Mth101 Final Term
Highlighted Lectures:23-45
By Sir Ray Afzal

Mt

Lecture # 23

Maximum and Minimum Values of Functions

- Absolute Extrema
- Finding Absolute Extrema for a continuous function
- Summary of extreme behaviors of functions over
- Applied maximum and minimum problems
- Problems involving finite closed intervals
- Problems involving intervals that are non-finite and closed

Absolute Extrema

Previously we talked about relative maxima, relative minima of functions.
These were like the highest mountain and the deepest valley in a given vicinity or neighborhood.
Now we will talk about absolute maximum and minimum values of functions.
These are like the highest peak in a mountain range, and the deepest valley.

DEFINITION 4.6.1

If $f(x_0) \geq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute maximum value or simply the maximum value of f.

DEFINITION 4.6.2

If $f(x_0) \leq f(x)$ for all x in the domain of f, then $f(x_0)$ is called the absolute minimum value or simply the minimum value of f.

Absolute maximum means that the value is the maximum one over the entire domain of the function.
Absolute minimum means that the value is the minimum one over the entire domain of the function.
If we think of the Earth's surface as defining some function, then its absolute maximum will be Mt. Everest, and absolute minimum will be the Marianna Trench in the Pacific Ocean near Hawaii.

Example

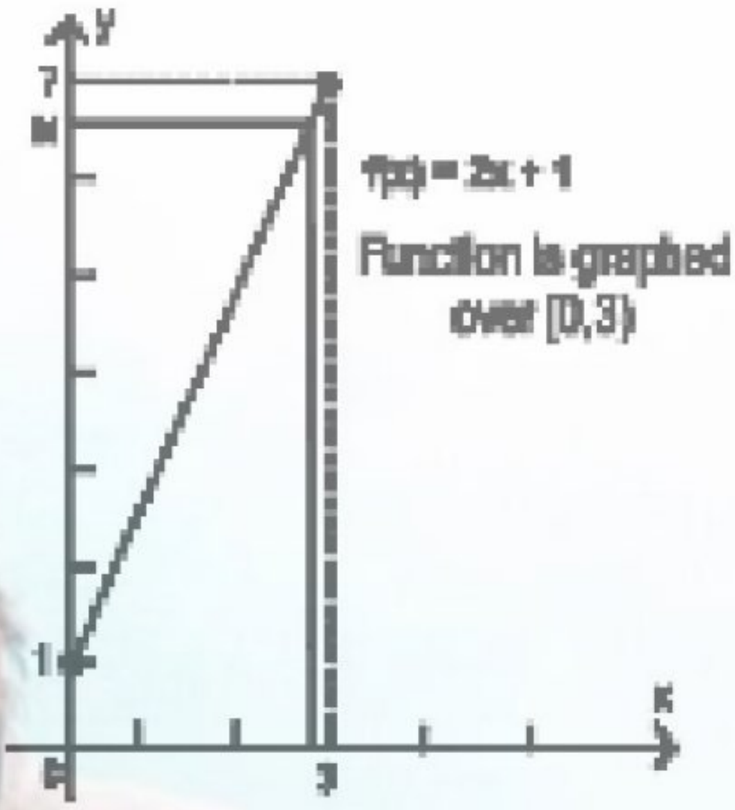
Consider the following picture of the graph of f (x) = 2x+1 on the interval [0,3).
The minimum value is 1 at
x = 0. But is there a maximum?

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No. Because the function is
defined on the interval [0,3) which excludes the point x = 3.
So note that you can get very close to 7 as the maximum value as you get very close to x = 3, but this is in a limiting process, and you can always get more closer to 7, and yet never EQUAL 7! So 7 looks like a max, but its NOT!



The question of interest given a function f(x) is:
Does f (x) has a maximum (minimum) value?
If f (x) has a maximum (minimum) , what is it?
If f (x) has a maximum value, where does it occur?

THEOREM 4.6.4
(Extreme-Value Theorem)

If a function f is continuous on a closed interval [a,b], then f has both a maximum value and a minimum value on [a,b]

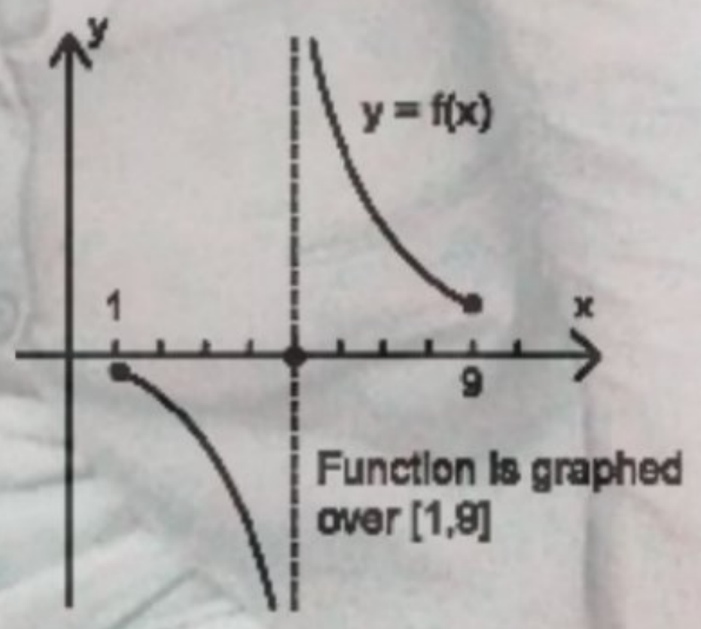
Won't prove this as its difficult. Just use it. This theorem doesn't tell us what the max and the min are, just the conditions on a function which will make it have a max or min

