11: Tangent Lines and Rates of Change

Key terms: tangent line, secant line, average rate of change, instantaneous rate of change

The tangent line to $y = f(x)$ at $x = a$ is the limiting position of secant lines through $(a, f(a))$ and $(a+h, f(a+h))$ as $h \rightarrow 0$. Its slope - the instantaneous rate of change - is the limit of difference quotients $[f(a+h) - f(a)]/h$. Average velocity over $[t_1, t_2]$ is displacement/time; instantaneous velocity is the limit as the interval shrinks.

- Slope of tangent at a : $m = \lim_{h \rightarrow 0} [f(a+h) - f(a)]/h$.
- The tangent line equation: $y - f(a) = m(x - a)$; the normal line has slope $-1/m$.
- Example: $f(x) = x^2$ at $a = 1$: $m = \lim [(1+h)^2 - 1]/h = \lim (2h + h^2)/h = 2$; tangent: $y - 1 = 2(x - 1)$.

Lecture 12: The Derivative: Definition and Notation

Key terms: derivative, differentiability, Leibniz notation, dy/dx

The derivative $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)]/h$ gives the slope of the tangent (and instantaneous rate of change) at every x where the limit exists. Notation: $f'(x)$, dy/dx , $D_x y$, y' . If $f'(a) > 0$ the function is increasing at a ; if $f'(a) < 0$ it is decreasing there.

- Differentiability implies continuity, but not conversely: $|x|$ is continuous at 0 but not differentiable there (corner).
- Non-differentiable points: corners/cusps, vertical tangents, discontinuities.
- Higher derivatives: f'' is the derivative of f' - acceleration is the second derivative of position.

EXAM POINTER: Use the limit definition to differentiate $f(x) = 1/x$ or $\sqrt{x}$ - a guaranteed exam question testing the definition itself.

Lecture 13: Differentiation Rules: Power, Sum, Product, Quotient

Key terms: power rule, product rule, quotient rule

From the definition we prove the working rules: $d/dx [x^n] = n x^{(n-1)}$ (power rule); derivatives of sums/differences and constant multiples split termwise; the product rule $(fg)' = f'g + fg'$; the quotient rule $(f/g)' = (f'g - fg')/g^2$. Learn the quotient rule as 'low d-high minus high d-low over low squared'.

- Combine rules: differentiate term by term, then apply product/quotient where functions multiply or divide.
- Simplify before differentiating when possible: e.g. $(x^3 + 2x)/x = x^2 + 2$.
- Example: $d/dx [x^2 \sin x] = 2x \sin x + x^2 \cos x$.

EXAM POINTER: Speed and accuracy with product and quotient rules decide your marks in every differentiation question - drill them.

Lecture 14: Derivatives of Trigonometric Functions

Key terms: derivatives of sin, cos, tan, sec, csc, cot

$\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ (from the limit definition using $\sin x/x \rightarrow 1$). The rest follow by the quotient rule: $\frac{d}{dx} \tan x = \sec^2 x$, $\frac{d}{dx} \cot x = -\csc^2 x$, $\frac{d}{dx} \sec x = \sec x \tan x$, $\frac{d}{dx} \csc x = -\csc x \cot x$. Note the cofunctions (cos, cot, csc) take a minus sign.

- Mnemonic: derivatives of 'co-' functions get a negative sign.
- Combine with chain rule: $\frac{d}{dx} \sin(3x) = 3 \cos(3x)$.
- Example: $\frac{d}{dx} [x \tan x] = \tan x + x \sec^2 x$ (product rule).

Lecture 15: The Chain Rule

Key terms: chain rule, outside-inside, composite functions

The chain rule differentiates compositions: $\frac{d}{dx} f(g(x)) = f'(g(x)) * g'(x)$ - derivative of the outside evaluated at the inside, times the derivative of the inside. In Leibniz form, $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$. Apply repeatedly for triple compositions; it is the single most-used differentiation rule.

