Masters

final term

# LECTURE NO. 19

- Relative Frequency Definition of Probability
- Axiomatic Definition of Probability
- Laws of Probability
  - Rule of Complementation
  - Addition Theorem

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## THE RELATIVE FREQUENCY DEFINITION OF PROBABILITY ('A POSTERIORI' DEFINITION OF PROBABILITY)

based on past experience and experiments

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If a random experiment is repeated a large number of times, say n times, under identical conditions and if an event A is observed to occur m times, then the probability of the event A is defined as the LIMIT of the relative frequency m/n as n tends to infinitely.
Symbolically, we write

$$P(A) = \lim_{n\to\infty} \frac{m}{n}$$

The definition assumes that as n increases indefinitely, the ratio m/n tends to become stable at the numerical value P(A).
The relationship between relative frequency and probability can also be represented as follows:

Relative Frequency $\rightarrow$ Probability mcq
as $n \rightarrow \infty$

As its name suggests, the relative frequency definition relates to the relative frequency with which are event occurs in the *long run*. In situations where we can say that an experiment has been repeated a very large number of times, the *relative frequency definition* can be applied.
As such, this definition is very useful in those practical situations where we are interested in computing a probability in numerical form but where the classical definition cannot be applied.(Numerous real-life situations are such where various possible outcomes of an experiment are NOT equally likely). This type of probability is also called empirical probability as it is based on *EMPIRICAL* evidence i.e. on *OBSERVATIONAL* data.
It can also be called ***STATISTICAL PROBABILITY*** for it is this very probability that forms the basis of mathematical statistics. mcq
Let us try to understand this concept by means of two examples:

- from a coin-tossing experiment and
- From data on the numbers of boys and girls born.

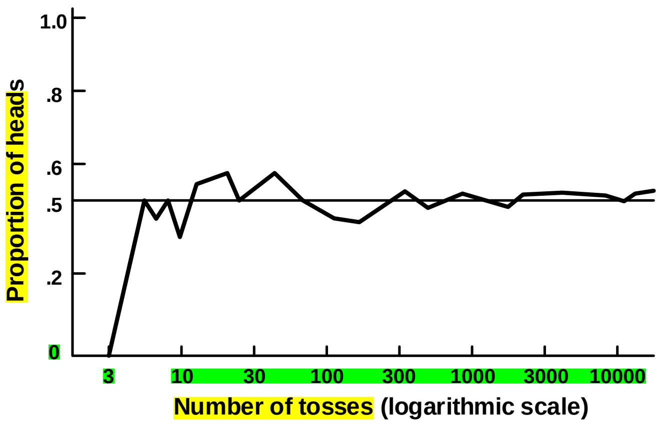
5067/10,000 = 0.5067

4933/10,000 = 0.4933

**EXAMPLE-1**

Coin-Tossing:
No one can tell which way a coin will fall but we expect the proportion of leads and tails after a large no. of tosses to be nearly equal. An experiment to demonstrate this point was performed by Kerrich in Denmark in 1946. He tossed a coin 10,000 times, and obtained altogether 5067 heads and 4933 tails.
The behavior of the proportion of heads throughout the experiment is shown as in the following figure:

**The proportion; of heads in a sequence of tosses of a coin (Kerrich, 1946):**



As you can see, the curve fluctuates *widely* at first, but begins to settle down to a more or less *stable* value as the number of spins increases. It seems reasonable to suppose that the fluctuations would continue to diminish if the experiment were continued *indefinitely*, and the proportion of heads would cluster more and more *closely* about a *limiting* value which would be *very near*, if not exactly, one-half. This hypothetical *limiting* value is the (statistical) probability of heads.
Let us now take an example closely related to our *daily* lives --- that relating to the sex ratio:-
In this context, the first point to note is that it has been known since the eighteenth century that in *reliable* birth statistics based on sufficiently *large* numbers (in at least some parts of the world), there is always a slight *excess* of boys, Laplace records that, among the 215,599 births in thirty districts of France in the years 1800 to 1802, there were 110,312 boys and 105,287 girls. The proportions of boys and girls were thus 0.512 and 0.488 respectively (indicating a slight *excess* of boys over girls).In a *smaller* number of births one would, however, expect considerable *deviations* from these proportions. This point can be illustrated with the help of the following example:

**EXAMPLE-2**

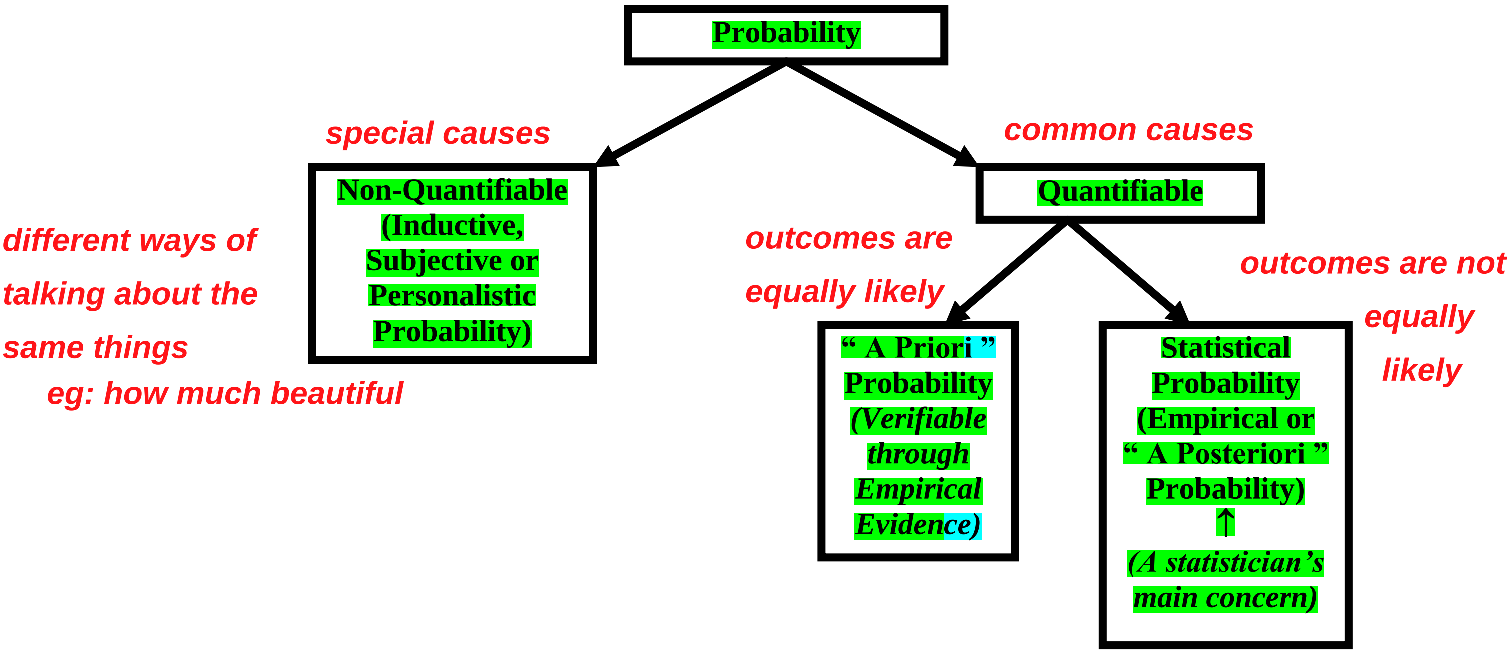
The following table shows the proportions of male births that have been worked out for the major regions of England as well as the rural districts of Dorset (for the year 1956).

**Proportions of Male Births in various Regions and Rural Districts of England in 1956**

| Region of England | Proportion of Male Births | Rural Districts of Dorset | Proportion of Male Births |
|---|---|---|---|
| Northern | .514 | Beaminster | .38 |
| E. & W. Riding | .513 | Blandford | .47 |
| North Western | .512 | Bridport | .53 |
| North Midland | .517 | Dorchester | .50 |
| Midland | .514 | Shaftesbury | .59 |
| Eastern | .516 | Sherborne | .44 |
| London and S. Eastern | .514 | Sturminster | .54 |
| Southern | .514 | Wareham and Purbeck | .53 |
| South Western | .513 | Wimborne & Cranborne | .54 |
| Whole country | .514 | All Rural District's of Dorset | .512 |

***(Source: Annual Statistical Review)***

As you can see, the figures for the rural districts of Dorset, based on about 200 births each, fluctuate between 0.38 and 0.59. While those for the major regions of England, which are each based on about 100,000 births, do not fluctuate much, rather, they range between 0.512 and 0.517 only. The larger sample size is clearly the reason for the greater constancy of the latter. We can imagine that if the sample were increased indefinitely, the proportion of boys would tend to a *limiting* value which is unlikely to differ much from 0.514, the proportion of male births for the *whole* country. This hypothetical *limiting* value is the (statistical) *probability* of a male birth. The overall discussion regarding the various ways in which probability can be defined is presented in the following diagram:




As far as *quantifiable* probability is concerned, in those situations where the various possible outcomes of our experiment are equally likely, we can compute the probability *prior* to actually conducting the experiment --- otherwise, as is generally the case, we can compute a probability only *after* the experiment has been conducted (and this is why it is also called 'a posteriori' probability*). Non-quantifiable probability is the one that is called Inductive Probability.*
It refers to the degree of belief which it is reasonable to place in a proposition on *given* evidence.
An important point to be noted is that it is difficult to express inductive probabilities numerically — to construct a numerical scale of inductive probabilities, with 0 standing for impossibility and for logical certainty. An important point to be noted is that it is difficult to express inductive probabilities numerically — to construct a numerical scale of inductive probabilities, with 0 standing for impossibility and for logical certainty. Most statisticians have arrived at the conclusion that inductive probability cannot, in general, he measured and, therefore cannot be use in the mathematical theory of statistics.
This conclusion is not, perhaps, very surprising since there seems no reason why rational degree of belief should be measurable any more than, say, degrees of beauty. Some paintings are very beautiful, some are quite beautiful, and some are ugly, but it would be observed to try to construct a numerical scale of beauty, on which Mona Lisa had a beauty value of 0.96.Similarly some propositions are highly probable, some are quite probable and some are improbable, but it does not seem possible to construct a numerical scale of such (inductive) probabilities .Because of the fact that inductive probabilities are not quantifiable and cannot be employed in a mathematical argument, this is the reason why the usual methods of statistical inference such as tests of significance and confidence interval are based entirely on the concept of statistical probability. Although we have discussed three different ways of defining probability, the most formal definition is yet to come.
This is The Axiomatic Definition of Probability.

## THE AXIOMATIC DEFINITION OF PROBABILITY

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This definition, introduced in 1933 by the Russian mathematician Andrei N. Kolmogrov, is based on a set of AXIOMS. Let S be a sample space with the sample points E1, E2, … Ei, …En. To each sample point, we assign a real number, denoted by the symbol P(Ei), and called the probability of Ei, that must satisfy the following basic axioms:

**Axiom 1:**

For any event Ei,
$0 < P(E_i) < 1.$

**Axiom 2:**

$P(S) = 1$
for the sure event S.

**Axiom 3:**
If A and B are mutually exclusive events (subsets of S), then

events are disjoint

$P(A \cup B) = P(A) + P(B).$

It is to be emphasized that According to the axiomatic theory of probability:

SOME probability defined as a non-negative real number is to be ATTACHED to each sample point Ei *such that the sum of all such numbers must equal ONE*. The ASSIGNMENT of probabilities may be based on past evidence or on some other underlying conditions. (If this assignment of probabilities is based on past evidence, we are talking about EMPIRICAL probability, and if this assignment is based on underlying conditions that ensure that the various possible outcomes of a random experiment are EQUALLY LIKELY, then we are talking about the CLASSICAL definition of probability. Let us consider another example:

**EXAMPLE**

Table given below shows the numbers of births in England and Wales in 1956 classified by (a) sex and (b) whether live born or stillborn.

*Table-1*

*Number of births in England and Wales in 1956 by sex and whether live- or still born*

*(Source Annual Statistical Review)*

| | Liveborn | Stillborn | Total |
|---|---|---|---|
| Male | 359,881 (A) | 8,609 (B) | 368,490 |
| Female | 340,454 (B) | 7,796 (D) | 348,250 |
| Total | 700,335 | 16,405 | 716,740 |

There are four possible events in this double classification:

- Male livebirth (denoted by A),
- Male stillbirth (denoted by B),
- Female livebirth (denoted by C)
- Female stillbirth (denoted by D),

The relative frequencies corresponding to the figures of Table-1 are given in Table-2:

**Table-2**

Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn

(Source *Annual Statistical Review*)

| | Liveborn | Stillborn | Total |
|---|---|---|---|
| Male | .5021 | .0120 | .5141 |
| Female | .4750 | .0109 | .4859 |
| Total | .9771 | .0229 | 1.0000 |

The total number of births is large enough for these relative frequencies to be treated for all practical purposes as *PROBABILITIES*.

Let us denote the compound events 'Male birth' and 'Stillbirth' by the letters M and S. Now a male birth occurs whenever either a male live birth or a male stillbirth occurs, and so the proportion of male birth, regardless of whether they are live-or stillborn, is equal to the sum of the proportions of these two types of birth; that is to say,

$$
\begin{aligned}
p(M) &= p(A \text{ or } B) = p(A) + p(B) \\
&= .5021 + .0120 = .5141
\end{aligned}
$$

Similarly, a stillbirth occurs whenever either a male stillbirth or a female stillbirth occurs and so the proportion of stillbirths, regardless of sex, is equal to the sum of the proportions of these two events:

$$
\begin{aligned}
p(S) &= p(B \text{ or } D) = p(B) + p(D) \\
&= .0120 + .0109 = .0229
\end{aligned}
$$

Let us now consider some basic LAWS of probability. These laws have important applications in solving probability problems.

**LAW OF COMPLEMENTATION**

If $\bar{A}$ is the complement of an event A relative to the sample space S, then

$$P(\bar{A}) = 1 - P(A).$$

Hence the probability of the complement of an event is equal to one minus the probability of the event. Complementary probabilities are very useful when we are wanting to solve questions of the type 'What is the probability that, in tossing two fair dice, at least one even number will appear?'

**EXAMPLE**

A coin is tossed 4 times in succession. What is the probability that at least one head occurs?

- The sample space S for this experiment consists of $2^4 = 16$ sample points (as each toss can result in 2 outcomes),and
- We assume that each outcome is equally likely.

If we let A represent the event that at least one head occurs, then A will consist of MANY sample points, and the process of computing the probability of this event will become somewhat cumbersome! So, instead of denoting this particular event by A, let us denote its complement i.e. "No head" by $\bar{A}$.

Thus the event $\bar{A}$ consists of the SINGLE sample point {TTTT}.

Therefore $P(\bar{A}) = 1/16$.

Hence by the law of complementation, we have

$$P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{16} = \frac{15}{16}.$$

The next law that we will consider is the Addition Law or the General Addition Theorem of Probability:

**ADDITION LAW**

If A and B are any two events defined in a sample space S, then

$P(A \cup B) = P(A) + P(B) – P(A \cap B)$

In words, this law may be stated as follows:

**"**If two events A and B are not mutually exclusive, then the probability that at least one of them occurs, is given by the sum of the separate probabilities of events A and B minus the probability of the joint event $A \cap B$.**"**

## LECTURE NO. 20

- Application of Addition Theorem
- Conditional Probability
- Multiplication Theorem

First of all, let us consider in some detail the Addition Law or the General Addition Theorem of Probability:

### ADDITION LAW

If A and B are any two events defined in a sample space S, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
In words, this law may be stated as follows:

"If two events A and B are not mutually exclusive, then the probability that at least one of them occurs, is given by the sum of the separate probabilities of events A and B minus the probability of the joint event $A \cap B$."

### EXAMPLE

If one card is selected at random from a deck of 52 playing cards, what is the probability that the card is a club or a face card or both?
Let A represent the event that the card selected is a club, B, the event that the card selected is a face card, and $A \cap B$, the event that the card selected is both a club and a face card. Then we need $P(A \cup B)$
Now $P(A) = 13/52$, as there are 13 clubs, $P(B) = 12/52$, as there are 12 faces cards,
$P(A \cap B) = 3/52$, since 3 of clubs are also face cards.
Therefore the desired probability is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 13/52 + 12/52 - 3/52 = 22/52.$$

### COROLLARY-1

If A and B are mutually exclusive events, then
$P(A \cup B) = P(A) + P(B)$ (Since $A \cap B$ is an impossible event, hence $P(A \cap B) = 0$)

### EXAMPLE

Suppose that we toss a pair of dice, and we are interested in the event that we get a total of 5 or a total of 11. What is the probability of this event?

### SOLUTION

*if they can not occur at the same time.*

In this context, the first thing to note is that 'getting a total of 5' and 'getting a total of 11' are mutually exclusive events. Hence, we should apply the special case of the addition theorem. If we denote 'getting a total of 5' by A, and 'getting a total of 11' by B, then $P(A) = 4/36$ (since there are four outcomes favorable to the occurrence of a total of 5), and $P(B) = 2/36$ (since there are two outcomes favorable to the occurrence of a total of 11).
Hence the probability that we get a total of 5 or a total of 11 is given by

*( 1,2,3,4,5,6) ( 1,2,3,4,5,6)*

$$P(A \cup B) = P(A) + P(B) = 4/36 + 2/36 = 6/36 = 16.67\%.$$

*(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6) ,(5, 1), (5, 2), (5, 3), (5, 4), (5, 5),(5,6),(6,1),(6,2), (6.3),(6,4),(6,5),(6,6)*

### COROLLARY-2

If $A_1, A_2 \ldots A_k$ are k mutually exclusive events, then the probability that one of them occurs, is the sum of the probabilities of the separate events, i.e.

$$P(A_1 \cup A_2 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + \ldots + P(A_k).$$

Let us now consider an interesting example to illustrate the way in which probability problems can be solved:

### EXAMPLE

Three horses A, B and C are in a race; A is twice as likely to win as B and B is twice as likely to win as C. What is the probability that A or B wins?
Evidently, the events mentioned in this problem are not equally likely.
Let $P(C) = p$
Then $P(B) = 2p$ as B is twice as likely to win as C.
Similarly
$P(A) = 2P(B) = 2(2p) = 4p$
In this problem, we assume that no two of the horses A, B and C cannot win the race together (i.e. the race cannot end in a draw).
Hence, the events A, B and C are mutually exclusive.

Since A, B and C are mutually exclusive and collectively *exhaustive*, therefore the sum of their probabilities must be equal to 1.

Thus

$p + 2p + 4p = 1$

or $p = 1/7$

$\therefore$ $P(C) = 1/7$,

$P(B) = 2(1/7) = 2/7$,

and $P(A) = 4(1/7) = 4/7$.

Hence

$P(A \cup B) = P(A) + P(B)$

$= 4/7 + 2/7$

$= 6/7$.

Having discussed the addition theorem in some detail, we would now like to discuss the Multiplication Theorem. But, before we are in a position to take up the multiplication theorem, we need to consider the concept of conditional probability.

## CONDITIONAL PROBABILITY

The sample space for an experiment must often be changed when some additional information pertaining to the outcome of the experiment is received.

The effect of such information is to REDUCE the sample space by excluding some outcomes as being impossible which BEFORE receiving the information were believed possible. The probabilities associated with such a reduced sample space are called conditional probabilities.

The following example illustrates the concept of conditional probability

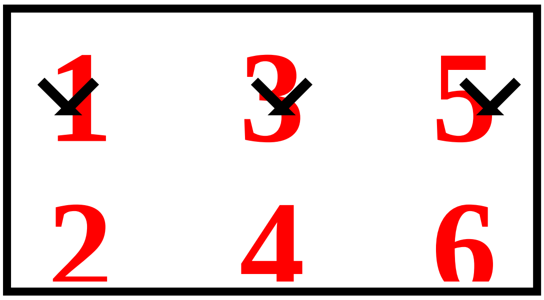
## EXAMPLE

Suppose that we toss a fair die. Then the sample space of this experiment is S = {1, 2, 3, 4, 5, 6}. Suppose we wish to know the probability of the outcome that the die shows 6 (say event A).Also, suppose that, before seeing the outcome, we are told that the die shows an EVEN number of dots (say event B).

Then the information that the die shows an even number excludes the outcomes 1, 3 and 5, and thereby reduces the original sample space to a sample space that consists of three outcomes 2, 4 and 6,

i.e. the reduced sample space is

B = {2, 4, 6}.




**(The sample space is reduced.)**

Then, the desired probability in the reduced sample space B is 1/3.

(since each outcome in the reduced sample space is EQUALLY LIKELY). This probability 1/3 is called the conditional probability of the event A because it is computed under the CONDITION that the die has shown an even number of dots. In other words,

P(die shows 6/die shows even numbers)

= 1/3,

(Where the vertical line is read as given that and the information following the vertical line describes the conditioning event).

Sometimes, it is not very convenient to compute a conditional probability by first determining the number of sample points that belong to the reduced sample space.

In such a situation, we can utilize the following *alternative method of computing a conditional probability* *the probability of an event occurring given that another event has already occurred*

**CONDITIONAL PROBABILITY**

If A and B are two events in a sample space S and if P(B) is not equal to zero, then the conditional probability of the event A given that event B has occurred, written as P(A/B), is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Where P (B) > 0
(If P (B) = 0, the conditional probability P(A/B) remains undefined.)
Similarly

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

where P(A) > 0.
It should be noted that P(A/B) SATISFIES all the basic axioms of probability, namely:

- 0 < P(A/B) < 1.
- **ii)** P( S/B) = 1
- P(A1∪A2/B) = P(A1/B) + P(A2/B) (provided that the events A1 and A2 are mutually exclusive).

Let us now apply this concept to a real-world example

**EXAMPLE-2**

At a certain elementary school in a Western country, the school-record of the past ten years shows that 75% of the students come from a two-parent home and that 20% of the students are low-achievers and belong to two-parent homes. What is the probability that such a randomly selected student will be a low achiever GIVEN THAT he or she comes from a two-parent home?

**SOLUTION**

Let A denote a low achiever and B a student from a two-parent home. Applying the *relative frequency definition* of probability, we have

P(B) = 0.75 and P(A ∩ B) = 0.20.
Thus, we obtain

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.20}{0.75} = 0.27$$

**MULTIPLICATION THEOREM OF PROBABILITY**

It is interesting to note that the multiplication theorem is obtained very conveniently from the formula of conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

As discussed earlier, the conditional probability of A given that B has occurred has already been defined as:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \text{ Where P (B) > 0}$$

Multiplying both sides by P (B), we get

P (A ∩ B) = P (B) . P (A/B).

And if we interchange the roles of A and B, we obtain:
P (A ∩ B) = P (A) P (B/A),
Provided P(A) > 0.

**MULTIPLICATION LAW**

If A and B are any two events defined in a sample space S, then
P (A ∩ B)
= P (A) P (B/A), provided P (A) > 0,
= P (B) P (A/B) provided P (B) > 0.
(The second form is easily obtained by interchanging A and B. This is called the GENERAL rule of multiplication of probabilities. It can be stated as follows:

## MULTIPLICATION LAW

"The probability that two events A and B will both occur is equal to the probability that one of the events will occur multiplied by the conditional probability that the other event will occur given that the first event has already occurred."
Let us apply the concept of multiplication theorem to an example

### EXAMPLE

A box contains 15 items, 4 of which are defective and 11 is good. Two items are selected. What is the probability that the first is good and the second defective?
Let A represent the event that the first item selected is good, and B, the event that the second items is defective.
Then we need to calculate the probability of the JOINT event $A \cap B$ by the rule
$P(A \cap B) = P(A)P(B/A)$.
We have:

| Type of Item | No. of Items |
|---|---|
| Defective | 4 |
| Good | 11 |
| Total | 15 |

Since all the items are equally likely to be chosen, hence
$P(A) = 11/15$.
Given the event A has occurred, there remain 14 items of which 4 are defective. Therefore the probability of selecting a defective item after a good item has been selected is 4/14 i.e.

$$P(B/A) = 4/14.$$

Hence

$$\begin{aligned} P(A \cap B) &= P(A)\,P(B/A) \\ &= 11/15 \times 4/14 \\ &= 44/210 \\ &= 0.16. \end{aligned}$$

In this lecture, the concepts of the Addition Theorem and the Multiplication Theorem of probability have been discussed in some detail.
In order to **differentiate** between the situation where the addition theorem is applicable and the situation where the multiplication theorem is applicable, the main point to keep in mind is that whenever we wish to compute the probability that *either* A occurs *or* B occurs, we should think of the Addition Theorem, where as, whenever we wish to compute the probability that *both* A *and* B occur, we should think of the Multiplication Theorem.

Masters

# LECTURE NO. 21

- Independent and Dependent Events
- Multiplication Theorem of Probability for Independent Events
- Marginal Probability

Before we proceed the concept of independent versus dependent events, let us review the Addition and Multiplication Theorems of Probability that were discussed in the last lecture.
To this end, let us consider an interesting example that illustrates the application of both of these theorems in one problem:

## EXAMPLE

A bag contains 10 white and 3 black balls. Another bag contains 3 white and 5 black balls. Two balls are transferred from first bag and placed in the second, and then one ball is taken from the latter.
What is the probability that it is a white ball?
In the beginning of the experiment, we have:

| Colour of Ball | No. of Balls in Bag A | No. of Balls in Bag B |
|---|---|---|
| White | 10 | 3 |
| Black | 3 | 5 |
| Total | 13 | 8 |

Let A represent the event that 2 balls are drawn from the first bag and transferred to the second bag. Then A can occur in the following three mutually exclusive ways:
A1 = 2 white balls are transferred to the second bag.
A2 = 1 white ball and 1 black ball are transferred to the second bag.
A3 = 2 black balls are transferred to the second bag.
Then, the total number of ways in which 2 balls can be drawn out of a total of 13 balls is $\binom{13}{2}$.

And, the total number of ways in which 2 white balls can be drawn out of 10 white balls is $\binom{10}{2}$.

Thus, the probability that two white balls are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_1)=\binom{10}{2}\div\binom{13}{2}=\frac{45}{78},$$

Similarly, the probability that one white ball and one black ball are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_2)=\binom{10}{1}\binom{3}{1}\div\binom{13}{2}=\frac{30}{78},$$

And, the probability that two black balls are selected from the first bag containing 13 balls (in order to *transfer* to the second bag) is

$$P(A_3)=\binom{3}{2}\div\binom{13}{2}=\frac{3}{78}.$$

AFTER having transferred 2 balls from the first bag, the second bag contains
**i)** 5 white and 5 black balls (if 2 white balls are transferred)

| Colour of Ball | No. of Balls in Bag A | No. of Balls in Bag B |
|---|---|---|
| White | 10 – 2 = 8 | 3 + 2 = 5 |
| Black | 3 | 5 |
| Total | 13 – 2 = 11 | 8 + 2 = 10 |

Hence: P(W/A1) = 5/10

**ii)** 4 white and 6 black balls (if 1 white and 1 black ball are transferred)

| Colour of Ball | No. of Balls in Bag A | No. of Balls in Bag B |
|---|---|---|
| White | 10 – 1 = 9 | 3 + 1 = 4 |
| Black | 3 – 1 = 2 | 5 + 1 = 6 |
| Total | 13 – 2 = 11 | 8 + 2 = 10 |

Hence: P(W/A2) = 6/10

**iii)** 3 white and 7 black balls (if 2 black balls are transferred)

| Colour of Ball | No. of Balls in Bag A | No. of Balls in Bag B |
|---|---|---|
| White | 10 | 3 |
| Black | 3 – 2 = 1 | 5 + 2 = 7 |
| Total | 13 – 2 = 11 | 8 + 2 = 10 |

Hence: P(W/A3) = 3/10

Let W represent the event that the WHITE ball is drawn from the second bag after having transferred 2 balls from the first bag.

Then $P(W) = P(A1 \cap W) + P(A2 \cap W) + P(A3 \cap W)$

Now $P(A1 \cap W) = P(A1)\ P(W/A1)$

$= 45/78 \times 5/10$

$= 15/52$

$P(A2 \cap W) = P(A2)\ P(W/A2)$

$= 30/78 \times 4/10$

$= 2/13$,

And

$P(A3 \cap W) = P(A3)\ P(W/A3)$

$= 3/78 \times 3/10$

$= 3/260$.

Hence the required probability is

P (W)

$= P(A1 \cap W) + P(A2 \cap W) + P(A3 \cap W)$

$= 15/52 + 2/13 + 3/260$

$= 59/130$

$= 0.45$

**INDEPENDENT EVENTS**

Two events A and B in the same sample space S, are defined to be independent (or statistically independent) if the probability that one event occurs, is not affected by whether the other event has or has not occurred, that is

$P(A/B) = P(A)$ and $P(B/A) = P(B)$. It then follows that two events A and B are independent if and only if

$P(A \cap B) = P(A)\ P(B)$

and this is known as the *special case* of the Multiplication Theorem of Probability.

**RATIONALE**

According to the multiplication theorem of probability, we have:

$$P(A \cap B) = P(A).\ P(B/A)$$

Putting P(B/A) = P(B), we obtain

$$P(A \cap B) = P(A)\ P(B)$$

The events A and B are defined to be *DEPENDENT* if $P(A \cap B) \neq P(A) \times P(B)$. MCQ

This means that the occurrence of one of the events in some way affects the probability of the occurrence of the other event. MCQ Speaking of independent events, it is to be emphasized that two events that are independent, can NEVER be mutually exclusive. MCQ

**EXAMPLE**

Two fair dice, one red and one green, are thrown. Let A denote the event that the red die shows an even number and let B denote the event that the green die shows a 5 or a 6. Show that the events A and B are independent.

The sample space S is represented by the following 36 outcomes:

S = {(1, 1), (1, 2), (1, 3), (1, 5), (1, 6);
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6);
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6);
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6);
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6);
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6) }

Since

A represents the event that red die shows an even number, and B represents the event that green die shows a 5 or a 6,

Therefore $A \cap B$ represents the event that red die shows an even number and green die shows a 5 or a 6.

Since A represents the event that red die shows an even number, hence P(A) = 3/6. Similarly, since B represents the event that green die shows a 5 or a 6, hence P(B) = 2/6.

Now, in order to compute the probability of the joint event $A \cap B$, the first point to note is that, in all, there are 36 possible outcomes when we throw the two dice together, i.e.

S = (1, 1), (1, 2), (1, 3), (1, 5), (1, 6);
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6);
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6);
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6);
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6);
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6)

*Event A represents the red die showing an even number, which consists of the outcomes:*
*(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)*
*(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)*
*(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)*

The joint event $A \cap B$ contains only 6 outcomes out of the 36 possible outcomes. *There are a total of 18 outcomes in event A.P(A)=18/36=1/2*

These are (2, 5), (4, 5), (6, 5), (2, 6), (4, 6), and (6, 6).

and $P(A \cap B) = 6/36$. *=1/6*

*Event B represents the green die showing a 5 or a 6, which consists of the outcomes:*
*(1, 5), (1, 6),(2, 5), (2, 6),(3, 5), (3, 6),(4, 5), (4, 6).(5, 5), (5, 6),(6, 5), (6,6)*

*There are a total of 12 outcomes in event B. P(B)=12/36 =1/3*

Now

P (A) P (B)
$= 3/6 \times 2/6$ *1/2 X 1/3 =1/6*
= 6/36
$= P(A \cap B)$.

Therefore the events A and B are independent. Let us now go back to the example pertaining to live births and stillbirths that we considered in the last lecture, and try to determine whether or not sex of the baby and nature of birth are independent.

**EXAMPLE**

Table-1 below shows the numbers of births in England and Wales in 1956 classified by (a) sex and (b) whether live born or stillborn.

*Table-1*

Number of births in England and Wales in 1956 by sex and whether live- or still born
(Source *Annual Statistical Review*)

| | Liveborn | Stillborn | Total |
|---|---|---|---|
| Male | 359,881 (A) | 8,609 (B) | 368,490 |
| Female | 340,454 (B) | 7,796 (D) | 348,250 |
| Total | 700,335 | 16,405 | 716,740 |

There are four possible events in this double classification:

- Male live birth,
- Male stillbirth,
- Female live birth, and
- Female stillbirth.

The corresponding relative frequencies are given in Table-2.

*Table-2*

Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn
*(Source Annual Statistical Review)*

| | Liveborn | Stillborn | Total |
|---|---|---|---|
| Male | .5021 | .0120 | .5141 |
| Female | .4750 | .0109 | .4859 |
| Total | .9771 | .0229 | 1.0000 |

As discussed in the last lecture, the total number of births is large enough for these relative frequencies to be treated for all practical purposes as *PROBABILITIES*. The compound events 'Male birth' and 'Stillbirth' may be represented by the letters M and S. If M represents a male birth and S a stillbirth, we find that

$$\frac{n(M \;\; and \;\; S)}{n(M)} = \frac{8609}{368490} = 0.0234$$

This figure is the proportion — and, since the sample size is large, it can be regarded as the *probability* — of males who are still born — in other words, the *CONDITIONAL* probability of a stillbirth *given that* it is a male birth. In other words, the probability of stillbirths *in* males. The corresponding proportion of stillbirths among females is

$$\frac{7796}{348258} = 0.0224.$$

These figures should be contrasted with the OVERALL, or *UNCONDITIONAL*, proportion of stillbirths, which is

$$\frac{16405}{716740} = 0.0229.$$

We observe that the conditional probability of stillbirths among boys is slightly *HIGHER* than the overall proportion. Where as the conditional proportion of stillbirths among girls is slightly *LOWER* than the overall proportion. It can be concluded that sex and stillbirth are statistically *DEPENDENT*, that is to say, the *SEX* of a baby yet to be born *has* an effect, (although a small effect), on its chance of being stillborn. The example, that we just considered point out the concept of *MARGINAL PROBABILITY*.
Let us have another look at the data regarding the live births and stillbirths in England and Wales:
Table-2Proportion of births in England and Wales in 1956 by sex and whether live- or stillborn (Source *Annual Statistical Review*)

| | Liveborn | Stillborn | Total |
|---|---|---|---|
| Male | .5021 | .0120 | .5141 |
| Female | .4750 | .0109 | .4859 |
| Total | .9771 | .0229 | 1.0000 |

And, the figures in Table-2 indicate that the probability of male birth is 0.5141, whereas the probability of female birth is 0.4859.Also, the probability of live birth is 0.9771, where as the probability of stillbirth is 0.0229. And since these probabilities appear in the margins of the Table, they are known as *Marginal Probabilities*. According to the above table, the probability that a new born baby is a male and is live born is 0.5021 whereas the probability that a new born baby is a male and is stillborn is 0.0120.Also, as stated earlier; the probability that a new born baby is a male is 0.5141, and, CLEARLY, 0.5141 = 0.5021 + 0.0120. Hence, it is clear that the joint probabilities occurring in any row of the table *ADD UP* to yield the corresponding *marginal* probability. If we reflect upon this situation carefully, we will realize that this equation is totally in accordance with the Addition Theorem of Probability for mutually exclusive events.

P (male birth) = P(male live-born *or* male stillborn)
= P (male live-born) + P (male stillborn)
= 0.5021 + 0.0120
= 0.5141

**EXAMPLE**

P (stillbirth/male birth)
P (male birth *and* stillbirth)/P(male birth)
=0.0120/0.5141
= 0.0233

Masters

## LECTURE NO. 22

- Bayes' Theorem
- Discrete Random Variable
    - Discrete Probability Distribution
    - Graphical Representation of a Discrete Probability Distribution
    - Mean, Standard Deviation and Coefficient of Variation of a Discrete Probability Distribution
    - Distribution Function of a Discrete Random Variable.

First of all, let us discuss the BAYES' THEOREM. This theorem deals with conditional probabilities in an interesting way:

**BAYES' THEOREM** *it is used to calculate the conditional probability*

If events $A_1$, $A_2$… $A_k$ form a PARTITION of a sample space S (that is, the events Ai are mutually exclusive and exhaustive (i.e. their union is S)),
and if B is any other event of S such that it can occur ONLY IF ONE OF THE Ai OCCURS, then for any i,

$$P(A_i/B)=\frac{P(A_i)P(B/A_i)}{\sum_{i=1}^{k} P(A_i)P(B/A_i)},$$

**for i = 1, 2, …, k.**

Stated differently:
**BAYES' THEOREM:**
If $A_1$, $A_2$... and $A_k$ are mutually exclusive events *of which one must occur*, then

$$P(A_i|B)=\frac{P(A_i).P(B|A_i)}{P(A_1).P(B|A_1)+P(A_2).P(B|A_2)+....+P(A_k).P(B|A_k)}$$

If k = 2, we obtain:
Bayes' Theorem for two mutually exclusive events A1 and A2:

$$P(A_i|B)=\frac{P(A_i).P(B|A_i)}{P(A_1).P(B|A_1)+P(A_2).P(B|A_2)}$$ +

Where i = 1, 2.

In other words

$$P(A_1|B)=\frac{P(A_1).P(B|A_1)}{P(A_1).P(B|A_1)+P(A_2).P(B|A_2)}$$

and

$$P(A_2|B)=\frac{P(A_2).P(B|A_2)}{P(A_1).P(B|A_1)+P(A_2).P(B|A_2)}$$

**EXAMPLE**

In a developed country where cars are tested for the emission of pollutants, 25 percent of all cars emit excessive amounts of pollutants. When tested, 99 percent of all cars that emit excessive amounts of pollutants will fail, but 17 percent of the cars that do not emit excessive amounts of pollutants will also fail. What is the probability that a car that fails the test actually *emits* excessive amounts of pollutants?

**SOLUTION**

Let A1 denote the event that it emits EXCESSIVE amounts of pollutants, and let A2 denote the event that a car does NOT emit excessive amounts of pollutants. (In other words, A2 is the complement of A1.)
Also, let B denote the event that a car *FAILS* the test.
The first thing to note is that any car will either *emit* or *not* emit excessive amounts of pollutants. In other words, A1 and A2 are mutually exclusive and exhaustive events i.e. A1 and A2 form a PARTITION of the sample space S.
Hence, we are in a position to apply the Bayes' theorem.
We need to calculate P(A1|B), and, according to the Bayes' theorem:

$$P(A_1|B)=\frac{P(A_1).P(B|A_1)}{P(A_1).P(B|A_1)+P(A_2).P(B|A_2)}$$

Now, according to the data given in this problem: *25%=25/100 =0.25*

P (A1) = 0.25,

P (A2) = 0.75 (as A2 is simply the complement of A1),

P (B|A1) = 0.99,
and

P (B|A2) = 0.17
Substituting the above values in the Bayes' theorem, we obtain:

$$
\begin{aligned}
P(A_1 | B) &= \frac{P(A_1).P(B | A_1)}{P(A_1).P(B | A_1) + P(A_2).P(B | A_2)} \\
&= \frac{(0.25)(0.99)}{(0.25)(0.99) + (0.75)(0.17)} \\
&= \frac{0.2475}{0.2475 + 0.1275} \\
&= \frac{0.2475}{0.3750} \\
&= 0.66
\end{aligned}
$$

This is the probability that a car which fails the test ACTUALLY emits excessive amounts of pollutants. The example that we just considered pertained to the simplest case when we have only two mutually exclusive and exhaustive events A1 and A2.
As stated earlier, the Bayes' theorem can be extended to the case of three, four, five or more mutually exclusive and exhaustive events.
Let us consider another example: In the following example, check the percentages of defective bolts from the recorded lecture.

**EXAMPLE**

In a bolt factory, 25% of the bolts are produced by machine A, 35% are produced by machine B, and the remaining 40% are produced by machine C. Of their outputs, 2%, 4% and 5% respectively are defective bolts. If a bolt is selected at random and found to be defective, what is the probability that it came from machine A? *2% =0.02*

In this example, we realize that "a bolt is produced by machine A", "a bolt is produced by machine B" and "a bolt is produced by machine C" represent three mutually exclusive and exhaustive events i.e. we can regard them as A1, A2 and A3. The event "defective bolt" represents the event B. Hence, in this example, we need to determine P(A1/B).
The students are encouraged to work on this problem on their own, in order to understand the application and significance of the Bayes' Theorem. This brings us to the *END* of the discussion of various *basic* concepts of probability. We now begin the discussion of a *very important* concept in mathematical statistics, i.e., the concept of *PROBABILITY DISTRIBUTIONS*.
As stated in the very beginning of this course, there are two types of quantitative variables --- the discrete variable, and the continuous variable. Accordingly, we have the discrete probability distribution as well as the continuous probability distribution.
We begin with the discussion of the discrete probability distribution.
In this regard, the first concept that we need to consider is the concept of Random variable.

**RANDOM VARIABLE**

Such a numerical quantity whose value is determined by the outcome of a random experiment is called a random variable.
For example, if we toss three dice together, and let X denote the number of heads, then the random variable X consists of the values 0, 1, 2, and 3. Obviously, in this example, X is a discrete random variable. Let us now discuss the concept of discrete probability distribution in detail with the help of the following example:
Example:
If a biologist is interested in the number of petals on a particular flower, this number may take the values 3, 4, 5, 6, 7, 8, 9, and each one of these numbers will have its own probability.
Suppose that upon observing a large no. of flowers, say 1000 flowers, of that particular species, the following results are obtained:

| No. of Petals X | f |
|---|---|
| 3 | 50 |
| 4 | 100 |
| 5 | 200 |
| 6 | 300 |
| 7 | 250 |
| 8 | 75 |
| 9 | 25 |
| | 1000 |

Since 1000 is quite a large number, hence the proportions f/∑f can be regarded as probabilities and hence we can write

| **No. of Petals X** | **P(x)** |
|---|---|
| $x_1 = 3$ | 0.05 |
| $x_2 = 4$ | 0.10 |
| $x_3 = 5$ | 0.20 |
| $x_4 = 6$ | 0.30 |
| $x_5 = 7$ | 0.25 |
| $x_6 = 8$ | 0.075 |
| $x_7 = 9$ | 0.025 |
| | 1 |

**PROPERTIES OF A DISCRETE PROBABILITY DISTRIBUTION**

- (1) $0 \le P(X_i) \le 1$ for each Xi (i = 1, 2, … 7)
- (2) $\sum p(X_i) = 1$

And, since the number of petals on a leaf can only be a *whole* number, hence the variable X is known as a *discrete* random variable, and the probability distribution of this variable is known as a D*ISCRETE* probability distribution.
In other words, Any discrete variable that is associated with a random experiment, and attached to whose various values are various probabilities (Such that $\sum_{i=1}^{\infty} P(X_i) = 1$)

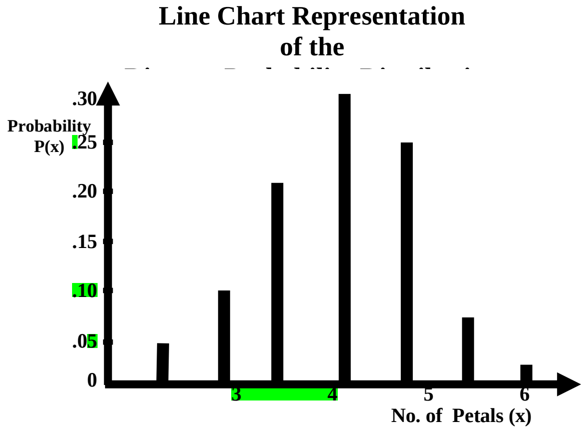
is known as a Discrete Random Variable, and its probability distribution is known as a Discrete Probability Distribution. Just as we can depict a frequency distribution graphically, we can draw the GRAPH of a probability distribution.

**EXAMPLE**

Going back to the probability distribution of the number of petals on the flowers of a particulars species, i.e.:

| No. of Petals X | P(x) |
|---|---|
| $x_1 = 3$ | 0.05 |
| $x_2 = 4$ | 0.10 |
| $x_3 = 5$ | 0.20 |
| $x_4 = 6$ | 0.30 |
| $x_5 = 7$ | 0.25 |
| $x_6 = 8$ | 0.075 |
| $x_7 = 9$ | 0.025 |
| | 1 |

This distribution can be represented in the form of a line chart.




Evidently, this particular probability distribution is approximately symmetric. In addition, this graph clearly shows that, just as in the case of a frequency distribution, every discrete probability distribution has a *CENTRAL* point and a *SPREAD*. Hence, similar to a frequency distribution, the discrete probability distribution has a *MEAN* and a *STANDARD DEVIATION*. How do we *calculate* the mean and the standard deviation of a probability distribution?
Let us first consider the computation of the MEAN:
We know that in the case of a frequency distribution such as

| X | f |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 4 |
| 4 | 2 |
| 5 | 1 |

the mean is given by

$$\overline{X} = \frac{\sum fX}{\sum f} = \frac{\sum Xf}{\sum f}$$

In case of a discrete probability distribution, such as the one that we have been considering i.e

| No. of Petals X | P(x) |
|---|---|
| $x_1 = 3$ | 0.05 |
| $x_2 = 4$ | 0.10 |
| $x_3 = 5$ | 0.20 |
| $x_4 = 6$ | 0.30 |
| $x_5 = 7$ | 0.25 |
| $x_6 = 8$ | 0.075 |
| $x_7 = 9$ | 0.025 |
| | 1 |

the mean is given by:

$$\mu = E(X) = \frac{\sum XP(X)}{\sum p(X)} = \frac{\sum XP(X)}{1} = \sum XP(X)$$

Hence we construct the column of XP(X), as shown below:

| No. of Petals x | P(x) | xP(x) |
|---|---|---|
| $x_1 = 3$ | 0.05 | 0.15 |
| $x_2 = 4$ | 0.10 | 0.40 |
| $x_3 = 5$ | 0.20 | 1.00 |
| $x_4 = 6$ | 0.30 | 1.80 |
| $x_5 = 7$ | 0.25 | 1.75 |
| $x_6 = 8$ | 0.075 | 0.60 |
| $x_7 = 9$ | 0.025 | 0.225 |
| Total | 1 | 5.925 |

Hence $\mu = E(X) = \sum XP(X) = 5.925$ i.e. the mean of the given probability distribution is 5.925. In other words, considering a very large number of flowers of that particular species, we would expect that, on the average, a flower contains 5.925 petals --- or, *rounding* this number, 6 petals. This interpretation points to the reason why the mean of the probability distribution of a random variable X is technically called the EXPECTED VALUE of the random variable X. ("Given that the probability that the flower has 3 petals is 5%, the probability that the flower has 4 petals is 10%, and so *ON*, we *EXPECT* that on the *average* a flower contains 5.925 petals.)

## COMPUTATION OF THE STANDARD DEVIATION

Just as in case of a frequency distribution, we have

$$S = \sqrt{\frac{\sum f\left(X-\bar{X}\right)^2}{\sum f}}$$

$$= \sqrt{\frac{\sum fX^2}{\sum f} - \left(\frac{\sum fX}{\sum f}\right)^2} = \sqrt{\frac{\sum X^2 f}{\sum f} - \left(\frac{\sum Xf}{\sum f}\right)^2}$$

Similarly, in case of a probability distribution, we have

$$\sigma = \text{S.D.}(X) = \sqrt{\frac{\sum X^2 P(X)}{\sum P(X)} - \left[\frac{\sum XP(X)}{\sum P(X)}\right]^2}$$

$$= \sqrt{\sum X^2 P(X) - \left[\sum XP(X)\right]^2}$$

$$\left(\because \quad \sum P(X) = 1\right)$$

In the above example

| No. of Petals x | P(x) | xP(x) | $x^2P(x)$ |
|---|---|---|---|
| $x_1 = 3$ | 0.05 | 0.15 | 0.45 |
| $x_2 = 4$ | 0.10 | 0.40 | 1.60 |
| $x_3 = 5$ | 0.20 | 1.00 | 5.00 |
| $x_4 = 6$ | 0.30 | 1.80 | 10.80 |
| $x_5 = 7$ | 0.25 | 1.75 | 12.25 |
| $x_6 = 8$ | 0.075 | 0.60 | 4.80 |
| $x_7 = 9$ | 0.025 | 0.225 | 2.025 |
| Total | 1 | 5.925 | 36.925 |

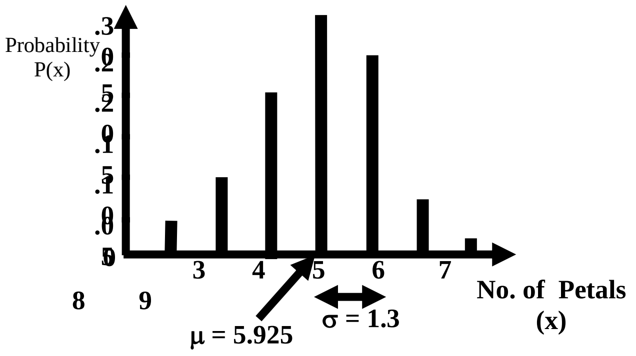
Hence

$$S.D.(X) = \sqrt{36.925 - (5.925)^2}$$

$$= \sqrt{36.925 - 35.106}$$

$$= \sqrt{1.819} = 1.3$$

**Graphical Representation:**




Now that we have both the mean and the standard deviation, we are in a position to compute the *coefficient of variation* of this distribution:

Coefficient of Variation

$$C.V. \quad = \frac{\sigma}{\mu} \times 100$$

$$= \frac{1.3}{5.925} \times 100$$

$$= 21.9\ \%$$

Let us consider another example to understand the concept of discrete probability distribution.