Example

Note that in the previous example we saw a function f (x) = 2x+1 which was defined over the interval [0, 3). This one is a continuous function on that interval, but had no maximum because the interval was not closed!

Example

Here is another function graphed over the interval [1,9]. Although the interval is closed, the function not continuous on this interval as we can see from the graph, and so has no maximum or minimum values on that interval .



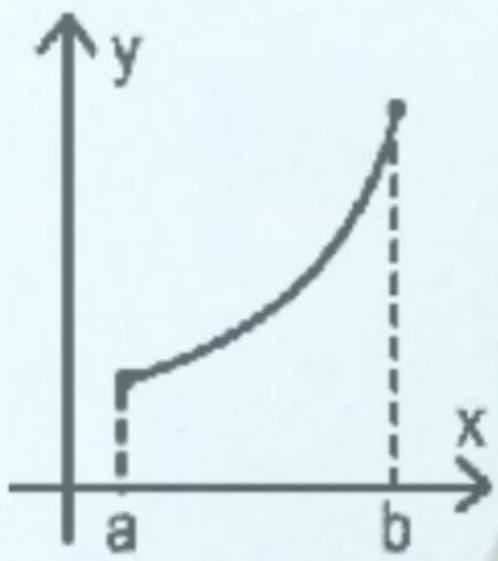
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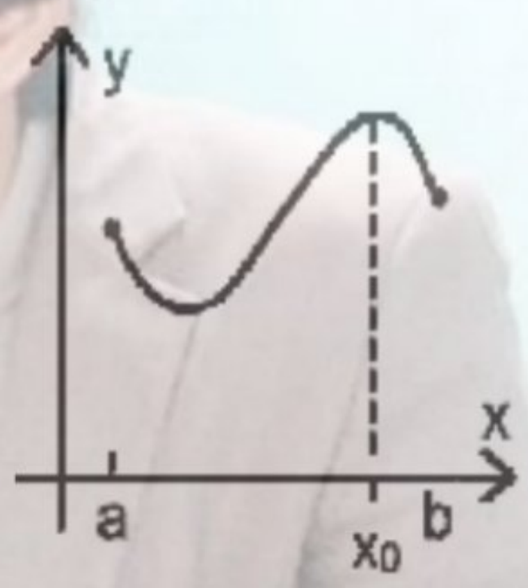
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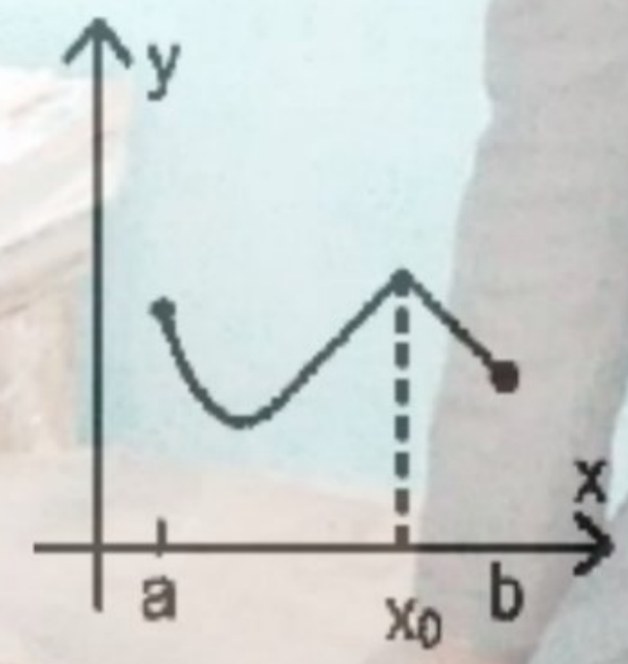
THEOREM 4.6.5

