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HIGHLIGHTED HANDOUT

MTH5211

Mechanics II

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Mechanics II

HANDOUT

This course extends classical mechanics: central forces and orbits, rigid-body dynamics, Lagrangian and Hamiltonian formulations, small oscillations, special relativity, and the mechanics of continuous media.

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MODULE 1: CENTRAL FORCES AND ORBITAL MOTION

Learning Objectives

- Reduce the two-body problem to an equivalent one-body problem
- Derive the orbit equation for central forces
- Analyze the inverse-square law: Kepler's laws
- Treat scattering in a central potential

Reduction of the Two-Body Problem

For two masses m_1, m_2 interacting via a **central force** $F(r) \hat{r}$, introduce the **center of mass** R and relative coordinate $r = r_1 - r_2$. The CM moves uniformly; the relative motion obeys $\mu \ddot{r} = F(r) \hat{r}$ with **reduced mass** $\mu = m_1 m_2 / (m_1 + m_2)$. Angular momentum $L = \mu r^2 \dot{\theta}$ is conserved (central force $\Rightarrow$ no torque), confining motion to a plane. Energy $E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$: the **effective potential** $V_{\text{eff}} = V(r) + \frac{L^2}{2\mu r^2}$ adds the **centrifugal barrier** — orbits are understood by reading this one-dimensional potential diagram.

The Orbit Equation and Kepler's Laws

With $u = 1/r$, the orbit equation is $d^2u/d\theta^2 + u = -\mu F(1/u)/(L^2 u^2)$. For the **inverse-square law** $F = -k/r^2$, this gives $u = (\mu k/L^2)(1 + e \cos \theta)$ — **conic sections**: $e < 1$ ellipse, $e = 1$ parabola, $e > 1$ hyperbola. Kepler's laws follow: (1) elliptical orbits with the center of force at a focus; (2) **equal areas in equal times** ($r^2 \dot{\theta} = \text{const}$ — angular momentum conservation); (3) $T^2 \propto a^3$. Bound orbits ($E < 0$) are closed only for inverse-square and Hooke's-law forces (**Bertrand's theorem**).

Scattering

For $E > 0$ (unbound), particles scatter: the **impact parameter** b and scattering angle θ are related by the orbit geometry; the **differential cross-section** $d\sigma/d\Omega$ counts particles scattered into each solid angle. **Rutherford scattering** (Coulomb repulsion): $d\sigma/d\Omega = (k/4E)^2 \csc^4(\theta/2)$ — the formula whose experimental confirmation established the nuclear atom. Cross-sections translate microscopic force laws into measurable angular distributions.

Key Terms & Definitions

Reduced mass	$\mu = m_1 m_2 / (m_1 + m_2)$; reduces two-body to one-body motion.
Effective potential	$V(r)$ plus the centrifugal term $L^2/(2\mu r^2)$.
Eccentricity	e : shape parameter of the conic orbit.
Cross-section	Effective target area for a scattering process.

Example: Reading the Effective Potential

For gravity, $V_{\text{eff}} = -k/r + l^2/(2\mu r^2)$: at small r the centrifugal barrier dominates (no fall to center for $l \neq 0$); the minimum is the circular orbit; E below zero but above the minimum gives bounded elliptical oscillation between turning points. One curve predicts every orbit type.

Module Summary

- Two-body $\rightarrow$ one-body via reduced mass; angular momentum conserved.
- Orbit equation yields conics for inverse-square forces; Kepler's laws follow.
- Bertrand's theorem: only inverse-square and Hooke give closed bound orbits.
- Scattering: impact parameter $\rightarrow$ Rutherford cross-section.

Exam Pointers (Very Important)

- ★ Derive the orbit equation for motion under a central force.
- ★ Show that Kepler's second law is conservation of angular momentum.
- ★ Derive $T^2 \propto a^3$ for circular orbits.
- ★ Define impact parameter and differential cross-section.