- General power rule: $\frac{d}{dx} [u^n] = n u^{(n-1)} * u'$.
- Example: $\frac{d}{dx} (3x^2 + 1)^5 = 5(3x^2+1)^4 * 6x = 30x(3x^2+1)^4$.
- Example: $\frac{d}{dx} \sin^2(x) = 2 \sin x \cos x = \sin(2x)$.

EXAM POINTER: Most chain-rule errors come from forgetting to multiply by the derivative of the inside - always do the 'times u-prime' step explicitly.

Lecture 16: Implicit Differentiation

Key terms: implicit differentiation, related rates preparation

When y is defined implicitly by an equation like $x^2 + y^2 = 25$, differentiate both sides with respect to x , treating y as a function of x : every y -term gets a chain-rule factor $\frac{dy}{dx}$ (e.g. $\frac{d}{dx}[y^2] = 2y \frac{dy}{dx}$). Then solve the resulting linear equation for $\frac{dy}{dx}$.

- $\frac{d}{dx} [xy] = y + x \frac{dy}{dx}$ (product rule plus chain rule).
- Second derivatives: differentiate $\frac{dy}{dx}$ implicitly again.
- Example: $x^2 + y^2 = 25 \rightarrow 2x + 2y y' = 0 \rightarrow y' = -x/y$. At $(3, 4)$: $y' = -3/4$.

Lecture 17: Derivatives of Logarithmic and Exponential Functions

Key terms: $d/dx \ln x = 1/x$, $d/dx e^x = e^x$, logarithmic differentiation

$d/dx \ln x = 1/x$ and $d/dx e^x = e^x$; with the chain rule, $d/dx \ln(u) = u'/u$ and $d/dx e^u = e^u \cdot u'$. Logarithmic differentiation handles products, quotients and powers elegantly: take $\ln$ of both sides, differentiate, solve for y' . It is essential for functions like x^x : $\ln y = x \ln x$ gives $y' = x^x (\ln x + 1)$.

- $d/dx \log_a x = 1/(x \ln a)$; $d/dx a^x = a^x \ln a$.
- Use log differentiation for $y = (\text{complicated product/quotient})^{\text{power}}$.
- Example: $y = x^3 (2x+1)^2/(x-1)$: $\ln y = 3 \ln x + 2 \ln(2x+1) - \ln(x-1)$, then $y'/y = 3/x + 4/(2x+1) - 1/(x-1)$.

EXAM POINTER: Logarithmic differentiation of x^x type functions is a classic 'show that' question - memorise the steps.

Lecture 18: Related Rates

Key terms: related rates, implicit differentiation with respect to time

In related-rates problems, several quantities change with time and are linked by a geometric equation. Strategy: (1) draw and label, (2) write the relating equation, (3) differentiate implicitly with respect to t , (4) substitute the known instantaneous values (never before differentiating!), (5) solve for the unknown rate.

- Common relations: $A = \pi r^2$, $V = (4/3) \pi r^3$, Pythagoras $x^2 + y^2 = z^2$, similar triangles.
- Units: the answer's units are (units of quantity) per (unit of time).
- Example: spherical balloon $V = (4/3)\pi r^3$, $dV/dt = 100 \text{ cm}^3/\text{s}$; find dr/dt when $r = 5$: $dV/dt = 4 \pi r^2 dr/dt \rightarrow dr/dt = 100/(100 \pi) = 1/\pi \text{ cm/s}$.

EXAM POINTER: The number-one error is substituting the instant's values before differentiating - differentiate first, substitute after.

Lecture 19: Linear Approximations and Differentials

Key terms: linearisation, differentials, error estimation

Near $x = a$, $f(x) \approx f(a) + f'(a)(x - a)$ - the tangent line approximates the curve. Differentials: $dy = f'(x) dx$ estimates the change in y for a small change dx . This gives quick estimates ($\sqrt{4.02} \approx 2 + (1/4)(0.02) = 2.005$) and error propagation formulas used in science and engineering.

- Approximation improves as x gets closer to a ; error depends on f'' (curvature).
- Relative error in y approx $|f'(x)/f(x)|$ times absolute error in x .
- Example: estimate $(1.03)^5$ via $f(x) = x^5$ at $a = 1$: $1 + 5(0.03) = 1.15$ (true 1.159).