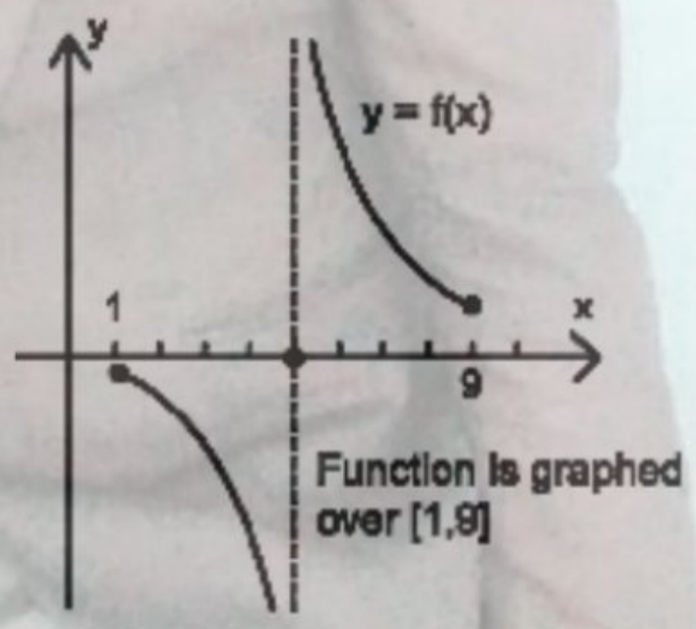
If a function f has an extreme value (either a maximum or a minimum) on an open interval(a,b), then the extreme value occurs at a critical point of f.

Step 1: Find the critical points of *f* in (a,b)
Step 2: Evaluate *f* at all the critical points and the endpoint a and b
Step 3: The largest of the values in Step 2 is the maximum value of *f* on [a,b] and the smallest is the minimum.









Example

Find the maximum and the minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval [1, 5]
Since *f* is a polynomial, it's continuous and differentiable on the interval (1, 5)

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$f'(x) = 6x^2 - 30x + 36 = 0$

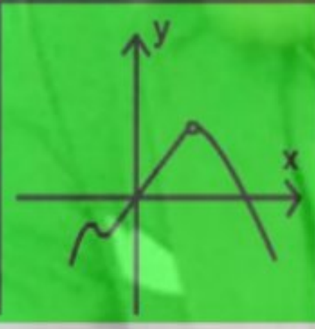
$f'(x) = (x-3)(x-2) = 0$

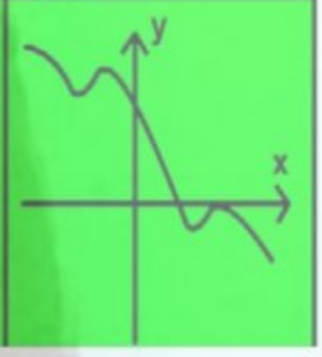
So f' is 0 at x = 2 and x = 3. Max or min will occur at these two points or at the end points.

Evaluate the function at the critical points and the endpoints and we see that max is 55 at x = 5 and min is 23 at x = 1

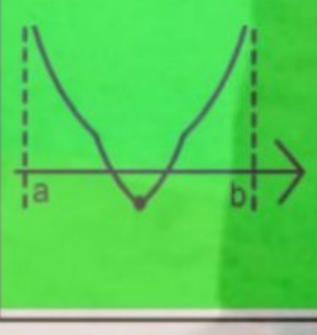
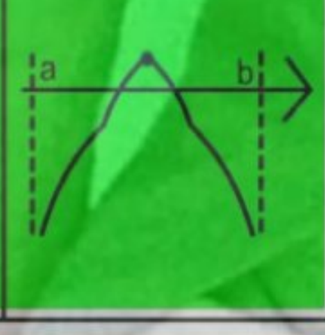
We want to know the max and min over $(-\infty, +\infty)$

Here is how to find them for continuous functions

Limits	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	f has a minimum but no maximum on $(-\infty,+\infty)$	f has a maximum but no minimum on $(-\infty,+\infty)$
Graph		

Limits	$\lim_{x\to-\infty} f(x) = -\infty$ $\lim_{x\to+\infty} f(x) = +\infty$	$\lim_{x\to-\infty} f(x) = +\infty$ $\lim_{x\to+\infty} f(x) = -\infty$
Conclusion if f is continuous	f has neither a maximum nor a minimum on $(-\infty,+\infty)$	f has neither a maximum nor a minimum on $(-\infty,+\infty)$
Graph		

Summary of extreme behaviors of functions over (a,b)

Limits	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has a minimum but no maximum on (a,b)	f has a maximum but no minimum on (a,b)
Graph		

Limits	$\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to b^-} f(x) = +\infty$	$\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has neither a maximum nor a minimum on (a,b)	f has neither a maximum nor a minimum on (a,b)
Graph		

Find the max and min values if any, of the function $f(x) = x^4 + 2x^3 - 1$ on the interval $(-\infty, +\infty)$

This is a continuous function on the given interval and

$\lim_{x\to+\infty} (x^4 + 2x^3 - 1) = +\infty$ and $\lim_{x\to-\infty} (x^4 + 2x^3 - 1) = +\infty$

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So *f* has a minimum but no maximum on (-inf, +inf). By Theorem 4.6.5, the min must occur at a critical point. So

$$f'(x) = 4x^3 + 6x^2 = 2x^2(2x+3) = 0$$

This gives x = 0, and x = -3/2 as the critical points. Evaluating gives min = -43/16 at x = -3/2

Applied maximum and minimum problems

We will use what we have learnt so far to do some applied problems in OPTIMIZATION. Optimization is the way efficiency is got in business, machines and even in nature in terms of animals competing for resources.