MODULE 2: RIGID-BODY DYNAMICS

Learning Objectives

- Compute centers of mass and inertia tensors
- Apply Euler's equations for torque-free motion
- Analyze the symmetric top
- Use the parallel/perpendicular axis theorems

Inertia Tensor

For extended bodies, angular momentum $\mathbf{L} = \mathbf{I} \cdot \boldsymbol{\omega}$ with the **inertia tensor** $I_{ij} = \int (r^2 \delta_{ij} - x_i x_j) dm$.

Principal axes diagonalize $\mathbf{I}$ (eigenvectors); eigenvalues are the **principal moments**. Theorems:

parallel axis $I = I_{cm} + Md^2$; **perpendicular axis** (lamina) $I_z = I_x + I_y$. Kinetic energy $T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega}$.

Choosing principal axes at the CM simplifies every rigid-body problem — always do this first.

Euler's Equations

In the body frame: $I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = N_1$ (and cyclic). **Torque-free motion**: energy and $|\mathbf{L}|$ conserved; the polhode rolls on the invariable plane. **Stability**: rotation about the largest and smallest principal axes is stable; about the **intermediate axis it is unstable** (tennis-racket/Dzhanibekov effect).

For an axisymmetric body ($I_1 = I_2$), torque-free precession gives **body cone rolling on space cone**.

The Heavy Symmetric Top

A top with fixed point: Lagrangian in Euler angles (θ, ϕ, ψ) gives conserved p_ψ (spin), p_ϕ (azimuthal angular momentum), and energy. **Steady precession** at fixed θ ; **nutation** (nodding) when energy exceeds the steady-precession value; **sleeping top** (vertical, fast spin) stable above a critical spin rate.

The **gyroscope**: torque $\perp \mathbf{L}$ causes precession $\Omega = \tau/L$, not toppling — the basis of gyrocompasses and inertial guidance.

Key Terms & Definitions

Principal axes	Axes diagonalizing the inertia tensor.
Polhode	Path of $\boldsymbol{\omega}$ on the inertia ellipsoid (torque-free motion).
Nutation	Nodding oscillation of a top's axis.
Gyroscopic precession	$\Omega = \tau/L$: torque causes precession, not falling.

Example: The Tennis-Racket Theorem

Toss a book spinning about each principal axis: spins about the longest and shortest axes stay clean; about the intermediate axis it flips wildly mid-air. Euler's equations predict exactly this — intermediate-axis instability, demonstrated on the ISS as the Dzhanibekov effect.

Module Summary

- Inertia tensor $\rightarrow$ principal axes/moments; parallel/perpendicular axis theorems.
- Euler's equations; intermediate-axis instability.
- Symmetric top: steady precession, nutation, sleeping top.
- Gyroscope: torque $\perp$ L gives precession — basis of navigation instruments.

Exam Pointers (Very Important)

- ★ State and prove the parallel-axis theorem.
- ★ Derive Euler's equations for a rigid body.
- ★ Explain intermediate-axis instability.
- ★ Analyze steady precession of the symmetric top.

MODULE 3: LAGRANGIAN MECHANICS

Learning Objectives

- Apply D'Alembert's principle and generalized coordinates
- Derive and use the Euler-Lagrange equations
- Exploit cyclic coordinates and conservation laws
- Solve constrained systems (pendulum, Atwood, beads)

From Newton to Lagrange

Generalized coordinates q_i respect constraints automatically (e.g., angle θ for a pendulum — the string constraint built in). **D'Alembert's principle** (virtual work of applied + inertial forces vanishes) leads to the **Euler-Lagrange equations**: $d/dt(\partial L/\partial \dot{q}_i) - \partial L/\partial q_i = 0$, with $L = T - V$. Advantages: no constraint forces to eliminate, any coordinates, scalar formalism. A **cyclic coordinate** ($\partial L/\partial q_i = 0$) gives a conserved **conjugate momentum** $p_i = \partial L/\partial \dot{q}_i$.

Conservation Laws

Cyclic in a translation direction $\rightarrow$ **momentum conserved**; cyclic in an angle $\rightarrow$ **angular momentum conserved**; no explicit time dependence $\rightarrow$ **energy (Jacobi integral) conserved**. This is the elementary form of **Noether's theorem**: continuous symmetries $\Leftrightarrow$ conservation laws. Identifying cyclic coordinates before computing saves enormous labor — symmetry first, algebra second.