**EXAMPLE**

a) Find the probability distribution of the sum of the dots when two fair dice are thrown
b) Use the probability distribution to find the probabilities of obtaining (i) a sum that is greater than 8, and (ii) a sum that is greater than 5 but less than or equal to 10.

**SOLUTION**

a) The sample space S is represented by the following 36 outcomes:

S = {(1, 1), (1, 2), (1, 3), (1, 5), (1, 6);
(2, 1), (2, 2), (2, 3), (2, 5), (2, 6);
(3, 1), (3, 2), (3, 3), (3, 5), (3, 6);
(4, 1), (4, 2), (4, 3), (4, 5), (4, 6);
(5, 1), (5, 2), (5, 3), (5, 5), (5, 6);
(6, 1), (6, 2), (6, 3), (6, 5), (6, 6) }

Since each of the 36 outcomes is equally likely to occur, therefore each outcome has probability 1/36.
Let X be the random variable representing the sum of dots which appear on the dice. Then the values of the r.v. are 2, 3, 4… 12.
The probabilities of these values are computed as below:

$f(2) = P(X = 2) = P[\{1, 1\}] = \frac{1}{36}$, as there is only one outcome resulting in a sum of 2,

$$f(3) = P(X = 3) = P[\{(1, 2), (2,1)\}] = \frac{2}{36},$$

$$f(4) = P(X = 4) = P[\{(1, 3), (2, 2), (3,1)\}] = \frac{3}{36},$$

Similarly

$$f(5) = \frac{4}{36}, f(6) = \frac{5}{36}, f(7) = \frac{6}{36}, f(8) = \frac{5}{36}, f(9) = \frac{4}{36},$$

$$f(10) = \frac{3}{36}, f(11) = \frac{2}{36} \text{ and } f(12) = \frac{1}{36}.$$

Therefore the desired probability distribution of the r.v X is

| $x_i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x_i)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

The probabilities in the above table clearly indicate that if we draw the line chart of this distribution, we will obtain a triangular-shaped graph. The students are encouraged to draw the graph of this probability distribution, in order to be able to develop a visual picture in their minds.

b) Using the probability distribution, we get the required probabilities as follows:

i) P(a sum that is greater than 8)

$$= P(X > 8)$$

$$= P(X=9) + P(X=10) + P(X=11) + P(X=12)$$

$$= f(9) + f(10) + f(11) + f(12)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$$

ii) P(a sum that is greater than 5 but less than or equal to 10)

$$= P(5 < X \le 10)$$

$$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= f(6) + f(7) + f(8) + f(9) + f(10)$$

$$= \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} = \frac{23}{36}.$$

Next, we consider the concept of the DISTRIBUTION FUNCTION of a discrete random variable:

**DISTRIBUTION FUNCTION**

The distribution function of a random variable X, denoted by F(x), is defined by $F(x) = P(X < x)$.
The function F(x) gives the probability of the event that X takes a value LESS THAN OR EQUAL TO a specified value x. The distribution function is abbreviated to *d.f.* and is also called the *cumulative distribution function (cdf)* as it is the cumulative probability function of the random variable X from the smallest value upto a *specific* value x.
Let us illustrate this concept with the help of the same example that we have been considering --- that of the probability distribution of the sum of the dots when two fair dice are thrown. As explained earlier, the probability distribution of this example is:

| $x_i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x_i)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

The term 'distribution function' implies the cumulating of the probabilities similar to the cumulation of frequencies in the case of the frequency distribution of a discrete variable.

| $x_i$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x_i)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |
| $F(x_i)$ | $\frac{1}{36}$ | $\frac{3}{36}$ | $\frac{6}{36}$ | $\frac{10}{36}$ | $\frac{15}{36}$ | $\frac{21}{36}$ | $\frac{26}{36}$ | $\frac{30}{36}$ | $\frac{33}{36}$ | $\frac{35}{36}$ | $\frac{36}{36}$ |

If we are interested in finding the probability that we obtain a sum of five or less, the column of cumulative probabilities immediately indicates that this probability is 10/36.

Masters

# LECTURE NO. 23

- Graphical Representation of the Distribution Function of a Discrete Random Variable
- Mathematical Expectation
- Mean, Variance and Moments of a Discrete Probability Distribution
- Properties of Expected Values

First, let us consider the concept of the DISTRIBUTION FUNCTION of a discrete random variable.

## DISTRIBUTION FUNCTION

The distribution function of a random variable X, denoted by F(x), is defined by F(x) = P(X < x). The function F(x) gives the probability of the event that X takes a value LESS THAN OR EQUAL TO a specified value x. The distribution function is abbreviated to d.f. and is also called the cumulative distribution function (cdf) as it is the cumulative probability function of the random variable X from the smallest value up to a specific value x.

## EXAMPLE

Find the probability distribution and distribution function for the number of heads when 3 balanced coins are tossed. Depict both the probability distribution and the distribution function graphically. Since the coins are balanced, therefore the equally probable sample space for this experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}.

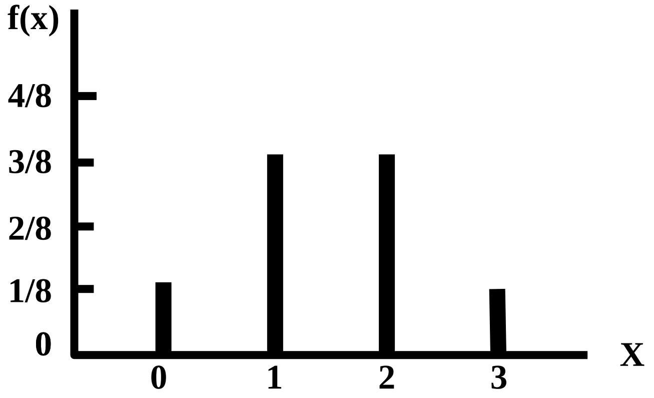
Let X be the random variable that denotes the number of heads.
Then the values of X are 0, 1, 2 and 3.
And their probabilities are:

f (0) = P(X = 0)
= P [{TTT}] = 1/8
f(1) = P (X = 1)
= P [{HTT, THT, TTH}] = 3/8
f(2) = P (X = 2)
= P [{HHT, HTH, THH}] = 3/8
f(2) = P (X = 3)
= P [{HHH}] = 1/8

Expressing the above information in the tabular form, we obtain the desired probability distribution of X as follows:

| Number of Heads ($x_i$) | Probability $f(x_i)$ |
|---|---|
| 0 | $\frac{1}{8}$ |
| 1 | $\frac{3}{8}$ |
| 2 | $\frac{3}{8}$ |
| 3 | $\frac{1}{8}$ |
| Total | 1 |

The line chart of the above probability distribution is as follows:




In order to obtain the distribution function of this random variable, we compute the cumulative probabilities as follows:

| Number of Heads ($x_i$) | Probability $f(x_i)$ | Cumulative Probability $F(x_i)$ |
|---|---|---|
| 0 | $\frac{1}{8}$ | $\frac{1}{8}$ |
| 1 | $\frac{3}{8}$ | $\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$ |
| 2 | $\frac{3}{8}$ | $\frac{4}{8}+\frac{3}{8}=\frac{7}{8}$ |
| 3 | $\frac{1}{8}$ | $\frac{7}{8}+\frac{1}{8}=1$ |

Hence the desired distribution function is

$$F(x)=\begin{cases} 0, & for \quad x<0 \\ \frac{1}{8}, & for \;\; 0 \le x < 1 \\ \frac{4}{8}, & for \;\; 1 \le x < 2 \\ \frac{7}{8}, & for \;\; 2 \le x < 3 \\ 1, & for \quad x \ge 3 \end{cases}$$

Why has the distribution function been expressed in this manner? The answer to this question is:

**INTERPRETATION**

If $x < 0$, we have $P(X < x) = 0$, the reason being that it is not possible for our random variable X to assume value less than zero.(The minimum number of heads that we can have in tossing three coins is zero.)
If $0 < x < 1$, we note that it is not possible for our random variable X to assume any value between zero and one. (We will have no head or one head but we will NOT have 1/3 heads or 2/5 heads!)
Hence, the probabilities of all such values will be zero, and hence we will obtain a situation which can be explained through the following table:

| Number of Heads $(x_i)$ | Probability $f(x_i)$ | Cumulative Probability $F(x_i)$ |
|---|---|---|
| **0** | $\frac{1}{8}$ | $\frac{1}{8}$ |
| **0.2** | **0** | $\frac{1}{8}+0=\frac{1}{8}$ |
| **0.4** | **0** | $\frac{1}{8}+0=\frac{1}{8}$ |
| **0.6** | **0** | $\frac{1}{8}+0=\frac{1}{8}$ |
| **0.8** | **0** | $\frac{1}{8}+0=\frac{1}{8}$ |
| **1** | $\frac{3}{8}$ | $\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$ |

The above table clearly shows that the probability that X is LESS THAN any value lying between zero and 0.9999… will be equal to the probability of X = 0 i.e. For $0 < x < 1$,

Similarly, $$P(X<x) = P(X=0) = \frac{1}{8};$$

- For $1 < x < 2$, we have

$$P(X<x) = P(X=0) + P(X=1)$$
$$= \frac{1}{8}+\frac{3}{8}=\frac{4}{8};$$

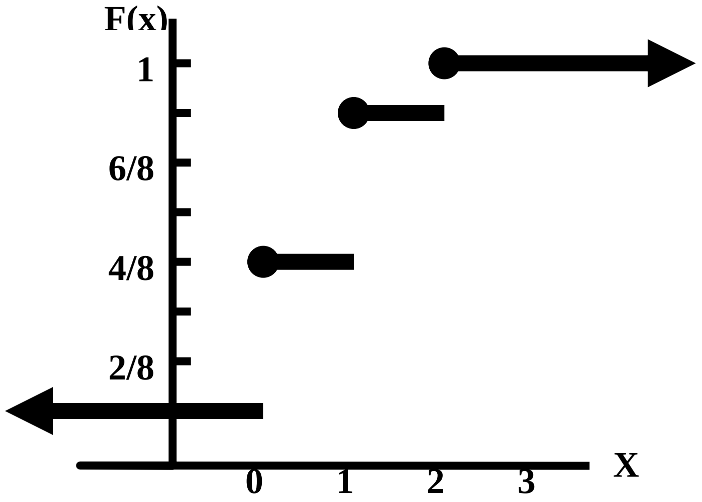
- For $2 < x < 3$, we have

$$P(X<x) = P(X=0) + P(X=1) + P(X=2)$$
$$= \frac{1}{8}+\frac{3}{8}+\frac{3}{8}=\frac{7}{8};$$

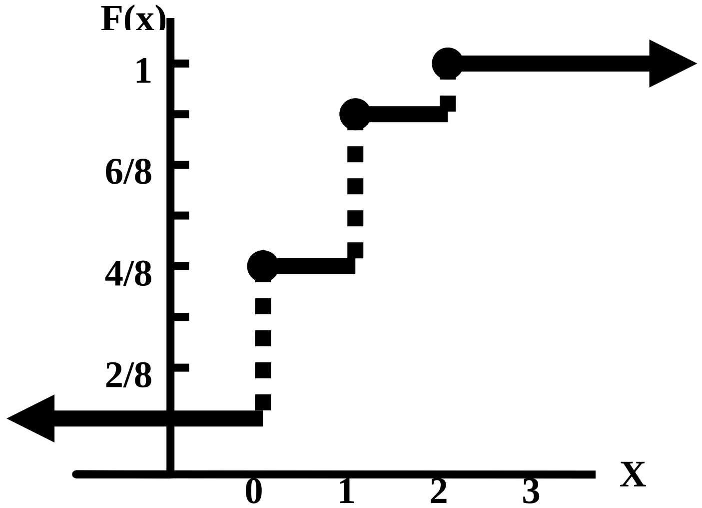
And, finally, for $x > 3$, we have

$$P(X<x) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= \frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{8}=\frac{8}{8}=1.$$

Hence, the graph of the DISTRIBUTION FUNCTION is as follows:




As this graph resembles the steps of a staircase, it is known as a step function. It is also known as a jump function (as it takes jumps at integral values of X).In some books, the graph of the distribution function is given as shown in the following figure:




In what way do we interpret the above distribution function from a REAL-LIFE point of view? If we toss three balanced coins, the probability that we obtain at the most one head is 4/8, the probability that we obtain at the most two heads is 7/8, and so on. Let us consider another interesting example to illustrate the concepts of a discrete probability distribution and its distribution function:

**EXAMPLE**

A large store places its last 15 clock radios in a clearance sale. Unknown to any one, 5 of the radios are defective. If a customer tests 3 different clock radios selected at random, what is the probability distribution of X, where X represent the number of defective radios in the sample?

**SOLUTION**

We have:

| Type of Clock Radio | Number of Clock Radios |
|---|---|
| Good | 10 |
| Defective | 5 |
| Total | 15 |

The total number of ways of selecting 3 radios out of 15 is $\binom{15}{3}$. 15C3

Also, the total number of ways of selecting 3 good radios (and no defective radio) is $\binom{10}{3}\binom{5}{0}$.
Hence, the probability of X = 0 is

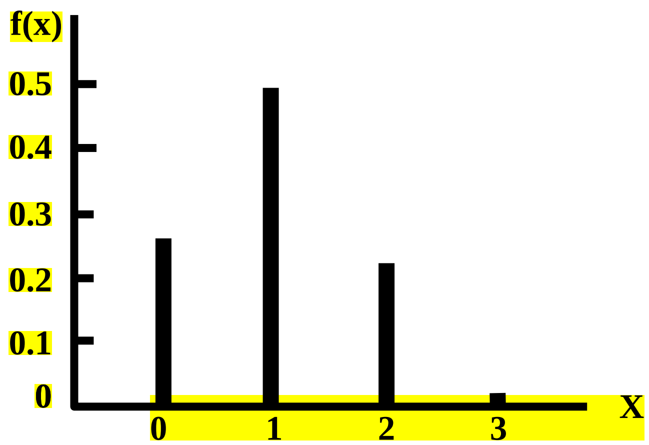
$$\frac{\binom{10}{3}\binom{5}{0}}{\binom{15}{3}} = 0.26.$$

The probabilities of X = 1, 2, and 3 are computed in a similar way. Hence, we obtain the following probability distribution:

| Number of defective clock radios in the sample X | Probability f(x) |
|---|---|
| 0 | 0.26 |
| 1 | 0.49 |
| 2 | 0.22 |
| 3 | 0.02 |
| Total | 0.99 ≈1 |

The line chart of this distribution is:

**LINE CHART**




As indicated by the above diagram, it is not necessary for a probability distribution to be symmetric; it can be positively or negatively skewed. The distribution function of the above probability distribution is obtained as follows:

| Number of defective clock radios in the sample X | f(x) | F(x) |
|---|---|---|
| 0 | 0.26 | 0.26 |
| 1 | 0.49 | 0.75 |
| 2 | 0.22 | 0.97 |
| 3 | 0.02 | 0.99 ≈ 1 |
| Total | 0.99 ≈ 1 | |

**INTERPRETATION**

The probability that the sample of 3 clock radios contains at the most one defective radio is 0.75, the probability that the sample contains at the most two defective radios is 0.97, and so on.
Let a discrete random variable X have possible values x1, x2, …, xn with corresponding probabilities f(x1), f(x2), …, f(xn) such that Σf(xi) =1. Then the mathematical expectation or the expectation or the expected value of X, denoted by E(x), is defined as
E(X) = x1f(x1) + x2f(x2) + … + xnf(xn) imp

$$= \sum_{i=1}^{n} x_i f(x_i),$$

E(X) is also called the mean of X and is usually denoted by the letter μ.

The expression $E(X)=\sum_{i=1}^{n} x_i f(x_i)$

may be regarded as a weighted mean of the variable's possible values x1, x2, …,xn, each being weighted by the respective probability.
In case the values are equally likely,

$$E(X)=\frac{1}{n}\sum x_i,$$

Which represents the ordinary arithmetic mean of the n possible values
It should be noted that E(X) is the average value of the random variable X over a very large number of trials.

**EXAMPLE**
If it rains, an umbrella salesman can earn $ 30 per day. If it is fair, he can lose $ 6 per day. What is his expectation if the probability of rain is 0.3?

**SOLUTION**

Let X represents the number of dollars the salesman earns. Then X is a random variable with possible values 30 and –6, (where -6 corresponds to the fact that the salesman loses), and the corresponding probabilities are 0.3 and 0.7 respectively. Hence, we have:

| EVENT | AMOUNT EARNED ($) x | PROBABILITY P(x) |
|---|---|---|
| Rain | 30 | 0.3 |
| No Rain | –6 | 0.7 |
| | Total | 1 |

In order to compute the expected value of X, we carry out the following computation

| EVENT | AMOUNT EARNED ($) x | PROBABILITY P(x) | xP(x) |
|---|---|---|---|
| Rain | 30 | 0.3 | 9.0 |
| No Rain | –6 | 0.7 | -4.2 |
| | Total | 1 | 4.8 |

Hence
E(X) = $ 4.80 per day
i.e. on the average, the salesman can expect to earn 4.8 dollars per day.
Until now, we have considered the mathematical expectation of the random variable X.

But, in many situations, we may be interested in the mathematical expectation of some FUNCTION of X:

## EXPECTATION OF A FUNCTION OF A RANDOM VARIABLE

Let H(X) be a function of the random variable X. Then H(X) is also a random variable and also has an expected value, (as any function of a random variable is also a random variable). If X is a discrete random variable with probability distribution f(x), then, since H(X) takes the value $H(x_i)$ when $X = x_i$, the expected value of the function H(X) is

$E[H(X)] = H(x_1) f(x_1) + H(x_2)f(x_2) + \ldots + H(x_n) f(x_n)$

$$= \sum_i H(x_i) f(x_i),$$

Provided the series converges absolutely. Again, if $H(X) = (X - \mu)^2$, where $\mu$ is the population mean, then

$$E(X - \mu)^2 = \Sigma(x_i - \mu)^2 f(x).$$

We call this expected value the variance and denote it by Var (X) or $\sigma^2$.

And, since

$$E(X - \mu)^2 = E(X^2) - [E(X)]^2,$$

hence the short cut formula for the variance is

$$\sigma^2 = E(X^2) - [E(X)]^2$$

The positive square root of the variance, a before, is called the standard deviation. More generally, if $H(X) = X^k$, k = 1, 2, 3… then

$$E(X^k) = \Sigma x_i^k f(x)$$

which we call the kth moment about the origin of the random variable X and we denote it by $\mu'_k$. Similarly, if $H(X) = (X - \mu)^k$, k = 1, 2, 3, …, then we get an expected value, called the kth moment about the mean of the random variable X, which we denote by $\mu_k$. That is:

$\mu_k = E(X - \mu)^k = \Sigma(x_i - \mu)^k f(x)$

The skewness of a probability distribution is often measured by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}$$

and kurtosis by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}.$$

These moment-ratios assist us in determining the Skewness and kurtosis of our probability distribution in exactly the same way as was discussed in the case of frequency distributions.

## PROPERTIES OF MATHEMATICAL EXPECTATION

The important properties of the expected values of a random variable are as follows:

- If c is a constant, then E(c) = c. Thus the expected value of a constant is constant itself. This point can be understood easily by considering the following interesting example: Suppose that a very difficult test was given to students by a professor, and that every student obtained 2 marks out of 20! It is obvious that the mean mark is also 2. Since the variable 'marks' was a constant, therefore its expected value was equal to itself.
- If X is a discrete random variable and if a and b are constants, then E(aX + b) = a E(X) + b.

**EXAMPLE**

Let X represents the number of heads that appear when three fair coins are tossed. The probability distribution of X is:

| X | P(x) |
|---|---|
| 0 | 1/8 |
| 1 | 3/8 |
| 2 | 3/8 |
| 3 | 1/8 |
| Total | 1 |

The expected value of X is obtained as follows:

| x | P(x) | xP(x) |
|---|---|---|
| 0 | 1/8 | 0 |
| 1 | 3/8 | 3/8 |
| 2 | 3/8 | 6/8 |
| 3 | 1/8 | 3/8 |
| Total | 1 | 12/8=1.5 |

Hence, $E(X) = 1.5$

Suppose that we are interested in finding the expected value of the random variable 2X+3.Then we carry out the following computations:

| x | 2x+3 | P(x) | (2x+3)P(x) |
|---|---|---|---|
| 0 | 3 | 1/8 | 3/8 |
| 1 | 5 | 3/8 | 15/8 |
| 2 | 7 | 3/8 | 21/8 |
| 3 | 9 | 1/8 | 9/8 |
| | Total | 1 | 48/8=6 |

Hence $E(2X+3) = 6$ It should be noted that

$E(2X+3) = 6= 2(1.5) + 3= 2E(X) + 3$

i.e. $E (aX + b) = a E(X) + b.$

Masters

# LECTURE NO. 24

- Chebychev's Inequality
- Concept of Continuous Probability Distribution
- Mathematical Expectation, Variance & Moments of a Continuous Probability Distribution

We begin with the discussion of the concept of the Chebychev's Inequality in the case of a discrete *probability* distribution

**Chebychev's Inequality**

If X is a random variable having mean μ and variance σ2 > 0, and k is any positive constant, then the probability that a value of X falls within k standard deviations of the mean is at least

That is:

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2},$$

Alternatively, we may state Chebychev's theorem as follow: Given the probability distribution of the random variable X with mean μ and standard deviation σ, the probability of the observing a value of X that differs the μ by k or more standard deviations cannot exceed 1/k2. As indicated earlier, this inequality is due to the Russian mathematician P.L. Chebychev (1821-1894), and it provides a means of understanding *how the standard deviation measures variability about the mean* of a random variable. It holds for all probability distributions having finite mean and variance.

Let us apply this concept to the example of the number of petals on the flowers of a particular species that we considered earlier:

**EXAMPLE**

If a biologist is interested in the number of petals on a particular flower, this number may take the values 3, 4, 5, 6, 7, 8, 9, and each one of these numbers will have its own probability

The probability distribution of the random variable X is:

| No. of Petals X | P(x) |
|---|---|
| $x_1 = 3$ | 0.05 |
| $x_2 = 4$ | 0.10 |
| $x_3 = 5$ | 0.20 |
| $x_4 = 6$ | 0.30 |
| $x_5 = 7$ | 0.25 |
| $x_6 = 8$ | 0.075 |
| $x_7 = 9$ | 0.025 |
| | 1 |

The mean of this distribution is:

$\mu = E(X) = \sum XP(X) = 5.925 \cong 5.9$

And the standard deviation of this distribution is:

$\sigma =$

$$S.D.(X) = \sqrt{36.925 - (5.925)^2}$$

formula on page165

$$= \sqrt{36.925 - 35.106}$$

$$= \sqrt{1.819} = 1.3$$

$1/2^2$

According to the Chebychev's inequality, the probability is at least 1 - 1/22 = 1 - 1/4 = 3/4 = 0.75 that X will lie between μ - 2σ and μ + 2σ i.e. between 5.9 - 2(1.3) and 5.9 + 2(1.3) i.e. between 3.3 and 8.5

Let us have another look at the probability distribution:

| No. of Petals X | P(x) |
|---|---|
| $x_1 = 3$ | 0.05 |
| $x_2 = 4$ | 0.10 |
| $x_3 = 5$ | 0.20 |
| $x_4 = 6$ | 0.30 |
| $x_5 = 7$ | 0.25 |
| $x_6 = 8$ | 0.075 |
| $x_7 = 9$ | 0.025 |
| | 1 |

According to this distribution, the probability that X lies between 3.3 and 8.5 is
0.10 + 0.20 + 0.30 + 0.25 + 0.075
= 0.925
which is *greater* than 0.75(*AS* indicated by the Chebychev's inequality).
Finally, and most importantly, we will use the concepts in Chebychev's Rule and the Empirical Rule to build the foundation for statistical inference-making. The method is illustrated in next example.

**EXAMPLE**

Suppose you invest a fixed sum of money in each of five business ventures. Assume you know that 70% of such ventures are successful, the outcomes of the ventures are independent of one another, and the probability distribution for the number, x, of successful ventures out of five is:

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| P(x) | .002 | .029 | .132 | .309 | .360 | .168 |

**a)** Find $\mu = E(X)$.
Interpret the result.
**b)**Find

$$\sigma = \sqrt{E\left[(X-\mu)^2\right]}.$$

Interpret the result.
**c)** Graph P(x).
**d)** Locate $\mu$ and the interval $\mu + 2\sigma$ on the graph. Use either Chebychev's Rule or the Empirical Rule to approximate the probability that x falls in this interval. Compare this result with the actual probability.
**e)** Would you expect to observe fewer than two successful ventures out of five?

**SOLUTION**

a) Applying the formula,
$\mu = E(X) = \sum xP(x)$
= 0(.002) +1(.029) + 2(.132) + 3(.309) + 4 (.360) + 5(.168)
= 3.50

**INTERPRETATION**

On average, the number of successful ventures out of five will equal 3.5. (It should be remembered that this expected value has meaning only when the experiment – investing in five business ventures – is repeated a large number of times.)
**b)** Now we calculate the variance of X:
We know that
$\sigma 2 = E[(X - \mu)2] = \sum(x - \mu)2\ P(x)$
Hence, we will need to construct a column of x - $\mu$:

| x | P(x) | x-μ | $(x-\mu)^2$ | $(x-\mu)^2P(x)$ |
|---|---|---|---|---|
| 0 | .002 | –3.5 | 12.25 | 0.02 |
| 1 | .029 | –2.5 | 6.25 | 0.18 |
| 2 | .132 | –1.5 | 2.25 | 0.30 |
| 3 | .309 | –0.5 | 0.25 | 0.08 |
| 4 | .360 | +0.5 | 0.25 | 0.09 |
| 5 | .168 | +1.5 | 2.25 | 0.38 |
| | | | Total | 1.05 |

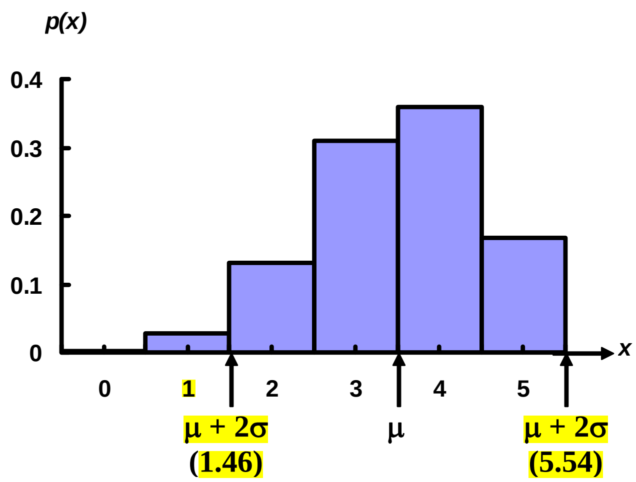
Thus, the variance is σ2 = 1.05 and the standard deviation is

$$\sigma = \sqrt{\sigma^2} = \sqrt{1.05} = 1.02$$

This value measures the spread of the probability distribution of X, the number of successful ventures out of five.

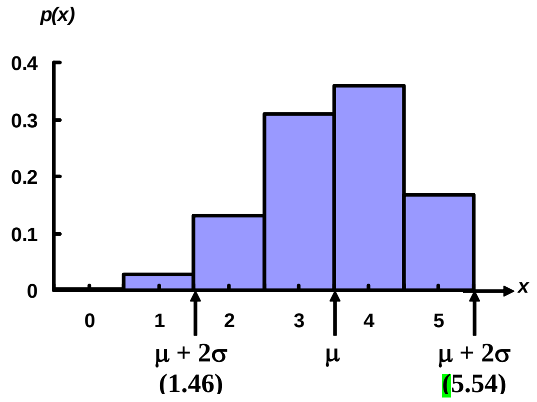
**c)** The graph of P(x) is shown in the following figure with the mean μ and the interval

μ + 2σ =3.50+2(1.02)
=3.50+2.04
= (1.46, 5.54) shown on the graph.




Note particularly that μ = 3.5 locate the *centre* of the probability distribution. Since this distribution is a theoretical relative frequency distribution that is moderately mound-shaped, we expect (from Chebychev's Rule) at least 75% and, more likely (from the Empirical Rule), approximately 95% of observed x values to fall in the interval μ + 2σ ------ that is, between 1.46 and 5.54.

It can be seen from the above figure that the actual probability that X falls in the interval μ + 2σ includes the sum of P(x) for the values
X = 2, X = 3, X = 4, and X = 5.

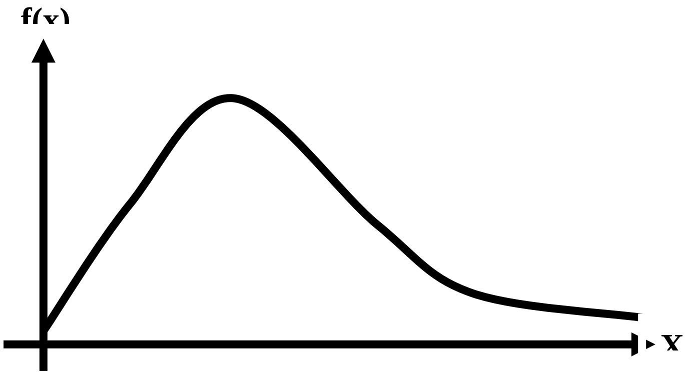



This probability is P(2) + P(3) + P(4) + P(5)
= .132 +.309 + .360 + .168
= .969.
Therefore, 96.9% of the probability distribution lies within 2 standard deviations of the mean. This percentage is *CONSISTENT* with both the Chebychev's rule and the Empirical Rule.

**d)** Fewer than two successful ventures out of five implies that x = 0 or x = 1. Since both these values of x lie outside the interval μ + 2σ, we know from the Empirical Rule that such a result is unlikely (with approximate probability of only .05).The exact probability, P(x < 1), is P(0) + P(1) = .002 + .029 = .031.
Consequently, in a *single* experiment where we invest in five business ventures, we would *not* expect to observe fewer than two successful ones. The *key* question: What is the ***significance*** of the Chebychev's Inequality and the Empirical Rule?
The answer to this question is that both these rules assist us in having a certain *IDEA* regarding amount of data lying between the mean minus a certain number of standard deviations and mean plus that same number of standard deviations. Given any data-set, the moment we compute the mean and standard deviation, we *HAVE* an idea regarding the two points (i.e. mean minus two standard deviations, and mean plus two standard deviations) between which the *BULK* of our data lies. If our data-set is hump-shaped, we obtain this idea through the *Empirical Rule*, and if we don't have any reason to believe that our data-set is hump-shaped, then we obtain this idea through the Chebychev's Rule
We now begin the discussion of CONTINUOUS RANDOM VARIABLES – quantities that are measurable. As stated in the very first lecture, continuous variables result from measurement, and can therefore take any value within a certain range. For example, the height of a normal Pakistani adult male may take any value between 5 feet 4 inches and 6 feet. The temperature at a place, the amount of rainfall, time to failure for an electronic system, etc. are all *examples* of continuous random variable. Formally speaking, a continuous random variable can be defined as follows:

**CONTINUOUS RANDOM VARIABLE**

A random variable X is defined to be continuous if it can assume every possible value in an interval [a, b], a < b, where a and b may be –∞ and +∞ respectively. The function f(x) is called the *probability density function*, imp abbreviated to *p.d.f.*, or simply density function of the random variable X. A continuous probability distribution looks something like this:



A p.d.f. has the following properties:

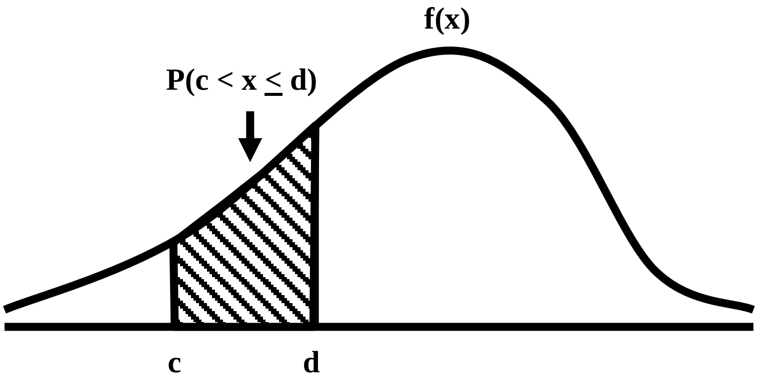
**i) $f(x) \geq 0$, for all x**

**ii)** $$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

**iii)** The probability that X takes on a value in the interval [c, d], c < d is given by:

$$P(c < x < d) = \int_{c}^{d} f(x)\,dx$$

which is the area under the curve y = f(x) between X = c and X = d, as shown in the following figure:




The *TOTAL* area under the curve is 1. In other words:

- f(x) a non-negative function,
- The integration takes place over all possible values of the random variable X *between the specified limits*, and
- The probabilities are given by appropriate areas under the curve.

Since

$$P(X = k) = \int_{k}^{k} f(x)\ dx = 0,$$

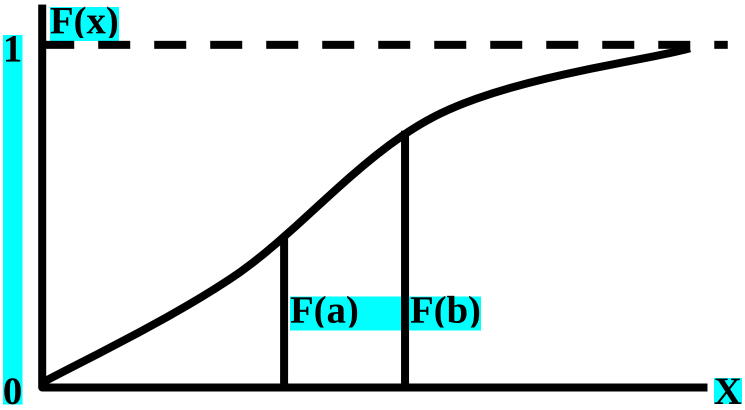
It should therefore be noted that the probability of a continuous random variable X taking any *particular* value k is always *zero*. That is why probability for a continuous random variable is measurable only over a given interval.
Further, since for a continuous random variable X, P(X = x) = 0 for every x, the following four probabilities are regarded as the same:

P(c < X < d), P(c < X < d),
P(c < X < d) and P(c < X d).

They may be different for a discrete random variable. The values (expressed as intervals) of a continuous random variable and their associated probabilities can be expressed by means of a formula.
We now discuss the *distribution function* of a continuous random variable.

**CONTINUOUS RANDOM VARIABLE**

A random variable X may also be defined as continuous if its *distribution function* F(x) is continuous and is differentiable everywhere except at isolated points in the given range. In contrast with the graph of the distribution function of a discrete variable, the graph of F(x) in the case of a continuous variable has no jumps or steps but is a *continuous* function for all x-values, as shown in the following figure:




Since F(x) is a non-decreasing function of x, we have

**i)** f(x) > 0,

**ii)** $F(x) = \int_{-\infty}^{x} f(x)\,dx,$ for all x.

The relationship between f(x) and F(x) is as follows: f(x) is obtained by finding the derivative of F(x), i.e.

$$\frac{d\,F(x)}{dx} = f(x)$$

**EXAMPLE**

**a)** Find the value of k so that the function f(x) defined as follows, may be a density function

$$f(x) = kx, \; 0 < x < 2$$
$$= 0, \text{ elsewhere}$$

**b)** Compute P(X = 1).

**c)** Compute P(X > 1).

**d)** Compute the distribution function F(x).

**e)** $P(X < 1/2 \;/\; 1/3 < X < 2/3)$

**SOLUTION**

a) The function f(x) will be a density function, if

i) f(x) > 0 for every x, and

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

The first condition is satisfied when k > 0. The second condition will be satisfied, if

$$\int_{-\infty}^{\infty} f(x)\,dx = 1,$$

$$\text{i.e. if} \quad 1 = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$\text{i.e. if} \quad 1 = \int_{-\infty}^{0} 0\,dx + \int_{0}^{2} kx\,dx + \int_{2}^{\infty} 0\,dx$$

$$\text{i.e. if} \quad 1 == 0 + \left[k\frac{x^2}{2}\right]_0^2 + 0 = 2k$$

We had

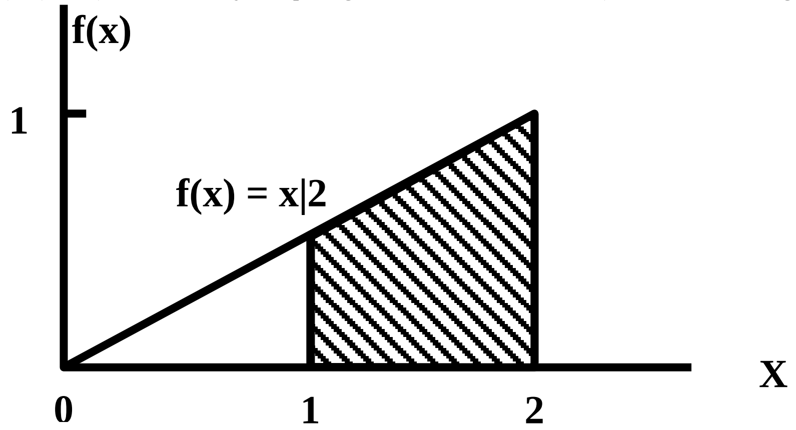
This gives k = 1/2

f(x) = kx, 0 < x < 2
= 0, elsewhere
and since we have obtained
k = 1/2, hence:

$$f(x)=\begin{cases}\frac{x}{2}, & \text{for } 0\le x\le 2\\ 0, & \text{elsewhere}\end{cases}$$

b) Since f(x) is continuous probability function, therefore (X = 1) = 0.

c) P(X > 1) is obtained by computing the area under the curve (in this case, a straight line) between X=1 and X=2:




This area is obtained as follows:

$$\begin{aligned}P(X>1) &= \text{area of shaded region}\\ &=\int_1^2 f(x)\,dx\\ &=\int_1^2 \frac{x}{2}\,dx=\left[\frac{x^2}{4}\right]_1^2=\frac{3}{4}\end{aligned}$$

d) To compute the distribution function, we need to find:

$$\mathbf{F(x) = P(X < x) =} \int_{-\infty}^{x} f(x)\,dx$$

We do so *step by step*, as shown below:

**For any x such that -∞ < x ≤ 0,**

$$\mathbf{F(x) =} \int_{-\infty}^{x} 0\,dx = 0,$$

**If 0 < x ≤ 2, we have**

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_0^x\left(\frac{x}{2}\right)dx=\left[\frac{x^2}{4}\right]_0^x=\frac{x^2}{4},$$

**and, finally, for x > 2 we have**

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_0^2\frac{x}{2}\,dx+\int_0^2 0\,dx=1$$

Hence

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ \dfrac{x^2}{4}, & \text{for } 0 \le x \le 2 \\ \mathbf{1}, & \textbf{for } \mathbf{x > 2.} \end{cases}$$

We will discuss the computation of the conditional probability

$P(X < 1/2 \; / \; 1/3 < X < 2/3)$ ***include in next lecture 25***

# LECTURE NO. 25

- Mathematical Expectation, Variance & Moments of a Continuous Probability Distribution
- BIVARIATE Probability Distribution

In the last lecture, we were dealing with an example of a continuous probability distribution in which we were interested in computing a conditional probability. We now discuss this particular concept

**EXAMPLE**

**a)** Find the value of k so that the function f(x) defined as follows, may be a density function

f(x) = kx, 0 < x < 2
= 0, elsewhere

**b)** Compute P(X = 1).
**c)** Compute P(X > 1).
**d)** Compute the distribution function F(x).
**e)** $P\left(X < 1/2 \;/\; 1/3 < X < 2/3\right)$

**SOLUTION**

We had
f(x) = kx, 0 < x < 2
= 0, elsewhere
and we obtained k = 1/2.

Hence:

$$f(x) = \begin{cases} \frac{x}{2}, & \text{for } 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

e) Applying the definition of conditional probability, we get

$$P\left(X \le \tfrac{1}{2} \mid \tfrac{1}{3} \le X \le \tfrac{2}{3}\right) = \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)} = \frac{\int_{1/3}^{1/2} \frac{x}{2}\,dx}{\int_{1/3}^{2/3} \frac{x}{2}\,dx}$$

$$= \left[\frac{x^2}{4}\right]_{1/3}^{1/2} \div \left[\frac{x^2}{4}\right]_{1/3}^{2/3}$$

The above example was of the simplest case when the graph of our continuous probability distribution is in the form of a straight line.

Let us now consider a slightly more *complicated* situation.

**EXAMPLE**

A continuous random variable X has the d.f. F(x) as follows:

$$F(x) = 0, \quad \text{for } x < 0,$$
$$= \frac{2x^2}{5}, \quad \text{for } 0 < x \le 1,$$
$$= -\frac{3}{5} + \frac{2}{5}\left(3x - \frac{x^2}{2}\right), \quad \text{for } 1 < x \le 2,$$
$$= 1 \quad \text{for } x > 2.$$

Find the p.d.f. and P(|X| < 1.5).

**SOLUTION**

By definition, we have $f(x) = \frac{d}{dx} F(x).$

Therefore $f(x) = \frac{4x}{5}$ for $0 < x \le 1$

*The derivative of a constant term is always zero.*

$= \frac{2}{5}(3 - x)$ for $1 < x \le 2$

$= 0$ elsewhere.

Let us now discuss the mathematical expectation of *continuous* random variables through the following example:

**EXAMPLE**

Find the expected value of the random variable X having the p.d.f

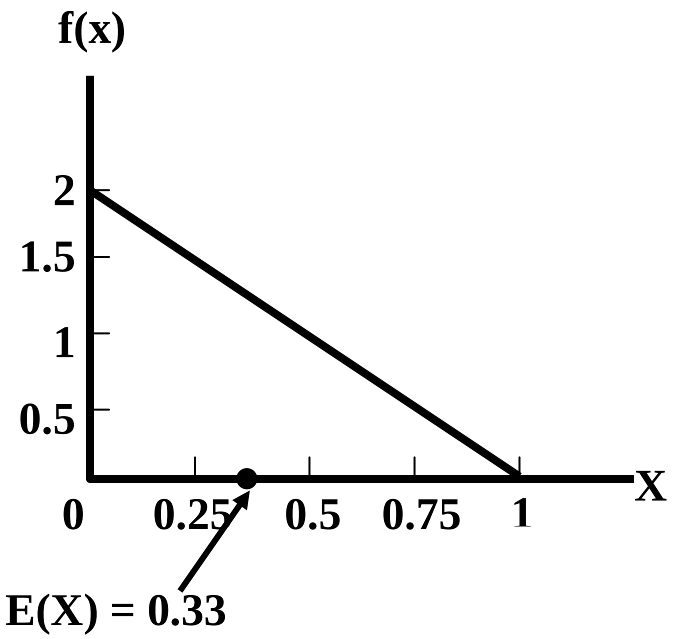
$f(x) = 2(1 - x),\quad 0 < x < 1$

$= 0$, elsewhere

**SOLUTION**

Now
$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$
$$= 2\int_{0}^{1} x(1 - x)\, dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}$$

As indicated earlier, the term 'expected value' implies the *mean* value. The graph of the above probability density function and its *mean* value are presented in the following figure:




Suppose that we are interested in verifying the properties of mathematical expectation that are valid in the case of univariate probability distributions. In the last lecture, we noted that if X is a discrete random variable and if a and b are constants, then

E (aX + b) = a E(X) + b.

This property is equally valid in the case of continuous probability distributions. In this example, suppose that a = 3 and b = 5. Then, we wish to verify that

E(3X + 5) = 3 E(X) + 5.

The right-hand-side of the above equation is:

3 E(X) + 5 = 3(   ) + 5 = 1 + 5 = 6

In order to compute the left-hand-side, we proceed as follows:

$$E(3X+5) = 2\int_0^1 (3x+5)(1-x)dx$$

$$= 2\int_0^1 \left(5-2x-3x^2\right)dx$$

$$= 2\left[5x - x^2 - x^3\right]_0^1$$

$$= 2[5-1-1] = 2(3) = 6.$$

Since the left-hand-side is equal to the right-hand-side, therefore the property is verified.

**SPECIAL CASE**

We have

E(aX + b) = a E(X) + b.

If b = 0, the above property takes the following simple form:

E(aX) = a E(X).

Next, let us consider the computation of the *moments* and *moment-ratios* in the case of a continuous probability distribution:

**EXAMPLE**

A continuous random variable X has the p.d.f.

$$f(x) = \frac{3}{4}x(2-x),\ 0 \le x \le 2.$$

$$= 0, \qquad \text{otherwise}$$

Find the first four moments about the mean and the moment-ratios.

We first calculate the moments about origin as

$$\mu'_1 = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \frac{3}{4}\int_0^2 x\left(2x - x^2\right) dx = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{16}{12}\right] = 1;$$

$$\mu'_2 = E\left(X^2\right) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$$

$$= \frac{3}{4}\int_0^2 x^2\left(2x - x^2\right) dx = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5};$$

$$\mu'_3 = E\left(X^3\right) = \int_{-\infty}^{\infty} x^3 f(x)\, dx$$

$$= \frac{3}{4}\int_0^2 x^3\left(2x - x^2\right) dx = \frac{3}{4}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{5} - \frac{64}{6}\right] = \frac{3}{4}\left[\frac{64}{30}\right] = \frac{8}{5};$$

$$\mu'_4 = E\left(X^4\right) = \int_{-\infty}^{\infty} x^4 f(x)\, dx$$

$$= \frac{3}{4}\int_0^2 x^4\left(2x - x^2\right) dx = \frac{3}{4}\left[\frac{2x^6}{6} - \frac{x^7}{7}\right]_0^2$$

$$= \frac{3}{4}\left[\frac{64}{3} - \frac{128}{7}\right] = \frac{3}{4}\left[\frac{64}{21}\right] = \frac{16}{7}.$$

Next, we find the moments about the mean as follows:

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - (\mu'_1)^2 = \frac{6}{5} - (1)^2 = \frac{1}{5}$$

$$\mu_3 = \mu'_3 - 3\mu'_1\, \mu'_2 + 2(\mu'_1)^3$$

$$= \frac{8}{5} - 3(1)\left(\frac{6}{5}\right) + 2(1)^3 = \frac{8}{5} - \frac{18}{5} + 2 = 0;$$

$$\mu_4 = \mu'_4 - 4\mu'_1\, \mu'_3 + 6(\mu'_1)^2\, \mu'_2 - 3(\mu'_1)^4$$

$$= \frac{16}{7} - 4(1)\left(\frac{8}{5}\right) + 6(1)^2\left(\frac{6}{5}\right) - 3(1)^4$$

$$= \frac{16}{7} - \frac{32}{5} + \frac{36}{5} - 3 = \frac{3}{35}.$$

The first moment-ratio is

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0^2}{\left(\frac{1}{5}\right)^3} = 0.$$

This implies that this particular continuous probability distribution is *absolutely* symmetric
The second moment-ratio is

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{3}{35}}{\left(\frac{1}{5}\right)^2} = 2.14.$$

This implies that this particular continuous probability distribution may be regarded as playkurtic, i.e. flatter than the normal distribution.
The students are encouraged to draw the graph of this distribution in order to develop a visual picture in their minds.

We begin the concept of Bivariate probability distribution by introducing the term 'Joint Distributions':

## JOINT DISTRIBUTIONS

The distribution of two or more random variables which are observed simultaneously when an experiment is performed is called their *JOINT* distribution. It is customary to call the distribution of a single random variable as univariate. Likewise, a distribution involving two, three or many r.v.'s simultaneously is referred to as bivariate, trivariate or multivariate. A bivariate distribution may be discrete when the possible values of (X, U) are finite or count ably infinite. It is continuous if (X, Y) can assume all values in some non-countable set of the plane. A bivariate distribution is said mixed when one r.v. is discrete and the other is continuous.

## BIVARIATE PROBABILITY FUNCTION

Let X and Y be two discrete r.v.'s defined on the same sample space S, X taking the values x1, x2, …, xm and Y taking the values y1, y2, …, yn. Then the probability that X takes on the value xi and, at the same time, Y takes on the value, denoted by f(xi, yj) or pij, is defined to be the joint probability function or simply the joint distribution of X and Y.
Thus the joint probability function, also called the bivariate probability function f(x, y) is a function whose value at the point (xi, yj) is given byf(xi, yj) = P(X = xi and Y = yj),
i = 1, 2, …, m.
j = 1, 2, …, n.
The joint or bivariate probability distribution consisting of all pairs of values (xi, yj) and their associated probabilities f(xi, yj) i.e. the set of triples [xi, yj, f(xi, yj)] can either be shown in the following two-way table:

### Joint Probability Distribution of X and Y

| X\Y | $y_1$ | $y_2$ | … | $y_j$ | … | $y_n$ | $P(X = x_i)$ |
|---|---|---|---|---|---|---|---|
| $x_1$ | $f(x_1,y_1)$ | $f(x_1,y_2)$ | … | $f(x_1,y_j)$ | … | $f(x_1,y_n)$ | $g(x_1)$ |
| $x_2$ | $f(x_2,y_1)$ | $f(x_2,y_2)$ | … | $f(x_2,y_j)$ | … | $f(x_2,y_n)$ | $g(x_2)$ |
| ⋮ | ⋮ | | | | | ⋮ | ⋮ |
| $x_i$ | $f(x_i,y_1)$ | $f(x_i,y_2)$ | … | $f(x_i,y_j)$ | … | $f(x_i,y_n)$ | $g(x_i)$ |
| ⋮ | ⋮ | | | | | ⋮ | ⋮ |
| $x_m$ | $f(x_m,y_1)$ | $f(x_m,y_2)$ | … | $f(x_m,y_j)$ | … | $f(x_m,y_n)$ | $g(x_m)$ |
| $P(Y=y_j)$ | $h(y_1)$ | $h(y_2)$ | … | $h(y_j)$ | … | $h(y_n)$ | 1 |

or be expressed by mean of a formula for f(x, y). The probabilities f(x, y) can be obtained by substituting appropriate values of x and y in the table or formula. A joint probability function has the following properties:

## PROPERTIES

**i)** f(xi, yj)>0,for all(xi, yj),i.e. for i=1,2,…,m; j = 1, 2, …, n.

**ii)** $$\sum_i \sum_j f(x_i, y_j) = 1$$

## MARGINAL PROBABILITY FUNCTIONS

The point to be understood here is that, from the joint probability function for (X, Y), we can obtain the INDIVIDUAL probability function of X and Y. Such individual probability functions are called *MARGINAL* probability functions.