Lecture 20: Maximum and Minimum Values

Key terms: absolute extrema, local extrema, critical numbers, Extreme Value Theorem

An absolute maximum is the largest value of f on its domain; a local maximum is the largest nearby. Critical numbers are interior points where $f'(x) = 0$ or f' is undefined - local extrema can only occur there (or at endpoints). The Extreme Value Theorem: a continuous function on a closed interval $[a, b]$ attains an absolute max and min - find them by evaluating f at critical numbers and endpoints.

- Fermat's theorem: if f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.
- Procedure for $[a, b]$: find critical numbers in (a, b) , evaluate f there and at a, b ; largest/smallest wins.
- Example: $f(x) = x^3 - 3x^2$ on $[-1, 3]$: $f' = 3x(x-2)$, critical 0, 2; values $f(-1) = -4$, $f(0) = 0$, $f(2) = -4$, $f(3) = 0$: abs max 0, abs min -4.

EXAM POINTER: Closed-interval absolute extrema problems are mechanical - never forget to evaluate the endpoints. 

Lecture 21: The Mean Value Theorem and Its Consequences

Key terms: Rolle's theorem, Mean Value Theorem, increasing/decreasing test

Rolle's theorem: if f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then $f'(c) = 0$ for some c in (a, b) . The Mean Value Theorem generalises: some c in (a, b) has $f'(c) = [f(b) - f(a)]/(b - a)$ - the tangent is parallel to the secant. Consequences: if $f' > 0$ on an interval, f is increasing there; if $f' = 0$ everywhere, f is constant.

- MVT justifies speed-limit logic: average speed 120 km/h implies instantaneous speed 120 km/h at some moment.
- First-derivative test for local extrema follows from the increasing/decreasing test.
- Hypotheses matter: continuity on the closed interval, differentiability on the open interval.

EXAM POINTER: Be ready to verify the MVT hypotheses for a given function and find the guaranteed point c explicitly.

Lecture 22: First and Second Derivative Tests; Curve Sketching

Key terms: first derivative test, concavity, inflection points, second derivative test

First derivative test: f' changes + to - at c gives a local max; - to + gives a local min. Concavity: $f'' > 0$ means concave up (cup), $f'' < 0$ concave down (cap). Inflection points are where concavity changes ($f'' = 0$ or undefined, with a sign change). Second derivative test: $f'(c) = 0$ and $f''(c) > 0$ implies a local min; $f''(c) < 0$ a local max; $f''(c) = 0$ inconclusive.

- Sketching checklist: domain, intercepts, asymptotes, intervals of increase/decrease, local extrema, concavity, inflection points.
- Example: $f(x) = x^3 - 3x$: $f' = 3(x^2 - 1)$, local max at $x = -1$, local min at $x = 1$; $f'' = 6x$, inflection at 0.

EXAM POINTER: Full curve-sketching questions combine everything - organise your answer under the checklist headings for full marks.

Lecture 23: Optimization Problems

Key terms: optimisation, objective function, constraint

Strategy: (1) understand and draw, (2) write the quantity to maximise/minimise as a function of one variable using the constraint to eliminate others, (3) find the domain from physical sense, (4) use the closed-interval or first/second derivative test to identify the optimum, (5) answer the question asked with units.

- Classic types: fence/enclosure area, box volume, closest point, minimum cost/surface area, maximum profit.
- If the domain is an open interval, use the first or second derivative test rather than endpoint comparison.
- Example: rectangle of perimeter 100 with max area: $A = x(50 - x)$, $A' = 50 - 2x = 0 \rightarrow x = 25$, a 25x25 square, $A = 625$.

EXAM POINTER: Always verify it is a maximum (not minimum) with a derivative test, and state the final answer in the problem's words.

Lecture 24: Antiderivatives and the Indefinite Integral

Key terms: antiderivative, indefinite integral, constant of integration

F is an antiderivative of f if $F' = f$. Antiderivatives differ by a constant, so the indefinite integral is $\int f(x) dx = F(x) + C$. Basic formulas: $\int x^n dx = x^{(n+1)}/(n+1) + C$ ($n \neq -1$); $\int 1/x dx = \ln|x| + C$; $\int e^x dx = e^x + C$; $\int \sin x dx = -\cos x + C$; $\int \cos x dx = \sin x + C$; $\int \sec^2 x dx = \tan x + C$.