Worked Systems

Simple pendulum: $L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta \rightarrow \ddot{\theta} + (g/l) \sin \theta = 0$. **Spherical pendulum**: ϕ cyclic $\rightarrow$ angular momentum about vertical conserved; effective potential in θ . **Atwood machine**: one coordinate x , $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 - (m_1 - m_2)gx$. **Bead on rotating hoop**: symmetry breaking — above critical rotation speed the bead leaves the bottom (pitchfork bifurcation). **Double pendulum**: two coupled nonlinear equations — chaos from determinism.

Key Terms & Definitions

Generalized coordinates	Minimal coordinates respecting constraints.
Cyclic coordinate	Absent from L ; its conjugate momentum is conserved.
Noether's theorem	Symmetries imply conservation laws.
Jacobi integral	Energy-like conserved quantity for scleronomous systems.

Example: Symmetry First

For the spherical pendulum, noting ϕ is cyclic immediately gives $p_\phi = ml^2 \sin^2 \theta \dot{\phi} = \text{const}$ — reducing a 2-DOF problem to 1-DOF in an effective potential, before writing a single equation of motion. Spotting symmetry is the Lagrangian superpower.

Module Summary

- $L = T - V$; Euler-Lagrange equations in generalized coordinates.
- Cyclic coordinates $\rightarrow$ conserved momenta (Noether's theorem).
- Constraints built into coordinates — no constraint forces.
- Classic systems: pendulum, Atwood, bead on hoop, double pendulum.

Exam Pointers (Very Important)

- ★ Derive the Euler-Lagrange equations from D'Alembert's principle.
- ★ State Noether's theorem and give three examples.
- ★ Find the equations of motion for the spherical pendulum.
- ★ Explain the bead-on-rotating-hoop bifurcation.

MODULE 4: HAMILTONIAN MECHANICS

Learning Objectives

- Construct Hamiltonians via Legendre transform
- Use Hamilton's canonical equations
- Apply Poisson brackets and canonical transformations
- Use the Hamilton-Jacobi method

Hamilton's Equations

Legendre transform: $H(q,p,t) = \sum p_i \dot{q}_i - L$, with $p_i = \partial L / \partial \dot{q}_i$. **Canonical equations:** $\dot{q}_i = \partial H / \partial p_i$, $\dot{p}_i = -\partial H / \partial q_i$ — $2n$ first-order equations replacing n second-order ones. For scleronomous systems $H = T + V$ (total energy). Phase space (q,p) gives the **geometric** view: motion follows the Hamiltonian vector field; **Liouville's theorem** — phase-space volume is preserved (foundation of statistical mechanics).

Poisson Brackets

Poisson bracket $\{f,g\} = \sum (\partial f / \partial q_i \partial g / \partial p_i - \partial f / \partial p_i \partial g / \partial q_i)$; $\dot{\bullet} = \{f,H\} + \partial f / \partial t$. Brackets reveal structure: $\{q_i, p_j\} = \delta_{ij}$; conserved quantities satisfy $\{C,H\} = 0$; **angular momentum components** obey $\{L_i, L_j\} = \epsilon_{ijk} L_k$ — the algebra that becomes quantum commutation relations. **Canonical transformations** preserve the bracket structure (and Hamilton's form); **generating functions** produce them systematically.

Hamilton-Jacobi Theory

Seek a canonical transformation making H vanish: the **Hamilton-Jacobi equation** $H(q, \partial S / \partial q, t) + \partial S / \partial t = 0$ for Hamilton's principal function S . For time-independent H , separate $S = W(q) - Et$: the **time-independent HJ equation** $H(q, \partial W / \partial q) = E$. Complete integrals yield motion by differentiation. **Action-angle variables** for periodic systems: $J = \oint p dq$, angle advances uniformly — the starting point of **perturbation theory** and old quantum theory (Bohr-Sommerfeld quantization).