Let f(x, y) be the joint probability function of two discrete r.v.'s X and Y. Then the marginal probability function of X is defined as

$$g(x_i) = \sum_{j=1}^{n} f(x_i, y_j)$$

f(xi, y1) + f(xi, y2) + … + f(xi, yn)
as xi must occur either with y1 or y2 or … or yn
= P(X = xi);

that is, the individual probability function of X is found by adding over the rows of the two-way table. Similarly, the marginal probability function for Y is obtained by adding over the column as

$$h(y_j) = \sum_{i=1}^{m} f(x_i, y_j) = P(Y = y_j)$$

The values of the marginal probabilities are often written in the margins of the joint table as they are the row and column totals in the table. The probabilities in each marginal probability function add to 1.

## CONDITIONAL PROBABILITY FUNCTION

Let X and Y be two discrete r.v.'s with joint probability function f(x, y). Then the conditional probability function for X given Y = y, denoted as f(x|y), is defined by

$$\begin{aligned} f(x_i \mid y_j) &= P(X = x_i \mid Y = y_j) \\ &= \frac{P(X = x_i \text{ and } Y = y_j)}{P(Y = y_j)} \\ &= \frac{f(x_i, y_j)}{h(y_j)}, \\ &\text{for } i = 1, 2, \ldots, j = 1, 2, \ldots \end{aligned}$$

Where h(y) is the marginal probability, and h(y) > 0

It gives the probability that X takes on the value xi given that Y has taken on the value yj. The conditional probability f(xi | yj) is non-negative and (for a given fixed yj) adds to 1 on i and hence is a *probability function*. Similarly, the conditional probability function for Y given X = x is

$$\begin{aligned} f(y_j \mid x_i) &= P(Y = y_j \mid X = x_i) \\ &= \frac{P(Y = y_j \text{ and } X = x_i)}{P(X = x_i)} \\ &= \frac{f(x_i, y_j)}{g(x_i)}, \quad \text{where } g(x) > 0. \end{aligned}$$

## INDEPENDENCE

Two discrete r.v.'s X and Y are said to be statistically independent, if and only if, for all possible pairs of values (xi, yj) the joint probability function f(x, y) can be expressed as the *product* of the two marginal probability functions.
That is, X and Y are independent, if

$$\begin{aligned} f(x, y) &= P(X = x_i \text{ and } Y = y_j) \\ &= P(X = x_i).\ P(Y = y_j) \\ &\qquad \text{for all } i \text{ and } j. \\ &= g(x)\ h(y). \end{aligned}$$

It should be noted that the joint probability function of X and Y when they are *independent*, can be obtained by *MULTIPLYING* together their marginal probability functions.

## EXAMPLE

An urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at random from it. If X is the number of black balls and Y is the number of red balls selected, then find

i) the joint probability function f(x, y);
ii) P(X + Y ≤ 1);
iii) the marginal p.d. g(x) and h(y);
iv) the conditional p.d. f(x | 1),
v) P(X = 0 | Y = 1); and
vi) Are x and Y independent?

**i)** The sample space S for this experiment contains sample points. The possible values of X are 0, 1, and 2, and those for Y are 0, 1, and 2. The values that (X, Y) can take on are (0, 0), (0, 1), (1, 0), (1, 1), (0, 2) and (2, 0). We desire to find f(x, y) for each value (x, y).
The total number of ways in which 2 balls can be drawn out of a total of 8 balls is

$$\binom{8}{2} = \frac{8x7}{2} = 28.$$

Now f(0, 0) = P(X = 0 and Y = 0), where the event (X = 0 and Y = 0) represents that neither black nor red ball is selected, implying that the 2 selected are green balls. This event therefore contains $\binom{3}{0}\binom{2}{0}\binom{3}{2} = 3$ sample points,

and

$$f(0, 0) = P(X = 0 \text{ and } Y = 0) = 3/28$$

$$\text{Again } f(0, 1) = P(X = 0 \text{ and } Y = 1)$$

$$= P\text{ (none is black, 1 is red and 1 is green)}$$

$$= \frac{\binom{3}{0}\binom{2}{1}\binom{3}{1}}{28} = \frac{6}{28}$$

Similarly, $f(1, 1)$

$$= P(X = 1 \text{ and } Y = 1)$$

$$= P\text{(1 is black 1 is red and none is green)}$$

$$= \frac{\binom{3}{1}\binom{2}{1}\binom{3}{0}}{28} = \frac{6}{28}$$

Similar calculations give the probabilities of other values and the joint probability function of X and Y is given as:

Joint Probability Distribution

| X \ Y | 0 | 1 | 2 | $P(X = x_i)$ g(x) |
|---|---|---|---|---|
| 0 | 3/28 | 6/28 | 1/28 | 10/28 |
| 1 | 9/28 | 6/28 | 0 | 15/28 |
| 2 | 3/28 | 0 | 0 | 3/28 |
| $P(Y = y_j)$ h(y) | 15/28 | 12/28 | 1/28 | 1 |

## LECTURE NO. 26

- BIVARIATE Probability Distributions (Discrete and Continuous)
- Properties of Expected Values in the case of Bivariate Probability Distributions

In the last lecture we began the discussion of the example in which we were drawing 2 balls out of an urn containing 3 black, 2 red and 3 green balls, and you will remember that, in this example, we were interested in computing quite a few quantities.

**EXAMPLE**

An urn contains 3 black, 2 red and 3 green balls and 2 balls are selected at random from it. If X is the number of black balls and Y is the number of red balls selected, then find

i) the joint probability function f(x, y)
ii) $P(X + Y \leq 1)$
iii) the marginal p.d. g(x) and h(y)
iv) the conditional p.d. f(x | 1)
v) $P(X = 0 \mid Y = 1)$
vi) Are x and Y independent?

As indicated in the last lecture, using the rule of combinations in conjunction with the classical definition of probability, the probability of the first cell came out to be 3/28. By similar calculations, we obtain all the remaining probabilities, and, as such, we obtain the following bivariate table:

Joint Probability Distribution

| X \ Y | 0 | 1 | 2 | $P(X = x_i)$ g(x) |
|---|---|---|---|---|
| 0 | 3/28 | 6/28 | 1/28 | 10/28 |
| 1 | 9/28 | 6/28 | 0 | 15/28 |
| 2 | 3/28 | 0 | 0 | 3/28 |
| $P(Y = y_j)$ h(y) | 15/28 | 12/28 | 1/28 | 1 |

This joint p.d. of the two r.v.'s (X, Y) can be represented by the formula

$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{28} = \begin{array}{l} x=0,1,2 \\ y = 0,1,2 \\ 0 \leq x+y \leq 2. \end{array}$$

ii) To compute $P(X + Y < 1)$, we see that $x + y < 1$ for the cells (0, 0), (0, 1) and (1, 0).

***(0, 0), (0, 1), (1, 0), (1, 1), (0, 2) and (2, 0).***

Therefore

$P(X + Y < 1)$
$= f(0, 0) + f(0, 1) + f(1, 0)$
$= 3/28 + 6/28 + 9/28$
$= 18/28 = 9/14$

iii) The marginal p.d.'s are:

| x | 0 | 1 | 2 |
|---|---|---|---|
| g(x) | 10/28 | 15/28 | 3/28 |

| y | 0 | 1 | 2 |
|---|---|---|---|
| h(x) | 15/28 | 12/28 | 1/28 |

iv) By definition, the conditional p.d. f(x | 1) is

f(x | 1) = P(X = x | Y = 1)

$$= \frac{P(X = x \text{ and } Y = 1)}{P(Y = 1)} = \frac{f(x,1)}{h(1)}$$

Now

$$h(1) = \sum_{x=0}^{2} f(x,1)$$

$$= \frac{6}{28} + \frac{6}{28} + 0$$

$$= \frac{12}{28} = \frac{3}{7}$$

Therefore

$$f(x|1) = \frac{f(x,1)}{h(1)}$$

That is, $= \frac{3}{7} f(x,1), \quad x = 0,1,2$

$$f(0|1) = \frac{7}{3} f(0,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(1|1) = \frac{7}{3} f(1,1) = \left(\frac{7}{3}\right)\left(\frac{6}{28}\right) = \frac{1}{2}$$

$$f(2|1) = \frac{7}{3} f(2,1) = \left(\frac{7}{3}\right)(0) = 0$$

Hence the conditional p.d. of X given that Y = 1, is

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x\|1) | 1/2 | 1/2 | 0 |

vi) We find that f(0, 1) = 6/28,

$$g(0) = \sum_{y=0}^{2} f(0,y)$$

$$= \frac{3}{28} + \frac{6}{28} + \frac{1}{28} = \frac{10}{28}$$

$$h(1) = \sum_{x=0}^{2} f(x,1)$$

$$= \frac{6}{28} + \frac{6}{28} + 0 = \frac{12}{28}$$

v) Finally,

P(X = 0 | Y = 1)
= f(0 | 1) = 1/2

$$\text{Now} \quad \frac{6}{28} \neq \frac{10}{28} \times \frac{12}{28},$$

i.e. $\quad f(0,1) \neq g(0)h(1),$

therefore X and Y are ***NOT***

Satistically independent.

**CONTINUOUS BIVARIATE DISTRIBUTIONS**

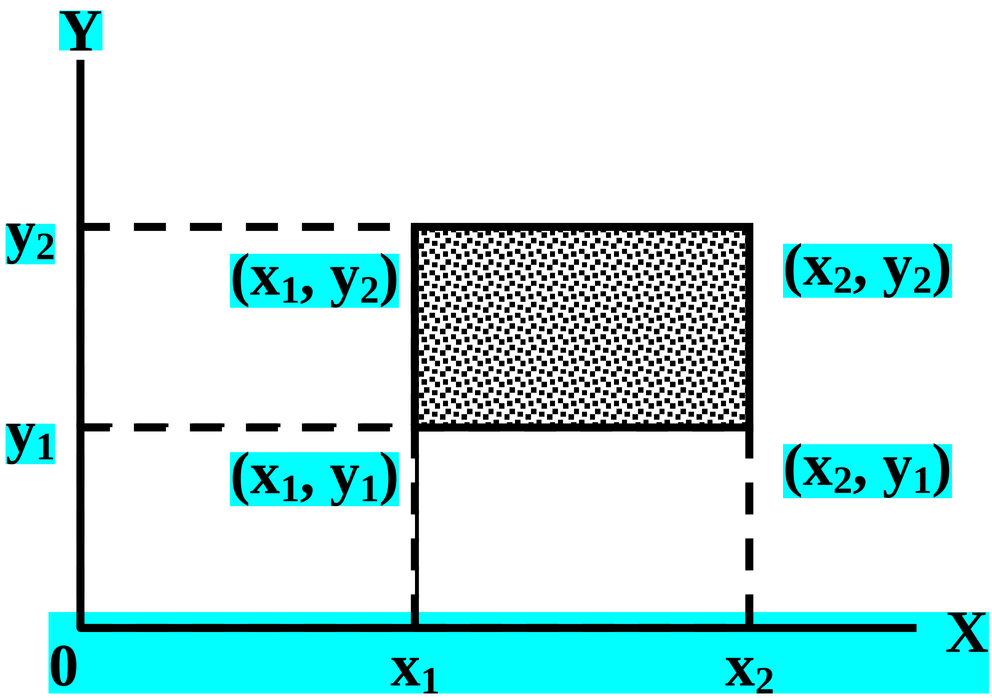
The bivariate probability density function of continuous r.v.'s X and Y is an integral function f(x,y) satisfying the following properties:

i) $f(x,y) \geq 0$ for all (x, y)

ii) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1,$ and

iii) $$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Let us try to understand the graphic picture of a bivariate continuous probability distribution: The region of the XY-plane depicted by the interval ($x1 < X < x2$; $y1 < Y < y2$) is shown graphically:




Just as in the case of a continuous univariate situation, the probability function f(x) gives us a curve under which we compute areas in order to find various probabilities, in the case of a continuous bivariate situation, the probability function f(x,y) gives a SURFACE and, when we compute the probability that our random variable X lies between x1 and x2 *AND*, simultaneously, the random variable Y lies between y1 and y2, we will be computing the *VOLUME* under the surface given by our probability function f(x, y) encompassed by this region. The MARGINAL p.d.f. of the continuous r.v. X is

and that of the r.v. Y is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$h(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

That is, the marginal p.d.f. of any of the variables is obtained by integrating out the *other* variable from the joint p.d.f. between the limits –∞ and +∞.The CONDITIONAL p.d.f. of the continuous r.v. X given that Y takes the value y, is defined to be

$$f(x\,|\,y)=\frac{f(x,y)}{h(y)},$$

where f(x,y) and h(y) are respectively the joint p.d.f. of X and Y, and the marginal p.d.f. of Y, and h(y) > 0. Similarly, the conditional p.d.f. of the continuous r.v. Y given that X = x, is

$$f(y\,|\,x)=\frac{f(x,y)}{g(x)},$$

provided that g(x) > 0

It is worth noting that the conditional p.d.f's satisfy all the requirements for the UNIVARIATE density function.

**FINALLY**

Two continuous r.v.'s X and Y are said to be Statistically Independent, if and only if their joint density f(x,y) can be factorized in the form f(x,y) = g(x)h(y) for all possible values of X and Y.

**EXAMPLE**

Given the following joint p.d.f

$$f(x,y)=\frac{1}{8}(6-x-y), 0\le x\le 2;\ 2\le y\le 4,$$
$$=0,\quad \text{elsewhere}$$

a) Verify that f(x,y) is a joint density function.

b) Calculate $P\left(X\le\frac{3}{2}, Y\le\frac{5}{2}\right)$,

c) Find the marginal p.d.f. g(x) and h(y).

d) Find the conditional p.d.f. f(x | y) and f(y | x).

**SOLUTION**

a) The joint density f(x,y) will be a p.d.f *if*

(i) f(x,y) > 0 and

(ii)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=1$$

Now f(x,y) is clearly greater than zero for all x and y in the given region, and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy=\frac{1}{8}\int_{0}^{2}\int_{2}^{4}(6-x-y)\,dy\,dx$$

$$=\frac{1}{8}\int_{0}^{2}\left[6y-xy-\frac{y^2}{2}\right]_2^4 dx$$

$$=\frac{1}{8}\int_{0}^{2}(6-2x)dx=\frac{1}{8}\left[6x-x^2\right]_0^2$$

$$=\frac{1}{8}[12-4]=1$$

Thus f(x,y) *has* the properties of a joint p.d.f.

b) To determine the probability of a value of the r.v. (X, Y) falling in the region X < 3/2, Y < 5/2,

We find
$$P\left(X \le \frac{3}{2}, Y \le \frac{5}{2}\right)$$

$$= \int_{x=0}^{\frac{3}{2}} \int_{y=2}^{\frac{5}{2}} \frac{1}{8}(6-x-y)\, dy\, dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left[6y - xy - \frac{y^2}{2}\right]_2^{\frac{5}{2}} dx$$

$$= \frac{1}{8}\int_0^{\frac{3}{2}} \left(\frac{15}{8} - \frac{x}{2}\right) dx = \frac{1}{64}\left[15x - 2x^2\right]_0^{\frac{3}{2}} = \frac{9}{32}$$

c) The marginal p.d.f. of X is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy, \qquad -\infty < x < \infty$$

$$= \frac{1}{8}\int_2^4 (6 - x - y)\, dy, \qquad 0 \le x \le 2$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4 \qquad 0 \le x \le 2$$

$$= \frac{1}{4}(3 - x), \qquad 0 \le x \le 2$$

$$= 0, \qquad x < 0 \ \underline{\text{OR}}\ x \ge 2$$

Similarly, the marginal p.d.f. of Y is

$$h(y) = \frac{1}{8}\int_0^2 (6 - x - y)dx, \ 2 \le y \le 4$$

$$= \frac{1}{4}(5 - y), \qquad 2 \le y \le 4$$

= 0, elsewhere.

d) The conditional p.d.f. of X given Y = y, is

$$f(x \mid y) = \frac{f(x,y)}{h(y)}, \text{ where } h(y) > 0$$

Hence

$$f(x|y) = \frac{\left(\frac{1}{8}\right)(6-x-y)}{\left(\frac{1}{4}\right)(5-y)} = \frac{6-x-y}{2(5-y)}, \ 0 \le x \le 2$$

and the conditional p.d.f. of Y given X = x, is

$$f(y|x)=\frac{f(x,y)}{g(x)}, \text{ where } g(x) > 0$$

Hence

$$f(y|x)=\frac{\left(\frac{1}{8}\right)(6-x-y)}{\left(\frac{1}{4}\right)(3-x)}=\frac{6-x-y}{2(3-x)},\quad 2\le y\le 4$$

Next, we consider *two* important properties of mathematical expectation which are valid in the case of *BIVARIATE* probability distributions:

**PROPERTY NO. 1**

The expected value of the sum of any two random variables is equal to the *sum* of their expected values, i.e.
E(X + Y) = E(X) + E(Y).
The result also holds for the *difference* of r.v.'s i.e.

E(X – Y) = E(X) – E(Y).

**PROPERTY NO. 2**

The expected value of the product of two *independent* r.v.'s is equal to the *product* of their expected values, i.e.
E(XY) = E(X) E(Y).
It should be noted that these properties are valid for *continuous* random variable's in which case the summations are replaced by *integrals.*

**EXAMPLE**

Let X and Y be two discrete r.v.'s with the following joint p.d

| y \ x | 2 | 4 |
|---|---|---|
| 1 | 0.10 | 0.15 |
| 3 | 0.20 | 0.30 |
| 5 | 0.10 | 0.15 |

Find E(X), E(Y), E(X + Y), and E(XY).

**SOLUTION**

To determine the expected values of X and Y, we first find the marginal p.d. g(x) and h(y) by adding over the columns and rows of the two-way table as below:

| y \ x | 2 | 4 | h(y) |
|---|---|---|---|
| 1 | 0.10 | 0.15 | 0.25 |
| 3 | 0.20 | 0.30 | 0.50 |
| 5 | 0.10 | 0.15 | 0.25 |
| g(x) | 0.40 | 0.60 | 1.00 |

Now $E(X) = \sum xj\ g(xj)$

$= 2 \times 0.40 + 4 \times 0.60$

$= 0.80 + 2.40 = 3.2$

$E(Y) = \sum yi\ h(yi)$

$= 1 \times 0.25 + 3 \times 0.50 + 5 \times 0.25$

$= 0.25 + 1.50 + 1.25$

$= 3.0$

Hence

$E(X) + E(Y) = 3.2 + 3.0 = 6.2$

In order to compute E(XY) directly, we apply the formula:

$$E(X+Y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$$

$$E(XY) = \sum_i \sum_j (x_i y_j) f(x_i, y_j)$$

## LECTURE NO. 27

- Properties of Expected Values in the case of Bivariate Probability Distributions (*Detailed* discussion)
- Covariance & Correlation
- Some Well-known Discrete Probability Distributions:
    - Discrete Uniform Distribution
    - An Introduction to the Binomial Distribution

**EXAMPLE**

Let X and Y be two discrete r.v.'s with the following joint p.d.

| x \ y | 1 | 3 | 5 |
|---|---|---|---|
| 2 | 0.10 | 0.20 | 0.10 |
| 4 | 0.15 | 0.30 | 0.15 |

**Find** E(X), E(Y), E(X + Y), and E(XY).

**SOLUTION**

To determine the expected values of X and Y, we first find the marginal p.d. g(x) and h(y) by adding over the columns and rows of the two-way table as below:

| x \ y | 1 | 3 | 5 | g(x) |
|---|---|---|---|---|
| 2 | 0.10 | 0.20 | 0.10 | 0.40 |
| 4 | 0.15 | 0.30 | 0.15 | 0.60 |
| h(y) | 0.25 | 0.50 | 0.25 | 1.00 |

Now $E(X) = \sum x_i\, g(x_i)$
$= 2 \times 0.40 + 4 \times 0.60$
$= 0.80 + 2.40 = 3.2$

$E(Y) = \sum y_j\, h(y_j)$
$= 1 \times 0.25 + 3 \times 0.50 + 5 \times 0.25$
$= 0.25 + 1.50 + 1.25$
$= 3.0$

Hence
$E(X) + E(Y) = 3.2 + 3.0 = 6.2$

$$E(X+Y) = \sum_i \sum_j \left(x_i + y_j\right) f\left(x_i, y_j\right)$$

$= (2 + 1)(0.10) + (2 + 3)(0.20) +$
$(2 + 5)(0.10) + (4 + 1)(0.15) +$
$(4 + 3)(0.30) + (4 + 5)(0.15)$
$= 0.30 + 1.00 + 0.70 + 0.75 +$
$2.10 + 1.35 = 6.20$
$= E(X) + E(Y)$

In order to compute E(XY) directly, we apply the formula:

$$E(XY) = \sum_i \sum_j \left(x_i y_j\right) f\left(x_i, y_j\right)$$

In this example,

$$E(XY) = \sum_i \sum_j \left(x_i y_j\right) f\left(x_i, y_j\right)$$

= (2 × 1) (0.10) + (2 × 3) (0.20) + (2 × 5) (0.10) + (4 × 1) (0.15) + (4 × 3) (0.30) + (4 × 5) (0.15)
= 9.6

Now
E(X) E(Y)
= 3.2 × 3.0
= 9.6
Hence E(XY) = E(X) E(Y) implying that X and Y are independent.
This was the discrete situation; let us now consider an example of the *continuous* situation:

**EXAMPLE**
Let X and Y be independent r.v.'s with joint p.d.f.

$$f(x, y) = \frac{x\left(1+3y^2\right)}{4}, \quad 0 < x < 2,\ 0 < y < 1$$
$$= 0, \quad \text{elsewhere.}$$

Find E(X), E(Y), E(X + Y) and E(XY). To determine E(X) and E(Y), we first find the marginal p.d.f. g(x) and h(y) as below:

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\,dy = \int_0^1 \frac{x\left(1+3y^2\right)}{4}\,dy$$

$$= \frac{1}{4}\left[xy + xy^3\right]_0^1 = \frac{x}{2}, \quad \text{for } 0 < x < 2.$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y)\,dx$$

$$= \int_0^2 \frac{x\left(1+3y^2\right)}{4}\,dx = \frac{1}{4}\left[\frac{x^2}{2} + 3xy^2\right]_0^2$$

$$= \frac{1}{2}\left(1+3y^2\right), \quad \text{for } 0 < y < 1.$$

Hence

$$E(X) = \int_{-\infty}^{\infty} x\,g(x)\,dx$$

$$= \int_0^2 x\left(\frac{x}{2}\right)dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}, \text{and}$$

$$E(Y) = \int_{-\infty}^{\infty} y\,h(y)\,dy = \frac{1}{2}\int_0^1 y\left(1+3y^2\right)dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + \frac{3y^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{3}{4}\right] = \frac{5}{8},$$

And

$$E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\,f(x,y)dx\,dy$$

$$\int_0^2\int_0^1(x+y)\,\frac{x\left(1+3y^2\right)}{4}dy\,dx$$

$$=\int_0^2\int_0^1\frac{x^2+3x^2y^2}{4}dx\,dy+\int_0^2\int_0^1\frac{xy+3xy^3}{4}dy\,dx$$

$$=\int_0^2\frac{1}{4}\left[x^2y+x^2y^3\right]_0^1dx+\int_0^2\frac{1}{4}\left[\frac{xy^2}{2}+\frac{3xy^4}{4}\right]_0^1dx$$

$$=\int_0^2\frac{1}{4}\left(2x^2\right)dx+\int_0^2\frac{1}{4}\left(\frac{x}{2}+\frac{3x}{4}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}\right]_0^2=\frac{1}{4}\left[\frac{x^2}{4}+\frac{3x^2}{8}\right]_0^2$$

$$=\frac{4}{3}+\frac{5}{8}=\frac{47}{24},\text{ and}$$

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xy\;f(x,y)dx\,dy$$

$$=\int_0^2\int_0^1(xy)\,\frac{x\left(1+3y^2\right)}{4}dy\,dx=\int_0^2\int_0^1\frac{x^2y+3x^2y^3}{4}dy\,dx$$

$$=\int_0^2\frac{1}{4}\left[\frac{x^2y^2}{2}+\frac{3x^2y^4}{4}\right]_0^1dx=\int_0^2\frac{1}{4}\left(\frac{5x^2}{4}\right)dx=\frac{1}{4}\left[\frac{5x^3}{12}\right]_0^2=\frac{5}{6}$$

It should be noted that

**i)** E(X) + E(Y) = 4/3 + 5/8
= 47/24 = E(X + Y), and

**ii)** E(X) E(Y) = (4/3) (5/8)
= 5/6 = E(XY).

Hence, the two properties of mathematical expectation valid in the case of bivariate probability distributions are verified.

**COVARIANCE OF TWO RANDOM VARIABLES**

The covariance of two r.v.'s X and Y is a numerical measure of the extent to which their values tend to increase or decrease *together*. It is denoted by σXY or Cov (X, Y), and is defined as the expected value of the product

[X – E(X)] [Y – E(Y)]. That is
Cov (X, Y) = E {[X – E(X)] [Y – E(Y)]}
And the short cut formula is:

Cov (X, Y) = E(XY) – E(X) E(Y).

If X and Y are independent, then
E(XY) = E(X) E(Y), and
Cov (X, Y) = E(XY) – E(X) E(Y)
= 0
It is very important to note that covariance is zero when the r.v.'s X and Y are independent but its converse is not generally true. The covariance of a r.v. with itself is obviously its variance.

## CORRELATION CO-EFFICIENT OF TWO RANDOM VARIABLES

Let X and Y be two r.v.'s with non-zero variances σ2X and σ2Y. Then the correlation coefficient which is a measure of linear relationship between X and Y, denoted by ρXY (the Greek letter rho) or Corr(X, Y), is defined as

$$\rho_{XY} = \frac{E[X - E(X)][Y - E(X)]}{\sigma_X \sigma_Y} = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}}$$

If X and Y are independent r.v.'s, then ρXY will be zero but zero correlation does not necessarily imply independence.

## EXAMPLE

From the following joint p.d. of X and Y, find Var(X), Var(Y), Cov(X,Y) and ρ.

| x \ y | 0 | 1 | 2 | 3 | g(x) |
|---|---|---|---|---|---|
| 0 | 0.05 | 0.05 | 0.10 | 0 | 0.20 |
| 1 | 0.05 | 0.10 | 0.25 | 0.10 | 0.50 |
| 2 | 0 | 0.15 | 0.10 | 0.05 | 0.30 |
| h(y) | 0.10 | 0.30 | 0.45 | 0.15 | 1.00 |

Now
E(X) = Σxig(xi)
= 0 × 0.20 + 1 × 0.50 +2 × 0.30
= 0 + 0.50 + 0.60 = 1.10
E(Y) = Σyjh(yj)
= 0 × 0.10 + 1 × 0.30 + 2 × 0.45 + 3 × 0.15
= 0 + 0.30 + 0.90 + 0.45 = 1.65
E(X2) = Σxi2g(xi)
= 0 × 0.20 + 1 × 0.50 + 4 × 0.30
= 1.70
E(Y2) = Σyj2h(yj)
= 0 × 0.10 + 1 × 0.30 + 4 × 0.45 + 9 × 0.15
= 3.45

Thus
Var(X) = E(X2) – [E(X)]2
= 1.70 – (1.10)2 = 0.49,
and

Var(Y) = E(Y2) – [E(Y)]2
= 3.45 – (1.65)2 = 0.7275

Again:
E(XY) $= \sum_i \sum_j (x_i y_j) f(x_i, y_j)$
= 1 × 0.10 + 2 × 0.15 + 2 × 0.25 + 4 × 0.10 + 3 × 0.10 + 6 × 0.05

= 0.10 + 0.30 + 0.50 + 0.40 + 0.30 + 0.30
= 1.90

$\therefore$ Cov(X,Y) = E(XY) – E(X) E(Y)

= 1.90 – 1.10 × 1.65 = 0.085, and

$$\rho = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var}(Y)}}$$

$$= \frac{0.085}{\sqrt{(0.49)(0.7275)}} = \frac{0.085}{0.595}$$

$$= 0.14$$

Hence, we can say that there is a weak positive linear correlation between the random variables X and Y.

**EXAMPLE**

If f(x, y)
= x2 + xy/3 , 0 < x < 1, 0 < y < 2
= 0, elsewhere,

**Find**

Var(X), Var(Y) and Corr(X,Y)

**SOLUTION**

The marginal p.d.f.'s are

$$g(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy = 2x^2 + \frac{3}{2}x,$$

$$0 \le x \le 1$$

and

$$h(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dx = \frac{1}{3} + \frac{y}{6},$$

Now

$$0 \le y \le 2$$

$$E(X) = \int_{-\infty}^{\infty} xg(x)\, dx$$

$$= \int_0^1 x\left(2x^2 + \frac{2x}{3}\right) dx = \frac{13}{18},$$

$$E(Y) = \int_{-\infty}^{\infty} yh(y)\, dy$$

Thus

$$= \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{10}{9}.$$

$$\mathrm{Var}(X) = E[X - E(X)]^2$$

$$= \int_{-\infty}^{\infty} (x + \mu_x)^2 g(x)\, dx$$

$$= \int_0^1 \left(x - \frac{13}{18}\right)^2 \left(2x^2 + \frac{2x}{3}\right) dx = \frac{73}{1620}$$

$$\text{Var}(Y) = E[Y - E(Y)]^2$$
$$= \int_{-\infty}^{\infty} (y - \mu_y)^2 h(y)\, dy$$
$$= \int_0^2 \left(y - \frac{10}{9}\right)^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{26}{81}, \text{ and}$$

Cov(X, Y)

$$= E\{[X - E(X)]\,[Y - E(Y)]\}$$
$$= \int_0^1 \int_0^2 \left(x - \frac{13}{18}\right)\left(y - \frac{10}{9}\right)\left(x^2 + \frac{xy}{3}\right) dy\, dx$$
$$= \int_0^1 \left(-\frac{2}{9}x^3 + \frac{25}{81}x^2 - \frac{26}{243}x\right) dx = \frac{-1}{162}.$$

Hence

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$$
$$= \frac{-1/162}{\sqrt{(73/1620)(26/81)}}$$
$$= -0.05$$

Hence we can say that there is a *VERY* weak negative linear correlation between X and Y. In other words, X and Y are almost uncorrelated. This brings us to the end of the discussion of the BASIC concepts of discrete and continuous *Univariate* and *bivariate* probable. We now begin the discussion of some probability distributions that are WELL-KNOWN, and are encountered in *real-life* situations.

**DISCRETE UNIFORM DISTRIBUTION**

**EXAMPLE**

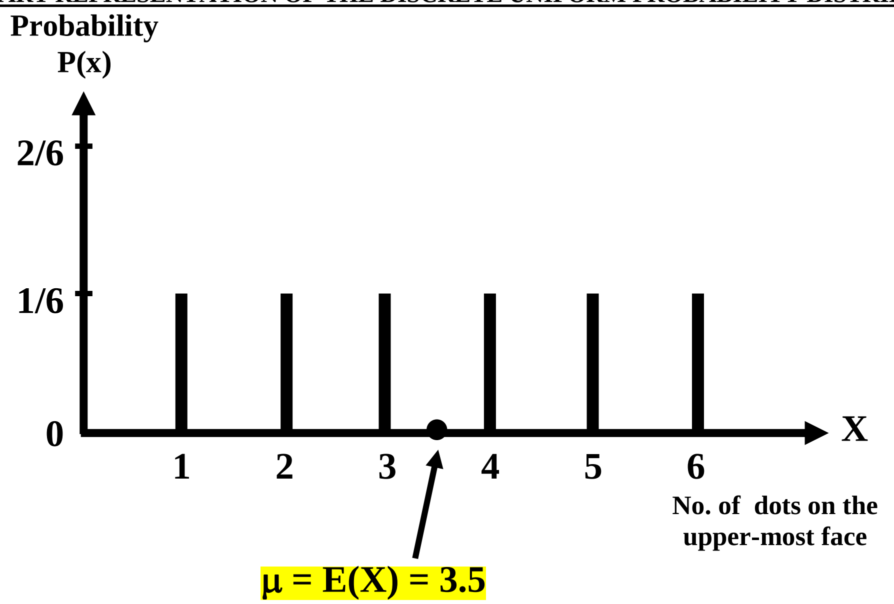
Suppose that we toss a fair die and let X denote the number of dots on the upper-most face. Since the die is *fair*, hence each of the X-values from 1 to 6 is equally likely to occur, and hence the probability distribution of the random variable X is as follows:

| X | P(x) |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |
| Total | 1 |

If we draw the line chart of this distribution, we obtain Line Chart Representation of the Discrete Uniform Probability Distribution


As all the vertical line segments are of equal height, hence this distribution is called a uniform distribution.
As this distribution is absolutely symmetrical, therefore the mean lies at the *exact centre* of the distribution i.e. the mean is equal to 3.5.

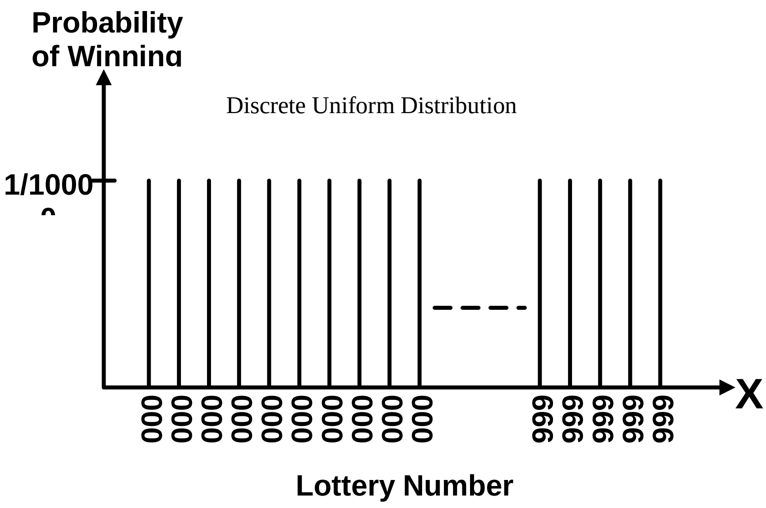
**LINE CHART REPRESENTATION OF THE DISCRETE UNIFORM PROBABILITY DISTRIBUTION**




What about the spread of this distribution? You are encouraged to compute the standard deviation as well as the coefficient of variation of this distribution on their own. Let us consider another interesting example.

**EXAMPLE**

The lottery conducted in various countries for purposes of money-making provides a good example of the discrete uniform distribution. Suppose that, in a particular lottery, as many as ten thousand lottery tickets are issued, and the numbering is 0000 to 9999. Since each of these numbers is *equally likely* to occur, hence we have the following situation:




**INTERPRETATION**

It reflects the fact that winning lottery numbers are selected by a random procedure which makes all numbers equally likely to be selected. The point to be kept in mind is that, whenever we have a situation where the various outcomes are equally likely, and of a form such that we have a random variable X with values 0, 1, 2, … or , as in the above example, 0000, 0001 …, 9999, we will be dealing with the discrete uniform distribution.

**BINOMIAL DISTRIBUTION**

The binomial distribution is a very important discrete probability distribution. It was discovered by James Bernoulli about the year 1700.We illustrate this distribution with the help of the following example:

**EXAMPLE**

Suppose that we toss a fair coin 5 times, and we are interested in determining the probability distribution of X, where X represents the number of heads that we obtain.
We note that in tossing a fair coin 5 times:

- every toss results in either a head or a tail,
- the probability of heads (denoted by p) is equal to ½ every time (in other words, the probability of heads remains *constant*),
- every throw is *independent* of every other throw, and
- the total number of tosses i.e. 5 is *fixed* in *advance*.

The above four points represents the *four basic* and vitally important *PROPERTIES* of a binomial experiment

**PROPERTIES OF A BINOMIAL EXPERIMENT**

- Every trial results in a success or a failure.
- The successive trials are independent.
- The probability of success, p, remains constant from trial to trial.
- The number of trials, n, is fixed in advanced.

Masters

# LECTURE NO. 28

- Binomial Distribution
- Fitting a Binomial Distribution to Real Data
- An Introduction to the Hyper geometric Distribution

The binomial distribution is a very important discrete probability distribution. We illustrate this distribution with the help of the following example:

**EXAMPLE**

Suppose that we toss a fair coin 5 times, and we are interested in determining the probability distribution of X, where X represents the number of heads that we obtain. We note that in tossing a fair coin 5 times:

- Every toss results in either a head or a tail,
- The probability of heads (denoted by p) is equal to ½ every time (in other words, the probability of heads remains *constant*),
- Every throw is *independent* of every other throw, and
- The total number of tosses i.e. 5 is *fixed* in *advance*.

The above four points represents the *four basic* and vitally important *PROPERTIES* of binomial experiment. Now, in 5 tosses of the coin, there can be 0, 1, 2, 3, 4 or 5 heads, and the no. of heads is thus a random variable which can take one of these six values. In order to compute the probabilities of these X-values, the formula is:

Binomial Distribution

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

Where
n = the total no. of trials
p = probability of success in each trial
q = probability of failure in each trial (i.e. q = 1 - p)
x = no. of successes in n trials.
x = 0, 1, 2, … n

The binomial distribution has two parameters, n and p. In this example, n = 5 since the coin was thrown 5 times, p = ½ since it is a fair coin, q = 1 – p = 1 – ½ = ½ Hence

$$P(X = x) = \binom{5}{x}\left(\tfrac{1}{2}\right)^x \left(\tfrac{1}{2}\right)^{5-x}$$

Putting x = 0

$$P(X = 0) = \binom{5}{0}\left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{5-0}$$

$$= \frac{5\,!}{0\,!\,5\,!}(1)\left(\tfrac{1}{2}\right)^5$$

$$= 1(1)\left(\tfrac{1}{2}\right)^5 = \frac{1}{32}$$

Putting x = 1

$$P(X = 1) = \binom{5}{1}\left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{5-1}$$

$$= \frac{5\,!}{1\,!\,4\,!}\left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^4$$

$$= \frac{(5)}{1}\left(\tfrac{1}{2}\right)$$

$$= 5\left(\tfrac{1}{2}\right)^5 = 5\left(\frac{1}{32}\right) = \frac{5}{32}$$

Similarly, we have:

$$P(X=2)=\binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{5-2}=\frac{10}{32}$$

$$P(X=3)=\binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3}=\frac{10}{32}$$

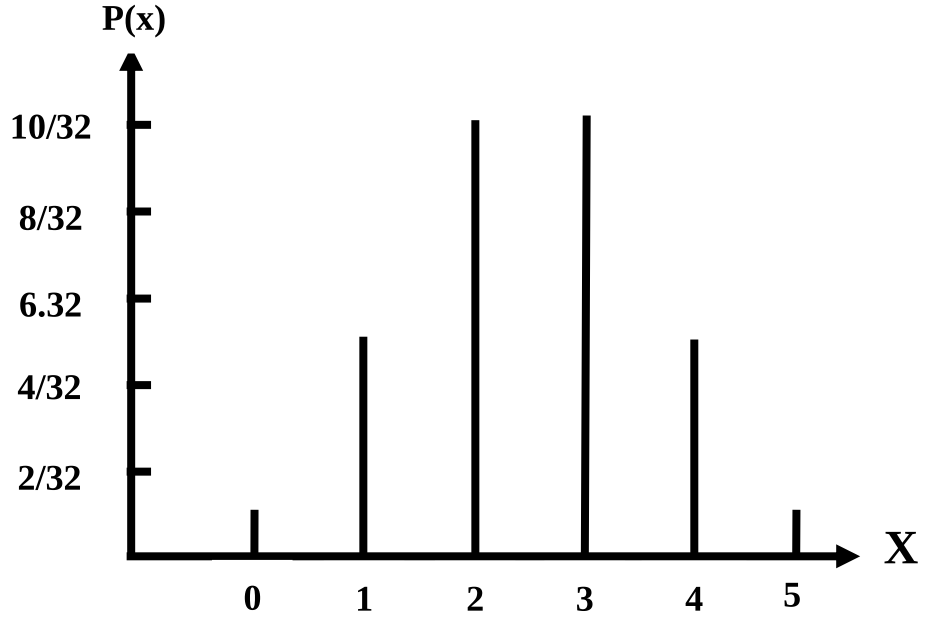
$$P(X=4)=\binom{5}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{5-4}=\frac{5}{32}$$

$$P(X=5)=\binom{5}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{5-5}=\frac{1}{32}$$

Hence, the binomial distribution for this particular example is as follows. Binomial Distribution in the case of tossing a fair coin five times:

| Number of Heads X | Probability P(x) |
|---|---|
| 0 | 1/32 |
| 1 | 5/32 |
| 2 | 10/32 |
| 3 | 10/32 |
| 4 | 5/32 |
| 5 | 1/32 |
| Total | 32/32 = 1 |

Graphical Representation of the above binomial distribution:




The next question is: What about the mean and the standard deviation of this distribution? We can calculate them just as before, using the formulas

Mean of X = E(X) = ∑XP(X)
Var(X) = ∑X2 P(X) – [∑XP(X)]2

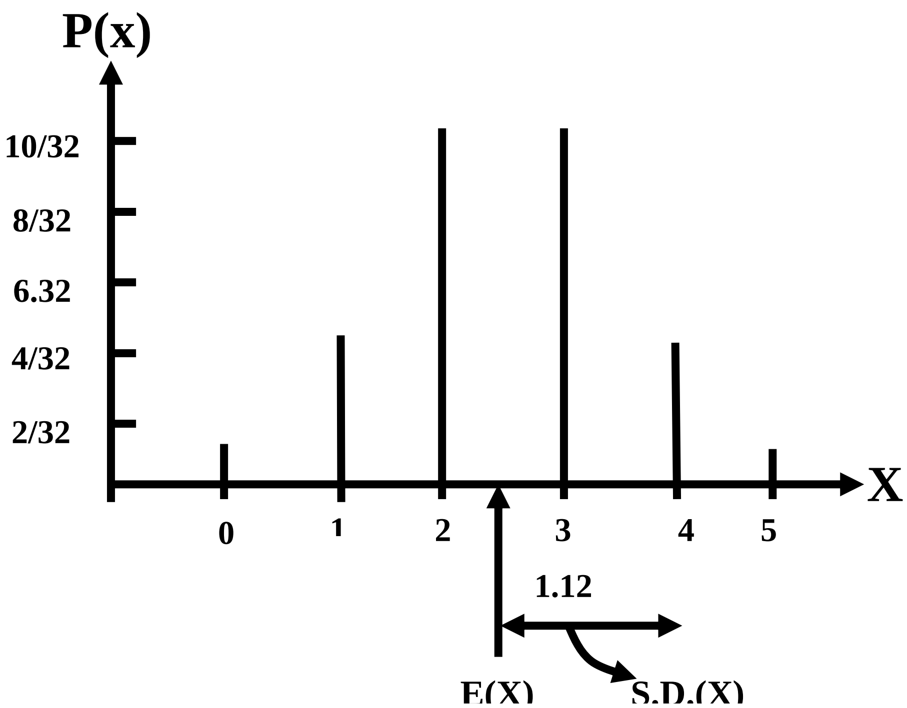
but it has been mathematically proved that for a binomial distribution given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

For a binomial distribution
E(X) = np
and Var(X) = npq
so that $S.D.(X) = \sqrt{npq}$
For the above example, n = 5, p = ½ and q = ½
Hence
Mean = E(X) = np = 5(½) = 2.5

$$\text{and S.D.(X)} = \sqrt{\text{npq}} = \sqrt{5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \sqrt{\left(\frac{5}{4}\right)} = \underline{1.12}$$

We would have got exactly the same answers if we had applied the LENGTHIER procedure.
E(X) = ∑XP(X) and Var X = ∑X2 P(X)-[∑XP(X)]2
Graphical Representation of the Mean and Standard Deviation of the Binomial Distribution (n=5, p=1/2)



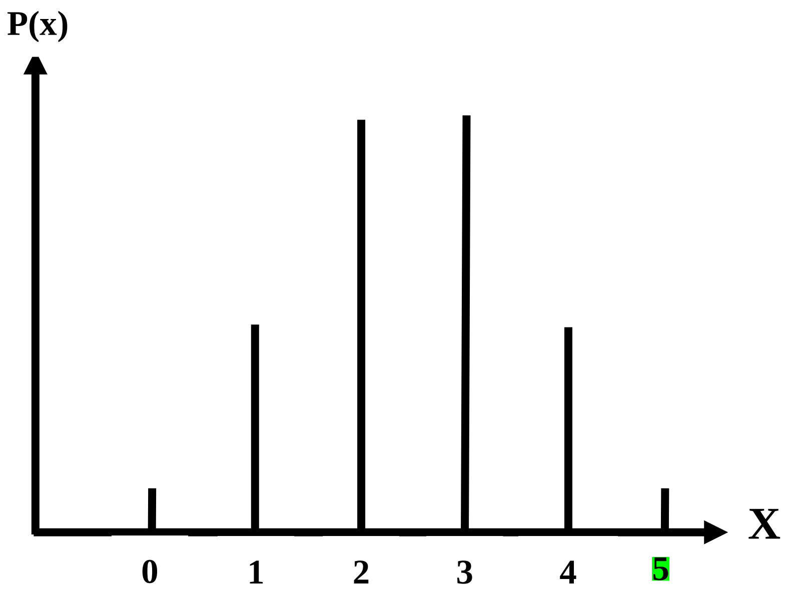

**WHAT DOES THIS MEAN?**

What this mean is that if 5 *fair* coins are tossed an *INFINITE* no. of times, sometimes we will get no head out of 5, sometimes/head… sometimes all 5 heads. But on the *AVERAGE* we should expect to get 2.5 heads in 5 tosses of the coin, or, a total of 25 heads in 50 tosses of the coin And 1.12 gives a measure of the possible *variability* in the various numbers of heads that can be obtained in 5 tosses. (As you know, in this problem, the number of heads can range from 0 to 5 had the coin been tossed 10 times, the no. of heads possible would vary from 0 to 10 and the standard deviation would probably have been different).

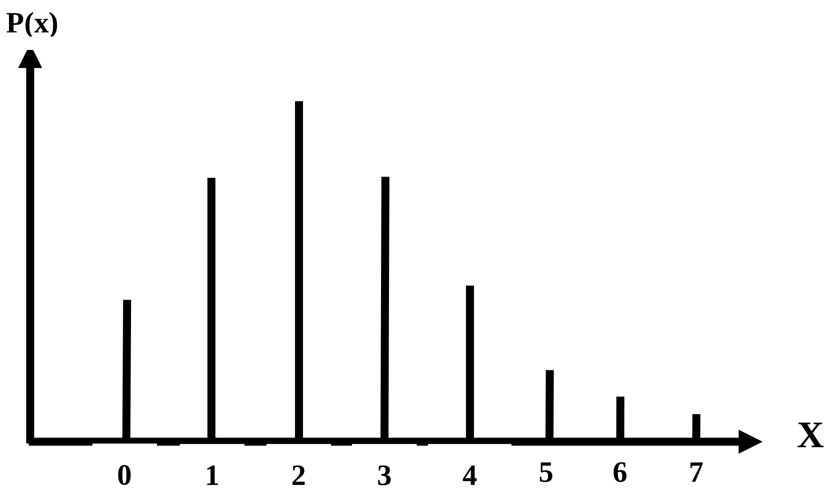
Coefficient of Variation:

$$\text{C.V.} = \frac{\sigma}{\mu} \times 100 = \frac{1.12}{2.5} \times 100 = 44.8\%$$

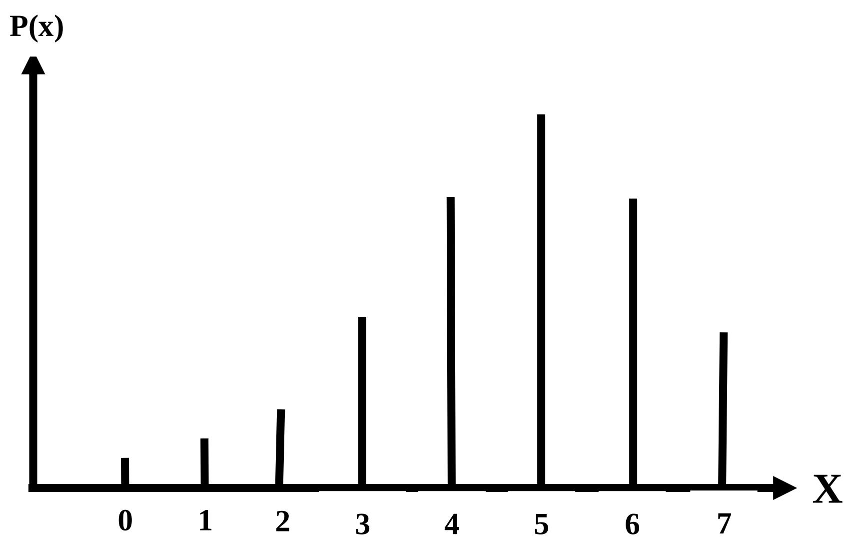
Note that the binomial distribution is not always symmetrical as in the above example. It will be symmetrical *only* when p = q = ½ (as in the above example).