Problems involving continuous functions and Finite closed intervals

These are problems where the function is defined over a closed interval. These problems always have a solution because it is guaranteed by the extreme values theorem.

Example

Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.

Let

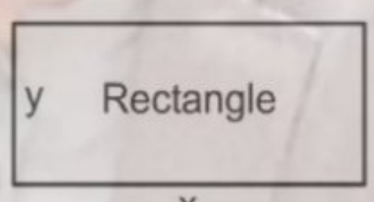
x = *length of the rectangle in feet*

y = *width of the rectangle in feet*

A = *area of the rectangle*

Perimeter = 100ft = 2x + 2y

y = 50 – x



A = xy

A = x(50 - x)

A = 50x - x²

Then A = xy

We want to maximize the area

$$A = x(50-x) = 50x - x^2$$

Perimeter = 100ft = 2x + 2y or y = 5 – x

Use this values of y in the equation A = xy to get A function of x

Because x represents a length, it cannot be negative and it cannot be a value that exceeds the perimeter of 100 ft. So we have the following constraints on x

$$0 \le x \le 50$$

So the question is of finding the max of $A = 50x - x^2$ on the interval [0,50]. By what we have seen so far, that max must occur at the end points of this interval or at a critical point

Critical points:

$$\frac{dA}{dx} = 50 - 2x = 0 \quad \Rightarrow \quad x = 25$$

So now we substitute x = 0, x = 25, and x = 50 into the function A to get the max 625 at the point x = 25.

Note that y = 25 also for x = 25. So the rectangle with perimeter 100 with the greatest area is a square with sides 25ft.

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Example

An open box is to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let

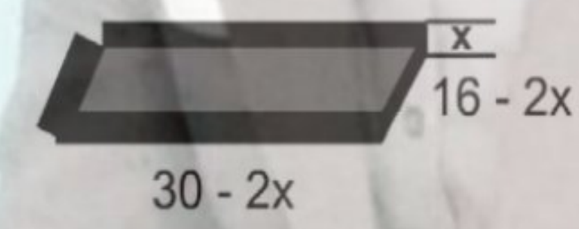
x = length of the sides of the squares to be cut out

V = volume of the resulting box.

We want to maximize the Volume V.

If we cut out the squares from the 4 corners of the cardboard, the resulting BOX will have dimensions

$(16 - 2x)$ by $(30 - 2x)$



$$V = (\text{length})(\text{width})(\text{Height})$$

$$= (16-2x)(30-2x)x$$

$$= 480x - 92x^2 + 4x^3$$

$$0 \le x \le 8$$

s

So want to find max of the function on [0,8].

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 0 \quad \Rightarrow \quad x = \frac{10}{3} \text{ and } x = 12$$

Because x = 12 is out of [0, 8], ignore it. Check V at the end points and at x = 10/3. We see then that V = 19600/27 is max when x = 10/3.

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Lecture # 24

Newton's Method, Rolle's Theorem, and the Mean Value Theorem

- Newton's method for approximating solutions to f(x)=0
- Some difficulties with Newton's method
- Rolle's theorem
- Mean Value Theorem

Newton's method for approximating solutions to f (x) =0

We have seen in algebra that the solution for the equation $ax+b=0$ is $x=-\frac{b}{a}$.

Similarly we have algebraic formulas for polynomial equation up to degree 5.
But there is no algebraic solution to the equation of the kind

$x-\cos(x)=0$

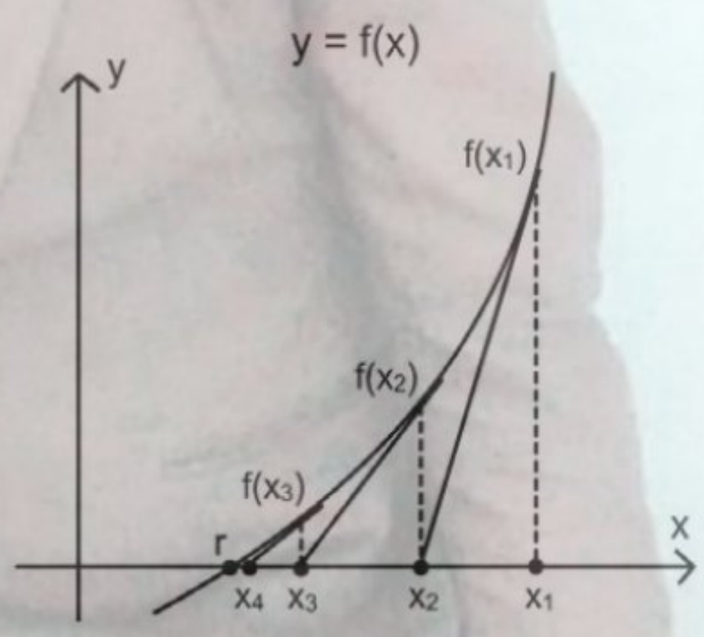
For this equation and many like it, we settle for approximate solutions. How do we approximate the solutions? There are many methods, and one of them is Newton's method. Here is how Newton's method works to find approximate solutions to equations.