Key Terms & Definitions

Phase space	$2n$ -dimensional space of (q, p) coordinates.
Poisson bracket	$\{f,g\}$; governs time evolution and canonical structure.
Canonical transformation	Preserves Hamilton's equations' form.
Action-angle variables	(J, θ) for periodic motion; $\dot{\theta} = \text{const.}$

Example: From Brackets to Quantum

The classical bracket $\{x, p\} = 1$ becomes the quantum commutator $[x, p] = i\hbar$. Hamiltonian mechanics isn't just a reformulation — its algebraic structure (Poisson brackets, canonical transformations) is the scaffolding on which quantum mechanics was built.

Module Summary

- $H = \sum p q' - L$; canonical equations $q' = \partial H / \partial p$, $p' = -\partial H / \partial q$.
- Liouville: phase volume preserved. Poisson brackets: $\{f, H\}$.
- Canonical transformations via generating functions.
- Hamilton-Jacobi: S with $H + \partial S / \partial t = 0$; action-angle for periodic systems.

Exam Pointers (Very Important)

- ★ Derive Hamilton's equations from the Lagrangian via Legendre transform.
- ★ Prove Liouville's theorem.
- ★ Show $\{L_i, L_j\} = \epsilon_{ijk} L_k$ for angular momentum.
- ★ Solve the harmonic oscillator by Hamilton-Jacobi.

MODULE 5: SMALL OSCILLATIONS AND NORMAL MODES

Learning Objectives

- Linearize about stable equilibrium
- Solve the eigenvalue problem for normal modes
- Apply to coupled pendulums and molecules
- Introduce damped and driven oscillations

Linearization

About stable equilibrium q_0 , expand: $T \approx \frac{1}{2} \sum T_{ij} \eta_i \eta_j$, $V \approx \frac{1}{2} \sum V_{ij} \eta_i \eta_j$ (η = small displacements; linear terms vanish at equilibrium). Equations: $\sum (T_{ij} \eta_j + V_{ij} \eta_j) = 0$. Seek $\eta = a e^{i\omega t} \rightarrow$ **generalized eigenvalue problem** $(V - \omega^2 T)a = 0$; secular equation $\det(V - \omega^2 T) = 0$ gives **normal frequencies** ω_k^2 . Stability requires all $\omega^2 > 0$ (V positive definite).

Normal Modes

Each eigenvalue gives a **normal mode** — a collective motion where all coordinates oscillate at one frequency. General motion = superposition of modes. Examples: **coupled pendulums** (in-phase and out-of-phase modes; beats when excited singly); **linear triatomic molecule** (symmetric stretch, antisymmetric stretch, bend — IR/Raman selection rules follow); loaded string $\rightarrow$ continuum limit gives the **wave equation**. Orthogonality of modes (with respect to T) decouples the energy.

Damping and Driving

Add damping $-b\dot{q}$ and driving $F \cos(\omega t)$: **resonance** at $\omega \approx \omega_0$ with amplitude $\propto 1/\text{damping}$ — bridges, buildings, and circuits must avoid it (or exploit it, as in MRI). **Q factor** measures sharpness. Parametric resonance (pumping a swing) and nonlinear corrections (Duffing) lie just beyond — the gateway to chaos theory.

Key Terms & Definitions

Normal mode	Collective oscillation at a single frequency.
Secular equation	$\det(V - \omega^2 T) = 0$; yields normal frequencies.
Resonance	Large response when driven near natural frequency.
Q factor	Sharpness of resonance; energy stored per cycle lost.

Example: Beats from Modes

Displace one of two coupled pendulums: energy sloshes back and forth (beats) — the motion looks complicated, but it's just the sum of in-phase and out-of-phase normal modes at slightly different frequencies. Normal modes are the 'Fourier basis' of vibration.

Module Summary

- Linearize: $(V - \omega^2 T)a = 0 \rightarrow$ normal frequencies and modes.
- General motion = superposition of normal modes.
- Applications: coupled pendulums, molecular vibrations, wave equation limit.
- Damping + driving $\rightarrow$ resonance; Q measures sharpness.

Exam Pointers (Very Important)

- ★ Set up the small-oscillation eigenvalue problem.
- ★ Find the normal modes of two coupled pendulums.
- ★ Explain beats as superposition of normal modes.
- ★ What is resonance and why does the Q factor matter?