**It is skewed to the right if p < q:**




**It is skewed to the left if p > q:**



But the degree of Skewness (or asymmetry) *decreases* as n increases. Next, we consider the *Fitting* of a Binomial Distribution to *Real* Data. We illustrate this concept with the help of the following example:

**EXAMPLE**

The following data has been obtained by tossing a *LOADED* die 5 times, and noting the number of times that we obtained a *six*. Fit a binomial distribution to this data.

| No. of Sixes | 0 | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|---|
| Frequency | 12 | 56 | 74 | 39 | 18 | 1 | 200 |

**SOLUTION**

To fit a binomial distribution, we need to find n and p.

Here n = 5, the largest x-value.

To find p, we use the relationship $\bar{x} = np$.

The rationale of this step is that, as indicated in the last lecture, the mean of a binomial *probability* distribution is equal to np, i.e.

$\mu = np$

But, here, we are not dealing with a *probability* distribution i.e. the entire *population* of all possible sets of throws of a loaded die --- we only have a *sample* of throws at our disposal.
As such, $\mu$ is not available to us, and all we can do is to replace it by its estimate $\bar{X}$.
Hence, our equation becomes $\bar{X} = np$.
Now, we have:

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{0 + 56 + 148 + 117 + 72 + 5}{200}$$

$$= \frac{398}{200} = 1.99$$

Using the relationship $\bar{x} = np$, we get 5p = 1.99 or p = 0.398. This value of p seems to indicate *clearly* that the die is not fair at all! (Had it been a fair die, the probability of getting a six would have been 1/6 i.e. 0.167; a value of p = 0.398 is *very* different from 0.167.) Letting the random variable X represent the number of sixes, the above calculations yield the fitted binomial distribution as

$$b(x;5,\ 0.398) = \binom{5}{x}(0.398)^x(0.602)^{5-x}$$

Hence the *probabilities* and *expected frequencies* are calculated as below:

| No. of Sixes (x) | Probability f(x) | | Expected frequency |
|---|---|---|---|
| 0 | $\binom{5}{0}q^5 = (0.602)^5$ | = 0.07907 | 15.8 |
| 1 | $\binom{5}{1}q^5 p = 5.(0.602)^4(0.398)$ | = 0.26136 | 52.5 |
| 2 | $\binom{5}{2}q^3 p^2 = 10.(0.602)^3(0.398)^2$ | = 0.34559 | 69.1 |
| 3 | $\binom{5}{3}q^2 p^3 = 10.(0.602)(0.398)^3$ | = 0.22847 | 45.7 |
| 4 | $\binom{5}{4}qp^4 = (0.602)(0.398)^4$ | = 0.07553 | 15.1 |
| 5 | $\binom{5}{5}p^5 = (0.398)^5$ | = 0.00998 | 2.0 |
| Total | | = 1.00000 | 200.0 |

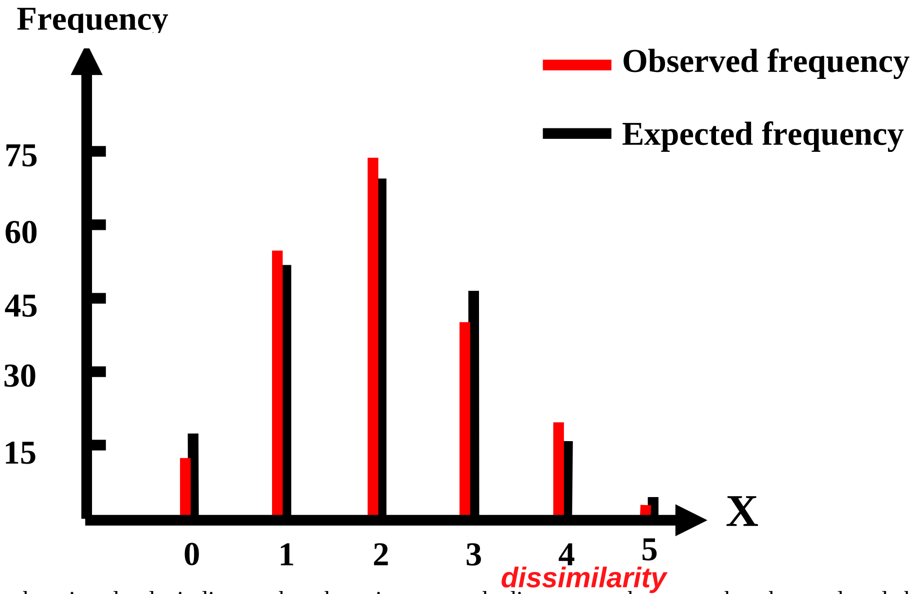
In the above table, the expected frequencies are obtained by multiplying each of the probabilities by 200.

In the entire above procedure, we are assuming that the given frequency distribution has the characteristics of the fitted theoretical binomial distribution, comparing the observed frequencies with the expected frequencies, we obtain:

| No. of Sixes x | Observed Frequency $f_0$ | Expected Frequency $f_e$ |
|---|---|---|
| 0 | 12 | 15.8 |
| 1 | 56 | 52.5 |
| 2 | 74 | 69.1 |
| 3 | 39 | 45.7 |
| 4 | 18 | 15.1 |
| 5 | 1 | 2.0 |
| Total | 200 | 200.0 |

The graphical representation of the observed frequencies as well as the expected frequencies is as follows:

**Graphical Representation of the Observed and Expected Frequencies:**




The above graph quite clearly indicates that there is not much discrepancy between the observed and the expected frequencies. Hence, we can say that it is a reasonably good fit.

There is a procedure known as the Chi-Square Test of Goodness of Fit which enables us to determine in a formal, mathematical manner whether or not the theoretical distribution fits the observed distribution reasonably well. This test comes under the realm of Inferential Statistics --- that area which we will deal with during the last 15 lectures of this course. Let us consider a *real-life* application of the binomial distribution:

**AN EXAMPLE FROM INDUSTRY**

Suppose that the past record indicates that the proportion of defective articles produced by this factory is 7%.And suppose that a law *NEWLY* instituted in this particular country states that there should not be more than 5% defective.

Suppose that the factory-owner makes the statement that his machinery has been *overhauled* so that the number of defectives has *DECREASED.*

In order to examine this claim, the relevant government department decides to send an inspector to examine a sample of 20 items.

What is the probability that the inspector will find 2 or more defective items in his sample (so that a fine will be imposed on the factory)?

**SOLUTION**

The first step is to identify the NATURE of the situation, If we study this problem closely, we realize that we are dealing with a binomial experiment because of the fact that all four properties of a binomial experiment are being fulfilled:

**PROPERTIES OF A BINOMIAL EXPERIMENT**

- Every item selected will either be defective (i.e. *success*) or not defective (i.e. *failure*)
- Every item drawn is independent of every other item
- The probability of obtaining a defective item i.e. 7% is the same (constant) for all items. (This probability figure is according to relative frequency definition of probability.
- The number of items drawn is fixed in advance i.e. 20 hence; we are in a position to apply the binomial formula

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

$$\mathrm{P}(\mathrm{X} = \mathrm{x}) = \binom{20}{\mathrm{x}} 0.07^{\mathrm{x}} 0.93^{20-\mathrm{x}}$$

Substituting n = 20 and p = 0.07, we obtain:

Now

$P(X > 2) = 1 - P(X < 2)$
$= 1 - [P(X = 0) + P(X = 1)]$

$$= 1 - \binom{20}{0} 0.07^0 0.93^{20-0} - \binom{20}{1} 0.07^1 0.93^{20-1}]$$

$$= 1 - 1 \times 1 \times 0.93^{20} - 20 \times 0.07 \times 0.93^{19}$$

$$= 1 - 0.234 - 0.353$$

$$= 0.413$$

$$= 41.3\%$$

Hence the probability is SUBSTANTIAL i.e. more than 40% that the inspector will find two or more defective articles among the 20 that he will inspect. In other words, there is CONSIDERABLE chance that the factory will be fined.
The point to be realized is that, generally speaking, whenever we are dealing with a 'success / failure' situation, we are dealing with what can be a binomial experiment. (For EXAMPLE, if we are interested in determining any of the following proportions, we are dealing with a BINOMIAL situation:

- Proportion of smokers in a city smoker → success, non-smokers → failure.
- Proportion of literates in a community → literacy rate, literate → success, illiterate → failure.
- Proportion of males in a city → *sex ratio*).

**HYPERGEOMETRIC PROBABILITY DISTRIBUTION**

There are many experiments in which the condition of independence is violated and the probability of success does not remain constant for all trials. Such experiments are called hyper geometric experiments.
In other words, a hyper geometric experiment has the following properties:

**PROPERTIES OF HYPERGEOMETRIC EXPERIMENT**

- The outcomes of each trial may be classified into one of two categories, success and failure.
- The probability of success changes on each trial.
- The successive trials are not independent.
- The experiment is repeated a fixed number of times.

The number of success, X in a hyper geometric experiment is called a hyper geometric random variable and its probability distribution is called the hyper geometric distribution. When the hyper geometric random variable X assumes a value x, the hyper geometric probability distribution is given by the formula

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}},$$

Where
N = number of units in the population,
n = number of units in the sample, and
k = number of successes in the population.
The hyper geometric probability distribution has three parameters N, n and k.
The hyper geometric probability distribution is appropriate when

- a random sample of size n is drawn *WITHOUT REPLACEMENT* from a *finite* population of N units;
- k of the units are of one kind (classified as success) and the remaining N – k of another kind (classified as failure).

# LECTURE NO. 29

- Hyper geometric Distribution
  (in some detail)
- Poisson Distribution
- Limiting Approximation to the Binomial
- Poisson Process
- Continuous Uniform Distribution

In the last lecture, we began the discussion of the HYPERGEOMETRIC PROBABILITY DISTRIBUTION. We now consider this distribution in some detail. As indicated in the last lecture, there are many experiments in which the condition of independence is violated and the probability of success does not remain constant for all trials. Such experiments are called hyper geometric experiments. In other words, a hyper geometric experiment has the following properties:

## PROPERTIES OF HYPERGEOMETRIC EXPERIMENT

- The outcomes of each trial may be classified into one of two categories, success and failure.
- The probability of success changes on each trial.
- The successive trials are not independent.
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where
N = number of units in the population,
n = number of units in the sample,
and
k = number of successes in the population.
The hyper geometric probability distribution has three parameters N, n and k. *mcq*

- The hyper geometric probability distribution is appropriate when
- a random sample of size n is drawn *WITHOUT REPLACEMENT* from a *finite* population of N units;
- k of the units are of one kind (classified as success) and the remaining N – k of another kind (classified as failure).

## EXAMPLE

The names of 5 men and 5 women are written on slips of paper and placed in a hat. Four names are drawn. What is the probability that 2 are men and 2 are women? Let us regard 'men' as success. Then X will denote the number of men. We have N = 5 + 5 = 10 names to be drawn from; Also, n = 4, (since we are drawing a sample of size 4 out of a 'population' of size 10) In addition, k = 5 (since there are 5 men in the population of 10). In this problem, the possible values of X are 0, 1, 2, 3, 4, i.e. n): The hyper geometric distribution is given by

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}},$$

Since N = 10, k = 5 and n = 4, hence, in this problem, the hyper geometric distribution is given by

$$P(X = x) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}$$

and the required probability,
i.e P(X = 2) is

$$P(X=2)=\frac{\binom{5}{2}\binom{5}{4-2}}{\binom{10}{4}}$$

$$=\frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$$

$$=\frac{10\times 10}{210}$$

$$=\frac{10}{21}$$

In other words, the probability is a little less than 50% that two of the four names drawn will be those of MEN. In the above example, just as we have computed the probability of X = 2, we could also have computed the probabilities of X = 0, X = 1, X = 3 and X = 4 (i.e. the probabilities of having zero, one, three *OR* four men among the four names drawn).The students are encouraged to compute these probabilities on their own, to check that the sum of these probabilities is 1, and to draw the line chart of this distribution.
Additionally, the students are encouraged to think about the *centre*, *spread* and *shape* of the distribution. Next, we consider some important *PROPERTIES* of the Hyper geometric Distribution:

**PROPERTIES OF THE HYPERGEOMETRIC DISTRIBUTION**

- The mean and the hyper geometric probability distribution are

$$\mu = n\frac{k}{N} \text{ and } \sigma^2 = n\frac{k}{N}\frac{N-k}{N}\frac{N-n}{N-1},$$

- If N becomes *indefinitely* large, the hyper geometric probability distribution tends to the *BINOMIAL* probability distribution.

The above property will be best understood with reference to the following important points:

- There are two ways of drawing a sample from a population, sampling with replacement, and sampling without replacement.
- Also, a sample can be drawn from either a finite population or an infinite population.

This leads to the following bivariate table: With reference to sampling, the various possible situations are:

| Population / Sampling | Finite | Infinite |
|---|---|---|
| With replacement | | |
| Without replacement | | |

The point to be understood is that, whenever we are sampling with replacement, the population remains undisturbed (because any element that is drawn at any one draw, is re-placed into the population before the next draw).Hence, we can say that the various trials (i.e. draws) are independent, and hence we can use the binomial formula. On the other hand, when we are sampling without replacement from a finite population, the constitution of the population changes at every draw (because any element that is drawn at any one draw is not re-placed into the population before the next draw). Hence, we cannot say that the various trials are independent, and hence the formula that is appropriate in this particular situation is the hyper geometric formula. *But*, if the population size is much larger than the sample size (so that we can regard it as an 'infinite' population), then we note that, although we are not re-placing any element that has been drawn back into the population, the population remains almost undisturbed. As such, we can assume that the various trials (i.e. draws) are independent, and, once again, we can apply the binomial formula.
In this regard, the generally accepted rule is that the *binomial* formula *can* be applied when we are drawing a sample from a finite population *without replacement* and the sample size n is not more than 5 percent of the population size N, or, to put it in another way, when n < 0.05 N.
*When n is greater than 5 percent of N*, the *hyper geometric* formula should be used.

Next, we discuss the Poisson Distribution.

**POISSON DISTRIBUTION**

The Poisson distribution is named after the French mathematician Sime'on Denis Poisson (1781-1840) who published its derivation in the year 1837.THE POISSON DISTRIBUTION ARISES IN THE FOLLOWING TWO SITUATIONS:

- It is a limiting approximation to the binomial distribution, when p, the probability of success is very small but n, the number of trials is so large that the product np = μ is of a moderate size;
- a distribution in its *own* right by considering a *POISSON PROCESS* where events occur *randomly* over a specified interval of *time* or *space* or *length*.

Such random events might be the number of typing errors per page in a book, the number of traffic accidents in a particular city in a 24-hour period, etc.

With regard to the *first* situation, if we assume that n goes to infinity and p approaches zero in such a way that μ = np remains constant, then the limiting form of the binomial probability distribution is

use in time or space

$$\lim_{\substack{n\to\infty \\ p\to 0}} b(x;n,p) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0,1,2,\ldots,\infty$$

number of success

where **e** = **2.71828.**

The Poisson distribution has only one parameter μ > 0.

The parameter μ may be interpreted as the mean of the distribution.

Although the theoretical requirement is that n should tend to infinity, and p should tend to zero, but in *PRACTICE*, generally, most statisticians use the Poisson approximation to the binomial when

p is 0.05 or less,

& n is 20 or more,

but *in fact*, the *LARGER* n is and the *SMALLER* p is, the *better* will be the approximation. We illustrate *this* particular application of the Poisson distribution with the help of the following example:

**EXAMPLE**

Two hundred passengers have made reservations for an airplane flight. If the probability that a passenger who has a reservation will not show up is 0.01, what is the probability that exactly three will not show up?

**SOLUTION**

Let us regard a "no show" as success. Then this is essentially a *binomial* experiment with n = 200 and p = 0.01. Since p is very small and n is considerably large, we shall apply the Poisson distribution, using

μ= np = (200) (0.01) = 2.

Therefore, if X represents the number of successes (not showing up), we have

$$P(X=3) = \frac{e^{-2}(2)^3}{3!}$$

$$= \frac{(0.1353)(8)}{3\times 2\times 1} = 0.1804$$

$$\left(\because e^{-2} = \frac{1}{(2.71828)^2} = 0.1353\right)$$

**POISSON PROCESS**

may be defined as a *physical* process governed at least in *part* by some *random* mechanism.

Stated differently a poisson process represents a situation where events occur *randomly* over a specified interval of *time* or *space* or *length*. Such random events might be the number of taxicab arrivals at an intersection per day; the number of traffic deaths per month in a city; the number of radioactive particles emitted in a given period; the number of flaws per unit length of some material; the number of typing errors per page in a book; etc.

The formula valid in the case of a Poisson process is:

$$P(X=x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},$$

where

$\lambda$ = average number of occurrences of the outcome of interest per unit of time,

t = number of time-units under consideration, and

x = number of occurrences of the outcome of interest in t units of time.

We illustrate this concept with the help of the following example:

**EXAMPLE**

Telephone calls are being placed through a certain exchange at random times on the average of four per minute. Assuming a Poisson Process, determine the probability that in a 15-second interval, there are 3 or more calls.

**SOLUTION**

**Step-1:** Identify the *unit* of time:
In this problem we take a minute as the unit of time.

**Step-2:** Identify$\lambda$, the *average* number of occurrences of the outcome of interest per unit of time,
In this problem we have the information that, on the average, 4 calls are received per minute, hence:

$\lambda = 4$

**Step-3:** Identify t, the number of time-units under consideration. In this problem, we are interested in a 15-second interval, and since 15 seconds are equal to 15/60 = ¼ minutes i.e. 1/4 units of time, therefore t = 1/4

**Step-4:** Compute $\lambda t$: In this problem,

$\lambda = 4$, &

t = 1/4,

Hence:

$\lambda t = 4 \times ¼ = 1$

**Step-5:** Apply the Poisson formula

$$P(X = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},$$

In this problem, since $\lambda t = 1$, therefore and since we are interested in 3 or more calls in a 15-second interval, therefore

P(X > 3) = 1 - P(X < 3)
= 1 - [P(X=0)+P(X=1)+P(X=2)]

$$= 1 - \sum_{x=0}^{2} \frac{e^{-}(1)^x}{x!}$$

$$= 1 - \sum_{x=0}^{2} \frac{(0.3679)(1)^x}{x!} \quad (\therefore\ e^{-1} = 0.3679)$$

$$= 1 - (0.91975) = 0.08025$$

Hence the probability is only 8% (i.e. a very low probability) that in a 15-second interval, the telephone exchange receives 3 or more calls.

**PROPERTIES OF THE POISSON DISTRIBUTION**

Some of the main properties of the Poisson distribution are given below:

- If the random variable X has a Poisson distribution with parameter $\mu$, then its mean and variance are given by $E(X) = \mu$ and $Var(X) = \mu$.
- (In other words, we can say that the mean of the Poisson distribution is *equal* to its variance.)
- The shape of the Poisson distribution is *positively skewed*. The distribution tends to be symmetrical as $\mu$ becomes *larger and larger*.

Comparing the Poisson distribution with the binomial, we note that, whereas the binomial distribution can be symmetric, positively skewed, *or* negatively skewed (depending on whether p = 1/2, p < 1/2, *or* p > 1/2), the Poisson distribution can never be negatively skewed.

## FITTING OF A POISSON DISTRIBUTION TO REAL DATA

Just as we discussed the fitting of the binomial distribution to real data in the last lecture, the Poisson distribution can *also* be fitted to real-life data. The procedure is very similar to the one described in the case of the fitting of the binomial distribution: The population mean μ is replaced by the sample mean $\bar{X}$, and the probabilities of the various values of X are computed using the Poisson formula. The *chi-square test of goodness of fit* enables us to determine whether or not it is a good fit i.e. whether or not the discrepancy between the expected frequencies and the observed frequencies is small. Next, we discuss some important mathematical points regarding Poisson distribution.

- **1)** The Poisson approximation to the binomial formula works well when
  $n > 20$ and $p < 0.05$.
- **2)** Suppose that the Poisson is used to approximate the binomial which, in *turn*, is being used to approximate the hyper geometric. Then the *Poisson* is being used to approximate the hyper geometric Putting the two approximation conditions *together*, the rule of *thumb* is that the Poisson distribution can be used to approximate the hyper geometric distribution when $n < 0.05N$, $n > 20$, and $p < 0.05$

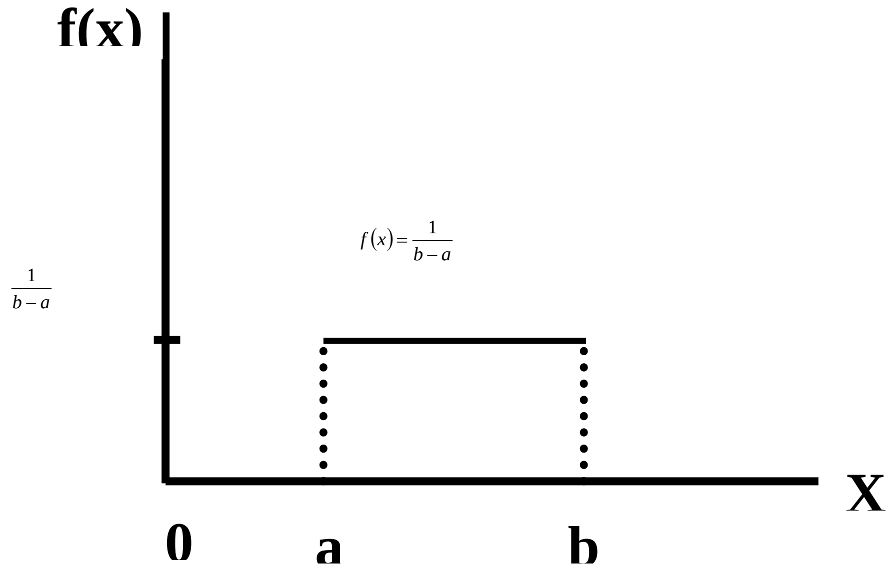
This brings to the *end* of the discussion of some of the most important and well-known Univariate discrete probability distributions. We now begin the discussion some of the well-known Univariate continuous probability distribution.
There are different types of continuous distributions e.g. the *uniform* distribution, the *normal* distribution, the *geometric* distribution, and the *exponential* distribution. Each one has its *own* shape and its *own* mathematical properties. In this course, we will discuss the uniform distribution and the normal distribution.
We begin with the continuous UNIFORM DISTRIBUTION (also known as the RECTANGULAR DISTRIBUTION).

## UNIFORM DISTRIBUTION

A random variable X is said to be uniformly distributed if its density function is defined as

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

The graph of this distribution is as follows




The above function is a *proper* probability density function because of the fact that:
**i)** Since a < b, therefore f(x) > 0

**ii)** $$\int_{-\infty}^{\infty} f(x)dx = \int_a^b \frac{1}{b-a}dx = \frac{1}{b-a}[x]_a^b = \frac{b-a}{b-a} = 1$$

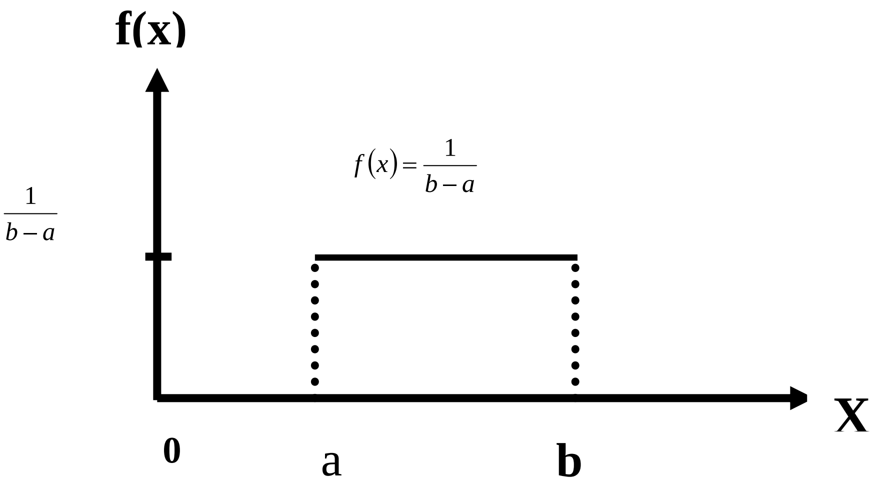
Since the shape of the distribution is like that of a rectangle, therefore the total area of this distribution can *also* be obtained from the simple formula:
Area of rectangle
= (Base) × (Height)

$$= (b-a) \times \left(\frac{1}{b-a}\right) = 1$$

Area under the Uniform Distribution

= Area of the rectangle
= (Base) × (Height)

$$=(b-a)\times\left(\frac{1}{b-a}\right)=1$$




The distribution derives its *name* from the fact that its density is constant or *uniform* over the interval [a, b] and is 0 elsewhere. It is also called the rectangular distribution because its total probability is confined to a rectangular region with base equal to (b – a) and height equal to 1/(b – a). The *parameters* of this distribution are a and b with

$$\mu=\frac{a+b}{2} \textbf{ and variance is } \sigma^2=\frac{(b-a)^2}{12}$$

**PROPERTIES OF THE UNIFORM DISTRIBUTION**

Let X has the uniform distribution over [a, b]. Then its mean is

The uniform probability distribution provides a *model* for continuous random variables that are *evenly distributed over a certain interval*. That is, a uniform random variable is one that is just *as* likely to assume a value in *one* interval as it is to assume a value in any *other* interval *of equal size*. There is *no clustering* of values around *any* value. Instead, there is an *even spread* over the *entire* region of possible values. As far as the *real-life application* of the uniform distribution is concerned, the point to be noted is that, for *continuous* random variables there is an *infinite* number of values in the sample space, but in *some* cases, *the values may appear to be equally likely*.

**EXAMPLE-1**

If a short exists in a 5 meter stretch of electrical wire, it may have an equal probability of being in any particular 1 centimeter segment along the line.

**EXAMPLE-2**

If a safety inspector plans to choose a time at random during the 4 afternoon work-hours to pay a surprise visit to a certain area of a plant, then each 1 minute time-interval in this 4 work-hour period will have an *equally likely* chance to being selected for the visit. Also, the uniform distribution arises in *the study of rounding off errors*, etc.

***EXAMPLE 1***

***we can calculate the probability of the short being in a specific segment using basic probability principles. Since the wire is 5 meters long and we have 1-centimeter segments, there are 500 segments in total (5 meters = 500 centimeters). If the probability is equal for each segment, the probability of the short being in any particular segment is 1/500.***

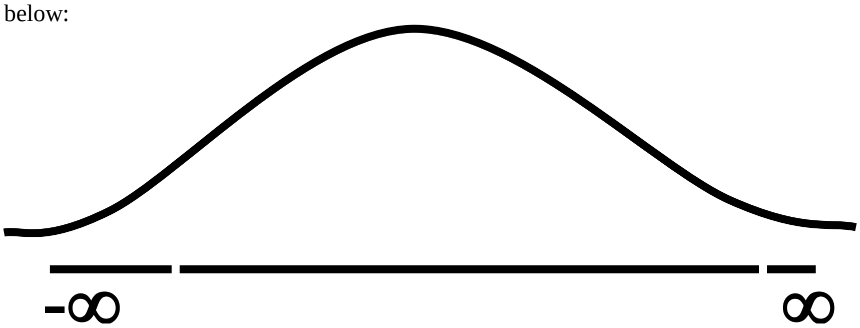
***EXAMPLE 1***

***In a 4-hour period, there are 240 minutes (4 hours * 60 minutes/hour = 240 minutes). Since each minute has an equal chance of being selected, the probability of the inspector arriving at any particular minute is 1/240.***

# LECTURE NO. 30

- Normal Distribution.
  - Mathematical Definition
  - Important Properties
- The Standard Normal Distribution
  - Direct Use of the Area Table
  - Inverse Use of the Area Table
- Normal Approximation to the Binomial Distribution

The normal distribution was discovered in 1733. The normal distribution has a bell-shaped curve of the type shown below:




Let us begin its detailed discussion by considering its formal MATHEMATICAL DEFINITION, and its main PROPERTIES.

## NORMAL DISTRIBUTION

A continuous random variable is said to be normally distributed with mean μ and standard deviation σ if its probability density function is given by

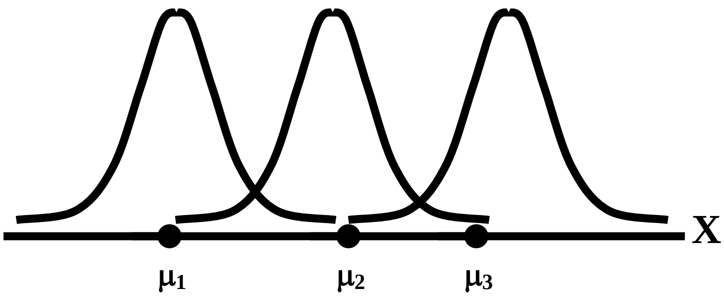
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, \quad -\infty < x < \infty \qquad \left(\begin{array}{l}\text{where} \\ \pi = 3.1416 \simeq 22/7, \\ e \simeq 2.71828\end{array}\right)$$

For any particular value of μ and any particular value of σ, giving different values to x and we obtain a set of ordered pairs (x, f(x)) that yield the bell-shaped curve given above. The formula of the normal distribution defines a *FAMILY* of distributions depending on the values of the two *parameters* μ and σ (as these are the two values that determine the shape of the distribution).

## PROPERTIES OF THE NORMAL DISTRIBUTION

### Property No. 1

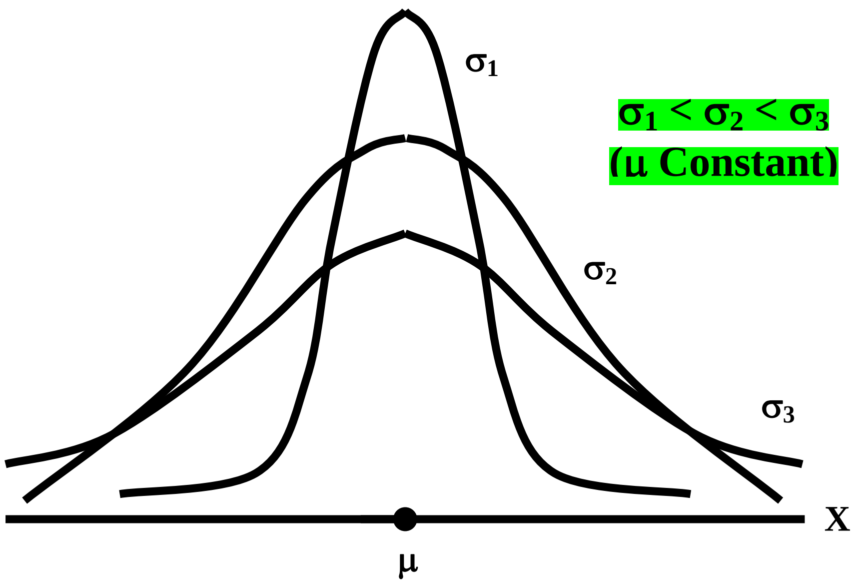
It can be mathematically proved that, for the normal distribution N(μ,σ2), μ represents the *mean*, and σ represents the *standard deviation* of the normal distribution. A change in the mean μ *shifts* the distribution to the left or to the right along the x-axis:




**$\mu_1 < \mu_2 < \mu_3$**

**(σ Constant)**

The different values of the standard deviationσ, (which is a measure of *dispersion*), determine the *flatness* or *peakedness* of the normal curve. In other words, achange in the standard deviation on σ *flattens* it or *compresses* it while leaving its centre in the same position:




**Property No. 2**

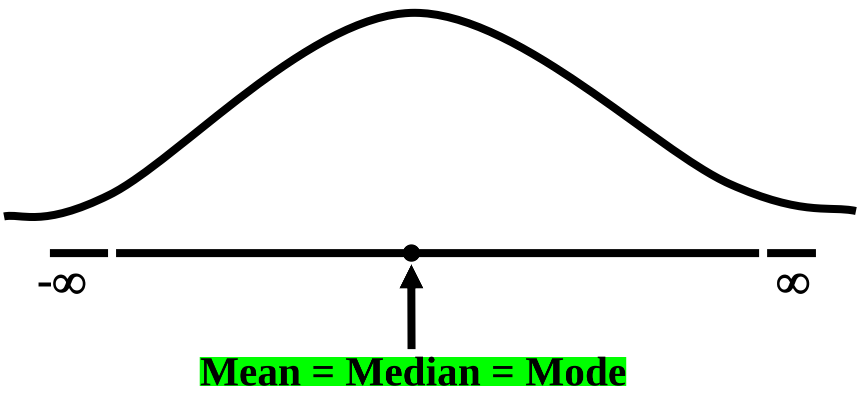
The normal curve is asymptotic to the x-axis as $x \to \pm\infty$.

**Property No. 3**

Because of the *symmetry* of the normal curve, 50% of the area is to the right of a vertical line erected at the mean, and 50% is to the left.(Since the total area under the normal curve from $-\infty$ to $+\infty$ is unity, therefore the area to the left of $\mu$ is 0.5 and the area to the right of $\mu$ is also 0.5.)

**Property No. 4**

The density function attains its maximum value at $x = \mu$ and falls off symmetrically on each side of $\mu$. This is why the mean, median and mode of the normal distribution are all equal to $\mu$.




**Property No. 5**

Since the normal distribution is absolutely symmetrical, hence $\mu_3$ , the third moment about the mean is zero.

**Property No. 6**

For the normal distribution, it can be mathematically proved that $\mu_4 = 3\sigma^4$

**Property No. 7**

The moment ratios of the normal distribution come out to be 0 and 3 respectively:

*Moment Ratios:*

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{0^2}{\left(\sigma^2\right)^3} = 0,$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{3\sigma^4}{\left(\sigma^2\right)^2} = 3$$

**NOTE**

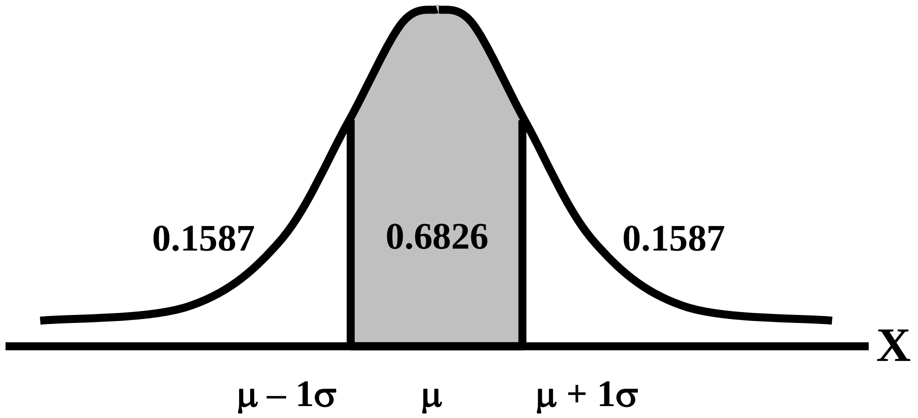
Because of the fact that, for the normal distribution, β2 comes out to be 3, this is why this value has been taken as a criterion for measuring the kurtosis of any distribution: The amount of peakedness of the *normal* curve has been taken as a *standard*, and we say that this particular distribution is masochistic. Any distribution for which β2 is greater than 3 is more peaked than the normal curve, and is called leptokurtic; Any distribution for which β2 is less than 3 is less peaked than the normal curve, and is called platykurtic.

**Property No. 8**

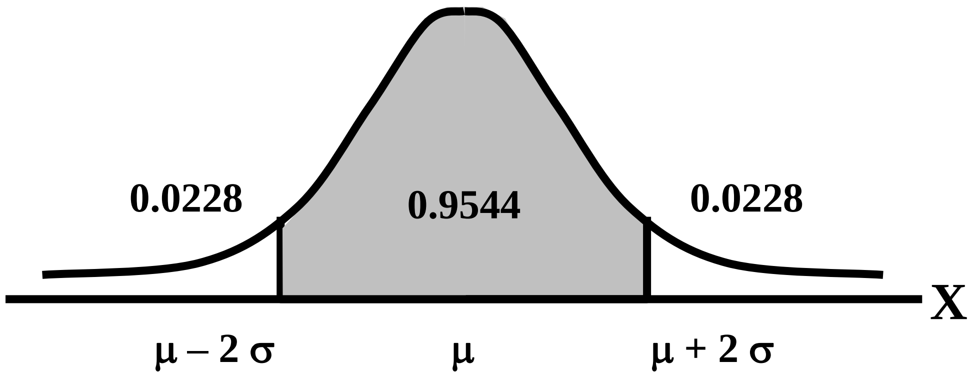
No matter what the values of μ and σ are, areas under the normal curve remain in certain *fixed* proportions within a *specified* number of standard deviations on either side of μ.

For the normal distribution:

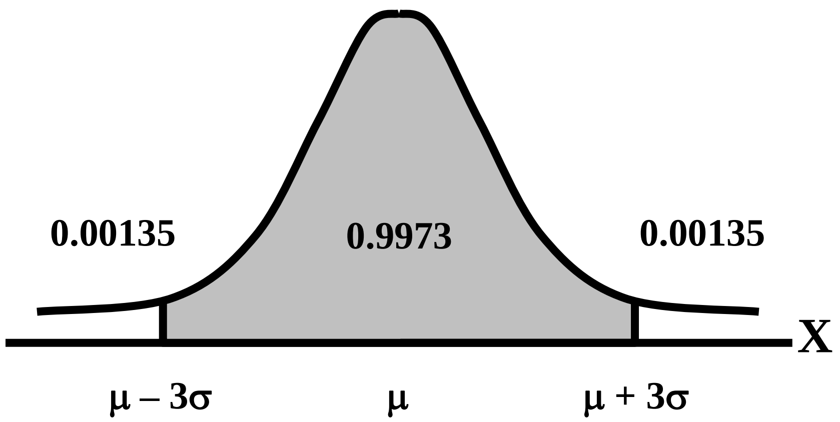
- The interval μ ± σ will always contain 68.26% of the total area.



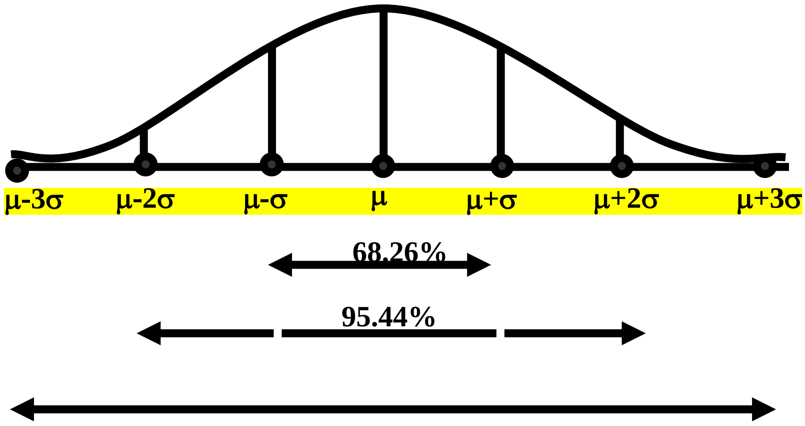

- The interval μ + 2σ will always contain 95.44% of the total area.




- The interval μ ± 3σ will always contain 99.73% of the total area.



Combining the above three results, we have:




At this point, the student are reminded of the Empirical Rule that was discussed during the first part of this course --- that on descriptive statistics. You will recall that, in the case of any approximately symmetric hump-shaped frequency distribution, approximately 68% of the data-values lie between $\bar{X}$ + S, approximately 95% between the $\bar{X}$ + 2S, and approximately 100% between $\bar{X}$ + 3S.You can now recognize the similarity between the empirical rule and the property given above. (In case a distribution is absolutely normal, the areas in the above-mentioned ranges are 68.26%, 95.44% and 99.73%; in case a distribution approximately normal, the areas in these ranges will be *approximately* equal to these percentages.)

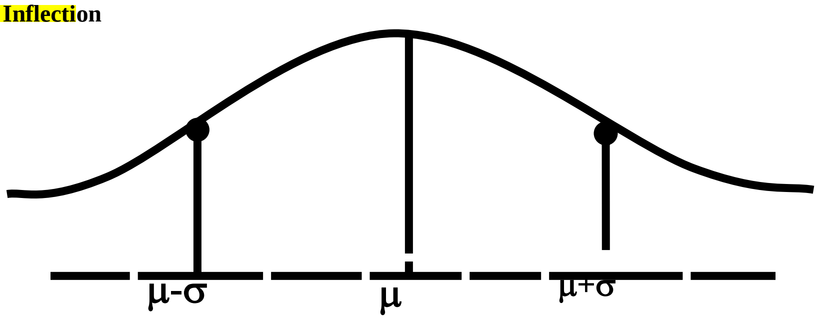
**Property No. 9**

The normal curve contains points of inflection (where the direction of concavity changes) which are equidistant from the mean. Their coordinates on the XY-plane are

$$\left(\mu-\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right) and \left(\mu+\sigma, \frac{1}{\sigma\sqrt{2\pi e}}\right)$$

respectively.

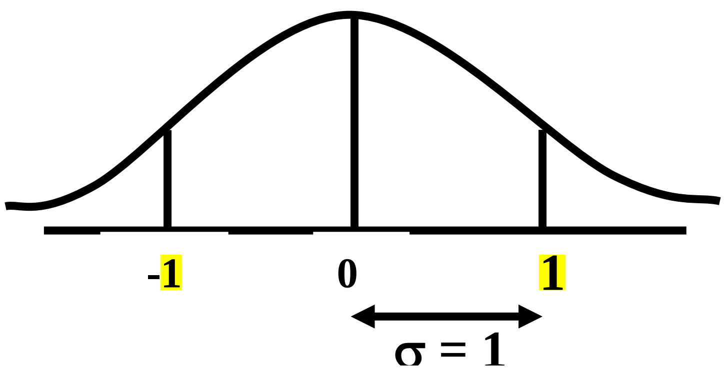
**Points of Inflection**




Next, we consider the concept of the Standard Normal Distribution:

**THE STANDARD NORMAL DISTRIBUTION**

A normal distribution whose mean is zero and whose standard deviation is 1 is known as the standard normal distribution.



This distribution has a very important role in *computing areas* under the normal curve. The *reason* is that the mathematical equation of the normal distribution is so complicated that it is not possible to find areas under the normal curve by ordinary integration. Areas under the normal curve have to be found by the more advanced method of *numerical integration*. The point to be noted is that areas under the normal curve have been computed for *that* particular normal distribution whose mean is zero and whose standard deviation is equal to 1, i.e. the standard normal distribution.

**Areas under the Standard Normal Curve**

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0159 | 0.0199 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| 0.1 | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0753 |
| 0.2 | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| 0.3 | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| 0.4 | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| 0.5 | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2083 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| 0.6 | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2380 | 0.2422 | 0.2454 | 0.2486 | 0.2518 | 0.2549 |
| 0.7 | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| 0.8 | 0.2881 | 0.2910 | 0.2939 | 0.2967 | 0.2995 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| 0.9 | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| 1.0 | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3531 | 0.3554 | 0.3577 | 0.3599 | 0.3621 |
| 1.1 | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3880 |
| 1.2 | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3990 | 0.3997 | 0.4015 |
| 1.3 | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| 1.4 | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| 1.5 | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4430 | 0.4441 |
| 1.6 | 0.4452 | 0.4463 | 0.4474 | 0.4485 | 0.4495 | 0.4505 | 0.4515 | 0.4525 | 0.4535 | 0.4545 |
| 1.7 | 0.4554 | 0.4564 | 0.4573 | 0.4582 | 0.4591 | 0.4599 | 0.4608 | 0.4616 | 0.4625 | 0.4633 |
| 1.8 | 0.4641 | 0.4649 | 0.4656 | 0.4664 | 0.4671 | 0.4678 | 0.4686 | 0.4693 | 0.4690 | 0.4706 |
| 1.9 | 0.4713 | 0.4719 | 0.4726 | 0.4732 | 0.4738 | 0.4744 | 0.4750 | 0.4758 | 0.4762 | 0.4767 |
| 2.0 | 0.4772 | 0.4778 | 0.4783 | 0.4788 | 0.4793 | 0.4798 | 0.4803 | 0.4808 | 0.4812 | 0.4817 |
| 2.1 | 0.4821 | 0.4826 | 0.4830 | 0.4834 | 0.4838 | 0.4842 | 0.4846 | 0.4850 | 0.4854 | 0.4857 |
| 2.2 | 0.4861 | 0.4865 | 0.4868 | 0.4871 | 0.4875 | 0.4878 | 0.4881 | 0.4884 | 0.4887 | 0.4890 |
| 2.3 | 0.4893 | 0.4896 | 0.4898 | 0.4901 | 0.4904 | 0.4906 | 0.4909 | 0.4911 | 0.4913 | 0.4916 |
| 2.4 | 0.4918 | 0.4920 | 0.4922 | 0.4925 | 0.4927 | 0.4929 | 0.4931 | 0.4932 | 0.4934 | 0.4936 |
| 2.5 | 0.4938 | 0.4940 | 0.4941 | 0.4943 | 0.4945 | 0.4946 | 0.4948 | 0.4949 | 0.4951 | 0.4952 |
| 2.6 | 0.4953 | 0.4955 | 0.4956 | 0.4957 | 0.4959 | 0.4960 | 0.4961 | 0.4962 | 0.4963 | 0.4964 |
| 2.7 | 0.4965 | 0.4966 | 0.4967 | 0.4968 | 0.4969 | 0.4970 | 0.4971 | 0.4972 | 0.4973 | 0.4974 |
| 2.8 | 0.4974 | 0.4975 | 0.4976 | 0.4977 | 0.4977 | 0.4978 | 0.4979 | 0.4980 | 0.4980 | 0.4981 |
| 2.9 | 0.4981 | 0.4982 | 0.4983 | 0.4983 | 0.4984 | 0.4984 | 0.4985 | 0.4985 | 0.4986 | 0.4986 |
| 3.0 | 0.49865 | 0.4987 | 0.4987 | 0.4988 | 0.4988 | 0.4989 | 0.4989 | 0.4989 | 0.4990 | 0.4990 |
| 3.1 | 0.49903 | 0.4991 | 0.4991 | 0.4991 | 0.4992 | 0.4992 | 0.4992 | 0.4992 | 0.4993 | 0.4993 |

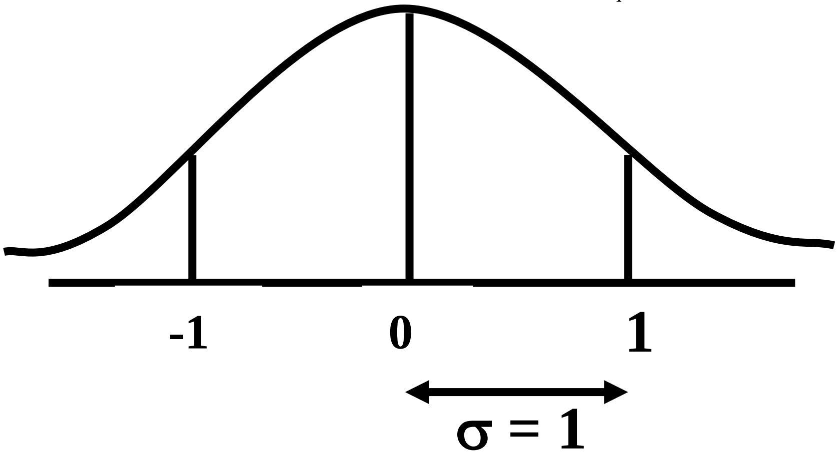
In any problem involving the normal distribution, the generally established procedure is that the normal distribution under consideration is *converted to* the standard normal distribution. This process is called *standardization*. The formula for converting N ($\mu$, $\sigma$) to N (0, 1) is:

**THE PROCESS OF STANDARDIZATION**

The standardization formula is:

$$Z = \frac{X - \mu}{\sigma}$$

If X is N ($\mu$, $\sigma$), then Z is N (0, 1). In other words, the standardization formula given above converts our normal distribution to the one whose mean is 0 and whose standard deviation is equal to 1.