- The $+ C$ is not optional - an indefinite integral without it is marked wrong.

- Initial-value problems: use $f(x_0) = y_0$ to solve for C .
- Example: $f'(x) = 3x^2 + 2$, $f(1) = 5$: $f(x) = x^3 + 2x + C$, $1 + 2 + C = 5 \rightarrow C = 2$.

Lecture 25: The Definite Integral and Area

Key terms: definite integral, Riemann sums, area under curve

The definite integral $\int_a^b f(x) dx$ is the limit of Riemann sums: partition $[a, b]$ into n subintervals, $\sum f(x_i^*) \Delta x$, let $n \rightarrow \infty$. For $f \geq 0$ it equals the area under the curve. Properties: linearity, additivity over adjacent intervals, reversal flips the sign, and $\int_a^a f(x) dx = 0$. The integrand's units times dx 's units give the integral's units.

- Left, right and midpoint sums approximate; more subintervals give better accuracy.
- If f is odd and limits symmetric: $\int_{-a}^a f(x) dx = 0$. If f is even: twice $\int_0^a f(x) dx$.
- Example: $\int_0^2 x^2 dx$ via geometry-free evaluation comes in the next lecture.

Lecture 26: The Fundamental Theorem of Calculus

Key terms: FTC Part 1, FTC Part 2, differentiation of integrals

FTC Part 1: if $g(x) = \int_a^x f(t) dt$ with f continuous, then $g'(x) = f(x)$ - differentiation undoes integration (use the chain rule when the upper limit is a function of x). FTC Part 2: $\int_a^b f(x) dx = F(b) - F(a)$ where F is any antiderivative - this is how definite integrals are actually computed.

- $\frac{d}{dx} \int_1^{x^2} \sin t dt = \sin(x^2) * 2x$ (chain rule!).
- Example: $\int_0^2 x^2 dx = [x^3/3]_0^2 = 8/3$.
- Net signed area: parts below the x -axis count negatively.

EXAM POINTER: FTC Part 1 with a non- x upper limit is an exam favourite - remember to multiply by the derivative of the limit.

Lecture 27: Integration by Substitution

Key terms: u-substitution, change of variables

Substitution reverses the chain rule: for $\int f(g(x)) g'(x) dx$, set $u = g(x)$, $du = g'(x) dx$, giving $\int f(u) du$. For definite integrals, change the limits to u -values (or substitute back before evaluating). Choose u as the 'inside' function whose derivative also appears.

- If the derivative of your u -choice is missing, try another u or another method.
- Definite example: $\int_0^1 2x(x^2+1)^3 dx$; $u = x^2+1$, $du = 2x dx$; limits $1 \rightarrow 2$: $\int_1^2 u^3 du = 15/4$.
- Trigonometric integrals like $\int \sin^3 x \cos x dx$ fall to $u = \sin x$.

Lecture 28: Areas Between Curves

Key terms: area between curves, top minus bottom, intersection points

The area between $y = f(x)$ (top) and $y = g(x)$ (bottom) from a to b is $\int_a^b [f(x) - g(x)] dx$. Find intersection points to get the limits; if the curves cross, split the integral at each crossing so the integrand stays nonnegative. When functions of y are easier, integrate with respect to y : $\int_c^d [\text{right} - \text{left}] dy$.

- Sketch first - the picture tells you which function is on top and where they cross.
- Example: $y = x^2$ and $y = x$ cross at 0 and 1; area = $\int_0^1 (x - x^2) dx = 1/6$.
- Total area enclosed always uses absolute value of each piece.

Lecture 29: Volumes by Slicing: Disks and Washers

Key terms: disk method, washer method, volumes of revolution

Revolve a region about an axis to get a solid of revolution. Disk method: $V = \int \pi [R(x)]^2 dx$ where R is the radius from curve to axis. Washer method (region with a hole): $V = \int \pi ([R_{\text{outer}}]^2 - [R_{\text{inner}}]^2) dx$. Revolve around the y -axis using dy , or use cylindrical shells (next lecture).