MODULE 6: SPECIAL RELATIVITY

Learning Objectives

- State Einstein's postulates and derive the Lorentz transform
- Explain time dilation, length contraction, simultaneity
- Use four-vectors and invariants
- Apply relativistic energy-momentum relations

Postulates and Lorentz Transformation

Postulates: (1) laws of physics identical in all inertial frames; (2) speed of light c is frame-independent. Consequences via the light-clock: **time dilation** $\Delta t = \gamma \Delta \tau$, **length contraction** $L = L_0/\gamma$, **relativity of simultaneity**. **Lorentz transformation:** $x' = \gamma(x - vt)$, $t' = \gamma(t - vx/c^2)$, $\gamma = 1/\sqrt{1-v^2/c^2}$. Galilean transform recovered as $v/c \rightarrow 0$. Spacetime diagrams (Minkowski): worldlines, light cones, invariant hyperbolae.

Four-Vectors

Four-vectors: $x^\mu = (ct, \mathbf{x})$, $p^\mu = (E/c, \mathbf{p})$. Lorentz-invariant **scalar product** $a \cdot b = a_\mu b^\mu = \mathbf{a} \cdot \mathbf{b} - a^0 b^0$. Invariants: **spacetime interval** $s^2 = c^2 t^2 - \mathbf{x}^2$ (timelike/lightlike/spacelike separation — causality lives here); **rest mass** from $p^\mu p_\mu = m^2 c^2$ giving **$E^2 = \mathbf{p}^2 c^2 + m^2 c^4$** . Four-velocity, four-force; electromagnetic field as tensor $F^{\mu\nu}$ unifying $\mathbf{E}$ and $\mathbf{B}$.

Relativistic Dynamics

Momentum $\mathbf{p} = \gamma m \mathbf{v}$; force $\mathbf{F} = d\mathbf{p}/dt$ (note: $\mathbf{F}$ not parallel to $\mathbf{a}$ in general). **Total energy** $E = \gamma mc^2 = \text{rest energy } mc^2 + \text{kinetic } (\gamma - 1)mc^2$. **Massless particles:** $E = pc$. Applications: **relativistic Doppler**, particle collisions (invariant-mass reconstruction discovers particles), **twin paradox** resolved (the traveling twin accelerates — asymmetry is physical), GPS clock corrections (SR + GR).

Key Terms & Definitions

Lorentz factor γ	$1/\sqrt{1 - v^2/c^2}$; measures relativistic effects.
Proper time	Time measured in the frame where events are co-located.
Invariant interval	$s^2 = c^2 t^2 - \mathbf{x}^2$; same in all frames.
Four-momentum	$p^\mu = (E/c, \mathbf{p})$; its norm gives rest mass.

Example: Muons Reach the Ground

Muons created 10 km up live only $2.2 \mu\text{s}$ — classically traveling 660 m. Time-dilated by $\gamma \approx 20$ at $0.999c$, they survive the trip: we detect them at sea level. Nature runs the relativity experiment continuously above our heads.

Module Summary

- Two postulates → Lorentz transform; dilation, contraction, simultaneity.
- Four-vectors; invariants: interval s^2 , rest mass, $E^2 = p^2c^2 + m^2c^4$.
- $p = \gamma mv$, $E = \gamma mc^2$; massless: $E = pc$.
- Applications: Doppler, collisions, twin paradox, GPS.

Exam Pointers (Very Important)

- ★ Derive time dilation from the light-clock thought experiment.
- ★ Derive the Lorentz transformation from the postulates.
- ★ Prove $E^2 = p^2c^2 + m^2c^4$ using four-momentum.
- ★ Resolve the twin paradox.

MODULE 7: CALCULUS OF VARIATIONS

Learning Objectives

- Derive the Euler-Lagrange equation variationally
- Solve brachistochrone and geodesic problems
- Handle constraints with Lagrange multipliers
- Connect to Fermat's and Hamilton's principles

The Variational Principle

Minimize $J[y] = \int F(x, y, y') dx$: the **Euler-Lagrange equation** $d/dx(\partial F/\partial y') - \partial F/\partial y = 0$ is the necessary condition (same form as mechanics with $x \rightarrow t$). **Hamilton's principle**: the true path extremizes the **action** $S = \int L dt$ — all of mechanics from one scalar functional. **Fermat's principle** (least time) gives Snell's law and mirages from the same mathematics.