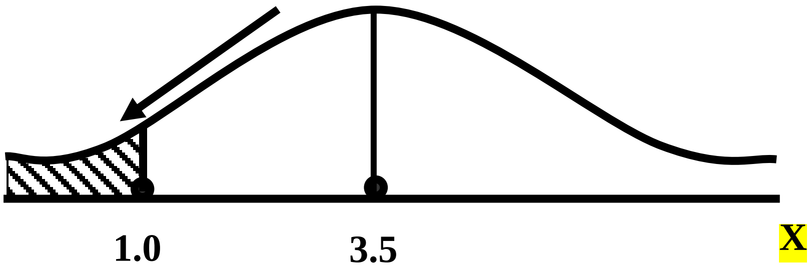

We illustrate this concept with the help of an interesting example:

**EXAMPLE**

The length of life for an automatic dishwasher is approximately normally distributed with a mean life of 3.5 years and a standard deviation of 1.0 years. If this type of dishwasher is guaranteed for 12 months, what fraction of the sales will require replacement?

**SOLUTION**

Since 12 months equal one year, hence we need to compute the fraction or *proportion* of dishwashers that will cease to function before a time-span of one year. In other words, we need to find the *probability* that a dishwasher fails before one year.



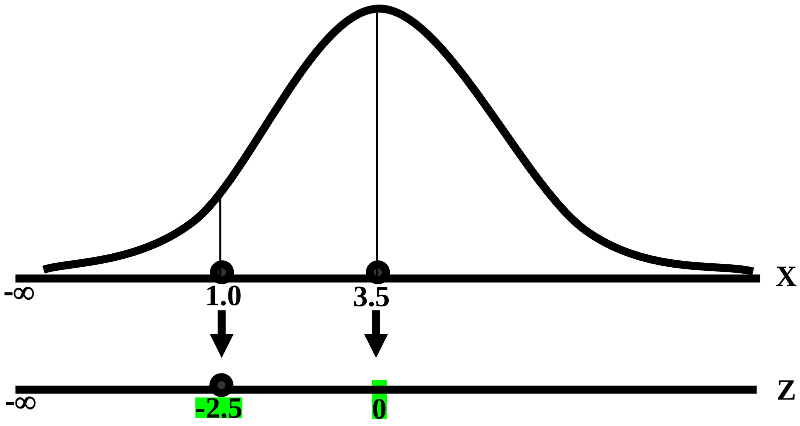

In order to find this area we nee to standardize normal distribution i.e. to convert N(3.5, 1) to N(0, 1):

The method is

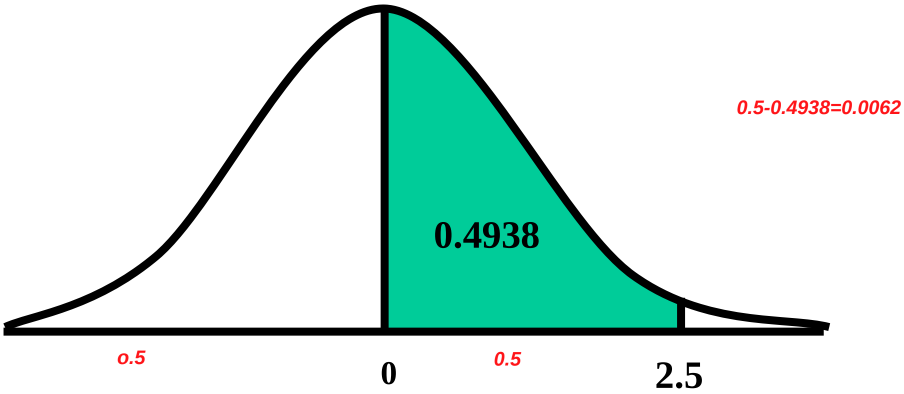
$$Z = \frac{X - \mu}{\sigma} = \frac{X - 3.5}{1.0}$$

The X-value representing the warranty period is 1.0 so

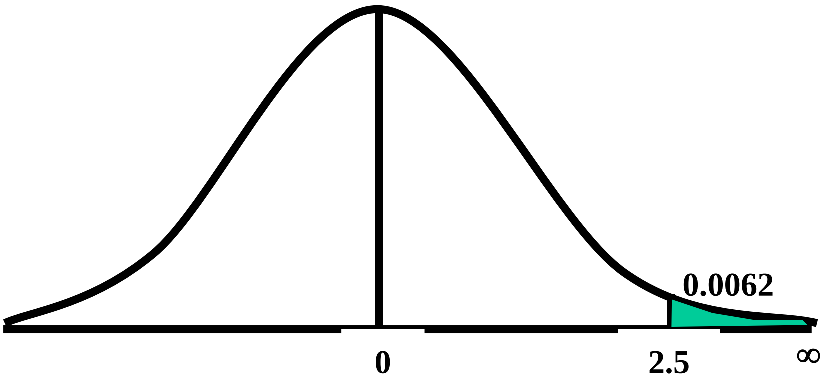
$$Z = \frac{1.0 - 3.5}{1.0} = \frac{-2.5}{1} = -2.5$$



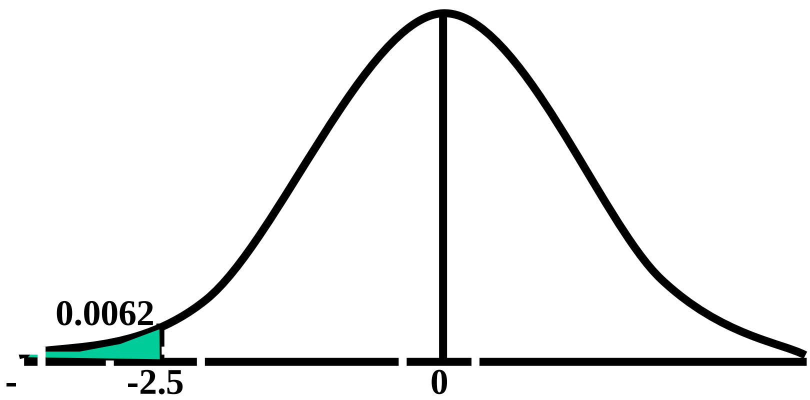

Now we need to find the area under the normal curve from z= -∞ to Z = -2.5. Looking at the area table of the standard normal distribution, we find that Area from 0 to 2.5 = 0.4938




**Hence:** The area from X = 2.5 to ∞ is 0.0062




But, this means that the area from -∞ to -2.5 is *also* 0.0062, as shown in the following figure:



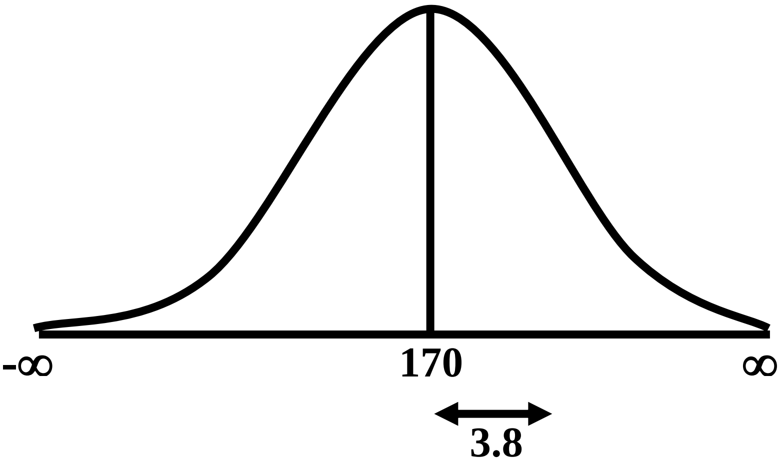
This means that the probability of a dishwasher lasting less than a year is 0.0062 i.e. 0.62% --- even less than 1%.Hence, the owner of the factory should be quite happy with the decision of placing a twelve-month guarantee on the dishwasher! Next, we discuss the Inverse use of the Table of Areas under the Normal Curve. In the above example, we were required to find a certain area against a given x-value. In some situations, we are confronted with just the opposite --- we are given certain areas, and we are required to find the corresponding x-values. We illustrate this point with the help of the following example:

**EXAMPLE**

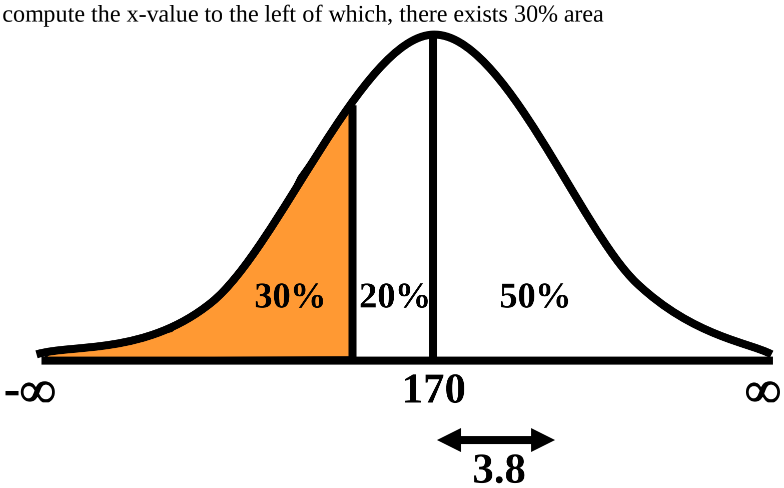
The heights of applicants to the police force in a certain country are normally distributed with mean 170 cm and standard deviation 3.8 cm. If 1000 persons apply for being inducted into the police force, and it has been decided that not more than 70% of these applicants will be accepted, (and the shortest 30% of the applicant are to be rejected), what is the minimum acceptable height for the police force?

**SOLUTION:**

We have:



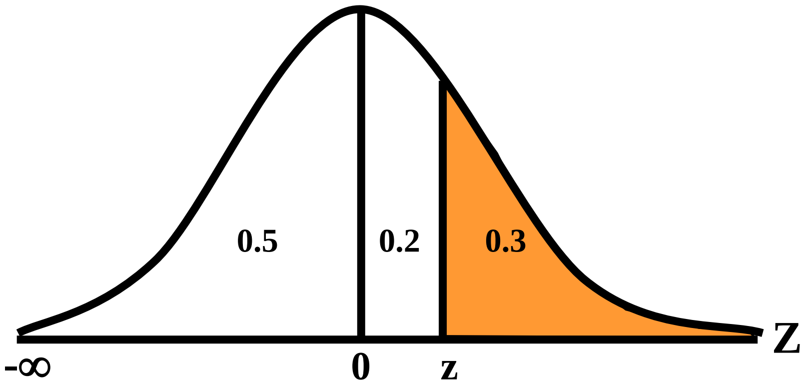

We need to compute the x-value to the left of which, there exists 30% area




The standardization formula can be re-written as

$$Z = \frac{X - \mu}{\sigma}$$

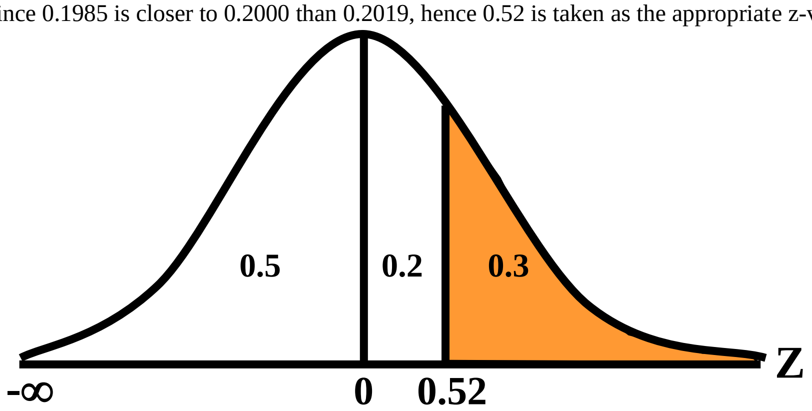
The Z value to the left of which there exists 30% area is obtained as follows.




By studying the figures inside the body of the area table of the standard normal distribution, we find that:

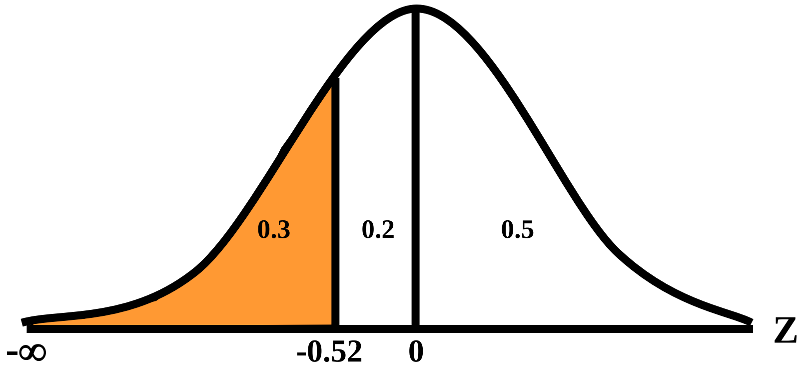
- The area between z = 0 and z = 0.52 is 0.1985, and
- The area between z = 0 and z = 2.53 is 0.2019

Since 0.1985 is closer to 0.2000 than 0.2019, hence 0.52 is taken as the appropriate z-value.




But, we are interested not in the upper 30% but the lower 30% of the applicants.

Hence, we have:




Since the normal distribution is absolutely symmetrical, hence the z-value to the left of which there exists 30% area (on the left-hand-side of the mean) will be at exactly the same distance from the mean as the z-value to the right of which there exists 30% area (on the right-hand-side of the mean).
Substituting z = -0.52 in the standardization formula, we obtain:

X = 170 + 3.8 Z
= 170 + 3.8 (-0.52)
= 170 - 1.976
= 168.024 168 cm

Hence, the minimum acceptable height for the police force is 168 cm. Just as binomial, Poisson and other discrete distributions can be *fitted* to *real-life* data; similarly, the normal distribution can also be *FITTED* to real data.
This can be done by equating $\mu$ to $\bar{X}$, the mean computed from the observed frequency distribution (based on sample data), and $\sigma$ to S, the standard deviation of the observed frequency distribution. Of course, this should be done only if

we are reasonably sure that the shape of the observed frequency distribution is quite similar to that of the normal distribution. (As indicated in the case of the fitting of the binomial distribution to real data), in order to *decide* whether or not our fitted normal distribution is a *reasonably good fit*, the *proper* statistical procedure is the Chi-square Test of Goodness of Fit.

## NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

The probability for a binomial random variable X to take the value x is

$$f(x) = \binom{n}{x} p^x q^{n-x},$$

$$\text{for } 0 \le x \le n \text{ and } q + p = 1.$$

The above formula becomes cumbersome to apply if n is LARGE. In such a situation, *as long as neither p nor q is close to zero*, we can compute the required probabilities by applying the normal approximation to the binomial distribution. The binomial distribution can be quite closely approximated by the normal distribution *when n is sufficiently large and neither p nor q is close to zero.* As a rule of *thumb,* the normal distribution provides a reasonable approximation to the binomial distribution *if both np and nq are equal to or greater than* 5, i.e.

np > 5 and nq > 5

**EXAMPLE:**

Suppose that a past record indicate that, in a particular province of an under-developed country, the death rate from Malaria is 20%. Find the probability that in a particular village of that particular province, the number of deaths is between 70 and 80 (inclusive) out of a total of 500 patients of Malaria.

**SOLUTION:**

Regarding 'death from Malaria' as success, we have

n = 500

and p = 0.20.

It is obvious that it is very cumbersome to apply the binomial formula in order to compute P(70 < X < 80).
In this problem,

np = 500(0.2) = 100 > > > 5, and nq = 500(0.8) = 400 > > > 5,

therefore we can *happily* apply the normal approximation to the binomial distribution. In order to apply the normal approximation to the binomial, we need to keep in mind the following two points:
**1)** The first point is: The mean and variance of the binomial distribution valid in our problem will be regarded as the mean and variance of the normal distribution that will be used to approximate the binomial distribution.
In this problem, we have:

$$\mu = np = 500 \times 0.20 = 100$$

and

$$\sigma^2 = npq = 500 \times 0.20 \times 0.80 = 80$$

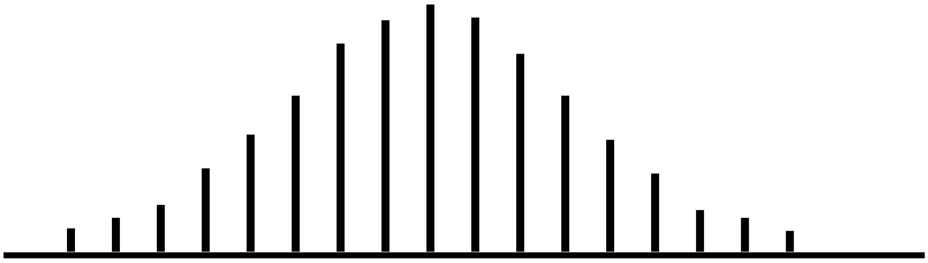
Hence

$$\sigma = \sqrt{npq} = \sqrt{80} = 8.94$$

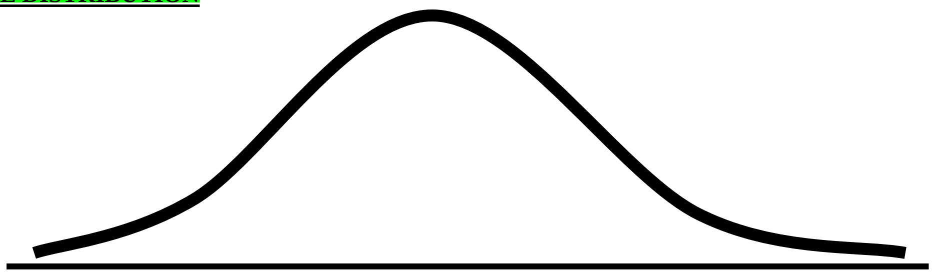
**2)** The second important point is:

We need to apply a correction that is known as the Continuity Correction. The rationale for this correction is as follows:
The binomial distribution is essentially a discrete distribution whereas the normal distribution is a continuous distribution i.e.:

**BINOMIAL DISTRIBUTION**

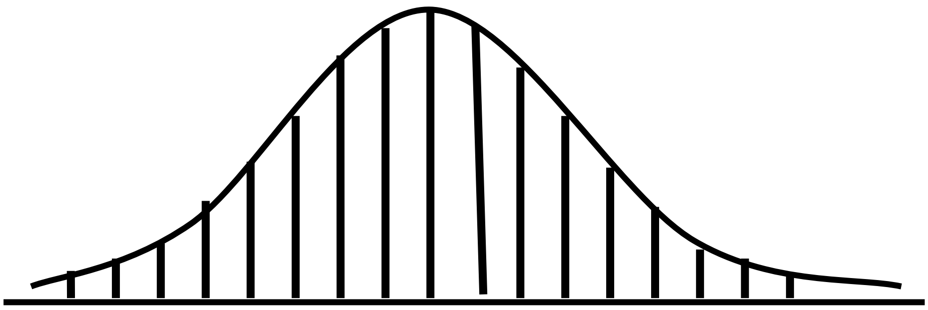


**NORMAL DISTRIBUTION**



In applying the normal approximation to the binomial, we have the following situation:

**THE NORMAL DISTRIBUTION *SUPERIMPOSED* ON THE BINOMIAL DISTRIBUTION**



But, the question arises: "How can a set of distinct vertical lines be replaced by a continuous curve?"
In order to overcome this problem, what we do is to replace every integral value x of our binomial random variable by an interval x - 0.5 to x + 0.5. By doing so, we will have the following situation. The x-value 70 is replaced by the interval 69.5 - 70.5, The x-value 71 is replaced by the interval 70.5 - 71.The x-value 72is replaced by the interval 71.5 - 72.5 …………..The x-value 80 is replaced by the interval 79.5 - 80.5

**Hence:**

Applying the continuity correction,

$P(70 < X < 80)$

is replaced by

$P(69.5 < X < 80.5)$.

Accordingly, the area that we need to compute is the area under the normal curve between the values 69.5 and 80.5.

It is left to the *students* to compute this area, and thus determine the required probability. (This computation involves a few steps.)

By doing so, the students will find that, in that particular village of that province, the probability that the number of deaths from Malaria in a sample of 500 lies between 70 and 80 (inclusive) is 0.0145 i.e. 1½%.

This brings us to the end of the second part of this course i.e. *Probability Theory*.

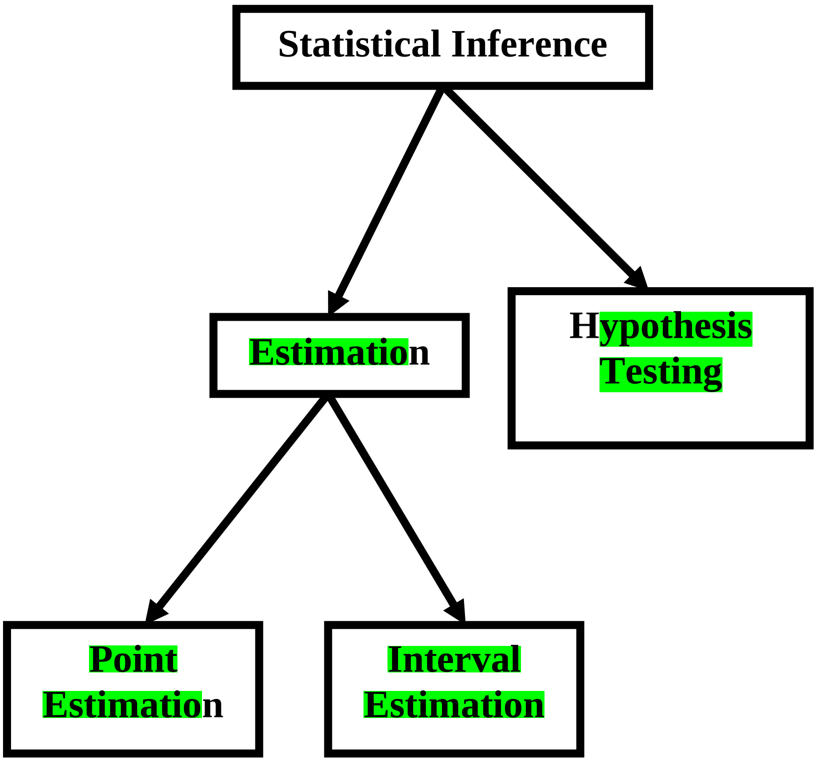
In the next lecture, we will begin the third and last portion of this course i.e. *Inferential Statistics* --- that area of Statistics which enables us to draw conclusions about various phenomena on the basis of data collected on sample basis.

# LECTURE NO. 31

- Sampling Distribution of $\overline{X}$
- Mean and Standard Deviation of the Sampling Distribution of $\overline{X}$
- Central Limit Theorem

## INFERENTIAL STATISTICS

That branch of Statistics which enables us to draw conclusions or inferences about various phenomena on the basis of real data collected on sample basis. In this regard, the first point to be noted is that statistical inference can be divided into two main branches --- estimation, and hypothesis-testing. Estimation itself can be further divided into two branches --- point estimation, and interval estimation




The second important point is that the concept of sampling distributions forms the basis for both estimation and hypothesis-testing,

## SAMPLING DISTRIBUTION

The probability distribution of any statistic (such as the mean, the standard deviation, the proportion of successes in a sample, etc.) is known as its sampling distribution. In this regard, the first point to be noted is that there are two ways of sampling --- sampling with replacement, and sampling without replacement. In case of a finite population containing N elements, the total number of possible samples of size n that can be drawn from this population with replacement is Nn.
In case of a finite population containing N elements, the total number of possible samples of size n that can be drawn from this population without replacement.
We illustrate the concept of the sampling distribution of $\binom{N}{n}$. with the help of the following example:

## EXAMPLE

Let us examine the case of an annual Ministry of Transport test to which all cars, irrespective of age, have to be submitted. The test looks for faulty breaks, steering, lights and suspension, and it is discovered after the first year that approximately the same numbers of cars have 0, 1, 2, 3, or 4 faults.
The above situation is equivalent to the following:
Let X denotes the number of faults in a car. Then X can take the values 0, 1, 2, 3, and 4, the probability of each of these X values is 1/5. Hence, we have the following probability distribution:

| No. of Faulty Items (X) | Probability f(x) |
|---|---|
| 0 | 1/5 |
| 1 | 1/5 |
| 2 | 1/5 |
| 3 | 1/5 |
| 4 | 1/5 |
| Total | 1 |

In order to compute the mean and standard deviation of this probability distribution, we carry out the following computations,

**MEAN AND VARIANCE OF THE POPULATION DISTRIBUTION**

$$\mu = E(X) = \sum xf(x) = 2$$

$$\sigma^2 = Var(X) = E(X)^2 - [E(X)]^2$$

$$= \sum x^2 f(x) - [\sum x\, f(x)]^2$$

$$= 6 - 2^2 = 6 - 4 = 2$$

Practically speaking, only a sample of the cars will be tested at any one occasion, and, as such, we are interested in considering the results that would be obtained if a sample of vehicles is tested. Let us consider the situation when only two cars are tested after being selected at the roadside by a mobile testing station. The following table gives all the possible situations:

NO. OF FAULTY ITEMS

| Second Car / First Car | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | (0,0) | (0,1) | (0,2) | (0,3) | (0,4) |
| 1 | (1,0) | (1,1) | (1,2) | (1,3) | (1,4) |
| 2 | (2,0) | (2,1) | (2,2) | (2,3) | (2,4) |
| 3 | (3,0) | (3,1) | (3,2) | (3,3) | (3,4) |
| 4 | (4,0) | (4,1) | (4,2) | (4,3) | (4,4) |

The above situation is equivalent to drawing all possible samples of size 2 from this probability distribution (i.e. the population) WITH REPLACEMENT. From the above list of 25 samples, we can work out all the possible sample means. These are indicated in the following table:

*x+y/2* SAMPLE MEANS

| Second Car / First Car | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
| 1 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
| 2. | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
| 3 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 |
| 4 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |

It is immediately evident that some of these possible samples mean occur several times. In view of this, it would seem reasonable and sensible to construct a frequency distribution from the sample means. This is given in the following table:

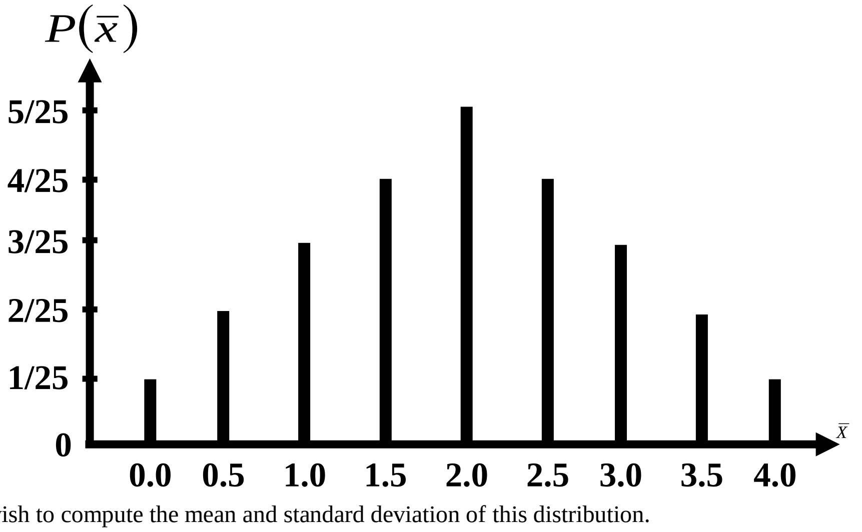
| Sample Mean | No. of Samples |
|---|---|
| $\bar{x}$ | f |
| 0.0 | 1 |
| 0.5 | 2 |
| 1.0 | 3 |
| 1.5 | 4 |
| 2.0 | 5 |
| 2.5 | 4 |
| 3.0 | 3 |
| 3.5 | 2 |
| 4.0 | 1 |
| Total | 25 |

If we divide each of the above frequencies by the total frequency 25, we obtain the probabilities of the various values of $\bar{X}$.(This is so because every one of the 25 possible situations is equally likely to occur, and hence the probabilities of the various possible values of $\bar{X}$ can be computed using the classical definition of probability i.e. m/n --- number of favorable outcomes divided by total number of possible outcomes). Hence, we obtain the following probability distribution:

| Sample Mean | No. of Samples | Probability |
|---|---|---|
| $\bar{x}$ | f | $P(\bar{X} = \bar{x})$ |
| 0.0 | 1 | 1/25 |
| 0.5 | 2 | 2/25 |
| 1.0 | 3 | 3/25 |
| 1.5 | 4 | 4/25 |
| 2.0 | 5 | 5/25 |
| 2.5 | 4 | 4/25 |
| 3.0 | 3 | 3/25 |
| 3.5 | 2 | 2/25 |
| 4.0 | 1 | 1/25 |
| Total | 25 | 25/25=1 |

The above is referred to as the SAMPLING DISTRIBUTION of the mean. The visual picture of the sampling distribution is as follows:

Sampling Distribution of $\bar{X}$ for n = 2




Next, we wish to compute the mean and standard deviation of this distribution.

As we are already aware, for the probability distribution of a random variable X, the mean is given by

**μ = E(X) = ∑x f(x) and the variance is given byσ2 = Var(X) = E(X2) - [E(X)]2**

The point to be noted is that, in case of the sampling distribution of $\bar{X}$, our random variable is not X but $\bar{X}$.

Hence, the mean and variance of our sampling distribution are given by

**MEAN AND VARIANCE OF THE SAMPLING DISTRIBUTION OF $\bar{X}$**

$$\mu_{\bar{x}} = E(\bar{X}) = \sum \bar{x} f(\bar{x})$$

$$\sigma^2_{\bar{x}} = Var(\bar{X}) = E(\bar{X})^2 - [E(\bar{X})]^2$$

$$= \sum \bar{x}^2 f(\bar{x}) - [\sum \bar{x} f(\bar{x})]^2$$

The square root of the variance is the standard deviation, and the standard deviation of a sampling distribution is termed as its standard error. In order to find the mean and standard error of the sampling distribution of $\bar{X}$ in this example, we carry out the following computations:

In order to find the mean and standard error of the sampling distribution of $\bar{X}$ in this example, we carry out the following computations:

| Sample Mean | **Probability** | | |
|---|---|---|---|
| $\bar{x}$ | $f(\bar{x})=P(\bar{X}=\bar{x})$ | $\bar{x}\, f(\bar{x})$ | $(\bar{x})^2 f(\bar{x})$ |
| 0.0 | 1/25 | 0 | 0 |
| 0.5 | 2/25 | 1/25 | 1/50 |
| 1.0 | 3/25 | 3/25 | 6/50 |
| 1.5 | 4/25 | 6/25 | 18/50 |
| 2.0 | 5/25 | 10/25 | 40/50 |
| 2.5 | 4/25 | 10/25 | 50/50 |
| 3.0 | 3/25 | 9/25 | 54/50 |
| 3.5 | 2/25 | 7/25 | 49/50 |
| 4.0 | 1/25 | 4/25 | 32/50 |
| Total | 25/25=1 | 50/25=2 | 250/50=5 |

Hence, in this example, we have:

$$\mu_{\bar{x}} = E(\bar{X}) = \sum \bar{x} f(\bar{x})$$

$$= 50/25 = 2$$

And

$$\sigma^2_{\bar{x}} = Var(\bar{X}) = E(\bar{X})^2 - [E(\bar{X})]^2$$

$$= \sum \bar{x}^2 f(\bar{x}) - [\sum \bar{x} f(\bar{x})]^2$$

$$= 5 - 2^2 = 5 - 4 = 1$$

$$\sigma_{\bar{x}} = \sqrt{\sigma^2_{\bar{x}}} = \sqrt{1} = 1$$

These computations lead to the following two very important properties of the sampling distribution of $\bar{X}$

**Property No.1**

In the case of sampling with replacement as well as in the case of sampling without replacement, we have:

$$\mu_{\bar{x}} = \mu$$

In this example:

$$\mu = 2$$

*and*

$$\mu_{\bar{x}} = 2$$

*Hence*

$$\mu_{\bar{x}} = \mu$$

**Property No.2**

In case of sampling with replacement:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

In this example:

$$\sigma = \sqrt{2}$$

$$\therefore \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2}}{\sqrt{2}} = 1$$

$$and \ \sigma_{\bar{x}} = 1$$

$$Hence \ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$$

***NOTE:***

In case of sampling without replacement from a finite population:

imp

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$$

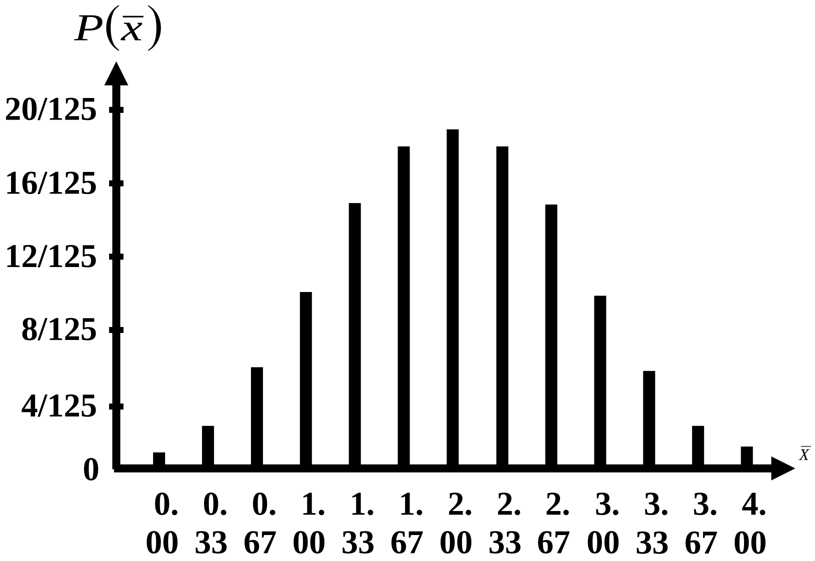
The factor $\sqrt{\frac{N-n}{N-1}}$

is known as the finite population correction (fpc). The point to be noted is that, if the sample size n is much smaller than the population size N, then is approximately equal to 1, and, as such, the fpc is not required. Hence, in sampling from a finite population, we apply the fpc only if the sample size is greater than 5% of the population size. Next, we consider the shape of the sampling distribution of $\bar{X}$. As indicated by the line chart, the above sampling distribution is absolutely symmetric and triangular. But let us consider what will happen to the shape of the sampling distribution with if the sample size is increased. If in the car tests instead of taking samples of 2 we had taken all possible samples of size 3, our sampling distribution would contain 53 = 125 sample means, and it would be in the following form:

SAMPLING DISTRIBUTION
FOR SAMPLES OF SIZE **3**

| $\bar{x}$ | No. of Samples | $f(\bar{x})$ |
|---|---|---|
| 0.00 | 1 | 1/125 |
| 0.33 | 3 | 3/125 |
| 0.67 | 6 | 6/125 |
| 1.00 | 10 | 10/125 |
| 1.33 | 15 | 15/125 |
| 1.67 | 18 | 18/125 |
| 2.00 | 19 | 19/125 |
| 2.33 | 18 | 18/125 |
| 2.67 | 15 | 15/125 |
| 3.00 | 10 | 10/125 |
| 3.33 | 6 | 6/125 |
| 3.67 | 3 | 3/125 |
| 4.00 | 1 | 1/125 |
| | 125 | 1 |

The graph of this distribution is as follows:
Sampling Distribution of $\bar{X}$ for n = 3

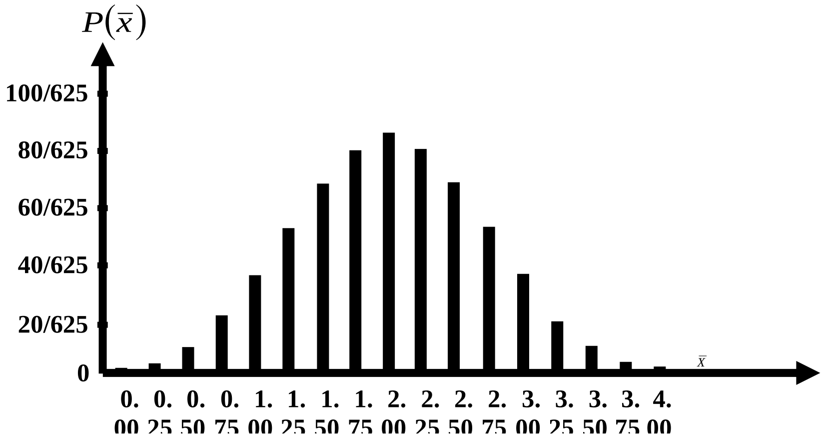



If in the car tests instead of taking samples of 2 we had taken all possible samples of size 4, our sampling distributions would contain 54 = 625 sample means, and it would be in the following form:

SAMPLING DISTRIBUTION
FOR SAMPLES OF SIZE 4

| $\bar{x}$ | No. of Samples | $f(\bar{x})$ |
|---|---|---|
| 0.00 | 1 | 1/625 |
| 0.25 | 4 | 4/625 |
| 0.50 | 10 | 10/625 |
| 0.75 | 20 | 20/625 |
| 1.00 | 35 | 35/625 |
| 1.25 | 52 | 52/625 |
| 1.50 | 68 | 68/625 |
| 1.75 | 80 | 80/625 |
| 2.00 | 85 | 85/625 |
| 2.25 | 80 | 80/625 |
| 2.50 | 68 | 68/625 |
| 2.75 | 52 | 52/625 |
| 3.00 | 35 | 35/625 |
| 3.25 | 20 | 20/625 |
| 3.50 | 10 | 10/625 |
| 3.75 | 4 | 4/625 |
| 4.00 | 1 | 1/625 |
| | 625 | 1 |

The graph of this distribution is as follows, Sampling Distribution of $\bar{X}$ for n = 4



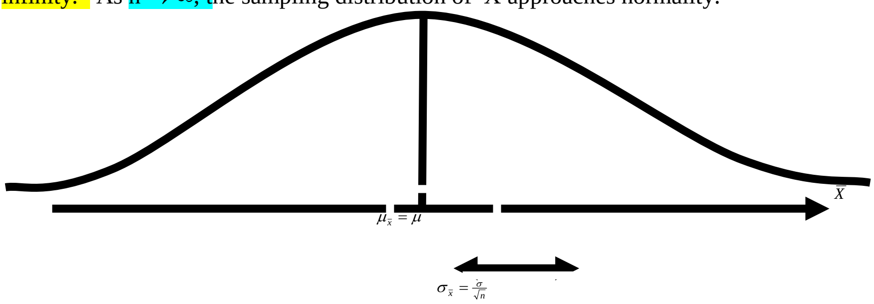

As in the case of the sampling distribution of $\bar{X}$ based on samples of size 2, each of these two distributions has a mean of 2 defective items. It is clear from the above figures that as larger samples are taken, the shape of the sampling distribution undergoes discernible changes.

In all three cases the line charts are symmetrical, but as the sample size increases, the overall configuration changes from a triangular distribution to a bell-shaped distribution. When relatively large samples are taken, this bell-shaped distribution assumes the form of a 'normal' distribution (also called the 'Gaussian' distribution), and this happens irrespective of the form of the parent population. (For example, in the problem currently under consideration, the population of defective items in a car is rectangular.)
This leads us to the following fundamentally important theorem:

## CENTRAL LIMIT THEOREM

The theorem states that:

**"**If a variable X from a population has mean μ and finite variance σ2, then the sampling distribution of the sample mean $\bar{X}$ approaches a normal distribution with mean μ and variance σ2/n as the sample size n approaches infinity.**"** As $n \to \infty$, the sampling distribution of $\bar{X}$ approaches normality.




Due to the Central Limit Theorem, the normal distribution has found a central place in the theory of statistical inference.(Since, in many situations, the sample is large enough for our sampling distribution to be approximately normal, therefore we can utilize the mathematical properties of the normal distribution to draw inferences about the variable of interest). The rule of thumb in this regard is that if the sample size, n, is greater than or equal to 30, then we can assume that the sampling distribution of $\bar{X}$ is approximately normally distributed. On the other hand, If the POPULATION sampled is normally distributed, then the sampling distribution of $\bar{X}$ will also be normal regardless of sample size. In other words, $\bar{X}$ will be normally distributed with mean μ and variance σ2/n.

# LECTURE NO. 32

- Sampling Distribution of $\hat{p}$
- Sampling Distribution of $\bar{X}_1 - \bar{X}_2$

We discussed the mean and the standard deviation of the sampling distribution, and, towards the end of the lecture, we consider the very important theorem known as the Central Limit Theorem. Let us now consider the *real-life* application of this concept with the help of an example:

**EXAMPLE**

A construction company has 310 employees who have an average annual salary of Rs.24,000.The standard deviation of annual salaries is Rs.5,000.
Suppose that the employees of this company launch a demand that the government should institute a law by which their average salary should be at least Rs. 24500, and, suppose that the government decides to check the validity of this demand by drawing a random sample of 100 employees of this company, and acquiring information regarding their present salaries. What is the probability that, in a random sample of 100 employees, the average salary will exceed Rs.24,500 (so that the government decides that the demand of the employees of this company is unfounded, and hence does not pay attention to the demand(although, in reality, it was justified))?

**SOLUTION**

The sample size (n = 100) is large enough to assume that the sampling distribution of $\bar{X}$ is approximately *normally* distributed with the following mean and standard deviation:
and standard deviation

$$\mu_{\bar{x}} = \mu = Rs.24{,}000.$$

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} \cdot \sqrt{\frac{N-n}{N-1}} = \frac{5000}{\sqrt{100}} \sqrt{\frac{310-100}{310-1}}$$

$$= Rs.412.20$$

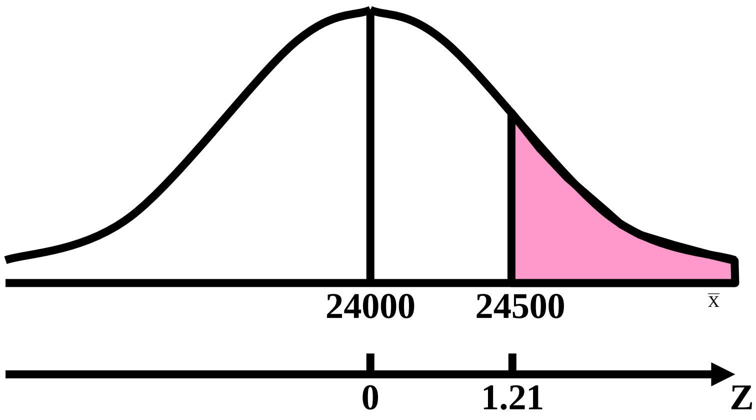
***NOTE:***

Here we have used finite population correction factor (fpc), because the sample size n = 100 is *greater than 5 percent* of the population size N = 310. Since $\bar{X}$ is approximately N (24000, 412.20), therefore

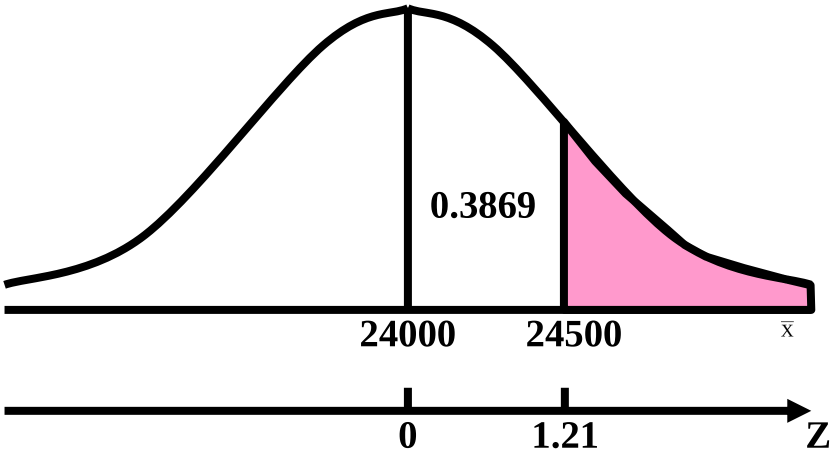
$$Z = \frac{\bar{X} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{X} - 24000}{412.20}$$

is approximately N(0, 1).We are required to evaluate P( $\bar{X}$ > 24,500).
At $\bar{x}$ = 24,500, we find that

$$z = \frac{24500 - 24000}{412.20} = 1.21$$

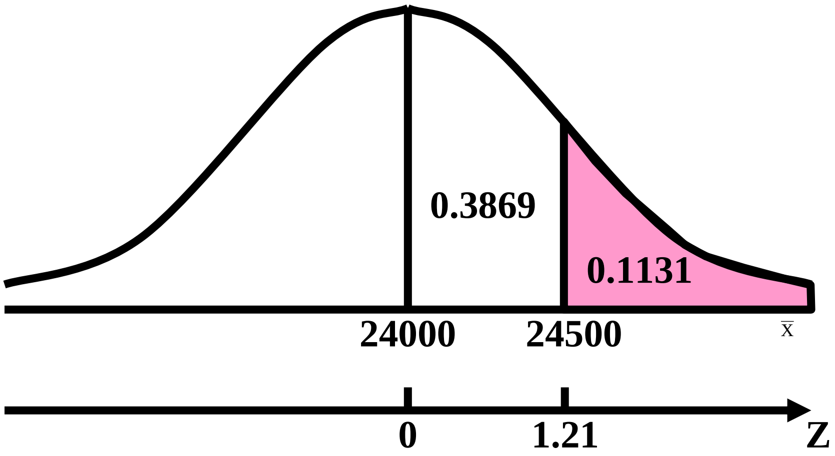



Using the table of areas under the standard normal curve, we find that the area between z = 0 and z = 1.21 is 0.3869.




Hence,

$P(\bar{X} > 24,500)$
$= P(Z > 1.21)$
$= 0.5 – P(0 < Z < 1.21)$
$= 0.5 – 0.3869 = 0.1131.$




Hence, the chances are only 11% that in a random sample of 100 employees from this particular construction company, the average salary will exceed Rs.24,500.In other words, the chances are 89% that, in such a sample, the average salary will *not* exceed Rs.24,500.

Hence, the chances are considerably *high* that the government *might* pay attention to the employees' demand.

## SAMPLING DISTRIBUTION OF THE SAMPLE PROPORTION

In this regard, the first point to be noted is that, whenever the elements of a population can be classified into *two* categories, technically called "success" and "failure", we may be interested in *the proportion of "successes"* in the population. If X denotes the number of successes in the population, then the proportion of successes in the population is given by

$$p = \frac{X}{N}.$$

Similarly, if we draw a sample of size n from the population, the proportion of successes in the sample is given by

$$\hat{p} = \frac{X}{n},$$

where X represents the number of successes in the sample.

It is interesting to note that X is a *binomial* random variable and the binomial parameter p is being called a proportion of successes here. The sample proportion has different values in different samples. It is obviously a random variable and has a probability distribution.

This probability distribution of the proportions of successes in all possible random samples of size n, is called the sampling distribution of $\hat{p}$.

We illustrate this sampling distribution with the help of the following examples:

**EXAMPLE-1**

A population consists of six values 1, 3, 6, 8, 9 and 12.Draw all possible samples of size n = 3 *without* replacement from the population and find the proportion of even numbers in each sample. Construct the sampling distribution of sample proportions and verify that

i) $\mu_{\hat{p}} = p$

ii) $\text{Var}(\hat{p}) = \frac{pq}{n} \cdot \frac{N-n}{N-1}$.

**SOLUTION**

The number of possible samples of size n = 3 that could be selected without replacement from a population of size N is

$\binom{6}{3} = 20.$

Let $\hat{p}$ represent the proportion of even numbers in the sample. Then the 20 possible samples and the proportion of even numbers are given as follows:

| Sample No. | Sample Data | Sample Proportion ($\hat{p}$) |
|---|---|---|
| 1 | 1, 3, 6 | 1/3 |
| 2 | 1, 3, 8 | 1/3 |
| 3 | 1, 3, 9 | 0 |
| 4 | 1, 3, 12 | 1/3 |
| 5 | 1, 6, 8 | 2/3 |
| 6 | 1, 6, 9 | 1/3 |
| 7 | 1, 6, 12 | 2/3 |
| 8 | 1, 8, 9 | 1/3 |
| 9 | 1, 8, 12 | 2/3 |
| 10 | 1, 9, 12 | 1/3 |
| 11 | 3, 6, 8 | 2/3 |
| 12 | 3, 6, 9 | 1/3 |
| 13 | 3, 6, 12 | 2/3 |
| 14 | 3, 8, 9 | 1/3 |
| 15 | 3, 8, 12 | 2/3 |
| 16 | 3, 9, 12 | 1/3 |
| 17 | 6, 8, 9 | 2/3 |
| 18 | 6, 8, 12 | 1 |
| 19 | 6, 9, 12 | 2/3 |
| 20 | 8, 9, 12 | 2/3 |

The sampling distribution of sample proportion is given below;

**SAMPLING DISTRIBUTION OF** $\hat{p}$ **:**

| $(\hat{p})$ | No. of Samples | Probability $f(\hat{p})$ | $\hat{p}\, f(\hat{p})$ | $\hat{p}^2 f(\hat{p})$ |
|---|---|---|---|---|
| 0 | 1 | 1/20 | 0 | 0 |
| 1/3 | 9 | 9/20 | 3/20 | 1/20 |
| 2/3 | 9 | 9/20 | 6/20 | 4/20 |
| 1 | 1 | 1/20 | 1/20 | 1/20 |
| $\sum$ | 20 | 1 | 10/20 | 6/20 |

**Now**

$$\mu_{\hat{p}} = \sum \hat{p}\, f(\hat{p}) = \frac{10}{20} = 0.5, \text{ and}$$

$$\sigma^2_{\hat{p}} = \sum \hat{p}^2 f(\hat{p}) - \left[\sum \hat{p}\, f(\hat{p})\right]^2$$

$$= \frac{2}{60} - \left(\frac{10}{20}\right)^2 = \frac{1}{20} = 0.05.$$

To verify the given relations, we first calculate the population proportion p.Thus:

$p = \frac{X}{N}$ ;Where X represents the number of even numbers in the population. In other words,

$$p = \frac{3}{6} = 0.5$$

Hence, we find that

$$\mu_{\hat{p}} = 0.5 = p\ ,$$

$$\frac{pq}{n} \cdot \frac{N-n}{N-1} = \frac{0.25}{3} \cdot \frac{6-3}{6-1}$$

$$= \frac{0.25}{5} = 0.05 = Var(\hat{p})$$

Hence, two properties of the sampling distribution of $\hat{p}$ are verified.