What does it mean for an equation to have a solution? or that f (x) = 0 ?
It means that we are looking for those x values, for which the corresponding y value or f (x) is 0.
This means that the solutions are those points where the graph of the function crosses the x axis.
Suppose that x = r is the solution we are looking for.
Let's approximate it by an initial guess called.
We draw a line tangent to the graph of the given function at the point.
If the tangent line is not parallel to the x-axis, then it will x_2 eventually intersect the x-axis at some point x_1 which will generally be closer to *r* than We will repeat the process, with a tangent line at x2 that meets x axis at x3, and so on…
This is Newton's method.

We need a formula for this method.
Note that the tangent line at x_1 has the equation

$y-f(x_1)=f'(x_1)(x_2-x_1)$

If $f'(x_1)\neq 0$, then the line meets the x axis at $(x_2,0)$



Plug this coordinate into above equation, we get

$-f(x_1)=f'(x_1)(x_2-x_1)$

$\Rightarrow x_2=x_1-\frac{f(x_1)}{f'(x_1)}$

Repeating this process for a third point $(x_3,0)$ gives

$-f(x_2)=f'(x_2)(x_3-x_2)$

$\Rightarrow x_3=x_2-\frac{f(x_2)}{f'(x_2)}$

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In general, then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

There is limiting process involved here for finding the solutions. We get as close as we like to the solution.

Example

The equation $x = \cos(x)$ has a solution between 0 and 1. Approximate it using Newton's method.

Rewrite as $x = \cos(x)$ so our function is $f(x) = x - \cos(x)$. The derivative is

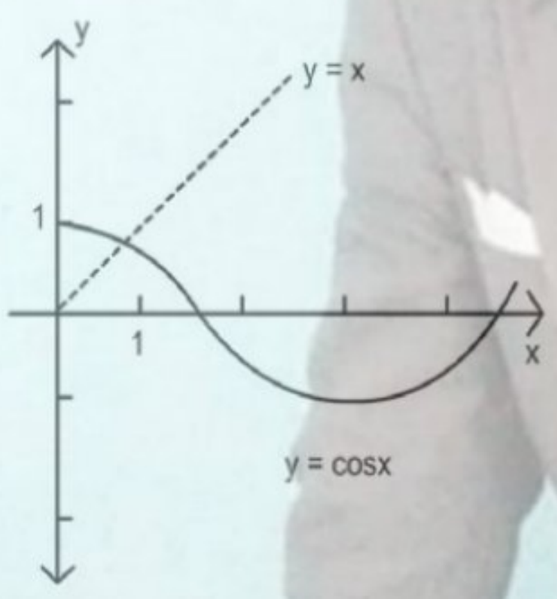
$$f'(x) = 1 + \sin(x)$$

So we have

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

As our approximation formula

Here is a graph of the situation. From the graph it looks like the solution is closer to 1 than 0. So we will use $x_1 = 1$.So we get the following approximations.



$$x_2 = x_1 - \frac{x_1 - \cos(x_1)}{1 + \sin(x_1)} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.7503$$

$$x_3 = x_2 - \frac{x_2 - \cos(x_2)}{1 + \sin(x_2)} = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1 + \sin(0.7503)} = 0.7391$$

You may continue if you will, but we will say that the solution is approximately $x \approx 0.7391$

Some difficulties with Newton's method

- Newton's method does not always work.
- If for some values of n $f'(x_n)$ then the formula for Newton's method involves division by 0 and we are out of business.

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- Such a case will occur if the tangent line for some approximation has slope 0 or is parallel to the x axis.
- Sometimes the approximations don't converge to a solution.

Consider the equation $x^{\frac{1}{3}} = 0$

The only solution is x = 0. Let's approximate it by Newton's Method with initial approx $x_1 = 1$. We get the following Formula

$$x_{n+1} = x_n - \frac{(x_n)^{\frac{1}{3}}}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} = -2x_n$$

Plug in $x_1 = 1$ and then the following approx to see that the values do not converge

Rolle's theorem

Rolle's Theorem says essentially that for a certain kind of function, if it crosses the x-axis at two point, then there is one point between those two points where the derivative of *f* is 0.

THEOREM 4.9.1
(Rolle's Theorem)

Let f be differentiable on (a,b) and continuous on [a,b] . If f(a) = f(b) = 0, then there is at least one point c in (a,b) where $f'(c) = 0$.