Classic Problems

Brachistochrone: fastest-descent curve is a **cycloid** (also tautochrone — equal-time from any start). **Geodesics**: shortest paths — straight lines on planes, **great circles** on spheres (basis of GR: free fall follows spacetime geodesics). **Minimal surfaces**: soap films. **Isoperimetric problem** (max area for given perimeter $\rightarrow$ circle) needs **Lagrange multipliers** for the constraint — the same multiplier technique used for non-holonomic constraints in mechanics.

Constraints and Generalizations

Holonomic constraints (relations among coordinates) vs. **non-holonomic** (non-integrable velocity constraints — rolling without slipping). Lagrange multipliers adjoin constraints to the functional. Multiple dependent variables, higher derivatives, and variable endpoints extend the machinery. The variational viewpoint unifies mechanics, optics, and field theory — the language of modern physics.

Key Terms & Definitions

Action	$S = \int L dt$; extremized by the true path.
Brachistochrone	Fastest-descent curve: a cycloid.
Geodesic	Locally shortest path on a surface/manifold.
Lagrange multiplier	Adjoins constraints to a variational problem.

Example: The Cycloid Wins

A bead sliding down a straight ramp vs. a cycloid ramp: the cycloid — which dips below the straight line first to gain speed — always wins, and any bead released anywhere on it reaches the bottom simultaneously (tautochrone). Huygens used this for the first accurate pendulum clock.

Module Summary

- $\delta J = 0 \rightarrow$ Euler-Lagrange equation; Hamilton's principle: $\delta \int L \, dt = 0$.
- Brachistochrone = cycloid; geodesics = great circles on spheres.
- Lagrange multipliers handle constraints (isoperimetric, non-holonomic).
- Variational principles unify mechanics, optics, field theory.

Exam Pointers (Very Important)

- ★ Derive the Euler-Lagrange equation from $\delta J = 0$.
- ★ Solve the brachistochrone problem.
- ★ Find geodesics on a sphere.
- ★ State Hamilton's principle and derive Lagrange's equations from it.

MODULE 8: MECHANICS OF CONTINUOUS MEDIA (INTRODUCTION)

Learning Objectives

- Describe the continuum approximation and deformation
- Define stress and strain tensors
- State the equations of elasticity and fluid motion
- Derive the wave equation for a string

From Particles to Continua

The **continuum approximation** smears discrete atoms into continuous fields: displacement $u(x,t)$, density $\rho(x,t)$, velocity $v(x,t)$. **Deformation**: strain tensor $\epsilon_{ij} = \frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i)$ (symmetric part of displacement gradient). **Stress tensor** σ_{ij} : force per area on each face — symmetric by angular-momentum balance. **Constitutive laws** link them: **Hooke's law** $\sigma = C:\epsilon$ for elastic solids; **Newtonian fluid** $\sigma = -pI + \eta(\nabla v + \nabla v^T)$ for viscous fluids.

Equations of Motion

Cauchy's equation: $\rho \ddot{u} = \nabla \cdot \sigma + f$. For elastic solids → **Navier equation**; for fluids, the material derivative gives **Navier-Stokes**: $\rho(\partial v / \partial t + v \cdot \nabla v) = -\nabla p + \eta \nabla^2 v + f$. Inviscid limit → **Euler equation**; incompressible + irrotational → **potential flow** and Bernoulli's equation. **Boundary conditions** (no-slip for viscous fluids) select the physical solution — and generate boundary layers.

Waves in Continua

Stretched string: transverse force balance → **wave equation** $\partial^2 u / \partial t^2 = c^2 \partial^2 u / \partial x^2$, $c = \sqrt{T/\mu}$ — d'Alembert's solution: superposed left/right traveling waves. Elastic solids carry **P (longitudinal) and S (transverse) waves** at different speeds — earthquake seismology reads Earth's interior from their arrivals. Surface waves, sound waves (compressible fluids), and dispersion complete the picture: **continuum mechanics is where mechanics becomes field theory**.