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}\,\frac{N-n}{N-1}},$$

The sampling distribution of $\hat{p}$ has the following important properties.

**PROPERTIES OF THE SAMPLING DISTRIBUTION OF** $\hat{p}$

**Property No. 1**

The mean of the sampling distribution of proportions, denoted by $\mu_{\hat{p}}$, is equal to the population proportion p, that is

$$\mu_{\hat{p}} = p.$$

**Property No. 2**

The standard deviation of the sampling distribution of proportions, called the *standard error* of $\hat{p}$ and denoted by $\sigma_{\hat{p}}$, is given as:

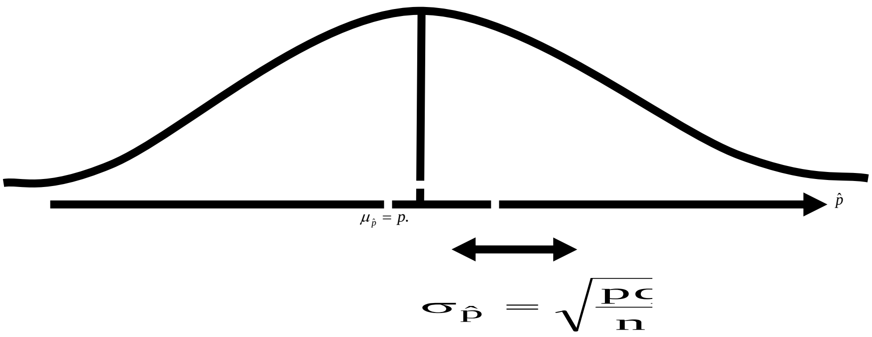
- a)
$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}},$$
- when the sampling is performed *with* replacement
- b) when sampling is done *without* replacement from a *finite* population
(As in the case of the sampling distribution of $\overline{X}$,is known as the finite population correction factor (fpc)

$$\sqrt{\frac{N-n}{N-1}},$$

**Property No. 3**

**SHAPE OF THE DISTRIBUTION**

The sampling distribution of $\hat{p}$ is the *binomial* distribution. However, for sufficiently *large* sample sizes, the sampling distribution of is approximately normal. As n → ∞, the sampling distribution of approaches normality.




As a rule of thumb, the sampling distribution of $\hat{p}$ will be approximately *normal* whenever both np and nq are equal to or greater than 5.Let us apply this concept to a real-world situation:

**EXAMPLE-2**

Ten percent of the 1-kilogram boxes of sugar in a large warehouse are underweight. Suppose a retailer buys a random sample of 144 of these boxes. What is the probability that at least 5 percent of the sample boxes will be underweight?

**SOLUTION**

Here the statistic is the sample proportion, the sample size (n = 144) is large enough to assume that the sample proportion is approximately normally distributed with mean

Mean of the sampling distribution of $\hat{p}$

$$\mu_{\hat{p}} = p = 0.10,$$

Standard Error of $\hat{p}$

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.10)(0.90)}{144}}$$

$$= \frac{0.3}{12} = 0.025.$$

Therefore, the sampling distribution of  is approximately N(0.10, 0.025); and hence

$$Z = \frac{\hat{p} - \mu_{\hat{p}}}{\sigma_{\hat{p}}} = \frac{\hat{p} - p}{\sqrt{pq/n}}$$

$$= \frac{\hat{p} - 0.10}{0.025}$$

is approximately N(0, 1).

We are required to find the probability that the proportion of underweight boxes in the sample is equal to or greater than 5% i.e., we require

$$P(\hat{p} \geq 0.05).$$

In this regard, a very important point to be noted is that, just as we use a *continuity correction* of + ½ whenever we consider the normal approximation to the binomially distributed random variable X, in *this* situation, since

$$\hat{p} = \frac{X}{n},$$

therefore, we need to use the following continuity correction; We need to use a *continuity correction* of $\pm \frac{1}{2n}$ in the case of the sampling distribution of $\hat{p}$.

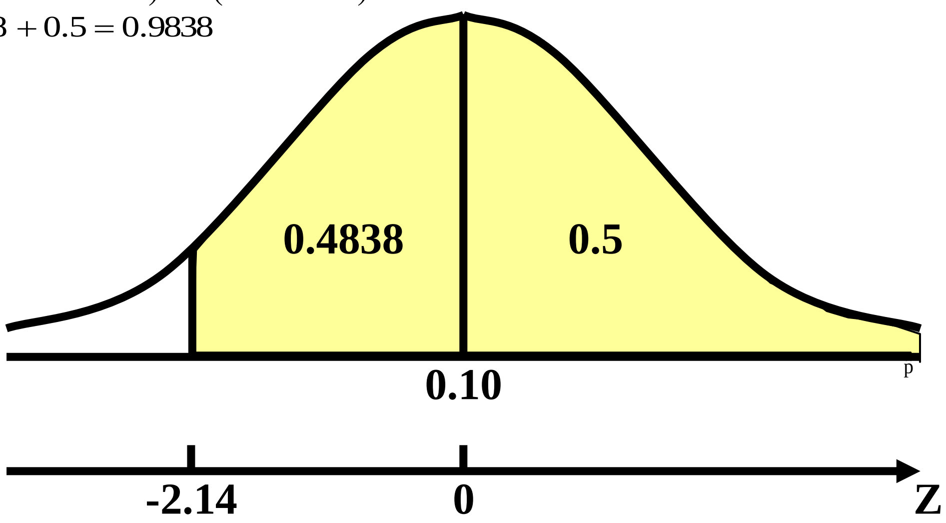
Applying the continuity correction in this problem, we have:

$$P(\hat{p} \geq 0.05) \Rightarrow P\left(\hat{p} \geq 0.05 - \frac{1}{(2)(144)}\right)$$

$$= P\left(\hat{p} \geq 0.05 - \frac{1}{288}\right)$$

$$= P\left(\frac{\hat{p} - 0.10}{0.025} \geq \frac{(0.05 - 1/288) - 0.10}{0.025}\right)$$

$$= P(Z \geq -2.14)$$
$$= P(-2.14 \leq Z \leq 0) + P(0 \leq Z \leq \infty)$$
$$= 0.4838 + 0.5 = 0.9838$$




Hence, the probability that at *least* 5% of the sample boxes are under-weight *is as high as* 98% .
The sampling distributions of $\bar{X}$ and $\hat{p}$ pertain to the situation when we are drawing all possible samples of a

particular size from one particular population. Next, we will discuss the case when we are dealing with all possible samples drawn from *two* populations, such that the samples from the two populations are *independent*.
In this regard, we will consider the sampling distributions of $\bar{X}_1 - \bar{X}_2$ and $\hat{p}_1 - \hat{p}_2$ :

We begin with the sampling distribution of $\bar{X}_1 - \bar{X}_2$ :

**SAMPLING DISTRIBUTION OF DIFFERENCES BETWEEN MEANS**

Suppose we have two distinct populations with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$ respectively.
Let *independent* random samples of sizes $n_1$ and $n_2$ be selected from the respective populations, and the differences $\bar{x}_1 - \bar{x}_2$ between the means of all possible pairs of samples be computed.
Then, a probability distribution of the differences $\bar{x}_1 - \bar{x}_2$ can be obtained. Such a distribution is called the sampling distribution of the differences of sample means $\bar{x}_1 - \bar{x}_2$. We illustrate the sampling distribution of $\bar{X}_1 - \bar{X}_2$ with the help of the following example.

**EXAMPLE**

imp

Draw all possible random samples of size n1 = 2 *with replacement* from a finite population consisting of 4, 6, similarly, draw all possible random samples of size n = 2 *with replacement* from another finite population consisting of 1, 2, 3.
**a)** Find the possible differences between the sample means of the two populations
**b)** Construct the sampling distribution of $\bar{X}_1 - \bar{X}_2$ and compute its mean and variance
**c)** Verify that

$$\mu_{\bar{x}_1-\bar{x}_2} = \mu_1 - \mu_2 \text{ and } \sigma^2_{\bar{x}_1-\bar{x}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_1}.$$

**SOLUTION**

Whenever we are sampling with replacement from a finite population, the total number of possible samples is Nn (where N is the population size, and n is the sample size).Hence, in this example, there are (3)2 = 9 possible samples which can be drawn with replacement from each population. These two sets of samples and their means are given below:

| From Population 1 | | | From Population 2 | | |
|---|---|---|---|---|---|
| Sample No. | Sample Value | $\bar{x}_1$ | Sample No. | Sample Value | $\bar{x}_2$ |
| 1 | 4, 4 | 4 | 1 | 1, 1 | 1.0 |
| 2 | 4, 6 | 5 | 2 | 1, 2 | 1.5 |
| 3 | 4, 8 | 6 | 3 | 1, 3 | 2.0 |
| 4 | 6, 4 | 5 | 4 | 2, 1 | 1.5 |
| 5 | 6, 6 | 6 | 5 | 2, 2 | 2.0 |
| 6 | 6, 8 | 7 | 6 | 2, 3 | 2.5 |
| 7 | 8, 4 | 6 | 7 | 3, 1 | 2.0 |
| 8 | 8, 6 | 7 | 8 | 3, 2 | 2.5 |
| 9 | 8, 8 | 8 | 9 | 3, 3 | 3.0 |

**a)** Since there are 9 samples from the first population as well as 9 from the second, hence, there are 81 possible combinations of $\bar{x}1$ and $\bar{x}2$
The 81 possible differences $\bar{x}1 - \bar{x}2$ are presented in the following table:

| $\bar{x}_2$ 1 | $\bar{x}_2$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 4 | 5 | 6 | 5 | 6 | 7 | 6 | 7 | 8 |
| 1.0 | 3.0 | 4.0 | 5.0 | 4.0 | 5.0 | 6.0 | 5.0 | 6.0 | 7.0 |
| 1.5 | 2.5 | 3.5 | 4.5 | 3.5 | 4.5 | 5.5 | 4.5 | 5.5 | 6.5 |
| 2.0 | 2.0 | 3.0 | 4.0 | 3.0 | 4.0 | 5.0 | 4.0 | 5.0 | 6.0 |
| 1.5 | 2.5 | 3.5 | 4.5 | 3.5 | 4.5 | 5.5 | 4.5 | 5.5 | 6.5 |
| 2.0 | 2.0 | 3.0 | 4.0 | 3.0 | 4.0 | 5.0 | 4.0 | 5.0 | 6.0 |
| 2.5 | 1.5 | 2.5 | 3.5 | 2.5 | 3.5 | 4.5 | 3.5 | 4.5 | 5.5 |
| 2.0 | 2.0 | 3.0 | 4.0 | 3.0 | 4.0 | 5.0 | 4.0 | 5.0 | 6.0 |
| 2.5 | 1.0 | 2.5 | 3.5 | 2.5 | 3.5 | 4.5 | 3.5 | 4.5 | 5.5 |
| 3.0 | 1.0 | 2.0 | 3.0 | 2.0 | 3.0 | 4.0 | 3.0 | 4.0 | 5.0 |

**b)**The sampling distribution of $\overline{X}_1 - \overline{X}_2$ is as follows:

| $\bar{x}_1 - \bar{x}_2 = d$ | Tally | f | Probability $f(\bar{x}_1 - \bar{x}_2) = f(d)$ | df (d) | $d^2$ f(d) |
|---|---|---|---|---|---|
| 1.0 | \| | 1 | 1/81 | 1/81 | 1.0/81 |
| 1.5 | \|\| | 2 | 2/81 | 3/81 | 4.5/81 |
| 2.0 | ~~\|\|\|\|~~ | 5 | 5/81 | 10/81 | 20.0/81 |
| 2.5 | ~~\|\|\|\|~~ \| | 6 | 6/81 | 15/81 | 37.5/81 |
| 3.0 | ~~\|\|\|\|~~ ~~\|\|\|\|~~ | 10 | 10/81 | 30/81 | 90.0/81 |
| 3.5 | ~~\|\|\|\|~~ ~~\|\|\|\|~~ | 10 | 10/81 | 35/81 | 122.5/81 |
| 4.0 | ~~\|\|\|\|~~ ~~\|\|\|\|~~ \|\|\| | 13 | 13/81 | 52/81 | 208.0/81 |
| 4.5 | ~~\|\|\|\|~~ ~~\|\|\|\|~~ | 10 | 10/81 | 45/81 | 202.5/81 |
| 5.0 | ~~\|\|\|\|~~ ~~\|\|\|\|~~ | 10 | 10/81 | 50/81 | 250.0/81 |
| 5.5 | ~~\|\|\|\|~~ \| | 6 | 6/81 | 33/81 | 181.5/81 |
| 6.0 | ~~\|\|\|\|~~ | 5 | 5/81 | 30/81 | 180.0/81 |
| 6.5 | \|\| | 2 | 2/81 | 13/81 | 84.5/81 |
| 7.0 | \| | 1 | 1/81 | 7/81 | 49.0/81 |
| Total | --- | 81 | 1 | 324/81 | 1431/81 |

Thus the mean and the variance are

$$\mu_{\bar{x}_1-\bar{x}_2} = \sum (\bar{x}_1 - \bar{x}_2)\, f(\bar{x}_1 - \bar{x}_2)$$
$$= \sum df(d) = \frac{324}{81} = 4, \quad and$$

$$\sigma^2_{\bar{x}_1-\bar{x}_2} = \sum d^2 f(d) - \left[\sum df(d)\right]^2$$
$$= \frac{1431}{81} - \left(\frac{324}{81}\right)^2 = \frac{53}{3} - 16 = \frac{5}{3} = 1.67$$

**c)** In order to verify the properties of the sampling distribution of $\overline{X}_1 - \overline{X}_2$ we first need to compute the mean and variance of the first population:

The mean and standard deviation of the first population are:

$$\mu_1 = \frac{4+6+8}{3} = 6, \text{ and}$$

$$\sigma_1^2 = \frac{(4-6)^2 + (6-6)^2 + (8-6)^2}{3} = \frac{8}{3}.$$

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{8}{3}.\frac{1}{2} + \frac{2}{3}.\frac{1}{2}$$

$$= \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$$

$$= 1.67$$

$$= \sigma^2_{\bar{x}_1 - \bar{x}_2}$$

The mean and variance of the second population are:

$$\mu_2 = \frac{1+2+3}{3} = 2, \text{ and}$$

$$\sigma_2^2 = \frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3} = \frac{2}{3}.$$

$$Now\ \mu_{\bar{x}_1 - \bar{x}_2} = 4 = 6 - 2 = \mu_1 - \mu_2,\ and$$

$$\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = \frac{8}{3}.\frac{1}{2} + \frac{2}{3}.\frac{1}{2}$$

$$= \frac{4}{3} + \frac{1}{3} = \frac{5}{3}$$

$$= 1.67$$

$$= \sigma^2_{\bar{x}_1 - \bar{x}_2}$$

Hence, two properties of the sampling distribution of $\bar{X}_1 - \bar{X}_2$ are satisfied. The sampling distribution of the differences $\bar{X}_1 - \bar{X}_2$ has the following properties:

**PROPERTIES OF THE SAMPLING DISTRIBUTION OF $\bar{X}_1 - \bar{X}_2$**

**Property No. 1:**

The mean of the sampling distribution of $\bar{X}_1 - \bar{X}_2$, denoted by $\mu_{\bar{X}_1 - \bar{X}_2}$, is equal to the difference between population means, that is

$$\mu_{\bar{X}_1 - \bar{X}_2} = \mu_1 - \mu_2$$

**Property No. 2:**

In case of sampling with or without replacement from two *infinite* populations, the standard deviation of the sampling distribution of $\bar{X}_1 - \bar{X}_2$ (i.e. *standard error* of $\bar{X}_1 - \bar{X}_2$), denoted by $\sigma_{\bar{X}_1 - \bar{X}_2}$, is given by

$$\sigma_{\bar{X}_1 - \bar{X}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

The above expression for the Standard Error of $\overline{X}_1 - \overline{X}_2$ also holds for finite population when sampling is performed *with* replacement. In case of sampling without replacement from a finite population, the formula for the standard error of will be suitably modified.

**Property No. 3:**

Shape of the distribution:

**a)** If the POPULATIONS are normally distributed, the sampling distribution of $\overline{X}_1 - \overline{X}_2$, regardless of sample sizes, will be *normal* with mean $\mu_1 - \mu_2$ and variance $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$.

In other words, the variable

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

is normally distributed with zero mean and unit variance.

**b)** If the POPULATIONS are *non-normal* and if *both* sample sizes are *large*, (i.e., greater than or equal to 30), then the sampling distribution of the differences between means is approximately a *normal* distribution by the Central Limit Theorem.

In this case too, the variable

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

will be approximately *normally* distributed with mean zero and variance one.

# LECTURE NO. 33

- Sampling Distribution of (continued)
- Point Estimation
- Desirable Qualities of a Good Point Estimator
  - Unbiasedness
  - Consistency

We illustrate the real-life application of the sampling distribution of $\bar{X}_1 - \bar{X}_2$ with the help of the following example:

**EXAMPLE**

Car batteries produced by company A have a mean life of 4.3 years with a standard deviation of 0.6 years. A similar battery produced by company B has a mean life of 4.0 years and a standard deviation of 0.4 years. What is the probability that a random sample of 49 batteries from company A will have a mean life of at least 0.5 years *more* than the mean life of a sample of 36 batteries from company B?

**SOLUTION**

We are given the following data:
Population A:
$\mu 1 = 4.3$ years, $\sigma 1 = 0.6$ years,
Sample size: n1 = 49
Population B:
$\mu 2 = 4.0$ years, $\sigma 2 = 0.4$ years,
Sample size: n2 = 36
Both sample sizes (n1 = 49, n2 = 36) are large enough to assume that the sampling distribution of the differences is approximately a normal such that: $\bar{X}_1 - \bar{X}_2$

**MEAN**

$$\mu_{\bar{x}_1-\bar{x}_2} = \mu_1 - \mu_2 = 4.3 - 4.0 = 0.3 \text{ years}$$

and standard deviation:

$$\sigma_{\bar{x}_1-\bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{0.36}{49} + \frac{0.16}{36}}$$

Thus the variable $= 0.1086$ *years*.

$$Z = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$
$$= \frac{\left(\bar{X}_1 - \bar{X}_2\right) - 0.3}{0.1086}$$

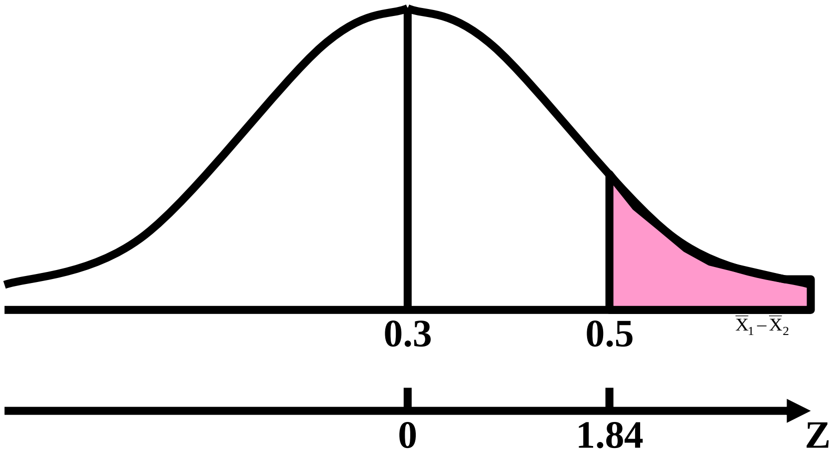
is approximately N (0, 1)
We are required to find the probability that the mean life of 49 batteries produced by company A will have a mean life of at least 0.5 years *longer* than the mean life of 36 batteries produced by company B, i.e.
We are required to find:

$$P\left(\bar{X}_1 - \bar{X}_2 \geq 0.5\right).$$

Transforming $\bar{X}_1 - \bar{X}_2 = 0.5$

to z-value, we find that:

$$z = \frac{0.5 - 0.3}{0.1086} = 1.84$$




Hence, using the table of areas under normal curve, we find:

$$\begin{aligned} P\left(\overline{X}_1 - \overline{X}_2 \geq 0.5\right) &= P(Z \geq 1.84) \\ &= 0.5 - P(0 < Z < 1.84) \\ &= 0.5 - 0.4671 \\ &= 0.0329 \end{aligned}$$

In other words, (given that the *real* difference between the mean lifetimes of batteries of company A and batteries of company B is
4.3 - 4.0 = *0.3* years), the probability that a *sample* of 49 batteries produced by company A will have a mean life of at least 0.5 years *longer* than the mean life of a *sample* of 36 batteries produced by company B, is only 3.3%.

## SAMPLING DISTRIBUTION OF THE DIFFERENCES BETWEEN PROPORTIONS

Suppose there are two *binomial* populations with proportions of successes p1 and p2 respectively. Let *independent* random samples of sizes $n_1$ and $n_2$ be drawn from the respective populations, and the differences $\hat{p}_1 - \hat{p}_2$ between the proportions of *all possible* pairs of samples be computed. Then, a probability distribution of the differences $\hat{p}_1 - \hat{p}_2$ can be obtained. Such a probability distribution is called the *sampling distribution* of the differences between the proportions $\hat{p}_1 - \hat{p}_2$ .We illustrate the sampling distribution of $\hat{p}_1 - \hat{p}_2$ with the help of the following example:

## EXAMPLE

It is claimed that 30% of the households in Community A and 20% of the households in Community B have at least one teenager. A simple random sample of 100 households from each community yields the following results:
What is the probability of observing a difference *this* large or *larger* if the claims are true?

$$\hat{p}_A = 0.34,\ \hat{p}_B = 0.13.$$

## SOLUTION

We assume that if the claims are true, the sampling distribution of $\hat{p}_A - \hat{p}_B$ is approximately normally distributed (as, in this example, both the sample sizes are large enough for us to apply the normal approximation to the binomial distribution).Since we are reasonably confident that our sampling distribution is approximately normally distributed, hence we will be finding any required probability by computing the relevant areas under our normal curve, and, in order to do so, we will first need to convert our variable $\hat{p}_A - \hat{p}_B$ to Z. In order to convert $\hat{p}_A - \hat{p}_B$ to Z, we need the values of $\mu_{\hat{p}_A - \hat{p}_B}$ as well as $\sigma_{\hat{p}_A - \hat{p}_B}$.
It can be mathematically proved that:

## PROPERTIES OF THE SAMPLING DISTRIBUTION OF $\hat{p}_1 - \hat{p}_2$

**Property No. 1:**

The mean of the sampling distribution of $\hat{p}_1 - \hat{p}_2$, denoted by $\mu_{\hat{p}_1 - \hat{p}_2}$, is equal to the difference between the population proportions, that is $\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2$.

**Property No. 2:**

The standard deviation of the sampling distribution of $\hat{p}_1 - \hat{p}_2$, (i.e. the standard error of $\hat{p}_1 - \hat{p}_2$) denoted by

$\sigma_{\hat{p}_1 - \hat{p}_2}$ is given by

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}},$$

where q = 1 – p
Hence, in this example, we have:

$$\mu_{\hat{p}_A - \hat{p}_B} = 0.30 - 0.20 = 0.10$$

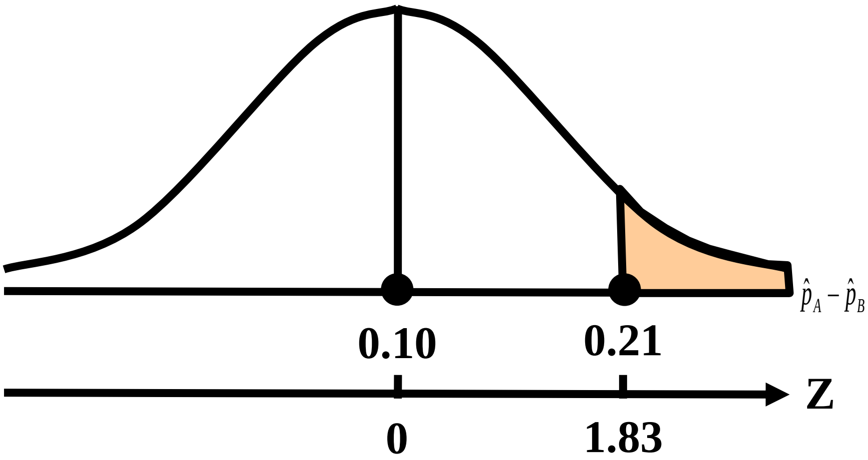
$$\sigma^2_{\hat{p}_A - \hat{p}_B} = \frac{(0.30)(0.70)}{100} + \frac{(0.20)(0.80)}{100} = 0.0037$$

The observed difference in sample proportions is

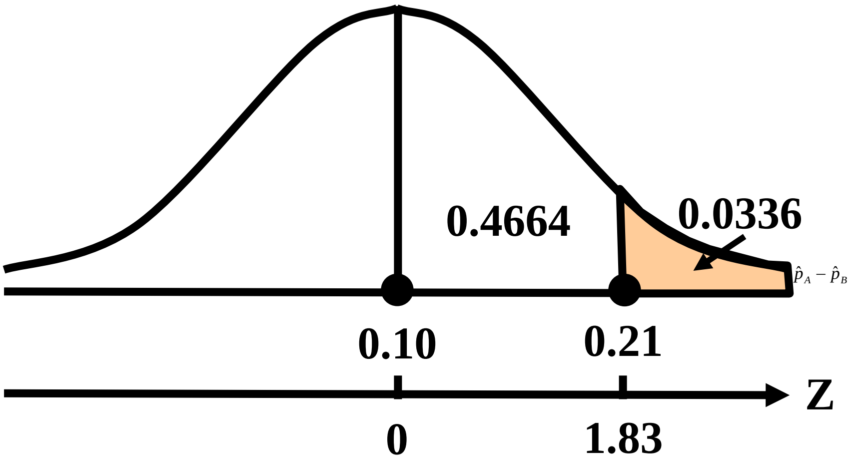
$$\hat{p}_A - \hat{p}_B = 0.34 - 0.13 = 0.21$$

The probability that we wish to determine is represented by the area to the right of 0.21 in the sampling distribution of $\hat{p}_A - \hat{p}_B$ .To find this area, we compute

$$z = \frac{0.21 - 0.10}{\sqrt{0.0037}} = \frac{0.11}{0.06} = 1.83$$




By consulting the Area Table of the standard normal distribution, we find that the area between z = 0 and z = 1.83 is 0.4664. Hence, the area to the right of z = 1.83 is 0.0336.
This probability is shown in following figure:

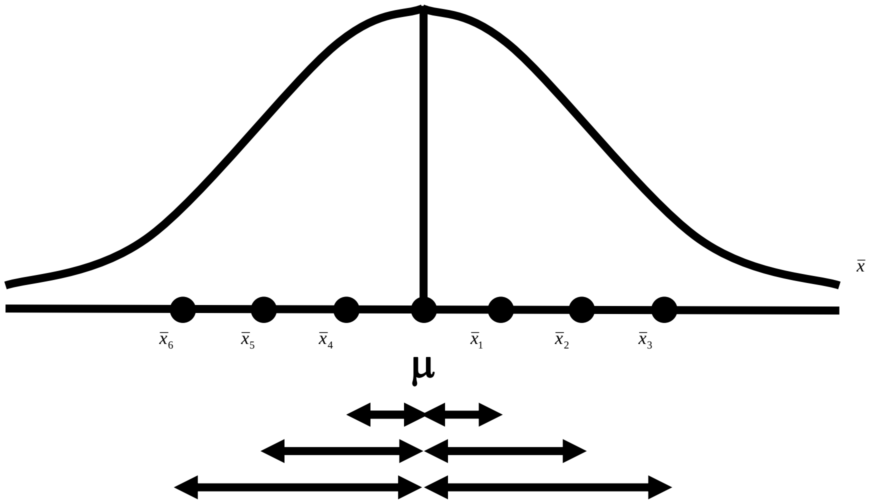


Thus, if the claim is true, the probability of observing a difference as larger as or larger than the actually observed is only 0.0336 i.e. 3.36%. The students are encouraged to try to *interpret* this result with reference to the situation at hand, as, in attempting to solve a statistical problem, it is very important not just to apply various formulae and obtain numerical results, but to *interpret* the results with reference to the problem under consideration. Does the result indicate that at least *one* of the two claims is untrue, or does it imply something *else*? Before we close the basic discussion regarding sampling distributions, we would like to draw the students' attention to the following two important points:

- We have discussed various sampling distributions with reference to the simplest technique of random sampling, i.e. *simple random sampling*.
- And, with reference to simple random sampling, it should be kept in mind that this technique of sampling is appropriate in that situation when the population is *homogeneous*.
- Let us consider the reason why the standard deviation of the sampling distribution of any statistic is known as its standard *error*:

To answer this question, consider the fact that any statistic, considered as an *estimate* of the corresponding population parameter, should be as *close* in magnitude to the parameter as possible. The difference between the value of the statistic and the value of the parameter can be regarded as an *error* --- and is called 'sampling error'. Geometrically, each one of these errors can be represented by *horizontal line segment* below the X-axis, as shown below

Sampling Distribution of




The above diagram clearly indicates that there are various magnitudes of this error, depending on how far or how close the values of our statistic are in different samples.

The standard deviation of $\bar{X}$ gives us a '*standard'* value of this error, and hence the term '*Standard* Error'.

Having presented the basic ideas regarding sampling distributions, we now begin the discussion regarding POINT ESTIMATION:

## POINT ESTIMATION

Point estimation of a population parameter provides as an estimate a *single* value calculated from the sample that is likely to be close in magnitude to the unknown parameter.

## DIFFERENCE BETWEEN 'ESTIMATE' AND 'ESTIMATOR' *mcq*

An *estimate* is a numerical value of the unknown parameter obtained by applying a rule or a formula, called an *estimator*, to a sample X1, X2, …, Xn of size n, taken from a population. In other words, an estimator stands for the *rule or method* that is used to estimate a parameter whereas an estimate stands for the *numerical value* obtained by substituting the sample observations in the rule or the formula.

For instance:

If $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population with mean μ, then $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an estimator of μ,

and $\bar{x}$, the *numerical value* of $\bar{X}$, is an estimate of μ (i.e. a point estimate of μ).

In general, the (the Greek letter θ) is customarily used to denote an unknown parameter that could be a mean, median, proportion or standard deviation, while an estimator of θ is commonly denoted by $\hat{\theta}$, or sometimes by T.

It is important to note that *an estimator* is always a *statistic* which is a *function* of the sample observations and hence *is a random variable* as the sample observations are likely to vary from sample to sample.

In other words:

In *repeated* sampling, an estimator is a *random variable*, and has a *probability distribution*, which is known as its *sampling distribution*. Having presented the basic definition of a point estimator, we now consider some *desirable qualities* of a good point estimator. In this regard, the point to be understood is that a point estimator is considered a good estimator if it satisfies *various criteria*. Three of these criteria are:

**DESIRABLE QUALITIES OF A GOOD POINT ESTIMATOR**

- unbiasedness **mcq**
- consistency
- efficiency

**UNBIASEDNESS**

An estimator is defined to be unbiased if the statistic used as an estimator has its expected value equal to the true value of the population parameter being estimated. In other words, let $\hat{\theta}$ be an estimator of a parameter $\theta$. Then $\hat{\theta}$ will be called an unbiased estimator if $E(\hat{\theta}) = \theta$. If $E(\hat{\theta}) \neq \theta$, the statistic is said to be a biased estimator

**EXAMPLE**

Let us consider the sample mean $\bar{X}$ as an estimator of the population mean $\mu$. Then we have $\theta = \mu$

and $\hat{\theta} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Now, we know that $E(\bar{X}) = \mu$

i.e. $E(\hat{\theta}) = \theta$.

Hence, $\bar{X}$ is an *unbiased* estimator of $\mu$. Let us illustrate the concept of unbiasedness by considering the example of the annual Ministry of Transport test that was presented in the last lecture:

**EXAMPLE**

Let us examine the case of an annual Ministry of Transport test to which all cars, irrespective of age, have to be submitted. The test looks for faulty breaks, steering, lights and suspension, and it is discovered after the first year that approximately the *same number* of cars have 0, 1, 2, 3, or 4 faults. The above situation is equivalent to the following: If we let X denote *the number of faults* in a car, then X can take the values 0, 1, 2, 3, and 4, and the *probability* of each of these X values is 1/5. Hence, we have the following probability distribution:

| No. of Faulty Items (X) | Probability f(x) |
|---|---|
| 0 | 1/5 |
| 1 | 1/5 |
| 2 | 1/5 |
| 3 | 1/5 |
| 4 | 1/5 |
| Total | 1 |

**MEAN OF THE POPULATION DISTRIBUTION**

$$\mu = E(X) = \sum x f(x) = 2$$

We are interested in considering the results that would be obtained if a *sample* of only two cars is tested.
You will recall that we obtained 52 = 25 different possible samples, and, computing the mean of each possible sample, we obtained the following sampling distribution of $\bar{X}$:

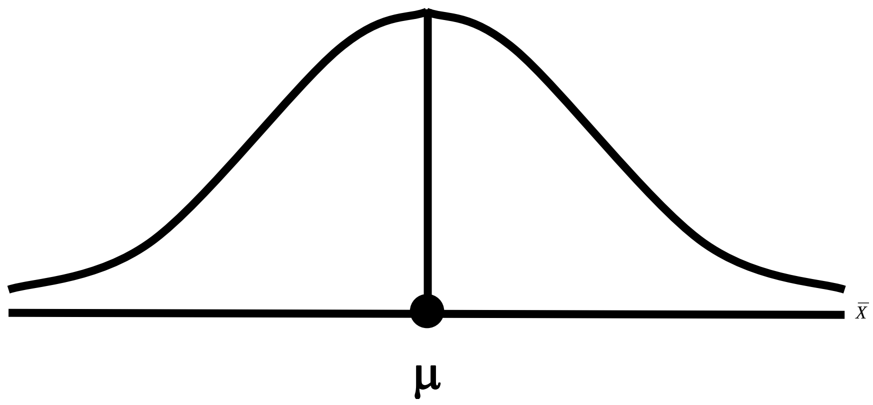
| Sample Mean | Probability |
|---|---|
| $\overline{X}$ | $P(\overline{X} = \overline{x})$ |
| 0.0 | 1/25 |
| 0.5 | 2/25 |
| 1.0 | 3/25 |
| 1.5 | 4/25 |
| 2.0 | 5/25 |
| 2.5 | 4/25 |
| 3.0 | 3/25 |
| 3.5 | 2/25 |
| 4.0 | 1/25 |
| Total | 25/25=1 |

We computed the mean of this sampling distribution, and found that the mean of the sample means i.e. comes out to be equal to 2 --- exactly the same as the mean of the population. We find that:

$$\mu_{\bar{x}} = \sum \bar{x}\ f(\bar{x}) = \frac{50}{25} = 2 = \mu$$

i.e the mean of the sampling distribution of $\overline{X}$ is equal to the population mean. By virtue of this property, we say that the sample mean is an *UNBIASED* estimate of the population mean. It should be noted that this property, *always* holds $\mu_{\overline{X}} = \mu$, *regardless* of the sample size. Unbiasedness is a property that requires that the probability distribution of $\theta$ be necessarily *centered* at the parameter$\theta$, irrespective of the value of n.

**VISUAL REPRESENTATION OF THE CONCEPT OF UNBIASEDNESS**




$E(\overline{X}) = \mu$ implies that the distribution of $\overline{X}$ is *centered* at μ.What this means is that, although many of the individual sample means are either *under-estimates* or *over-estimates* of the true population mean, in the long run, the over-estimates *balance* the under-estimates so that the *mean value* of the sample means comes out to be equal to the population mean.
Let us now consider some other estimators which possess the desirable property of being unbiased: The sample median is also an unbiased estimator of μ when the population is normally distributed (i.e. If X is normally distributed, then Also, as far as p, the *proportion of successes* in the sample is concerned, we have considering the binomial random variable X (which denotes the number of successes in n trials), we have:

$$E(\tilde{X}) = \mu.)$$

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n} E(X)$$

$$= \frac{np}{n} = p$$

Hence, the sample proportion is an *unbiased* estimator of the population parameter p. But As far as the sample variance S2 is concerned; it can be mathematically proved that E(S2) ≠ σ2.Hence, the sample variance S2 is a *biased* estimator of σ2.For any population parameter θ and its estimator $\hat{\theta}$, the quantity $E(\hat{\theta}) - \theta$ is known as the amount of *bias.*

This quantity is positive if $E(\hat{\theta}) > \theta,$ and is negative if $E(\hat{\theta}) < \theta,$ and, hence, the estimator is said to be positively biased when $E(\hat{\theta}) > \theta$ and negatively biased when $E(\hat{\theta}) < \theta$.Since unbiasedness is a desirable quality, we would like the sample variance to be an *unbiased* estimator of σ2.In order to achieve this end, the formula of the sample variance is *modified* as follows:

Modified formula for the sample variance:

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

Since E(s2) = σ2, hence s2 is an unbiased estimator of σ2.Why is unbiasedness consider a *desirable* property of an estimator? In order to obtain an answer to this question, consider the following: With reference to the estimation of the population mean μ, we note that, in an *actual* study, the probability is very high that the mean of our sample i.e. $\bar{X}$ will either be less than μ or more than μ.

Hence, in an actual study, we can never guarantee that our $\bar{X}$ will coincide with μ.

Unbiasedness implies that, although in an actual study, we *cannot* guarantee that our sample mean will coincide with μ, our estimation *procedure* (i.e. formula) is such that, in *repeated* sampling, the *average* value of our statistic *will* be equal to μ.

The next desirable quality of a good point estimator is *consistency*:

**CONSISTENCY**

An estimator $\hat{\theta}$ is said to be a consistent estimator of the parameter θ if, for any arbitrarily small positive quantity e,

$$\lim_{n\to\infty} P\left[\left|\hat{\theta} - \theta\right| \le e\right] = 1.$$

In other words, an estimator $\hat{\theta}$ is called a consistent estimator of θ if the probability that $\hat{\theta}$ is very close to θ, approaches unity with an increase in the sample size. It should be noted that consistency is a *large sample* property. Another point to be noted is that a consistent estimator *may or may not* be unbiased.

The sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, which is an unbiased estimator of μ, is a consistent estimator of the mean μ.The sample proportion $\hat{p}$ is also a consistent estimator of the parameter p of a population that has a binomial distribution. The median is *not* a consistent estimator of μ when the population has a skewed distribution. The sample variance

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2,$$

though a *biased* estimator, is a consistent estimator of the population variance σ2. Generally speaking, it can be proved that a statistic whose STANDARD ERROR *decreases* with an *increase* in the sample size, will be consistent.

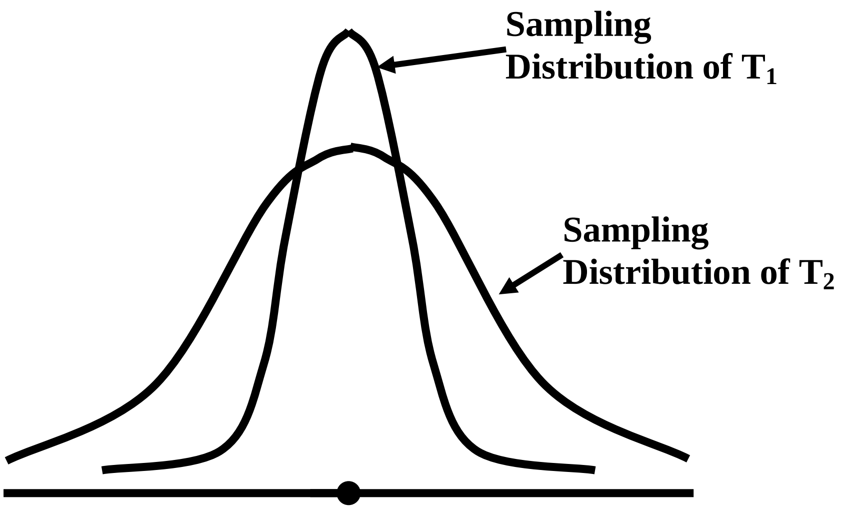
# LECTURE NO. 34

- Desirable Qualities of a Good Point Estimator:
  - Efficiency
- Methods of Point Estimation:
  - The Method of Moments
  - The Method of Least Squares
  - The Method of Maximum Likelihood
- Interval Estimation:
  - Confidence Interval for $\mu$

As a sample is only a *part* of the population, it is obvious that the *larger* the sample size, the *more* representative we expect it to be of the population from which it has been drawn. In agreement with the above argument, we will expect our estimator to be close to the corresponding parameter if the sample size is *large*. Hence, we will naturally be happy if the probability of our estimator being close to the parameter *increases* with an increase in the sample size. As *such*, consistency is a desirable property.
Another important desirable quality of a good point estimator is *EFFICIENCY*:

**EFFICIENCY**

An unbiased estimator is defined to be *efficient* if the variance of its sampling distribution is *smaller than* that of the sampling distribution of *any* other unbiased estimator of the same parameter. In other words, suppose that there are two unbiased estimators $T_1$ and $T_2$ of the same parameter $\theta$. Then, the estimator $T_1$ will be said to be *more efficient* than $T_2$ if $Var(T_1) < Var(T_2)$.
In the following diagram, since Var (T1) < Var (T2),
hence T1 is *more efficient* than T2 :




The *relative efficiency* of $T_1$ compared to $T_2$ (where both $T_1$ and $T_2$ are unbiased estimators) is given by the ratio

$$E_f = \frac{Var(T_2)}{Var(T_1)}.$$

And, if we multiply the above expression by 100, we obtain the relative efficiency in *percentage* form. It thus provides a criterion for *comparing* different unbiased estimators of a parameter. Both the sample mean and the sample median for a population that has a *normal* distribution, are unbiased and consistent estimators of $\mu$ but the variance of the sampling distribution of sample means is *smaller than* the variance of the sampling distribution of sample medians. Hence, the sample mean is more *efficient* than the sample median as an estimator of $\mu$. The sample mean may therefore be *preferred* as an estimator.
Next, we consider various *methods of point estimation*. A point estimator of a parameter can be obtained by several methods. We shall be presenting a brief account of the following three methods:

**METHODS OF POINTESTIMATION**

- The Method of Moments
- The Method of Least Squares
- The Method of Maximum Likelihood

These methods give estimates which may *differ* as the methods are based on different *theories* of estimation.

**THE METHOD OF MOMENTS**

The method of moments which is due to Karl Pearson (1857-1936), consists of calculating a few *moments* of the sample values and *equating* them to the corresponding moments of a population, thus getting *as many* equations as are needed to solve for the unknown parameters. The procedure is described below:
Let X1, X2, …, Xn be a random sample of size n from a population. Then the rth sample moment about zero is

$$m'_r = \frac{\sum X_i^r}{n}, \quad r = 1,2,\ldots$$

and the corresponding rth population moment is $\mu'_r$. We then *match* these moments and get *as many* equations as we need to solve for the unknown parameters. The following examples illustrate the method:

**EXAMPLE-1**

Let X be uniformly distributed on the interval (0, θ). Find an estimator of θ by the method of moments.

**SOLUTION**

The probability density function of the given uniform distribution is

$$f(x) = \frac{1}{\theta}, \quad 0 \le x \le \theta$$

Since the uniform distribution has only one parameter, (i.e. θ), therefore, in order to find the maximum likelihood estimator of θ by the method of moments, we need to consider only *one* equation.
The first sample moment about zero is

$$m'_1 = \frac{\sum X_i}{n}.$$

And, the first population moment about zero is

$$\mu'_1 = \int_0^\theta x.f(x)dx = \int_0^\theta x.\frac{1}{\theta}\,dx = \frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^\theta = \frac{\theta}{2}$$

Matching these moments, we obtain:

$$\frac{\sum X_i}{n} = \frac{\theta}{2} \text{ or } \theta = 2\overline{X}.$$

Hence, the moment estimator of θ is equal to $2\overline{X}$
i.e. $\hat{\theta} = 2\overline{X}.$

In other words, the moment estimator of θ is just twice the sample mean. It should be noted that, for the above uniform distribution, the mean is given by

$$\mu = \frac{\theta}{2}.$$

(This is so due to the *absolute* symmetry of the uniform distribution around the value $\frac{\theta}{2}$.)

Now, $\mu = \frac{\theta}{2}$ implies that $\theta = 2\mu$.
In other words, if we wish to have the *exact* value of θ, all we need to do is to multiply the population mean μ by 2.
Generally, it is not possible to determine μ, and all we can do is to draw a sample from the probability distribution, and compute the *sample* mean $\overline{X}$. Hence, naturally, the equation will be replaced by the equation
(As $2\overline{x}$ provides an *estimate* of θ, hence a 'hat' is placed on top of θ.)
It is interesting to note hat $2\overline{x}$ is *exactly* the same quantity as what we obtained as an estimate of θ by the method of moments!(The result obtained by the method of moments *coincides* with what we obtain through simple logic

**EXAMPLE-2**

Let X1, X2… Xn be a random sample of size n from a normal population with parameters μ and σ2. Find these parameters by the method of moments.

**SOLUTION**

Here we need two equations as there are *two* unknown parameters, μ and σ2. The first two sample moments about zero are

$$m'_1 = \frac{1}{n}\sum X_i = \overline{X} \text{ and } m'_2 = \frac{1}{n}\sum X_i^2.$$

The *corresponding* two moments of a normal distribution are

′μ1 = μ and ′μ2 = σ2 + μ2.

( σ2 = ′μ2 – ′μ12 = ′μ2 – μ2)

To get the desired estimators by the method of moments, we *match* them.

Thus, we have :

$$\mu = \frac{1}{n}\sum X_i \text{ and } \sigma^2 + \mu^2 = \frac{1}{n}\sum X_i^2$$

Solving the above equations *simultaneously*, we obtain:

$$\hat{\mu} = \frac{1}{n}\sum X_i = \overline{X}, \text{ and}$$

$$\hat{\sigma}^2 = \frac{\sum X_i^2}{n} - \overline{X}^2 = \frac{1}{n}\sum\left(X_i - \overline{X}\right)^2 = S^2.$$

as the moment estimators for μ and $\sigma^2$. A shortcoming of this method is that the moment estimators are, in general, *inefficient*.

## THE METHOD OF LEAST SQUARES

The method of Least Squares, which is due to *Gauss* (1777-1855) and *Markov* (1856-1922), is based on the theory of *linear* estimation. It is regarded as one of the *important* methods of point estimation. An estimator found by *minimizing* the sum of squared deviations of the sample values from some *function* that has been hypothesized as a *fit* for the data, is called the least squares estimator. The method of least-squares has already been discussed in connection with regression analysis that was presented in Lecture No. 15.

You will recall that, when fitting a straight line y = a+bx to real data, 'a' and 'b' were determined by *minimizing* the sum of squared deviations between the fitted line and the data-points.

The y-intercept and the slope of the fitted line i.e. 'a' and 'b' are *least-square estimates* (respectively) of the y-intercept and the slope of the *TRUE* line that would have been obtained by considering the entire population of data-points, and not just a sample.

## METHOD OF MAXIMUM LIKELIHOOD

*it can be satisfy all the qualities of good point estimator except unbasied*

The method of maximum likelihood is regarded as the *MOST important* method of estimation, and is the *most* widely used method. This method was introduced in 1922 by Sir Ronald A. Fisher (1890-1962).The mathematical technique of finding Maximum Likelihood Estimators is a bit *advanced*, and involves the concept of the Likelihood Function.

## RATIONALE OF THE METHOD OF MAXIMUM LIKELIHOOD (ML)

"To consider *every* possible value that the parameter might have, and for *each* value, compute the *probability* that the given sample would have occurred if that *were* the true value of the parameter. That value of the parameter for which the probability of a given sample is *greatest*, is chosen as an estimate." An estimate obtained by this method is called the maximum likelihood estimate (MLE). It should be noted that the method of maximum likelihood is applicable to *both* discrete and continuous random variables.

*the value of parameter which have the highest probability*

## EXAMPLES OF MLE's IN CASE OF DISCRETE DISTRIBUTIONS

**Example-1:**

For the Poisson distribution given by

$$P(X = x) = \frac{e^{-\mu}\mu^x}{x!},\ x = 0,1,2,......,$$

the MLE of μ is $\bar{X}$ (the sample mean).

**EXAMPLE-2**

For the geometric distribution given by the MLE of p is Hence, the MLE of p is equal to the reciprocal of the mean. $1/\bar{X}$

**EXAMPLE-3**
For the Bernoulli distribution given by

$$P(X = x) = p^x q^{1-x},\ x = 0,1\ ,$$

the MLE of p is (the sample mean).