Example

The function $f(x) = \sin(x)$ is continuous and differentiable everywhere, hence continuous on [0 , 2π] and differentiable on (0, 2π). Also, f (0) = sin (0) = 0 and

f(2π)=sin (2π) = 0

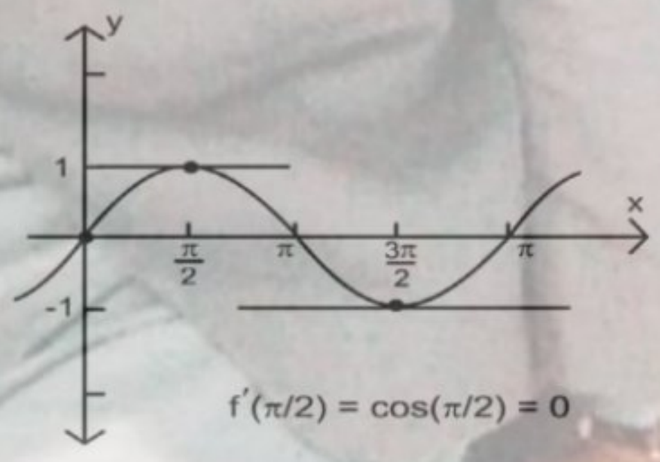
So the function satisfies the hypotheses of Rolle's Theorem. So there exists a point *c* in the interval (0 , 2π) such that

$$f'(c) = \cos(c) = 0$$

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Rightarrow$$

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$



f'(π/2) = cos(π/2) = 0

Here is a more tangible way to think of Roll's theorem

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I leave from Lahore to Islamabad.
When i start driving from Lahore, my velocity is 0, and when i reach Islamabad, my velocity is 0 as well.
Velocity is a continuous function on the interval [0 , 376].
Also, velocity is differentiable on (0 , 376) as its derivative acceleration is defined at each point on the velocity curve.
Hence during my drive from Lahore to ISB, there is some point on the motorway where the acceleration of the car was 0.
One could argue: What if you keep accelerating on the motorway!

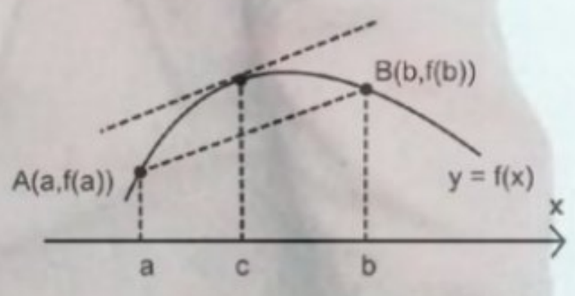
Mean Value Theorem
This says basically that under the right conditions, a function will have the same slope for the tangent line at a point as that of a certain secant line.

THEOREM 4.9.2
(Mean-Value Theorem)

Let f be differentiable on (a,b) and continuous on [a,b). Then there is at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of MVT
From this figure we have the following:
Slope of Secant line joining A and B:



v(x) = vertical distance between curve f(x) and secant line through A and B

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right]$$

Since f(x) is continuous [a , b] and differentiable on (a , b), so is v(x) by its formula involving f(x).
Also note that

$$v(a) = 0 \text{ and } v(b) = 0$$

So v(x) satisfies the assumptions of Roll's theorem on the interval [a , b]. So there is a point c in (a , b) such that v'(c)=0. But note that

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$$v'(x) = f'(x) - \left[\frac{f(b)-f(a)}{b-a}\right]$$

$$\Rightarrow v'(c) = f'(c) - \left[\frac{f(b)-f(a)}{b-a}\right]$$

So by this last formula, at the point where $v'(c) = 0$, we have

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



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Geometric Series

A geometric series is one of the form

$a + ar + ar^2 + ... + ar^{k-1} + ..$

Each term is obtained by multiplying the previous one by a constant number, *r*, and this number is called the ratio for the series.

Examples are

1+2+4+8+…

1+1+1+1+…

THEOREM 11.3.3

A geometric series

$a + ar + ar^2 + ... + ar^{k-1} + ...$ $(a \neq 0)$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\frac{a}{1-r} = a + ar + ar^2 + ... + ar^{k-1} + ...$$

Example

The series

$$5 + \frac{5}{4} + \frac{5}{4^2} + ... + \frac{5}{4^{k-1}} + ...$$

is a geometric series with $a = 5$ and $r = 1/4$. Since $|r| = 1/4 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$$

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Harmonic Series

A Harmonic series is of the type

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\ldots$$

It may look like this series converges as each successive term is smaller than the first, but in fact this series diverges!

Convergence Tests

Here are some tests that are used to determine if a series converges or diverges. These tests are applied on the kth term of a series (general term), and not on the nth partial sum as we did earlier to derive the definition of the sum of an infinite series.

THEOREM 11.4.1

(The Divergence Test)

a) If $\lim_{k\to+\infty} u_k \neq 0$, then the series $\sum u_k$ ***diverges***

b) If $\lim_{k\to+\infty} u_k = 0$, then the series $\sum u_k$ may either ***converge*** or ***diverge***.