- Radius = (curve) - (axis of rotation); be careful with axes like $y = -1$ or $x = 2$.
- Example: $y = \sqrt{x}$, $0 \leq x \leq 4$, about x -axis: $V = \int_0^4 \pi x dx = 8\pi$.
- Washer example: region between $y = x$ and $y = x^2$ about x -axis: $V = \pi \int_0^1 (x^2 - x^4) dx = 2\pi/15$.

Lecture 30: Volumes by Cylindrical Shells

Key terms: shell method, radius times height

The shell method is ideal when revolving around a vertical axis using dx (or horizontal axis using dy): $V = \int 2\pi (\text{radius})(\text{height}) dx$. Radius is the distance from the slice to the axis; height is the vertical extent of the region. Choose disks/washers when the slice is perpendicular to the axis, shells when it is parallel.

- Example: region under $y = x^2$, $0 \leq x \leq 2$, about y -axis: $V = \int_0^2 2\pi x \cdot x^2 dx = 8\pi$.
- For rotation about $x = -1$, radius = $x + 1$.
- Guideline: if integrating the 'natural' variable gives messy inverse functions, try shells.

Lecture 31: Exponential Growth, Logarithms and L'Hopital's Rule

Key terms: L'Hopital's rule, indeterminate forms 0/0 and ∞/∞

L'Hopital's rule: if $\lim f(x)/g(x)$ is $0/0$ or ∞/∞ , then it equals $\lim f'(x)/g'(x)$ provided the latter exists. Differentiate top and bottom separately (not the quotient rule!). Other indeterminate forms - $0 \cdot \infty$, $\infty - \infty$, 0^0 , 1^∞ , ∞^0 - are first converted to $0/0$ or ∞/∞ , often via logarithms.

- Check the form is indeterminate BEFORE applying - the commonest mistake.
- Example: $\lim_{x \rightarrow 0} (e^x - 1)/x = \lim e^x/1 = 1$.
- Example: $\lim_{x \rightarrow \infty} \ln x / x = \lim (1/x)/1 = 0$ - logs grow slower than powers.

EXAM POINTER: Rewrite 1^∞ forms as $e^{\{\dots\}}$ using $\ln$: $\lim (1 + 1/x)^x = e$ is the model solution.

Lecture 32: Sequences

Key terms: sequence, convergence, divergence, monotonic, bounded

A sequence $\{a_n\}$ is an ordered list of numbers. It converges to L if $a_n \rightarrow L$ as $n \rightarrow \infty$; otherwise it diverges (to ∞ , $-\infty$, or by oscillation). A convergent sequence must be bounded; a bounded monotone sequence always converges. Squeeze theorem for sequences and L'Hopital-style limit evaluation (treat n as continuous x) are the main tools.

- $a_n = (1 + 1/n)^n \rightarrow e$; $a_n = n^{1/n} \rightarrow 1$; $a_n = (-1)^n$ diverges by oscillation.
- Recursive sequences: find fixed points of the recursion for the limit, then prove convergence.
- Example: $a_n = (2n+1)/(3n-2) \rightarrow 2/3$ (divide by n).

Lecture 33: Infinite Series: Convergence Tests I

Key terms: infinite series, partial sums, geometric series, p-series, divergence test

The series $\sum a_n$ converges if its partial sums S_N approach a finite limit. Geometric series $\sum ar^n$ converges iff $|r| < 1$, to $a/(1-r)$. The divergence (nth-term) test: if a_n does not tend to 0, the series diverges - but $a_n \rightarrow 0$ does NOT guarantee convergence (harmonic series diverges). The p-series $\sum 1/n^p$ converges iff $p > 1$.

- Harmonic series $\sum 1/n$ diverges (integral test); $\sum 1/n^2 = \pi^2/6$ converges.
- Integral test: $\sum f(n)$ and $\int_1^\infty f(x) dx$ converge or diverge together (f positive decreasing).
- Example: $\sum_{n \geq 1} 3(1/2)^n = 3 \cdot (1/2)/(1-1/2) = 3$.