**EXAMPLES OF MLE's IN CASE OF *CONTINUOUS* DISTRIBUTIONS**
**Example-1**
For the exponential distribution given by

$$f(x) = \theta e^{-\theta x},\ x > 0,\ \theta > 0\ ,$$

the MLE of θ is (the reciprocal of the sample mean $\frac{1}{\bar{X}}$.)

**EXAMPLE-2**

For the *normal* distribution with parameters μ and σ2, the *joint* ML estimators of μ and σ2 is the sample mean and the sample variance S2*(which is not an unbiased estimator of σ2).*As indicated many times earlier, the *normal distribution* is encountered frequently in practice, and, in this regard, it is both interesting and important to note that, in the case of this *frequently* encountered distribution, the *simplest* formulae (i.e. the sample *mean* and the sample *variance*) fulfill the criteria of the relatively *advanced* method of maximum likelihood estimation !The last example among the five presented above (the one on the *normal* distribution) points to *another* important fact --- and that is : The Maximum Likelihood Estimators are consistent and efficient but *not necessarily unbiased*. (As we know, S2 is *not* an unbiased estimator of σ2.)

**EXAMPLE**

It is well-known that human weight is an approximately normally distributed variable. Suppose that we are interested in estimating the mean and the variance of the weights of adult males in one particular province of a country. A random sample of 15 adult males from this particular population yields the following weights (in pounds):

| | | | | |
|---|---|---|---|---|
| 131.5 | 136.9 | 133.8 | 130.1 | 133.9 |
| 135.2 | 129.6 | 134.4 | 130.5 | 134.2 |
| 131.6 | 136.7 | 135.8 | 134.5 | 132.7 |

Find the maximum likelihood estimates for θ1 = μ and θ2 = σ2.

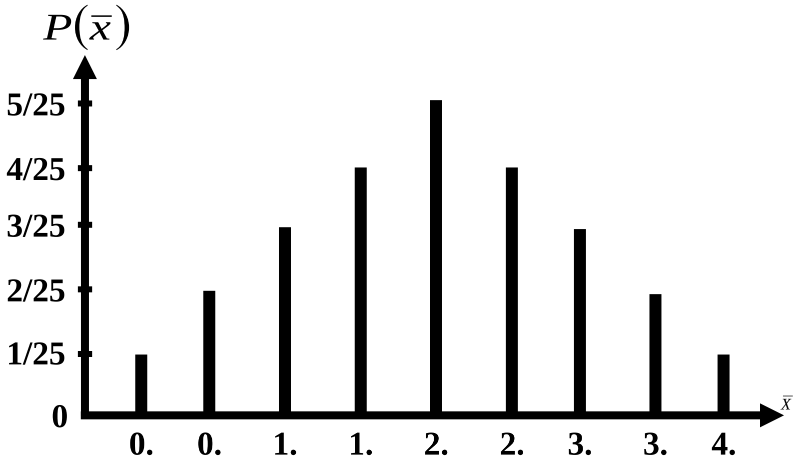
**SOLUTION**

The above data is that of a random sample of size 15 from N(μ, σ2). It has been mathematically proved that the joint maximum likelihood estimators of μ and σ2 are $\bar{X}$ and S2. We compute these quantities for this particular sample, and obtain $\bar{X}$ = 133.43, and S2 = 5.10 .These are the Maximum Likelihood Estimates of the mean and variance of the population of weights in this particular example. Having discussed the concept of point estimation in some detail, we now begin the discussion of the concept of *interval estimation*:
As stated earlier, whenever a *single* quantity computed from the sample acts as an estimate of a population parameter, we call that quantity a *point* estimate e.g. the sample mean is a point estimate of the population mean μ.
The *limitation* of point estimation is that we have no way of ascertaining how close our point estimate is to the true value (the parameter).
For example, we know that is an unbiased estimator of μ i.e. if we had taken all possible samples of a particular size from the population and calculated the mean of each sample, then the mean of the sample means would have been equal to the population mean (μ), but in an actual survey we will be selecting only one sample from the population and will calculate its mean .
We will have no way of ascertaining how close this particular is toμ. Whereas a point estimate is a single value that acts as an estimate of the population parameter, *interval estimation* is a procedure of estimating the unknown parameter which specifies a *range* of values within which the parameter is expected to lie. A *confidence interval* is an interval computed from the sample observations x1, x2….xn, with a statement of how *confident* we are that the interval *does* contain the population parameter.

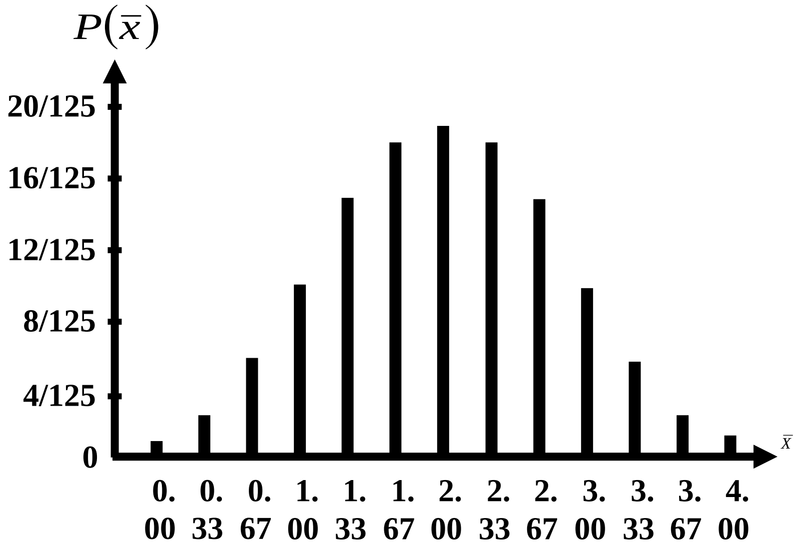
We develop the concept of interval estimation with the help of the example of the Ministry of Transport test to which all cars, irrespective of age, have to be submitted.

**EXAMPLE**

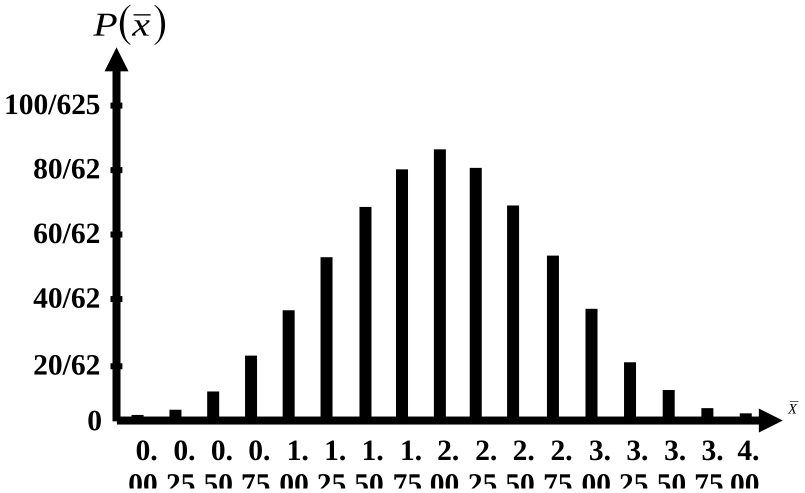
Let us examine the case of an annual Ministry of Transport test to which all cars, irrespective of age, have to be submitted. The test looks for faulty breaks, steering, lights and suspension, and it is discovered after the first year that approximately the *same number* of cars has 0, 1, 2, 3, or 4 faults. You will recall that when we drew all possible samples of size 2 from this uniformly distributed population, the sampling distribution of $\bar{X}$ was *triangular*:
Sampling Distribution of $\bar{X}$ for n = 2




But when we considered what happened to the shape of the sampling distribution with if the sample size is *increased*, we found that it was somewhat like a normal distribution:
Sampling Distribution of $\bar{X}$ for n = 3




And, when we increased the sample size to 4, the sampling distribution resembled a normal distribution even *more* closely, Sampling Distribution of $\bar{X}$ for n = 4




It is clear from the above discussion that as *larger* samples are taken, the shape of the sampling distribution of $\bar{X}$ undergoes *discernible changes*.

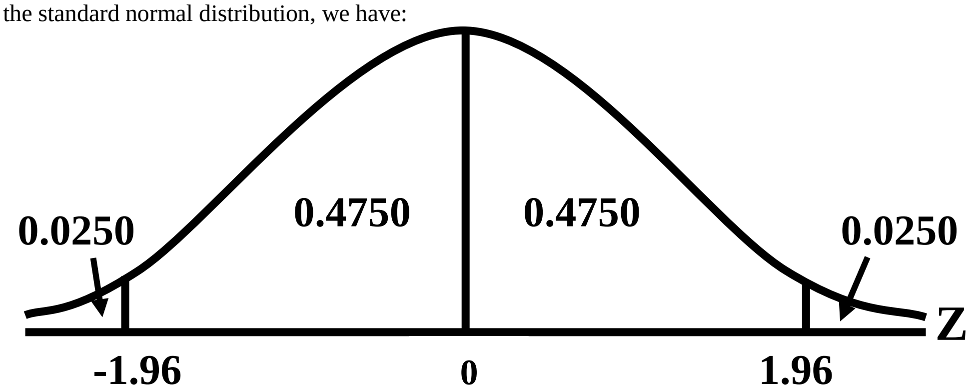
In all three cases the line charts are symmetrical, but as the sample size increases, the overall configuration changed from a triangular distribution to a *bell-shaped* distribution. In other words, for large samples, we are dealing with a *normal* sampling distribution of $\bar{X}$ .In other words: When sampling from an infinite population such that the sample size n is large, $\bar{X}$ is *normally distributed* with mean μ and variance $\frac{\sigma^2}{n}$

i.e. $\bar{X}$ is $N\left(\mu, \frac{\sigma^2}{n}\right)$.

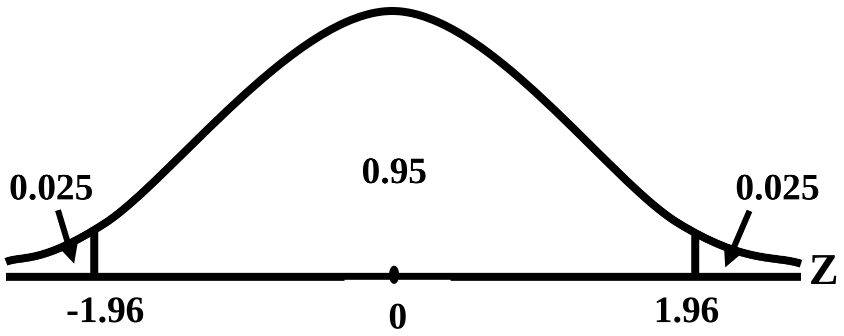
Hence, the standardized version of $\bar{X}$ i.e.

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

is normally distributed with mean 0 and variance 1 i.e. Z is N(0, 1). Now, for the standard normal distribution, we have: For the standard normal distribution, we have:




The above is equivalent to P(-1.96 < Z < 1.96)
= 0.4750 + 0.4750 = 0.95




In other words:

$$P\left(-1.96 \le \frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \le 1.96\right) = 0.95$$

The above can be re-written as:

$$P\left(-1.96\frac{\sigma}{\sqrt{n}} \le \overline{X}-\mu \le 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

Or

$$P\left(-\overline{X}-1.96\frac{\sigma}{\sqrt{n}} \le -\mu \le -\overline{X}+1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

or

$$P\left(\overline{X}+1.96\frac{\sigma}{\sqrt{n}} \ge \mu \ge \overline{X}-1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

or

$$P\left(\overline{X}-1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X}+1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

The above equation yields the 95% confidence interval for μ :
The 95% confidence interval for μ is

$$\left(\overline{X}-1.96\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X}+1.96\frac{\sigma}{\sqrt{n}}\right).$$

In other words, the 95% C.I. for μ is given by

$$\overline{X} \pm 1.96\frac{\sigma}{\sqrt{n}}$$

In a *real-life* situation, the population standard deviation is usually *not known* and hence it has to be *estimated*.
It can be mathematically proved that the quantity

$$s^2 = \frac{\sum\left(X-\overline{X}\right)^2}{n-1}$$

is an unbiased estimator of σ2 (the population variance). (just as the sample mean is an unbiased estimator of μ).
In this situation, the 95% Confidence Interval for μ is given by:

$$P\left(\overline{X}-1.96\frac{s}{\sqrt{n}} < \mu < \overline{X}+1.96\frac{s}{\sqrt{n}}\right) = 95\%$$

The points

$$\overline{X}-1.96\frac{s}{\sqrt{n}} \text{ and } \overline{X}+1.96\frac{s}{\sqrt{n}}$$

are called the lower and upper *limits* of the 95% confidence interval.

Masters

## LECTURE NO. 35

- Confidence Interval for μ (continued).
- Confidence Interval for μ1-μ2.

In the last lecture, we discussed the construction of the 95% confidence interval regarding the mean of a population i.e. μ.

**EXAMPLE-1**

Consider a car assembly plant employing something over 25,000 men. In planning its future labour requirements, the management wants an estimate of the number of days lost per man each year due to illness or absenteeism.
A random sample of 500 employment records shows the following situation:

| Number of Days Lost | Number of Employees |
|---|---|
| None | 48 |
| 1 or 2 | 43 |
| 3 or 4 | 90 |
| 5 or 6 | 186 |
| 7 or 8 | 78 |
| 9 to 12 | 34 |
| 13 to 20 | 21 |
| Total | 500 |

Construct a 95% confidence interval for the mean number of days lost per man each year due to illness or absenteeism.

**SOLUTION**

- The point estimate of μ is $\overline{X}$, which in this example comes out to be $\overline{X} = 5.38$ days
- In order to construct a confidence interval for μ, we need to compute s, which in this example comes out to be s = 3.53 days.

Hence, the 95% confidence interval for μ comes out to be

$$\left(5.38 - \frac{1.96 \times 3.53}{\sqrt{500}}, 5.38 + \frac{1.96 \times 3.53}{\sqrt{500}}\right)$$

or 5.38 ± 0.31 days
= 5.07 days to 5.69 days.

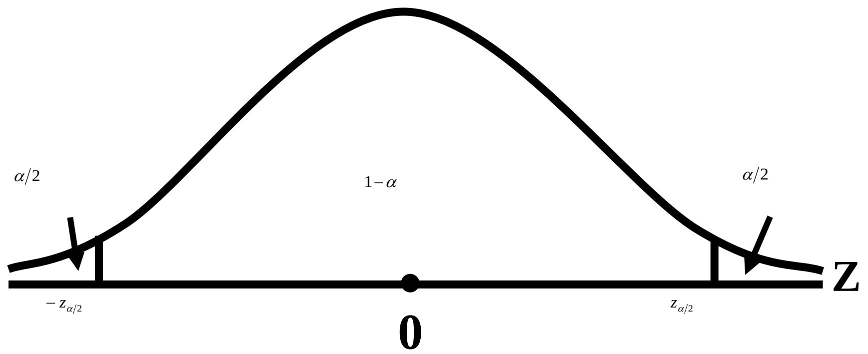
In other words, we can say that the mean number of days lost per man each year due to illness or absenteeism lies somewhere between 5.07 days and 5.69 days, and this statement is being made on the basis of 95% confidence. A very important point to be noted here is that we should be very careful regarding the interpretation of confidence intervals
When we set 1 - α = 0.95, it means that the probability is 95% that the interval

$$\text{from } \overline{X} - 1.96\frac{\sigma}{\sqrt{n}} \text{ to } \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}$$

will actually contain the true population mean μ.In other words, if we construct a large number of intervals of this type, corresponding to the large number of samples that we can draw from any particular population, then out of every 100 such intervals, 95 will contain the true population mean μ whereas 5 will not.
The above statement pertains to the overall situation in repeated sampling --- once a sample has actually been chosen from a population, $\overline{X}$ computed and the interval constructed, then this interval either contains μ, or does not contain μ. So, probability that our interval corresponding to sample values have actually occurred, is either one (i.e. cent per cent), or zero. The statement 95% probability is valid before any sample has actually materialized. In other words, we can say that our procedure of interval estimation is such that, in repeated sampling, 95% of the intervals will containμ. The above example pertained to the 95% confidence interval for μ.In general; the lower and upper limits of the confidence interval for μ are given by

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

Where the value of zα/2 depends on how much confidence we want to have in our interval estimate.




The above situation leads to the (1-α) 100% C.I. for μ . If (1-α) = 0.95, then zα/2 = 1.96 whereas , If (1-α) = 0.99, then zα/2 = 2.58 and If (1-α) = 0.90, then zα/2 = 1.645 .
(The above values of zα/2 are easily obtained from the area table of the standard normal distribution).An important to note is that, as indicated earlier, the above formula for the conference interval is valid when we are sampling from an infinite population in such a way that the sample size n is large. How large should n be in a practical situation?
The rule of thumb in this regard is that whenever n ≥ 30, we can use the above formula.

## CONFIDENCE INTERVAL FOR μ, THE MEAN OF AN INFINITE POPULATION

For large n (n ≥ 30), the confidence interval is given by

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $\overline{x} = \frac{\sum x}{n}$ is the sample mean

and

$$s = \sqrt{\frac{\sum (x - \overline{x})^2}{n-1}}$$

is the sample standard deviation.

**EXAMPLE-1**

The Punjab Highway Department is studying the traffic pattern on the G.T. Road near Lahore. As part of the study, the department needs to estimate the average number of vehicles that pass the Ravi Bridge each day. A random sample of 64 days gives X = 5410 and s = 680. Find the 90 per cent confidence interval estimate for μ, the average number of vehicles per day.

**SOLUTION**

The 90% confidence interval for μ is

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}},$$

where
$\overline{x}$ = 5410,
s = 680, n = 64 and z0.05 = 1.645.

Substituting these values, we obtain

$$5410 \pm (1.645)\left(\frac{680}{\sqrt{64}}\right)$$

or 5410 ± (1.645) ( 85)
or 5410 ± 139.8
or 5270.2 to 5549.8
or, rounding the above two figures correct to the nearest whole number, we have
5270 to 5550

Hence, we can say that the average number of vehicles that pass the Ravi bridge each day lies somewhere between 5270 and 5550, and this statement is being made on the basis of 90% confidence.

**EXAMPLE-2**

Suppose a car rental firm wants to estimate the average number of miles traveled per day by each of its cars rented in one particular city. A random sample of 110 cars rented in this particular city reveals that the mean travel distance per day is 85.5 miles, with a standard deviation of 19.3 miles.
Compute a 99% confidence interval to estimate$\mu$.

**SOLUTION**

Here, n = 110, $\overline{X}$ = 85.5, and S = 19.3. For a 99% level of confidence, a z-value of 2.575 is obtained.

$$\overline{X} - Z_{\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \overline{X} + Z_{\alpha/2}\frac{S}{\sqrt{n}}$$

$$85.5 - 2.575\frac{19.3}{\sqrt{110}} \le \mu \le 85.5 + 2.575$$

$$85.5 - 4.7 \le \mu \le 85.5 + 4.7$$

$$80.8 \le \mu \le 90.2$$

The point estimate indicates that the average number of miles traveled per day by a rental car in this particular city is 85.5. With 99% confidence, we estimate that the population mean is somewhere between 80.8 and 90.2 miles per day.
Next, we consider a very interesting and important way of interpreting a confidence interval. An Important Way of Interpreting a Confidence Interval, Because of the fact that

$$\sigma_{\overline{x}} \text{ is equal to } \frac{\sigma}{\sqrt{n}},$$

Hence,
$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ is equal to}$$
$$\overline{x} \pm z_{\alpha/2}\sigma_{\overline{x}}$$

(where $\sigma_{\overline{x}}$ represents the standard error of $\overline{X}$ ,Hence The C.I. for $\mu$ can be defined as X ± a certain number of standard errors of $\overline{X}$ . efining a Confidence Interval as:
"A point estimate plus/minus a few times the standard error of that estimate", The question arises: "How many times?"
The answer is:That depends on the level of confidence that we wish to have. In the case of 99% confidence, $z\alpha/2 \sim 2.5$, (so that, in this case, we can say that our confidence interval is

$$\overline{x} \pm 2\tfrac{1}{2}\sigma_{\overline{x}});$$

Similarly,
in the case of 95% confidence, $z\alpha/2 \sim 2$, (so that, in this case, we can say that our confidence interval is and so on.

$$\overline{x} \pm 2\sigma_{\overline{x}});$$

Another important point to be noted is that:
It is a matter of common sense that, in any situation, the narrower our confidence interval, the better.
(Ideally, the width of a confidence interval should be zero --- i.e. we should simply have a point estimate.)
It would be quite unwise to say: "I am 99.999% confident that the mean height of the adult males of this particular city lies somewhere between 4 feet and 12 feet." _!
The important question is: How do we achieve a narrow confidence interval with a high level of confidence?
To answer this question, we should have a closer look at the expression of the confidence interval:

$$\overline{x} \pm z_{\alpha/2}\sigma_{\overline{x}}$$

This expression shows clearly that if the quantity $z_{\alpha/2}\sigma_{\overline{x}}$is small, we will achieve a narrow confidence interval.

This quantity will be small if either $\sigma_{\overline{x}}$ is small or $z_{\alpha/2}$ is small.
Now,

$$\sigma_{\overline{x}} \text{ is equal to } \frac{\sigma}{\sqrt{n}},$$

and hence $\sigma_{\overline{x}}$ will be small if the sample size n is large.

On the other hand, $z_{\alpha/2}$ will be small if the level of confidence 1-α is relatively low. As far as the first point that of n being small is concerned, it should be noted that, in many real-life situations, due to practical constraints, we cannot increase the sample size beyond a certain limit.(We may not have the resources to be able to draw a relatively large sample --- our budget may be limited, the time-period at our disposal may be short, etc. As far as the second point, that of fixing a relatively low level of confidence, is concerned, this is in our own hands, and we can fix our level of confidence as low as we wish --- but, obviously, it will not make much sense to say; "I have estimated that the mean height of adult males of this particular city lies somewhere between 5 feet, 6 inches and 5 feet, 7 inches, and I am saying this with 20% confidence." _!
The gist of the above discussion is that, in any real-life situation, given a particular sample size, we need to strike a compromise between how low a level of confidence can we tolerate, or how wide an interval can we tolerate.
Next, we consider the confidence interval for the difference between two population means i.e. μ1-μ2:

**CONFIDENCE INTERVAL FOR THE DIFFERENCE BETWEEN THE MEANS OF TWO POPULATIONS**

For large samples drawn independently from two populations, the C.I. for μ1 – μ2 is given by

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where
Subscript 1 denotes the first population, and subscript 2 denotes the second population. We illustrate this concept with the help of a few examples:

**EXAMPLE-1:**

The means and variances of the weekly incomes in rupees of two samples of workers are given in the following table, the samples being randomly drawn from two different factories:

| Factory | Sample Size | Mean | Variance |
|---|---|---|---|
| A | 160 | 12.80 | 64 |
| B | 220 | 11.25 | 47 |

Calculate the 90% confidence interval for the real difference in the incomes of the workers from the two factories.

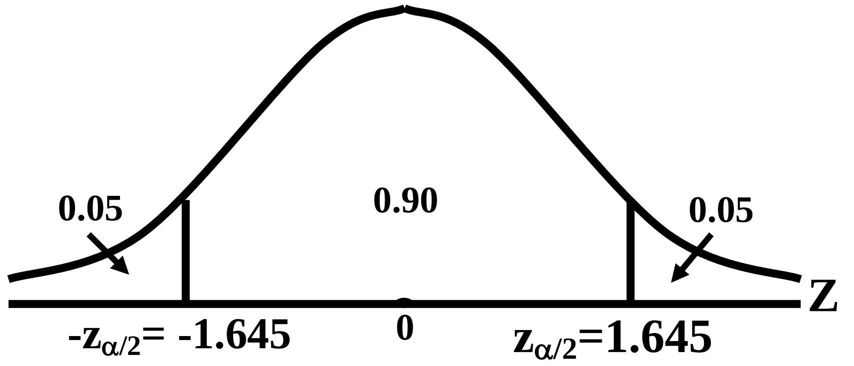
**SOLUTION**

**1.** If both n1 and n2 are large, the confidence limits are given by

$$\left(\bar{x}_1 - \bar{x}_2\right) \pm z_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

**2.** We know that

$z_{\alpha/2}$ = 1.645 for 90% confidence




**3.**Hence, Substituting the values in the formula, we obtain

$$(12.80 - 11.25) \pm 1.645\sqrt{\frac{64}{160} + \frac{47}{220}}$$

$$\sqrt{0.4 + 0.21}$$

or $1.55 \pm 1.645$

or $1.55 \pm 1.645 \sqrt{0.61}$
or $1.55 \pm 1.28$
or 0.27 and 2.83

Hence we can say that we are 90% confident that, on the average, the difference in the incomes of the workers from the two factories lies somewhere between Rs.0.27 and Rs.2.83.

**EXAMPLE-2**

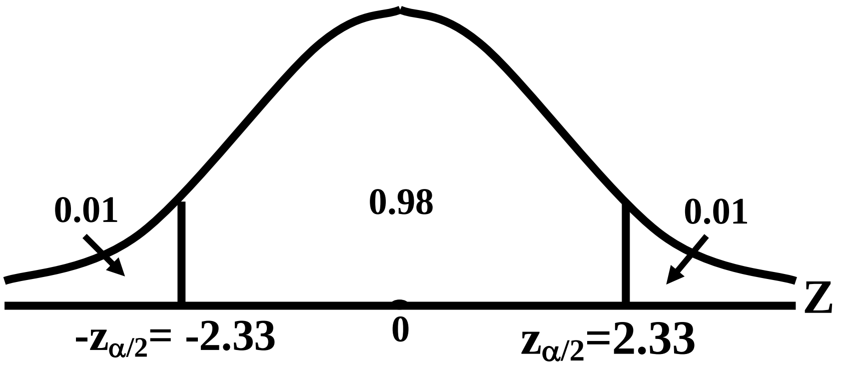
Suppose a study is conducted in a developed country to estimate the difference between middle-income shoppers and low-income shoppers in terms of the average amount saved on grocery bills per week by using coupons. Random samples of 60 middle-income shoppers and 80 low-income shoppers are taken, and their purchases are monitored for 1 week. The average amounts saved with coupons, as well as sample sizes and sample standard deviations are given below:

| Middle-Income Shoppers | Low-Income Shoppers |
|---|---|
| $n_1 = 60$ | $n_2 = 80$ |
| $\overline{X}_1 = \$5.84$ | $\overline{X}_2 = \$2.67$ |
| $S_1 = \$1.41$ | $S_2 = \$0.54$ |

Use this information to construct a 98% confidence interval to estimate the difference between the mean amounts saved with coupons by middle-income shoppers and low-income shoppers.

**SOLUTION:**

The value of $z_{\alpha/2}$ associated with a 98% level of confidence is 2.33.




Using this value, we can determine the confidence interval as follows:

$$(5.84 - 2.67) - 2.33\sqrt{\frac{1.41^2}{60} + \frac{0.54^2}{80}}$$
$$\leq \mu_1 - \mu_2$$
$$\leq (5.84 - 2.67) + 2.33\sqrt{\frac{1.41^2}{60} + \frac{0.54^2}{80}}$$

$$3.17 - 0.45 \leq \mu_1 - \mu_2 \leq 3.17 + 0.45$$
$$2.72 \leq \mu_1 - \mu_2 \leq 3.62$$

Hence, the 98% confidence interval for the difference between the mean amounts saved with coupons by middle-income shoppers and low-income shoppers is ($2.72, $3.62). The point estimate for the difference in mean savings is $3.17. Note that a zero difference in the population means of these two groups is unlikely, because the number zero is not in the 98% range. The data seems to provide a strong indication that, on the average, the middle income shoppers are saving a little more than the low income shoppers.

Masters

# LECTURE NO 36

- Large Sample Confidence Intervals for p and p1-p2
- Determination of Sample Size (with reference to Interval Estimation)
- Hypothesis-Testing (An Introduction)

In the last lecture, we discussed the construction and the interpretation of the confidence intervals for μ and μ1 - μ2. We begin today's lecture by focusing on the confidence intervals for p and p1-p2. First, we consider the confidence interval for p, the proportion of successes in a binomial population:

## CONFIDENCE INTERVAL FOR A POPULATION PROPORTION (P)

For a large sample drawn from a binomial population, the C.I. for p is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where

$\hat{p}$ = proportion of "successes" in the sample
n = sample size
$z\alpha/2$ = 1.96 for 95% confidence
= 2.58 for 99% confidence

(In a practical situation, the criterion for deciding whether or not n is sufficiently large is that if both np and nq are greater than or equal to 5, then we say that n is sufficiently large).We illustrate this concept with the help of a few examples:

### EXAMPLE-1

As a practical illustration, let us look at a survey of teenagers who have appeared in a juvenile court three times or more. A survey of 634 of these shows that 291 are orphans (one or both parents dead). What proportion of all teenagers with three or more appearances in court are orphans? The estimate is to be made with 99% confidence.

### SOLUTION

In this problem, we have n = 634, and

$\hat{p}$ =291/634 = 0.459,

$\hat{q} = 1 - \hat{p}$ = 0.541,

Hence, the 99% confidence limits for p are:

$$0.459 \pm 2.58\sqrt{\frac{0.459 \times 0.541}{634}}$$

$= 0.459 \pm 0.051$
= 0.408 and 0.510

Hence, we estimate that the percentage of teenagers of this type who are orphans lies between 40.8 per cent and 51.0 per cent. It should be noted that, in this problem, happily, the confidence interval has come out to be pretty narrow, and this is happening in spite of the fact that the level of confidence is very high ! This very desirable situation can be ascribed to the fact that the sample size of 634 is pretty large.

### EXAMPLE-2

After a long career as a member of the City Council, Mr. Scott decided to run for Mayor.
The campaign against the present Mayor has been strong with large sums of money spent by each candidate on advertisements. In the final weeks, Mr. Scott has pulled ahead according to polls published in a leading daily newspaper. To check the results, Mr. Scott's staff conducts their own poll over the weekend prior to the election. The results show that for a random sample of 500 voters 290 will vote for Mr. Scott. Develop a 95 percent confidence interval for the population proportion who will vote for Mr. Scott. Can he conclude that he will win the election?

### SOLUTION

We begin by estimating the proportion of voters who will vote for Mr. Scott. The sample included 500 voters and 290 favored Mr. Scott. Hence, the sample proportion is 290/500 = 0.58. The value 0.58 is a point estimate of the unknown population proportion p.
The 95% Confidence Interval for p is:

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= 0.58 \pm 1.96\sqrt{\frac{0.58(1-0.58)}{500}}$$
$$= 0.58 \pm 0.043$$
$$= (0.537,\ 0.623)$$

The end points of the confidence interval are 0.537 and 0.623. The lower point of the confidence interval is greater than 0.50. So, we conclude that the proportion of voters in the population supporting Mr. Scott is greater than 50 percent. He will win the election, based on the polling results.

**EXAMPLE-3**

A group of statistical researchers surveyed 210 chief executives of fast-growing small companies. Only 51% of these executives had a management-succession plan in place. A spokesman for the group made the statement that many companies do not worry about management succession unless it is an immediate problem. However, the unexpected exit of a corporate leader can disrupt and unfocused a company for long enough to cause it to lose its momentum.

Use the survey-figure to compute a 92% confidence interval to estimate the proportion of all fast-growing small companies that have a management-succession plan.

**SOLUTION**

The point estimate of the proportion of all fast-growing small companies that have a management-succession plan is the sample proportion found to be 0.51 for that particular sample of size 210 which was surveyed by the group of researchers. Realizing that the point estimate might change with another sample selection, we calculate a confidence interval, as follows:

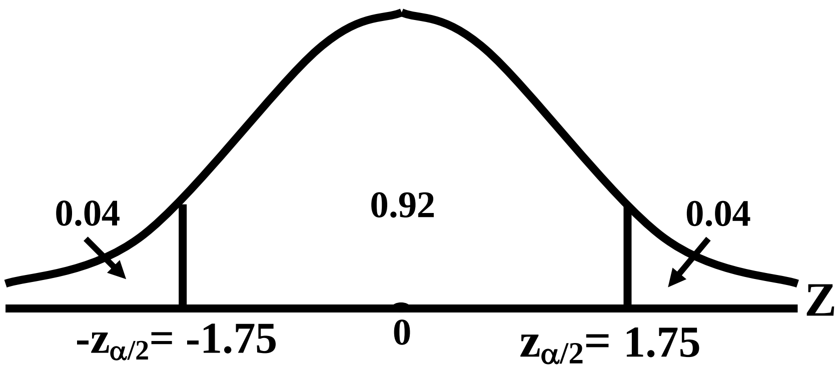
The value of n is 210;

$\hat{p}$ **is 0.51**

**and**

$$\hat{q} = 1 - \hat{p} = 0.49.$$

Because the level of confidence is 92%, the value of $Z_{.04} = 1.75$.




The confidence interval is computed as:

$$0.51 - 1.75\sqrt{\frac{(0.51)(0.49)}{210}} \le p$$
$$\le 0.51 + 1.75\sqrt{\frac{(0.51)(0.49)}{210}}$$
$$0.51 - 0.06 \le p \le 0.51 + 0.06$$
$$0.45 \le p \le 0.57$$
$$P(0.45 \le p \le 0.57) = 0.92.$$

**CONCLUSION**

It is estimated with 92% confidence that the proportion of the population of fast-growing small companies that have a management-succession plan is between 0.45 and 0.57.

Next, we consider the Confidence Interval for the difference in the population proportions (p1 – p2):

**CONFIDENCE INTERVAL FOR P1-P2**

For large samples drawn independently from two binomial populations, the C.I. for p1-p2 is given by

$$(\hat{p}_1 - \hat{p}_2) \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

where

subscript 1 denotes the first population, and subscript 2 denotes the second population.

We illustrate this concept with the help of an example:

**EXAMPLE**

In a poll of college students in a large university, 300 of 400 students living in students' residences (hostels) approved a certain course of action, whereas 200 of 300 students not living in students' residences approved it. Estimate the difference in the proportions favoring the course of action, and compute the 90% confidence interval for this difference.

**SOLUTION**

Let $\hat{p}_1$ be the proportion of students favouring the course of action in the first sample (i.e. the sample of resident students). And, let $\hat{p}_2$ be the proportion of students favouring the course of action in the second sample (i.e. the sample of students not residing in students' residences).

Then

$$\hat{p}_1 = \frac{300}{400} = 0.75,$$

And

$$\hat{p}_2 = \frac{200}{300} = 0.67.$$

$\therefore$ Difference in proportions

$= \hat{p}_1 - \hat{p}_2 =$ 0.75 – 0.67 = 0.08

The required level of confidence is 0.90. Therefore z0.05 = 1.645, and hence, the 90% confidence interval for p1 – p2 is

90% C.I. for p1-p2:

$$(\hat{p}_1 - \hat{p}_2) \pm (1.645)\sqrt{\frac{\hat{p}_1\ \hat{q}_1}{n_1} + \frac{\hat{p}_2\ \hat{q}_2}{n_2}}$$

$$or\ \ 0.08 \pm (1.645)\sqrt{\frac{(0.75)(0.25)}{400} + \frac{(0.67)(0.33)}{300}}$$

or 0.08 ± (1.645)

or 0.08 ± (1.645) (0.0347)

or 0.08 ± 0.057

or 0.023 to 0.137

Hence the 90 per cent confidence interval for p1 – p2 is (0.023, 0.137). In other words, on the basis of 90% confidence, we can say that the difference between the proportions of resident students and non-resident students who favor this particular course of action lies somewhere between 2.3% and 13.7%.Evidently, this seems to be a rather wide interval, even though the level of confidence is not extremely high. Hence, it is obvious that, in this example, sample sizes of 400 and 300 respectively, although apparently quite large, are not large enough to yield a desirably narrow confidence interval.

In the last lecture, we discussed the construction and interpretation of confidence intervals. Next, we consider the determination of sample size. In this regard, the first point to be noted is that, in any statistical study based on primary data, the first question is what is going to be the size of the sample that is to be drawn from the population of interest? We present below a method of finding the sample size in such a way that we obtain a desired level of precision with a desired level of confidence, first, we consider the determination of sample size in that situation when we are trying to estimate $\mu$, the population mean:

## Sample size for Estimating Population Mean

In deriving the 100(1–α) per cent confidence Interval for μ, we have the expression

$$P\left(-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{X}-\mu \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)=1-\alpha$$

which implies that the maximum allowable difference between $\bar{X}$ and μ is:

$$\left|\bar{x}-\mu\right| = z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $\frac{\sigma}{\sqrt{n}}$ is the standard error of $\bar{X}$ when sampling is performed with replacement of population is very large (infinite). The quantity $\left|\bar{x}-\mu\right|$ is also called the error of the estimator $\bar{X}$ and is denoted by e. Thus a 100(1–α) per cent error bound for estimating μ is given by $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. In other words, in order to have a 100(1–α) per cent \confidence that the error is estimating μ with $\bar{X}$ to be less than e, we need n such that

$$e = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

or $$\sqrt{n} = z_{\alpha/2}\frac{\sigma}{e}$$

or $$n = \left(\frac{z_{\alpha/2}\,\sigma}{e}\right)^2$$

Hence the desired sample size for being 100(1–α)% confident that the error in estimating μ will be less than e, when sampling is with replacement or the population is very large, is given by

$$n = \left(\frac{z_{\alpha/2}\,\sigma}{e}\right)^2$$

It is important to note that the population standard deviation σ is generally not known, and hence, its estimate is found either from past experience or from a pilot sample of size n > 30. In case of fractional result, it is always to be rounded to the next higher integer for the sample size.

**EXAMPLE**

A research worker wishes to estimate the mean of a population using a sample sufficiently large that the probability will be 0.95 that the sample mean will not differ from the true mean by more than 25 percent of the standard deviation. How large a sample should be taken?

**SOLUTION**

If the sample mean is not being allowed to differ from the true mean by more than 25% of σ with a probability of 0.95, then

$$e = \left|\bar{x}-\mu\right| = \frac{25\sigma}{100} = \frac{\sigma}{4}, \quad and \quad z_{\alpha/2} = 1.96.$$

Substituting these values in the formula

$$n = \left(\frac{z_{\alpha/2}\,\sigma}{e}\right)^2, \text{ we get}$$

$$n = \left(\frac{1.96\times\sigma}{\sigma/4}\right)^2 = 61.4656\ .$$

Hence the required sample size is 62, (the next higher integer), as the sample size cannot be fractional.

Next, we consider the determination of sample size in that situation when we are trying to estimate p, the proportion of successes in the population:

**SAMPLE SIZE FOR ESTIMATING POPULATION PROPORTION**

The large sample confidence interval for p is given by

$$\hat{p} = z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

This implies that $e = z_{\alpha/2}\sqrt{\frac{\hat{p}\hat{q}}{n}}$

Therefore, solving for n, we obtain

$$n = \frac{(z_{\alpha/2})^2 \hat{p}\hat{q}}{e^2}$$

Since the values of $\hat{p}$ and $\hat{q}$ are not known as the sample has not yet been selected, we therefore use an estimate $\hat{p}$ obtained from pilot sample information.

**EXAMPLE**

In a random sample of 75 axle shafts, 12 have a surface finish that is rougher than the specification will allow.
How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05?
Solution:

$$e = \left|\hat{p} - p\right| = 0.05,$$

Here

$$\hat{p} = \frac{12}{75} = 0.16,$$

$$\hat{q} = 1 - \hat{p} - 0.84 \text{ and } z_{0.025} = 1.96$$
$$(\because \alpha/2 = 0.025)$$

Substituting these values in the formula

$$n = \left(\frac{z_{\alpha/2}}{e}\right)^2 \hat{p}\hat{q},$$ we obtain

$$n = \left(\frac{1.96}{0.05}\right)^2 \times (0.16)(0.84) = 206.52$$

which, upon rounding upward, yields 207 as the desired sample size. As stated earlier, Inferential Statistics can be divided into two parts, estimation and hypothesis-testing. Having discussed the concepts of point and interval estimation in considerable detail, We now begin the discussion of Hypothesis-Testing:

**HYPOTHESIS-TESTING IS A VERY IMPORTANT AREA OF STATISTICAL INFERENCE**

It is a procedure which enables us to decide on the basis of information obtained from sample data whether to accept or reject a statement or an assumption about the value of a population parameter. Such a statement or assumption which may or may not be true is called a statistical hypothesis. We accept the hypothesis as being true, when it is supported by the sample data. We reject the hypothesis when the sample data fail to support it. It is important to understand what we mean by the terms 'reject' and 'accept' in hypothesis-testing. The rejection of a hypothesis is to declare it false. The acceptance of a hypothesis is to conclude that there is insufficient evidence to reject it. Acceptance does not necessarily mean that the hypothesis is actually true. The basic concepts associated with hypothesis testing are discussed below:

**NULL AND ALTERNATIVE HYPOTHESES**

**NULL HYPOTHESIS**

A null hypothesis, generally denoted by the symbol H0, is any hypothesis which is to be tested for possible rejection or nullification under the assumption that it is true.

A null hypothesis should always be precise such as 'the given coin is unbiased' or 'a drug is ineffective in curing a particular disease' or 'there is no difference between the two teaching methods'. The hypothesis is usually assigned a numerical value. For example, suppose we think that the average height of students in all colleges is 62″. This statement is taken as a hypothesis and is written symbolically as H0 : $\mu$ = 62″. In other words, we hypothesize that $\mu$ = 62″.

## ALTERNATIVE HYPOTHESIS

An alternative hypothesis is any other hypothesis which we are willing to accept when the null hypothesis H0 is rejected. It is customarily denoted by H1 or HA. A null hypothesis H0 is thus tested against an alternative hypothesis H1. For example, if our null hypothesis is H0 : $\mu$ = 62″, then our alternative hypothesis may be H1 : $\mu \neq$ 62″ or H1 : $\mu <$ 62″.

## LEVEL OF SIGNIFICANCE

The probability of committing Type-I error can also be called the level of significance of a test. Now, what do we mean by Type-I error? In order to obtain an answer to this question, consider the fact that, as far as the actual reality is concerned, H0 is either actually true, or it is false. Also, as far as our decision regarding H0 is concerned, there are two possibilities --- either we will accept H0, or we will reject H0. The above facts lead to the following table:

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="2">Decision</td></tr>
<tr><td>Accept $H_0$</td><td>Reject $H_0$ (or accept $H_1$)</td></tr>
<tr><td rowspan="2">True Situation</td><td>$H_0$ is true</td><td>Correct decision (No error)</td><td>Wrong decision (Type-I error)</td></tr>
<tr><td>$H_0$ is false</td><td>Wrong decision (Type-II error)</td><td>Correct decision (No error)</td></tr>
</table>

A close look at the four cells in the body of the above table reveals that the situations depicted by the top-left corner and the bottom right-hand corner are the ones where we are taking a correct decision. On the other hand, the situation depicted by the top-right corner and the bottom left-hand corner are the ones where we are taking an incorrect decision. The situation depicted by the top-right corner of the above table is called an error of the first kind or a Type I-error, while the situation depicted by the bottom left-hand corner is called an error of the second kind or a Type II-error.
In other words:

## TYPE-I AND TYPE-II ERRORS

*imp*

On the basis of sample information, we may reject a null hypothesis H0, when it is, in fact, true or we may accept a null hypothesis H0, when it is actually false. The probability of making a Type I error is conventionally denoted by $\alpha$ and that of committing a Type II error is indicated by $\beta$. In symbols, we may write

$\alpha$ = P (Type I error)
= P (reject H0|H0 is true),
$\beta$ = P (Type II error)
= P (accept H0|Ho is false).

Masters

# LECTURE NO. 37

- Hypothesis-Testing (continuation of basic concepts)
- Hypothesis-Testing regarding μ (based on Z-statistic)

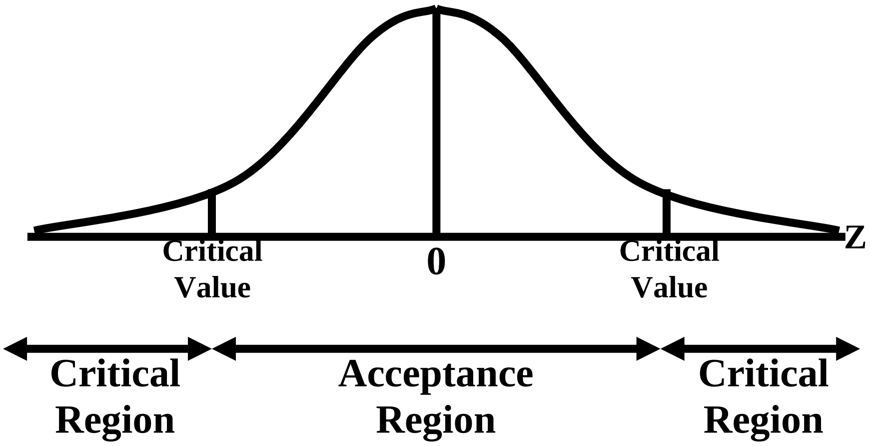
In the last lecture, we commenced the discussion of the concept of Hypothesis-Testing. We introduced the concepts of the Null and Alternative hypotheses as well as the concepts of Type-I and Type-II error. We now continue the discussion of the basic concepts of hypothesis-testing:

## TEST-STATISTIC

A statistic (i.e. a function of the sample data not containing any parameters), which provides a basis for testing a null hypothesis, is called a test-statistic. Every test-statistic has a probability distribution (i.e. sampling distribution) which gives the probability that our test-statistic will assume a value greater than or equal to a specified value OR a value less than or equal to a specified value when the null hypothesis is true.

## ACCEPTANCE AND REJECTION REGIONS

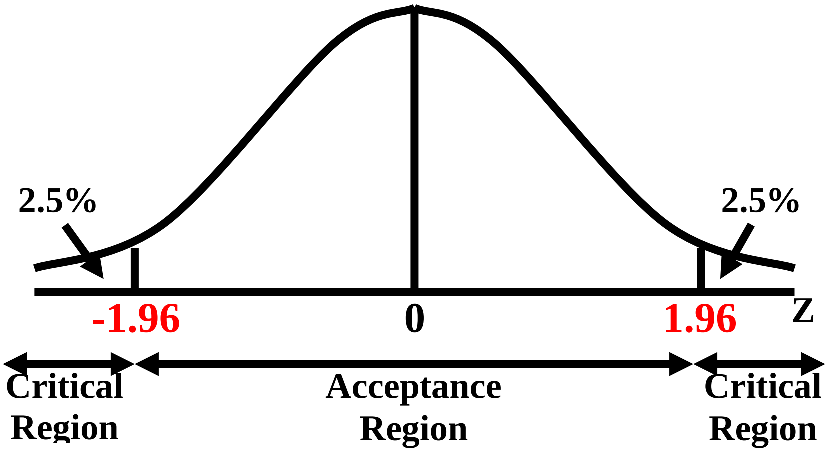
All possible values which a test-statistic may assume can be divided into two mutually exclusive groups: one group consisting of values which appear to be consistent with the null hypothesis (i.e. values which appear to support the null hypothesis), and the other having values which lead to the rejection of the null hypothesis. The first group is called the acceptance region and the second set of values is known as the rejection region for a test. The rejection region is also called the critical region. The value(s) that separates the critical region from the acceptance region, is called the critical value(s):




The critical value which can be in the same units as the parameter or in the standardized units, is to be decided by the experimenter. The most frequently used values of $\alpha$, the significance level, are 0.05 and 0.01, i.e. 5 percent and 1 percent. By $\alpha$ = 5%, we mean that there are about 5 chances in 100 of incorrectly rejecting a true null hypothesis.