This theorem tells us when a series diverges
The alternative form of part (a) is sufficiently important that we state it separately

THEOREM 11.4.2

If the series $\sum u_k$ converges, then

$$\lim_{k\to+\infty} u_k = 0$$

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Example

The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots\ldots + \frac{k}{k+1} + \ldots.$$

Diverges since

$$\lim_{k\to\infty} \frac{k}{k+1} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} = 1 \neq 0$$

Algebraic Properties of Infinite Series

THEOREM 11.4.3

a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum (u_k + v_k)$ and $\sum (u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

b) If c is a nonzero constant , then the series $\sum u_k$ and $\sum cu_k$ both *converge* or both *diverge*. In this case of *convergence*, the sums are related by

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

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c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots\cdots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \cdots\cdots$$

both converge or both diverge.

Example

Find the sum of the series

$$\sum_{k=1}^{\infty}\left(\frac{3}{4^k} - \frac{2}{5^{k-1}}\right)$$

Solution: The series

$$\sum_{k=1}^{\infty}\frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots\cdots\cdots$$

is a convergent series ($a = 3/4$, $r = 1/4$) and the series

$$\sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \cdots\cdots\cdots$$

is also a convergent series ($a = 2$, $r = 1/5$).

Thus, from Theorem 11.4.3(a) and 11.3.3 the given series converges and

$$\sum_{k=1}^{\infty}\frac{3}{4^k} - \sum_{k=1}^{\infty}\frac{2}{5^{k-1}} = \frac{3/4}{1-1/4} - \frac{2}{1-1/5} = -\frac{3}{2}$$

THEOREM 11.4.4

(The Integral Test)

Let $\sum u_k$ be a series with positive terms, and let f(x) be the function that results when k is replaced by x in the formula for u_k.

If f is decreasing and continuous on the interval $[a, +\infty]$, then

$$\sum_{k=1}^{\infty} u_k \text{ and } \int_a^{\infty} f(x)dx$$

both converge or both diverge.

This theorem allows us to study the convergence of a series by studying a related integral.

This integral is the improper integral over the interval $[1, +\infty)$ and the function is the kth term of the series.

It is interesting that the integral represents a continuous phenomenon, while the summation a discrete one!

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EXAMPLE

Use the integral test to determine whether the following series converge or diverge

$$\sum_{k=1}^{\infty}\frac{1}{k}$$

If we replace k by x in the general term 1/k, we obtain the function f(x) = 1/x.

Since

$$\int_1^{+\infty}\frac{1}{x}dx = \lim_{l\to+\infty}\int_1^{l}\frac{1}{x}dx$$

$$= \lim_{l\to+\infty}\left[\ln l - \ln 1\right] = +\infty$$

the integral diverges and consequently so does the series

p-Series

p-series or **hyper-harmonic series** is an infinite series of the form.

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = 1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$$

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\ldots \qquad where\ p=1$$

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}} = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots \qquad where\ p=\frac{1}{2}$$

The following theorem tells us whether a p-series converges or diverges

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THEOREM 11.4.5

(Convergence of p-series)

$$\sum_{k=1}^{\infty}\frac{1}{k^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots+\frac{1}{k^p}+\ldots\ldots$$

converges if $p>1$ and diverges if $0<p\leq 1$

EXAMPLE

$$1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\ldots\ldots+\frac{1}{\sqrt[3]{k}}+\ldots\ldots$$

Since $p=\frac{1}{3}<1$ so p-series diverges

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Lecture # 43

ADDITIONAL CONVERGENCE TESTS

In this section we shall develop some additional convergence tests for series with positive terms.

THEOREM 11.5.1
(The Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with nonnegative terms and suppose that

$a_1 \le b_1, a_2 \le b_2, a_3 \le b_3, \ldots\ldots\ldots, a_k \le b_k$

(a) If the "bigger series" $\sum b_k$ converges than the "smaller series" $\sum a_k$ also converges.

(b) If the "smaller series" $\sum a_k$ diverges, then the "bigger series" $\sum b_k$ also diverges.

This test basically compares two series with each other, and if the series in which each term is bigger than that of the other series converges, then so does the smaller one.

If the series in which each term is smaller than that of the other series diverges, then so does the bigger one.

We will see more of this later. For now, we shall use the comparison test to develop some other tests that are easier to apply.

THEOREM 11.5.2
(The Ratio Test)

Let $\sum u_k$ be a series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverge.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Use the ratio test to determine whether the following series converge or diverge.