EXAM POINTER: Always run the divergence test first - it is the cheapest way to prove divergence.

Lecture 34: Infinite Series: Convergence Tests II

Key terms: comparison tests, ratio test, root test, alternating series

Direct/limit comparison: compare with a known benchmark (geometric or p-series). Ratio test: $L = \lim |a_{n+1}/a_n|$; $L < 1$ converges absolutely, $L > 1$ diverges, $L = 1$ inconclusive - best for factorials and exponentials. Root test: $L = \lim |a_n|^{1/n}$ - best for nth powers. Alternating series test (Leibniz): for $\sum (-1)^n b_n$ with b_n decreasing to 0, the series converges (conditionally).

- Ratio test: $\sum n!/n^n$ and $\sum n!/10^n$ fall instantly.
- Absolute vs conditional convergence: absolute convergence implies convergence; alternating harmonic converges conditionally.
- Alternating series remainder: $|\text{error}| \leq \text{first omitted term } b_{N+1}$.

EXAM POINTER: Choose the test from the series' shape: factorials $\rightarrow$ ratio; nth powers $\rightarrow$ root; alternating signs $\rightarrow$ Leibniz; rational-like $\rightarrow$ comparison.

Lecture 35: Taylor and Maclaurin Series

Key terms: Taylor series, Maclaurin series, radius of convergence

The Taylor series of f at a is $\sum f^{(n)}(a)/n! (x-a)^n$; at $a = 0$ it is the Maclaurin series. Memorise: $e^x = \sum x^n/n!$, $\sin x = \sum (-1)^n x^{2n+1}/(2n+1)!$, $\cos x = \sum (-1)^n x^{2n}/(2n)!$, $1/(1-x) = \sum x^n (|x|<1)$, $\ln(1+x) = \sum (-1)^{n+1} x^n/n$. The radius of convergence comes from the ratio test; check endpoints separately for the interval of convergence.

- Substitution trick: $e^{-x^2} = \sum (-1)^n x^{2n}/n!$ - no fresh differentiation needed.
- Taylor polynomials approximate functions; the remainder estimates the error.
- Example: approximate $\sin(0.1)$ by $x - x^3/6$: $0.1 - 0.000167 = 0.099833$.

EXAM POINTER: Deriving a new series by substituting into a memorised one is faster and safer than differentiating from scratch.

Lecture 36: Revision: Midterm Practice Set

Attempt these under timed conditions, then verify each step against the lectures above.

1. Find the domain of $f(x) = \ln((x+1)/(x-2))$. (5 marks)
2. Evaluate $\lim_{x \rightarrow 0} (\tan x - x)/x^3$ using known limits or l'Hopital's rule. (5 marks)
3. Use the limit definition to find $f'(x)$ for $f(x) = \sqrt{x}$. (5 marks)
4. Differentiate $y = (x^2 + 1)^3 \sin(2x)$. (5 marks)

5. A ladder 5 m long slides with its foot moving at 0.4 m/s. How fast is the top descending when the foot is 3 m from the wall? (10 marks)

6. Find and classify all local extrema of $f(x) = x^4 - 4x^3$. (10 marks)

Lecture 37: Revision: Final-Term Practice Set

Comprehensive practice covering integration and series - the heart of the final term.

1. Evaluate $\int (3x^2 + 2x - 1/x) dx$ and $\int_1^4 (\sqrt{x} + 1/x^2) dx$. (10 marks)

2. Find the area enclosed by $y = x^2$ and $y = 2x$. (10 marks)

3. Use the washer method to find the volume when the region bounded by $y = x^2$ and $y = 4$ is revolved about the y-axis. (10 marks)

4. Evaluate $\lim_{x \rightarrow 0^+} x^x$. (5 marks)

5. Test $\sum_{n=1}^{\infty} n^2/2^n$ for convergence, naming the test used. (5 marks)

6. Find the Maclaurin series for $f(x) = x e^x$ up to the x^4 term. (10 marks)

End of Handout - Good luck with your exams!