Key Terms & Definitions

Stress / strain tensors	Force-per-area and deformation measures in a continuum.
Navier-Stokes	Momentum equation for viscous fluid flow.
Bernoulli's equation	Energy conservation along streamlines (inviscid).
d'Alembert solution	$u = f(x-ct) + g(x+ct)$ for the wave equation.

Example: Reading an Earthquake

A seismogram shows P waves first (fast, longitudinal), then S waves (slower, transverse), then surface waves. The S-P time gap gives the distance; S waves' absence through the core proved it liquid. Continuum wave mechanics lets us X-ray a planet we can never drill into.

Module Summary

- Continuum: displacement/density/velocity fields; stress and strain tensors.
- Hooke's law (solids), Newtonian fluid law; Navier-Stokes/Euler equations.
- Wave equation from the stretched string; P and S waves in solids.
- Continuum mechanics bridges particle mechanics and field theory.

Exam Pointers (Very Important)

- ★ Define the stress and strain tensors.
- ★ Derive the wave equation for a stretched string.
- ★ Write the Navier-Stokes equation and explain each term.
- ★ How do P and S waves reveal Earth's interior?

MODULE 9: NONLINEAR DYNAMICS AND CHAOS (OUTLOOK)

Learning Objectives

- Recognize nonlinearity: superposition fails
- Analyze fixed points and bifurcations
- Describe chaos: sensitivity and strange attractors
- Connect to the double pendulum and three-body problem

Beyond Linearity

Linear systems obey **superposition**; nonlinear ones don't — and most of nature is nonlinear (the pendulum's $\sin \theta$, fluid advection $\mathbf{v} \cdot \nabla \mathbf{v}$). **Phase-plane analysis**: fixed points classified by linearization (nodes, saddles, spirals, centers); **Lyapunov stability**. **Bifurcations**: qualitative change with parameters — saddle-node, pitchfork (bead on rotating hoop!), Hopf (birth of limit cycles). The **logistic map** $x_{n+1} = rx(1-x)$ shows the **period-doubling route to chaos**.

Chaos

Deterministic chaos: bounded, aperiodic motion with **sensitive dependence on initial conditions** (positive Lyapunov exponent) — the **Lorenz attractor** (weather), the driven Duffing oscillator, the **double pendulum**. **Strange attractors** have fractal structure. **KAM theory**: most quasi-periodic orbits survive small perturbations — explaining solar-system stability despite chaos. Poincaré sections tame the visualization.

Why It Matters

The **three-body problem** has no general closed-form solution — Poincaré's discovery birthed chaos theory. Practical impact: **predictability horizons** (weather ~2 weeks), chaos control, secure communications, cardiac arrhythmia analysis. Mechanics II ends where modern dynamics begins: **determinism without predictability** — Laplace's demon defeated by nonlinearity.

Key Terms & Definitions

Bifurcation	Qualitative change of dynamics at a parameter value.
Lyapunov exponent	Measures exponential divergence of nearby trajectories.
Strange attractor	Fractal attracting set of a chaotic system.
KAM theory	Persistence of quasi-periodic motion under perturbation.

Example: The Double Pendulum

Release a double pendulum twice from 'the same' position: within seconds the motions diverge completely — yet both obey deterministic equations. Sensitivity, not randomness: this tabletop device is the most visceral demonstration that determinism $\neq$ predictability.

Module Summary

- Nonlinearity breaks superposition; phase-plane + bifurcations organize behavior.
- Chaos: deterministic, bounded, aperiodic, sensitive — Lorenz, double pendulum.
- KAM theory explains persistence of order amid chaos.
- Three-body problem: no general solution — predictability horizons are fundamental.

Exam Pointers (Very Important)

- ★ Classify fixed points of a 2-D dynamical system.
- ★ Describe the period-doubling route to chaos in the logistic map.
- ★ What is a strange attractor? Give an example.
- ★ Why has the three-body problem no general closed-form solution?