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$

b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

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a)

$$\rho = \lim_{k\to+\infty} \frac{u_k + 1}{u_k} = \lim_{k\to+\infty} \frac{1/(k+1)!}{1/k!}$$

$$= \lim_{k\to+\infty} \frac{k!}{(k+1)!} = \lim_{k\to+\infty} \frac{k!}{(k+1)k!}$$

$$= \lim_{k\to+\infty} \frac{1}{(k+1)} = 0 < 1$$

The series converges.

b)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k}$$

$$= \frac{1}{2} \lim_{k\to+\infty} \frac{k+1}{k} = \frac{1}{2} < 1$$

The series converges

c)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k}$$

$$= \lim_{k\to+\infty} \frac{(k+1)^k}{k^k} = \lim_{k\to+\infty} \left(1 + \frac{1}{k}\right)^k$$

$$= e > 1$$

The series diverges

d)

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{2k!}$$

$$= \lim_{k\to+\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4}\right)$$

$$= \frac{1}{4} \lim_{k\to+\infty} (2k+2)(2k+1) = +\infty$$

The series diverges

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Determine whether the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots\ldots+\frac{1}{2k-1}+\ldots\ldots\ldots$$

converges or diverges.

The ratio test is of no help since

$$\rho=\lim_{k\to+\infty}\frac{u_{k+1}}{u_k}=\lim_{k\to+\infty}\frac{1}{2(k+1)-1}\cdot\frac{2k-1}{1}$$

$$=\lim_{k\to+\infty}\frac{2k-1}{2k+1}=1$$

However, the integral test proves that the series diverges since

$$\int_1^{+\infty}\frac{dx}{2x-1}=\lim_{l\to+\infty}\int_1^{l}\frac{dx}{2x-1}$$

$$=\lim_{l\to+\infty}\frac{1}{2}\ln(2x-1)\Big]_1^{l}=+\infty$$

So, this series diverges.

Sometimes the following result is easier to apply than ratio test

Theorem 11.5.3
(The Root Test)
Let $\sum u_k$ be a series with positive terms and suppose that

(a) If $\rho<1$, the series converges.
(d) If $\rho>1$ or $\rho=+\infty$, the series diverge.
(e) If $\rho=1$, the series may converge or diverge, so that another test must be tried.

Example

$$\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k$$

Use the root test to determine whether the following series converge or diverge.

Solution: The series diverges, since $\rho=\lim_{k\to\infty}\left(u_k\right)^{1/k}=\lim_{k\to\infty}\frac{4k-5}{2k+1}=2>1$

We have not really talked about the comparison test so far. To apply comparison test, we use an alternative version of this test that is easier to work with.

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INFORMAL PRINCIPLE 11.6.1

Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above informal principle to help guess whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$$

Solution: Deleting the constant 1 suggests that $\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$ & $\sum_{k=1}^{\infty} \frac{1}{2^k}$ behaves alike.

The modified series is a convergent geometric series, so the given series is likely to converge.

INFORMAL PRINCIPLE 11.6.2

If a polynomial in k appears as a factor in the numerator or denominator of u_k, all but the highest power of k in the polynomial may usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above principle to help guess whether the following series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ converge or diverge.

Solution: Deleting the term 2k suggests that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ & $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

behaves alike

Since the modified series is a convergent p-series (p=3/2) the given series is likely to converge.

The Limit Comparison Test

The following result can be used to establish convergence or divergence by examining the limit of the ratio of the general term of the series in question with general term of a series whose convergence properties are known.

THEOREM 11.6.3

(The Limit Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that $\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

Example

Use the limit comparison test to determine whether the following series $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$ converge or diverge.

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Solution: The given series behaves like the series $\sum_{k=1}^{\infty}\frac{1}{2k^2}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$

$a_k=\frac{1}{2k^2-k}$ $b_k=\frac{1}{2k^2}$ which is constant times a convergent p-series.

Thus the given series is likely to converge.

By applying theorem 11.6.3 with $\rho=\lim_{k\to+\infty}\frac{a_k}{b_k}=\lim_{k\to+\infty}\frac{2k^2}{2k^2-k}=\lim_{k\to+\infty}\frac{2}{2-1/k}=1$

Since ρ is finite and positive, it follows from Theorem 11.6.3 that the given series converge.

The Comparison Test

We shall now discuss some techniques for applying the comparison test (Theorem 11.5.1).

There are two basic steps required to apply the comparison test to a series $\sum u_k$ of positive terms:

- Guess whether the series $\sum u_k$ converges or diverges.
- Find a series that proves the guess to be correct. Thus if the guess is divergence, we must find a divergent series whose terms are "smaller" than the corresponding terms of $\sum u_k$, and if the guess is convergence,

we must find a convergent series whose terms are "bigger" than the corresponding terms of $\sum u_k$.

Example

$\sum_{k=1}^{\infty}\frac{1}{k-1/4}$

Use the comparison test to determine whether the following series converge or diverge.

Solution:

As $\frac{1}{k-1/4}>\frac{1}{k}$

Note that the series behaves like the divergent harmonic series, and hence is likely to diverge. Thus our goal is to find a divergent series that is "smaller" than the given series. We can do this by dropping the constant -1/4 in the denominator, thereby decreasing the size of the general term:

for k=1, 2, ……..

Thus the given series diverges by the comparison test, since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

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