## RELATIONSHIP BETWEEN THE LEVEL OF SIGNIFICANCE AND THE CRITICAL REGION

The level of significance acts as a basis for determining the CRITICAL REGION of the test. For example, if we are testing $H_0$: $\mu = 45$ against $H_1$: $\mu \neq 45$, our test statistic is the standard normal variable Z, and the level of significance is 5%, then the critical values are Z = +1.96 Corresponding to a level of significance of 5%, we have:




**ONE-TAILED AND TWO-TAILED TESTS**

A test, for which the entire rejection region lies in only one of the two tails – either in the right tail or in the left tail – of the sampling distribution of the test-statistic, is called a one-tailed test or one-sided test. A one-tailed test is used when the alternative hypothesis H1 is formulated in the following form:

$H_1 : \theta > \theta_0$

or

$H_1 : \theta < \theta_0$

For example, if we are interested in testing a hypothesis regarding the population mean, if n is large, and we are conducting a one-tailed test, then our alternative hypothesis will be stated as

$H_1 : \mu > \mu_0$

or

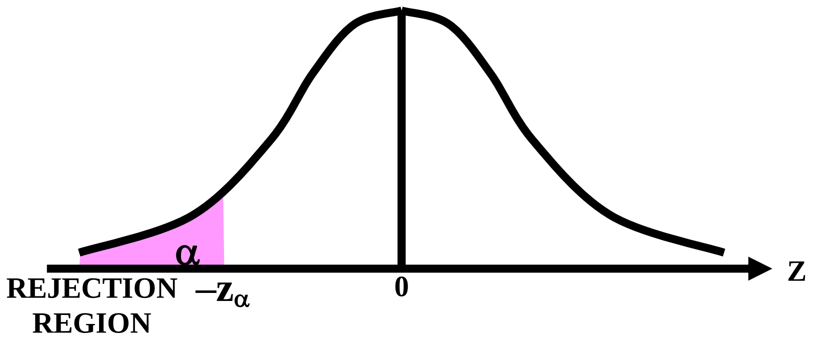
$H_1 : \mu < \mu_0$

In this case, the rejection region consists of either all z-values which are greater than $+ z_\alpha$ or less than $– z_\alpha$ (where $\alpha$ is the level of significance):

If $H_0 : \mu > \mu_0$

$H_1 : \mu < \mu_0$

Then (in case of large n):



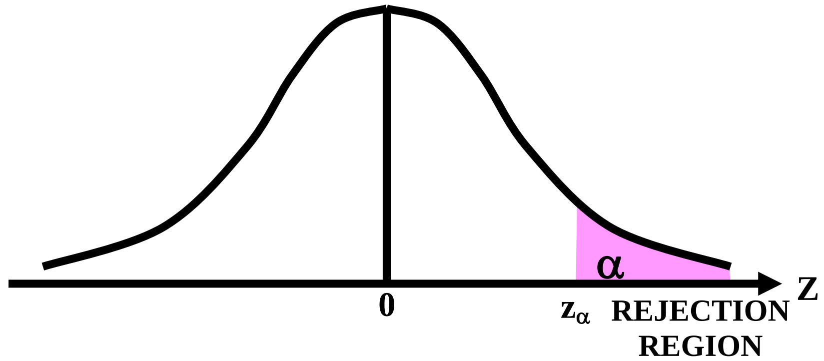

**REJECT H0 if $z < -z_\alpha$**

If

$H_0 : \mu < \mu_0$

$H_1 : \mu > \mu_0$

Then (in case of large n):




**REJECT H0 if z > zα/2**

If, on the other hand, the rejection region is divided equally between the two tails of the sampling distribution of the test-statistic, the test is referred to as a two-tailed test or two-sided test.

In this case, the alternative hypothesis H1 is set up as:

H1 : μ ≠ μ0

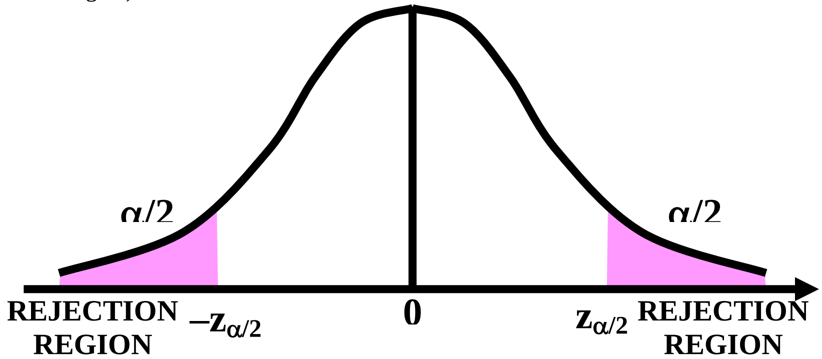
meaning thereby

H1 : μ < μ0 or μ > μ0

If H0 : μ = μ0

H1 : μ ≠ μ0

Then (in case of large n):




**REJECT H0 if z < –zα/2 or z > zα/2**

The location of critical region can be determined only after the alternative hypothesis H1 has been stated. It is important to note that the one-tailed and the two-tailed tests differ only in location of the critical region, not in the size. We illustrate the concept and methodology of hypothesis-testing with the help of an example:

**EXAMPLE**

A steel company manufactures and assembles desks and other office equipment at several plants in a particular country. The weekly production of the desks of Model A at Plant-I has a mean of 200 and a standard deviation of 16. Recently, due to market expansion, new production methods have been introduced and new employees hired. The vice president of manufacturing would like to investigate whether there has been a change in the weekly production of the desks of Model A. To put it another way, is the mean number of desks produced at Plant-I different from 200 at the 0.05 significance level? The mean number of desks produced last year (50 weeks, because the plant was shut down 2 weeks for vacation) is 203.5. On the basis of the above result, should the vice president conclude that the there has been a change in the weekly production of the desks of Model A.

**SOLUTION:**

We use the statistical hypothesis-testing procedure to investigate whether the production rate has changed from 200 per month.

**Step-1:**

Formulation of the Null and Alternative Hypotheses:

The null hypothesis is “The population mean is 200.”

The alternative hypothesis is ‘The mean is different from 200” or “The mean is not 200.”

These two hypotheses are written as follows:

H0 : μ = 200

H1 : μ ≠ 200

**Note:**

This is a two-tailed test because the alternative hypothesis does not state a direction. In other words, it does not state whether the mean production is greater than 200 or less than 200. The vice president only wants to find out whether the production rate is different from 200.

**Step-2:**

Decision Regarding the Level of Significance (i.e. the Probability of Committing Type-I Error):

Here, the level of significance is 0.05.

This is α, the probability of committing a Type-I error (i.e. the risk of rejecting a true null hypothesis).

**Step-3:**

Test Statistic (that statistic that will enable us to test our hypothesis):

The test statistic for a large sample mean is

$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

Transforming the production data to standard units (z values) permits the use of the area table of the standard normal distribution.

**Step-4:**

Calculations:

In this problem, we have n = 50, $\overline{X}$ = 203.5, and σ = 16.

Hence, the computed value of z comes out to be:

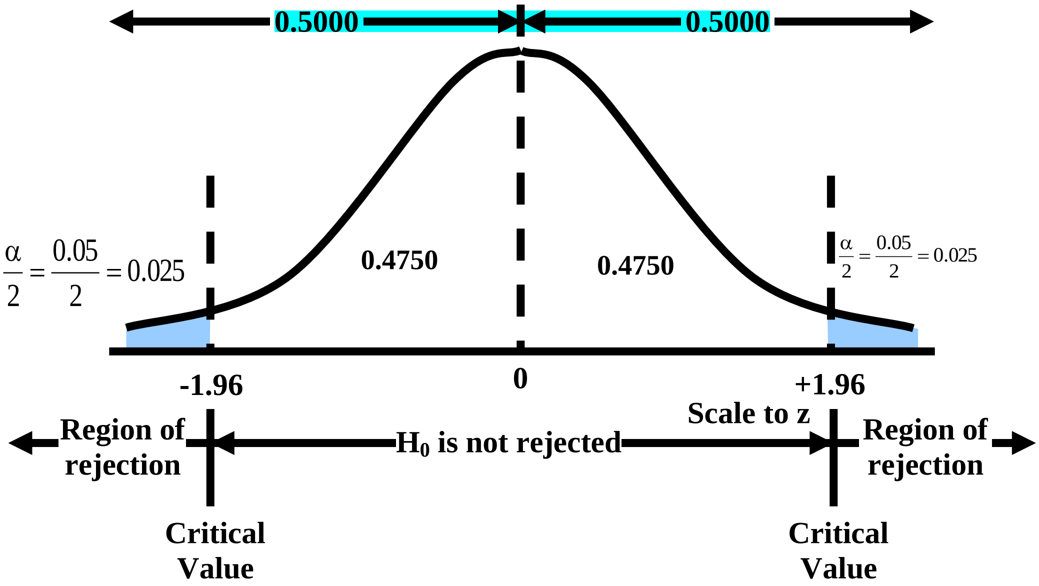
$$z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{203.5 - 200}{16/\sqrt{50}} = 1.55$$

**Step-5:**

Critical Region (that portion of the X-axis which compels us to reject the null hypothesis):Since this is a two-tailed test, half of 0.05, or 0.025, is in each tail.

The area where H0 is not rejected, located between the two critical values, is therefore 0.95.

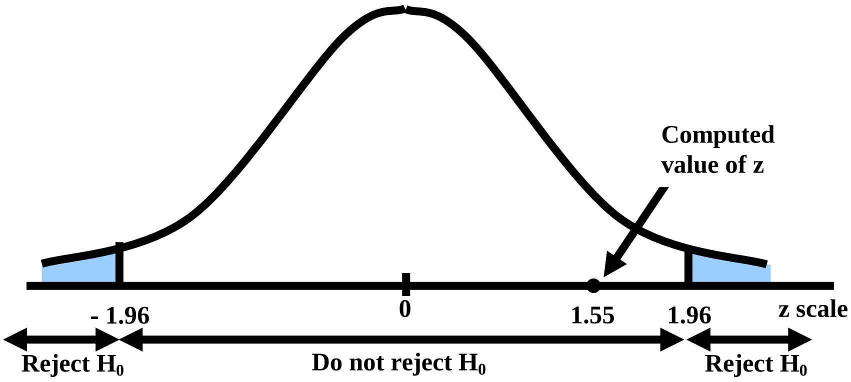
Applying the inverse use of the Area Table, we find that, corresponding to α = 0.05, the critical values are 1.96 and -1.96, as shown below:



**DECISION RULE FOR THE 0.05 SIGNIFICANCE LEVEL THE DECISION RULE IS, THEREFORE**

Reject the null hypothesis and accept the alternative hypothesis if the computed value of z is not between –1.96 and +1.96. Do not reject the null hypothesis if z falls between –1.96 and + 1.96.

Step-6:
Conclusion:
The computed value of z i.e. 1.55 lies between -1.96 and + 1.96, as shown below:




Because 1.55 lies between -1.96 and + 1.96, therefore, it does not fall in the rejection region, and hence H0 is not rejected. In other words, we conclude that the population mean is not different from 200. So, we would report to the vice president of manufacturing that the sample evidence does not show that the production rate at Plant-I has changed from 200 per week. The difference of 3.5 units between the historical weekly production rate and the production rate of last year can reasonably be attributed to chance. The above example pertained to a two-tailed test. Let us now consider a few examples of one-tailed tests:

**EXAMPLE**

A random sample of 100 workers with children in day care show a mean day-care cost of Rs.2650 and a standard deviation of Rs.500. Verify the department's claim that the mean exceeds Rs.2500 at the 0.05 level with this information.

**SOLUTION**
In this problem, we regard the department's claim, that the mean exceeds Rs.2500, as H1,
and regard the negation of this claim as H0.

Thus, we have

**i)** H0 : μ < 2500
H1 : μ > 2500 (exceeds 2500)

(Important Note: We should always regard that hypothesis as the null hypothesis which contains the equal sign.)

**ii)** We are given the significance level at $\alpha = 0.05$.

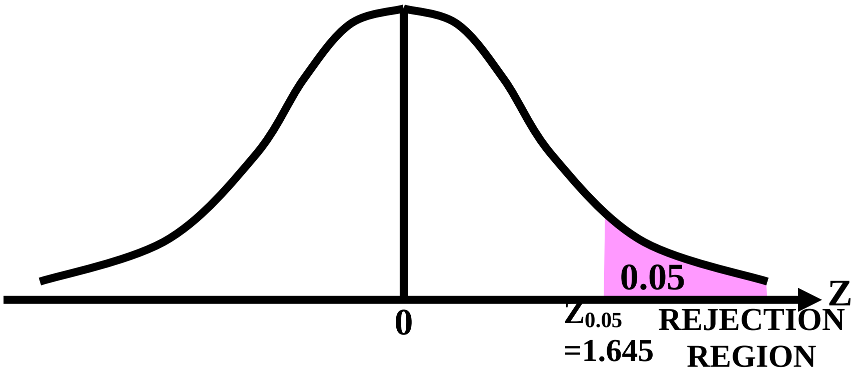
**iii)** The test-statistic, under H0 is

$$Z = \frac{\overline{X} - \mu_0}{S/\sqrt{n}},$$

which is approximately normal as n = 100 is large enough to make use of the central limit theorem.

**iv)** The rejection region is

Z > Z0.05 = 1.645




**v)** Computing the value of Z from sample information, we find

$$z = \frac{2650 - 2500}{500/\sqrt{100}} = \frac{150}{50} = 3$$

**vi) Conclusion:**
Since the calculated value z = 3 is greater than 1.645, hence it falls in the rejection region, and, therefore, we reject H0, and may conclude that the department's claim is supported by the sample evidence.

**An Interesting and Important Point:**

For $\alpha = 0.01$, $Z\alpha = 2.33$.

As our computed value of Z i.e. 3 is even greater than 2.33, the computed value of $\bar{X}$ is highly significant. (With only 1% chance of being wrong, the department's claim was correct).

# LECTURE NO. 38

Masters

- Hypothesis-Testing regarding $\mu_1 - \mu_2$ (based on Z-statistic)
- Hypothesis Testing regarding p (based on Z-statistic)

In the last lecture, we discussed the basic concepts involved in hypothesis-testing. Also, we applied this concept to a few examples regarding the testing of the population mean $\mu$. These examples pointed to the six main steps involved in any hypothesis-testing procedure.

## GENERAL PROCEDURE FOR TESTING HYPOTHESES

Testing a hypothesis about a population parameter involves the following six steps:

- State your problem and formulate an appropriate null hypothesis H0 with an alternative hypothesis H1, which is to be accepted when H0 is rejected.
- Decide upon a significance level of the test, $\alpha$, which is the probability of rejecting the Null Hypothesis if it is true.
- Choose a test-statistic such as the normal distribution, the t-distribution, etc. to test H0.
- Determine the rejection or critical region in such a way that the probability of rejecting the null hypothesis H0, if it is true, is equal to the significance level, $\alpha$. The location of the critical region depends upon the form of H1 (i.e. whether we are carrying out a one-tailed test or a two-tailed test). The critical value(s) will separate the acceptance region from the rejection region.
- Compute the value of the test-statistic from the sample data in order to decide whether to accept or reject the null hypothesis H0.
- Formulate the decision rule (i.e. draw a conclusion) as follows:
  a) Reject the null hypothesis H0, if the computed value of the test statistic falls in the rejection region.
  b) Accept the null hypothesis H0, otherwise.

**IMPORTANT NOTE**

It is very important to realize that when applying a hypothesis-testing procedure of the type explained above, we always begin by assuming that the null hypothesis is true.

**IMPORTANT NOTE:**

As s2 is an unbiased estimator of $\sigma$2 whereas S2 is a biased estimator, hence we would like to use this estimator whenever $\sigma$2 is unknown. However, when n is large, s2 is approximately equal to S2, as explained below:
We know that

$$s^2 = \frac{\sum(x-\bar{x})^2}{n-1} \Rightarrow \sum(x-\bar{x})^2 = (n-1)s^2$$

*whereas*

$$S^2 = \frac{\sum(x-\bar{x})^2}{n} \Rightarrow \sum(x-\bar{x})^2 = nS^2.$$

*Hence*

$$(n-1)s^2 = nS^2 \Rightarrow S^2 = \frac{(n-1)}{n}s^2 = \left(1-\frac{1}{n}\right)s^2$$

Now, as n $\to \infty$, $\frac{1}{n} \to 0.$

Hence, if n is large,

$$S^2 \simeq s^2.$$

Hence, in case of a large sample drawn from a population with unknown variance $\sigma$2, we may replace $\sigma$2 by S2.We now consider the case when we are interested in testing the equality of two population means.
We illustrate this situation with the help of the following example.

**EXAMPLE**

A survey conducted by a market-research organization five years ago showed that the estimated hourly wage for temporary computer analysts was essentially the same as the hourly wage for registered nurses. This year, a random sample of 32 temporary computer analysts from across the country is taken. The analysts are contacted by telephone and asked what rates they are currently able to obtain in the market-place A similar random sample of 34 registered nurses is taken. The resulting wage figures are listed in the following table:

| Computer Analysts | | | Registered Nurses | | |
|---|---|---|---|---|---|
| $ 24.10 | $25.00 | $24.25 | $20.75 | $23.30 | $22.75 |
| 23.75 | 22.70 | 21.75 | 23.80 | 24.00 | 23.00 |
| 24.25 | 21.30 | 22.00 | 22.00 | 21.75 | 21.25 |
| 22.00 | 22.55 | 18.00 | 21.85 | 21.50 | 20.00 |
| 23.50 | 23.25 | 23.50 | 24.16 | 20.40 | 21.75 |
| 22.80 | 22.10 | 22.70 | 21.10 | 23.25 | 20.50 |
| 24.00 | 24.25 | 21.50 | 23.75 | 19.50 | 22.60 |
| 23.85 | 23.50 | 23.80 | 22.50 | 21.75 | 21.70 |
| 24.20 | 22.75 | 25.60 | 25.00 | 20.80 | 20.75 |
| 22.90 | 23.80 | 24.10 | 22.70 | 20.25 | 22.50 |
| 23.20 | | | 23.25 | 22.45 | |
| 23.55 | | | 21.90 | 19.10 | |

Conduct a hypothesis test at the 2% level of significance to determine whether the hourly wages of the computer analysts are still the same as those of registered nurses.

**SOLUTION**

Hypothesis Testing Procedure:
**Step-1:**
Formulation of the Null and Alternative Hypotheses:

H0 : $\mu 1 - \mu 2 = 0$
HA : $\mu 1 - \mu 2 \neq 0$
(Two-tailed test)

**Step-2:**

Level of Significance:
$\alpha = 0.02$

**Step-3:**

Test Statistic:

$$Z = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

**Step-4:**

**Calculations:**
The sample size, sample mean and sample standard deviation for each of the two samples are given below:
Computer Analysts:

| | | |
|---|---|---|
| n1 | = | 32 |
| $\overline{X}$ 1 | = | $23.14 |
| S12 | = | 1.854 |

Registered Nurses:

n2 = 34
$\bar{X}2$ = \$21.99
S22 = 1.845

Since the sample sizes are larger than 30, hence, the unknown population variances σ12 and σ22 can be replaced by S12 and S22. Hence, our formula becomes:

$$Z = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

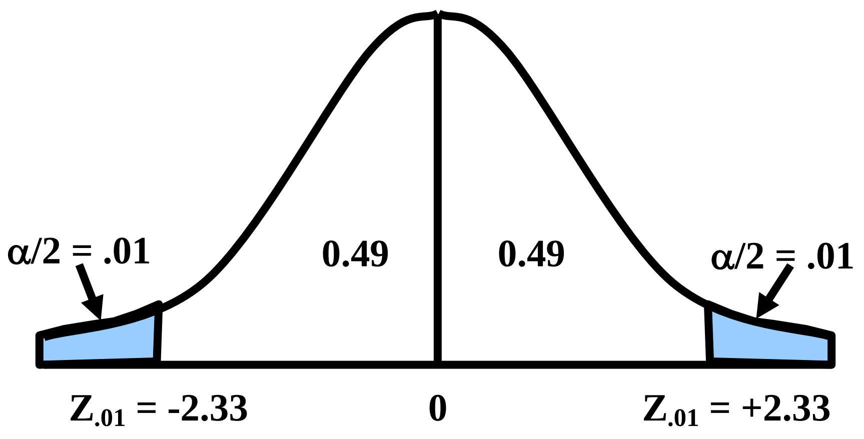
Hence, the computed value of Z comes out to be :

$$Z = \frac{(23.14 - 21.99) - (0)}{\sqrt{\frac{1.854}{32} + \frac{1.845}{34}}} = \frac{1.15}{0.335} = 3.43$$

**Step-5:**

**Critical Region:**

As the level of significance is 2%, and this is a two-tailed test, hence, we have the following situation:



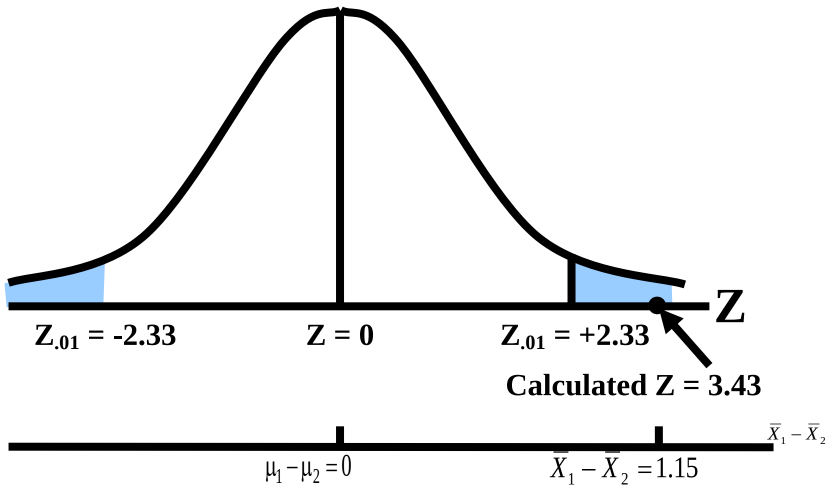

Hence, the critical region is given by

| Z | > 2.33

**Step-6:**

**Conclusion:**

As the computed value i.e. 3.43 is greater than the tabulated value 2.33, hence, we reject H0.



The researcher can say that there is a significant difference between the average hourly wage of a temporary computer analyst and the average hourly wage of a temporary registered nurse. The researcher then examines the sample means and uses common sense to conclude that, on the average, temporary computer analyst earn more than temporary registered nurses. Let us consolidate the above concept by considering another example:

**EXAMPLE**

Suppose that the workers of factory B believe that the average income of the workers of factory A exceeds their average income. A random sample of workers is drawn from each of the two factories, an the two samples yield the following information:

| Factory | Sample Size | Mean | Variance |
|---|---|---|---|
| A | 160 | 12.80 | 64 |
| B | 220 | 11.25 | 47 |

Test the above hypothesis?

**SOLUTION**

Let subscript 1 denote values pertaining to Factory A, and let subscript 2 denote values pertaining to Factory B. Then, we proceed as follows:
Hypothesis-testing Procedure:

**Step 1:**

H0 : μ1 < μ2 (or μ1 - μ2 < 0)
HA : μ1 > μ2 (or μ1 - μ2 > 0).

**Step 2:**
Level of significance
= 5%.

**Steps 3 & 4:**

$$Z = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{12.80 - 11.25}{\sqrt{\frac{64}{160} + \frac{47}{220}}}$$

$$= \frac{1.55}{\sqrt{0.61}} = \frac{1.55}{0.78} = 1.99$$

**Step 5:**
Critical Region:

Since it is a right-tailed test, hence the critical region is given by
Z > Z0.05
i.e. Z > 1.645

**Step 6:**
Conclusion:
Since 1.99 is greater than 1.645, hence H0 should be rejected in favour of HA. The sample evidence has consolidated the belief of the workers of factory B. Next, we consider the case when we are interested in conducting a test regarding p, the proportion of successes in the population. We illustrate this situation with the help of the following example:

**EXAMPLE**

A sociologist has a hunch that not more than 50% of the children who appear in a particular juvenile court three times or more are orphans. To test this hypothesis, a sample of 634 such children is taken and it is found that 341 of these children are orphans, (one or both parents dead). Test the above hypothesis using 1% level of significance.

**SOLUTION**

Hypothesis-testing Procedure:

**Step 1:**

H0 : $p < 0.50$
HA : $p > 0.50$
(one-tailed test)

**Step 2:**
Level of significance: $\alpha = 1\%$

**Step 3:**
Test statistic:

$$Z = \frac{X \pm \frac{1}{2} - n\ p_0}{\sqrt{n\ p_0(1 - p_0)}}$$

(where + ½ denotes the continuity correction)

**Step 4:**
Computation:

Here $np0 = 634\ (0.50) = 317$
and $X = 341$
Hence $X > np0$ so use $X - ½$

**So** $$Z = \frac{341 - \frac{1}{2} - 317}{\sqrt{634(0.50)(0.50)}} = \frac{23.5}{12.59}$$

**= 1.87**

**Step 5:**
Critical region:
Since $\alpha = 0.01$, hence the critical region is given by
$Z > 2.33$

**Step 6:**
Conclusion:
Since $1.87 < 2.33$,
Hence the computed Z does not fall in the critical region. Hence, we conclude that the sociologist's hunch is acceptable.

Masters

# LECTURE NO. 39

- Hypothesis Testing Regarding p1-p2 (based on Z-statistic)
- The Student's t-distribution
- Confidence Interval for μ based on the t-distribution

In the last lecture, we discussed hypothesis-testing regarding p, the proportion of successes in a binomial population. Next, we consider the case when we are interested in testing the equality of two population proportions. We illustrate this situation with the help of the following example:

**EXAMPLE**

A leading perfume company in a western country recently developed a new perfume which they plan to market under the name 'Fragrance'. A number of comparison tests indicate that 'Fragrance' has very good market potential. The Sales Departments of the company want to plan their strategy so as to reach and impress the largest possible segments of the buying public. One of the questions is whether the perfume is preferred by younger or older women. These are two independent populations, a population consisting of the younger women and a population consisting of the older women. A standard scent test will be used where each sampled woman is asked to sniff several perfumes, one of which is 'Fragrance', and indicate the one that she likes best.
A total of 100 young women were selected at random, and each was given the standard scent test. Twenty of the 100 young women chose 'Fragrance' as the perfume they liked best. Two hundred older women were selected at random, and each was given the same standard scent test. Of the 200 older women, 100 preferred 'Fragrance
Test the hypothesis that there is no difference between the proportions of younger and older women who prefer 'Fragrance'.

**SOLUTION**

We designate p1 as the proportion of younger women who prefer 'Fragrance' and p2 as the proportion of older women who prefer 'Fragrance'.
Hypothesis-Testing Procedure:
**Step-1**:
H0 : p1 = p2 (i.e. p1-p2 = 0)
(There is no difference between the proportions of young women and older women who prefer 'Fragrance'.)
H1 : p1 ≠ p2 (i.e. p1-p2 ≠ 0)
(The two proportions are not equal.)
**Step-2:**
Level of Significance
α = 0.05.
**Step-3:**
Test Statistic

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}_c \hat{q}_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where the combined or pooled proportion, $\hat{p}_c$, is given by:

$$\hat{p}_c = \frac{\text{Total number of successes in the two samples combined}}{\text{Total number of observations in the two samples combined}}$$

$$= \frac{X_1 + X_2}{n_1 + n_2}$$

This can also be written as

$$\hat{p}_c = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2}$$

which means that $\hat{p}_c$ is the weighted mean of $\hat{p}_1$ and $\hat{p}_2$, n1 and n2 acting as the weights.

Important Note:
In this example, as the hypothesized value of p1 - p2 is equal to zero; therefore both $\hat{p}_1$ *and* $\hat{p}_2$ are estimating the common population proportion p. Hence, we use the pooled proportion of the two samples to estimate p.
(The rationale is that the pooled estimator $\hat{p}_c$ is a better estimator of the common Population proportion p (as compared with $\hat{p}_1$ or $\hat{p}_2$), as it is based on n1 + n2 observations (i.e. based on a greater amount of information).

**Step-4:**
**Calculations:**
X1is the number of Preferring 'Fragrance' = 20. n1 is the number is the sample = 100.

$$\hat{p}_1 = \frac{X_1}{n_1} = \frac{20}{100} = 0.20$$

X2 is the number of preferring 'Fragrance' = 100. n2 is the number is the sample = 200.

$$\hat{p}_2 = \frac{X_2}{n_2} = \frac{100}{200} = 0.50$$

Now, the pooled or weighted proportion $\hat{p}_c$, is computed as follows:

$$\hat{p}_c = \frac{X_1 + X_2}{n_1 + n_2} = \frac{20 + 100}{100 + 200} = \frac{120}{300} = 0.40$$

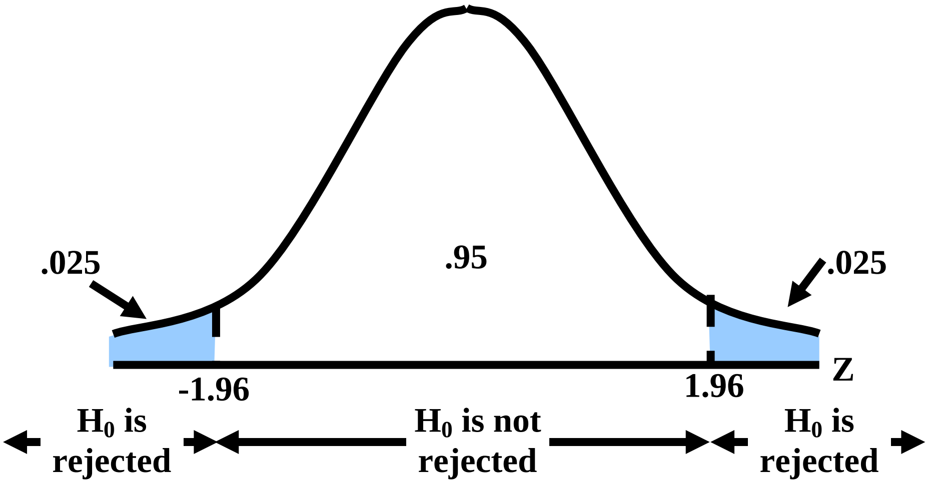
**Computation:**

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\hat{p}_c \hat{q}_c \left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{0.20 - 0.50}{\sqrt{(0.40)(0.60)\left(\frac{1}{100} + \frac{1}{200}\right)}}$$

$$= \frac{-0.30}{0.06} = -5.00$$

**Step-5:**
**Critical Region**
Since H1 does not state any direction (such as p1 < p2), the test is two-tailed. Thus, the critical values for the .05 level are –1.96 and + 1.96. Two-Tailed Test, Areas of Rejection and Non-rejection, .05 Level of Significance:




**Step-6:**

**CONCLUSION**
The computed z of –5.00 is in the area of rejection, that is, to the left of –1.96. Therefore, the null hypothesis is rejected at the .05 level of significance.

In other words, we conclude that the proportion of young women in the population who prefer 'Fragrance' is not equal to the proportion of older women in the population who prefer 'Fragrance'.(The difference between the two sample proportion i.e. 0.30 is so large that it is highly unlikely that such a large difference could be due to chance (i.e. attributable to sampling fluctuations).)
In fact, the value z = -5.00 is even larger than -2.58, the critical value lying on the left tail of the sampling distribution if $\alpha = 0.01$. As such, we can say that our statistic is highly significant.(In such a situation, the statistic is said to be highly significant because of the fact that we are allowing as small a risk of committing Type-I error as 1%.)
Now, consider another situation:
Suppose that the computed value of our test-statistic comes out to be such that it falls between -1.96 and -2.58. In such a situation, we will reject H0 at the 5% level of significance, but we cannot reject H0 at the 1% level. This means that, if we are willing to allow as much as 5% risk of committing type I error, then we say that we are going to reject H0. But if we are willing to allow only 1% risk of committing type I error, then we conclude that the sample does not provide sufficient evidence to reject H0.Going back to the example of the perfume, obviously, the company would be interested in determining, which category of women prefers this perfume in greater numbers than the other?
The data clearly indicates that the proportion of women who prefer this particular perfume is higher in the population of older women. (This is the reason why the computed value of our test-statistic has come out to be negative.) Let us consolidate the above ideas by considering another example:

**EXAMPLE**

A candidate for mayor in a large city believes that he appeals to at least 10 per cent more of the educated voters than the uneducated voters. He hires the services of a poll-taking organization, and they find that 62 of 100 educated voters interviewed support the candidate, and 69 of 150 uneducated voters support him at the 0.05 significance level.

**Step-1:**

The null and alternative hypothesis is
H0 : p1 – p2 > 0.10, and
H1 : p1 – p2 < 0.10, where p1 = proportion of educated voters, and p2 = proportion of uneducated voters.

**Step-2:**

Level of Significance:
$\alpha = 0.05.$

**Step-3:**

Test Statistic:
$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - 0.10}{\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1} + \frac{\hat{p}_2\hat{q}_2}{n_2}}}$$

which for large sample sizes, is approximately standard normal.

***Important Note***
In this example, as the hypothesized value of p1 - p2 is not equal to zero, therefore are note estimating the same quantity, and, as such, we do not use in the formula of the test statistic.

**Step-4:**

**Computation:**

$$\text{Here} \quad \hat{p}_1 = \frac{62}{100} = 0.62, \text{ so that } \hat{q}_1 = 0.38,$$

$$\hat{p}_2 = \frac{69}{150} = 0.46, \text{ so that } \hat{q}_2 = 0.54.$$

$$\text{Thus} \quad z = \frac{(0.62 - 0.46) - 0.10}{\sqrt{\frac{(0.62)(0.38)}{100} + \frac{(0.46)(0.54)}{150}}}$$

$$= \frac{0.06}{\sqrt{0.002356 + 0.001656}} = \frac{0.06}{0.063} = 0.95.$$

**Step-5:**

**Critical Region:**
As this is a one-tailed test, therefore the critical region is given by

$$Z < -z0.05 = -1.645$$

**Step-6:**
**Conclusion:**
Since the calculated value z = 0.95 does not fall in the critical region, so we accept the null hypothesis H0 : p1 – p2 > 0.10.The data seems to support the candidate's view.
Until now, we have discussed in considerable detail interval estimation and hypothesis-testing based on the standard normal distribution and the Z-statistic.
Next, we begin the discussion of interval estimation hypothesis-testing based on the t-distribution.

## t-DISTRIBUTION

We begin by presenting the formal definition of the t-distribution and stating some of its main properties:

### The Student's t-Distribution:

The mathematical equation of the t-distribution is as follows:

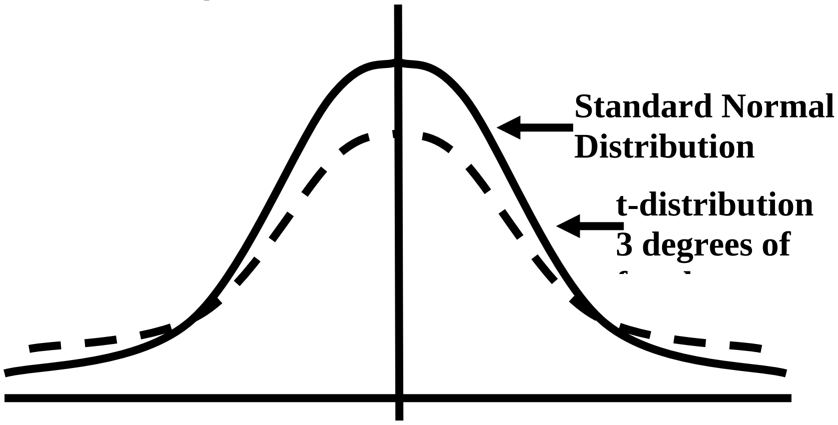
$$f(x) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{1}{2},\frac{\nu}{2}\right)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} \quad , -\infty < x < \infty$$

This distribution has only one parameter ν, which is known as the degrees of freedom of the t-distribution

## PROPERTIES OF STUDENT'S t-DISTRIBUTION

The t-distribution has the following properties:

**i)** The t-distribution is bell-shaped and symmetric about the value t = 0, ranging from – ∞ to ∞.
**ii)** The number of degrees of freedom determines the shape of the t-distribution.
Thus there is a different t-distribution for each number of degrees of freedom.
As such, it is a whole family of distributions.
The t-distribution, for small values of ν, is flatter than the standard normal distribution which means that the t-distribution is more spread out in the tails than is the standard normal distribution.




As the degrees of freedom increase, the t-distribution becomes narrower and narrower, until, as n tends to infinity, it tends to coincide with the standard normal distribution.
(The t-distribution can never become narrower than the standard normal distribution.)
**iii)** The t-distribution has a mean of zero, when $\nu \geq 2$. (The mean does not exist when $\nu = 1$.)
**iv)** The median of the t-distribution is also equal to zero.
**v)** The t-distribution is unimodal.The density of the distribution reaches its maximum at t = 0 and thus the mode of the t-distribution is t = 0.
(The students will recall that, for any hump-shaped symmetric distribution, the mean, median and mode are equal.)

**vi)** The variance of the t-distribution is given by $\sigma^2 = \frac{\nu}{\nu - 2}$ for $\nu > 2$.

It is always greater than1, the variance of the standard normal distribution. (This indicates that the t-distribution is more spread out than the standard normal distribution.)
For $\nu \le 2$, the variance does not exist. Next, we discuss the application of the t-distribution in statistical inference --- those situations where we need to carry out interval estimation and hypothesis - testing on the basis of the t-distribution. (Situations where the t-distribution is the appropriate sampling distribution)
With reference to interval estimation and hypothesis-testing about μ, it has been mathematically proved that, if the population from which the sample has been drawn is normally distributed, the population variance is unknown, and the sample size is small (less than 30), then the statistic

$$t = \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

$$\left(\text{where } s = \sqrt{\frac{\sum\left(X - \overline{X}\right)^2}{n-1}}\right)$$

Follows the t-distribution having n-1 degrees of freedom. First, we discuss the construction of a Confidence Interval for μ based on the t-distribution with the help of an example:

**<u>EXAMPLE</u>**

The masses, in grams, of thirteen ball bearings seen at random from a batch are
21.4, 23.1, 25.9, 24.7, 23.4, 24.5, 25.0, 22.5, 26.9, 26.4, 25.8, 23.2, 21.9
Calculate a 95% confidence interval for the mean mass of the population, supposed normal, from which these masses were drawn.

**<u>SOLUTION</u>**

The 95% confidence interval for the mean mass of the population μ, is given by

$$\overline{X} + t_{\alpha/2(n-1)} \frac{s}{\sqrt{n}}$$

(The derivation of the above confidence interval is very similar to that of the confidence interval for μ based on the Z-statistic.)
Now, in this problem, the sample mean $\overline{X}$ and s come out to be:

$$\overline{X} = \frac{\sum X}{n} = \frac{314.7}{13} = 24.21,$$

$$s = \sqrt{\frac{\sum\left(X - \overline{X}\right)^2}{n-1}} = \sqrt{\frac{1}{n-1}\left[\sum X^2 - \frac{\left(\sum X\right)^2}{n}\right]}$$

$$= \sqrt{\frac{1}{12}[7655.59 - 7618.16]} = \sqrt{\frac{37.43}{12}} = \sqrt{3.12} = 1.77$$

The question is: 'How do we find $t_{\alpha/2(n-1)}$ ?

For this purpose, we will need to consult the table of areas under the t-distribution:

**TABLE OF AREAS UNDER THE T-DISTRIBUTION**

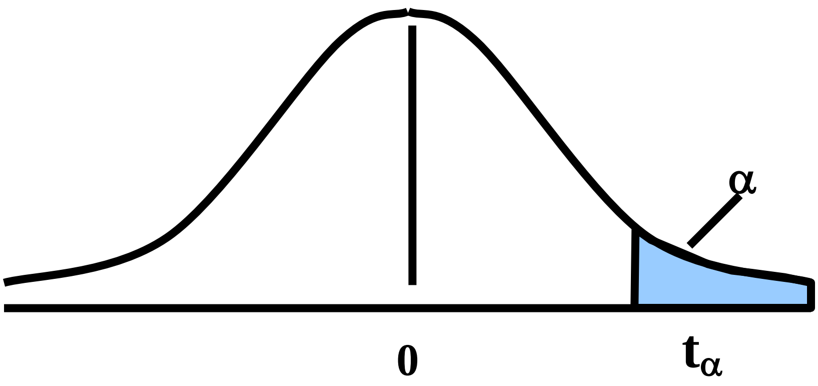
**Upper Percentage Points of the t-Distribution**

| ν \ α | 0.25 | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 |
|---|---|---|---|---|---|---|---|
| 1 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.310 |
| 2 | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.327 |
| 3 | 0.765 | 1.838 | 2.353 | 3.182 | 4.541 | 5.841 | 10.214 |
| 4 | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 |
| 5 | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 |
| 6 | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 |
| 7 | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.785 |
| 8 | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.501 |
| 9 | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.297 |
| 10 | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.144 |
| 11 | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 4.025 |
| 12 | 0.695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.930 |
| 13 | 0.694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.852 |
| 14 | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.787 |
| 15 | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.733 |

**Upper Percentage Points of the t-Distribution**

| ν \ α | 0.25 | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 |
|---|---|---|---|---|---|---|---|
| 16 | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.686 |
| 17 | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.646 |
| 18 | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.610 |
| 19 | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.579 |
| 20 | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.552 |
| 21 | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.527 |
| 22 | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.505 |
| 23 | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.485 |
| 24 | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.467 |
| 25 | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.450 |
| 26 | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.435 |
| 27 | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.421 |
| 28 | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.408 |
| 29 | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.396 |
| 30 | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.385 |
| 40 | 0.681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 3.307 |
| 60 | 0.679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 3.232 |
| 120 | 0.677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 3.160 |
| ∞ | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 3.090 |

The above table is an abridged version of the table by Fisher and Yates, and the entries in this table are values of $t_{\alpha,(\nu)}$ for which the area to their right under the t-distribution with $\nu$ degrees of freedom is equal to $\alpha$, as shown below:




Now, in this problem, since n – 1 = 12, and the desired level of confidence is 95%, therefore, the right-tail area is 2½%, and, hence, (using the t-table) we obtain

$$t_{0.025\,(12)} = 2.179$$

Substituting these values, we obtain the 95% confidence interval for μ as follows:

$$24.21 \pm 2.179\left(\frac{1.77}{\sqrt{13}}\right)$$

or $\quad 24.21 \pm 2.179\ (0.49)$

or $\quad 24.21 \pm 1.07$ or 23.14 to 25.28

Hence, the 95% confidence interval for the mean mass of the ball bearings calculated from the given sample is (23.1, 25.3) grams.

# LECTURE NO. 40



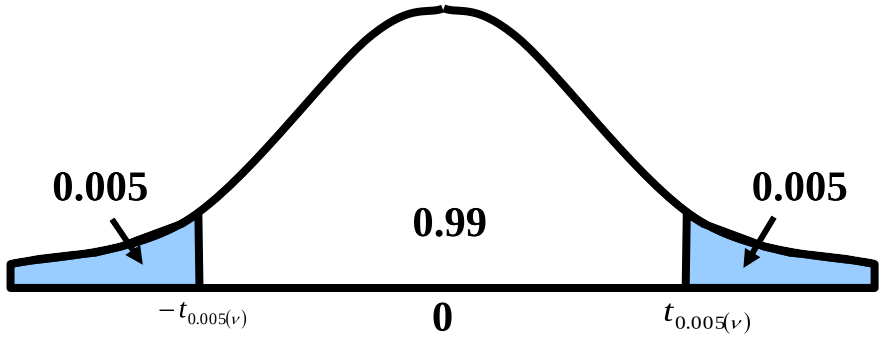
- Tests and Confidence Intervals based on the t-distribution

In the last lecture, we introduced the t-distribution, and began the discussion of statistical inference based on the t-distribution. In particular, we discussed the construction of the confidence interval for $\mu$ in that situation when we are drawing a small sample from a normal population having unknown variance $\sigma 2$. When the parent population is normal, the population variance is unknown, and the sample size n is small (less than 30), then the confidence interval for $\mu$ is given by

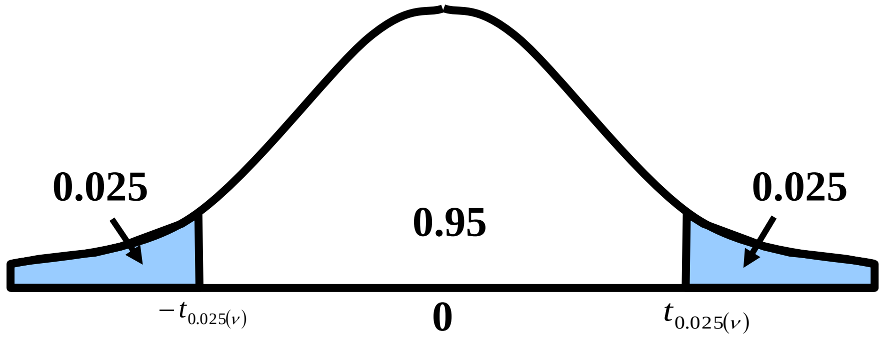
$$\bar{x} \pm t_{\alpha/2(n-1)} \frac{s}{\sqrt{n}}$$

where $\bar{x} = \frac{\sum x}{n}$ is the sample mean $s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$ is the sample standard deviation n = sample size and $t_{(\alpha/2,\, \nu)}$ is found by looking in the t-table under the appropriate value of $\alpha$ against $\nu = n - 1$;

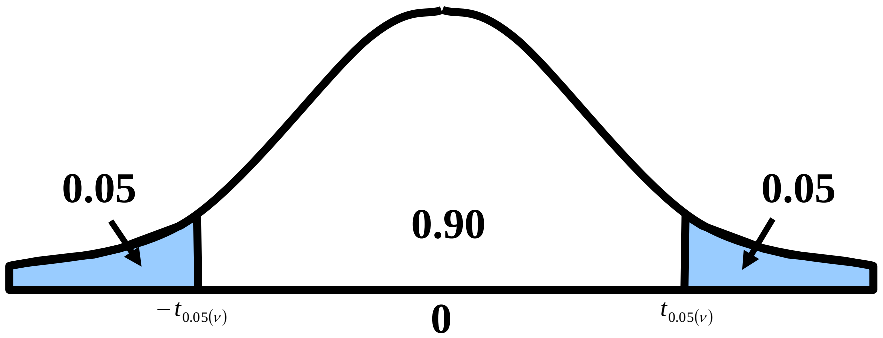
$\alpha/2$ = **0.005 if we desire 99% confidence:**




**$\alpha/2$ = 0.025 if we desire 95% confidence**:




**$\alpha/2$ = 0.05 if we desire 90% confidence:**




Next, we discuss hypothesis - testing regarding the mean of a normally distributed population for which $\sigma 2$ is unknown and the sample size is small (n < 30).
This procedure is illustrated through the following example:

**EXAMPLE-1**

Just as human height is approximately normally distributed, we can expect the heights of animals of any particular species to be normally distributed. Suppose that, for the past five years, a zoologist has been involved in an extensive research-project regarding the animals of one particular species. Based on his research-experience, the zoologist believes that the average height of the animals of this particular species is 66 centimeters. He selects a random sample of ten animals of this particular species, and, upon measuring their heights, the following data is obtained.
63, 63, 66, 67, 68, 69, 70, 70, 71, 71
In the light of these data, test the hypothesis that the mean height of the animals of this particular species is 66 centimeters.

**SOLUTION:**

Hypothesis-Testing Procedure:

**i)** We state our null and alternative hypotheses as
H0 : $\mu$ = 66 and H1 : $\mu \neq$ 66.

**ii)** We set the significance level at $\alpha$ = 0.05.
**iii)** Test Statistic:
The test-statistic to be used is

$$t = \frac{\overline{X} - \mu_0}{s/\sqrt{n}}$$

which, if H0 is true, has the t-distribution with n – 1 = 9 degrees of freedom.

*Important Note:*
As indicated in the previous discussion, we always begin by assuming that H0 is true.(The entire mathematical logic of the hypothesis-testing procedure is based on the assumption that H0 is true.)
**iv)** CALCULATIONS

| Individual No. | $x_i$ | $x_i^2$ |
|---|---|---|
| 1 | 63 | 3969 |
| 2 | 63 | 3969 |
| 3 | 66 | 4356 |
| 4 | 67 | 4489 |
| 5 | 68 | 4624 |
| 6 | 69 | 4761 |
| 7 | 70 | 4900 |
| 8 | 70 | 4900 |
| 9 | 71 | 5041 |
| 10 | 71 | 5041 |
| Total | 678 | 46050 |

Now $\overline{x} = \frac{\sum x_i}{n} = \frac{678}{10} = 67.8$ inches,

and $s^2 = \frac{1}{n-1}\sum\left(x_i - \overline{x}\right)^2 = \frac{1}{n-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right]$

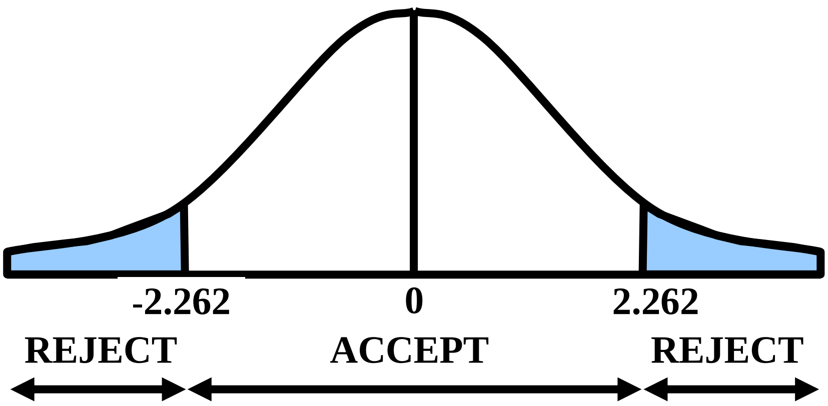
$$= \frac{1}{9}\left[46050 - 45968.4\right] = 9.0667,$$

$s = \sqrt{9.0667} = 3.01$ inches.

$$\therefore \quad t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$$
$$= \frac{67.8-66}{3.01/\sqrt{10}}$$
$$= \frac{(1.8)(3.1623)}{3.01}$$
$$= 1.89$$

**V)** Critical Region:

Since this is a two-tailed test, hence the critical region is given by
| t | > t0.025(9) = 2.262.




**vi)** Conclusion:

Since the computed value of t = 1.89 does not fall in the critical region, we therefore do not reject H0 and may conclude that the mean height of the animals of this particular species is 66 centimeters.

Next, we consider the construction of the confidence interval for µ1-µ2 in that situation when we are drawing small samples from two normally distributed populations having unknown but equal variances:
We illustrate this concept with the help of the following example:

**EXAMPLE:**

A record company executive is interested in estimating the difference in the average play-length of songs pertaining to pop music and semi-classical music. To do so, she randomly selects 10 semi-classical songs and 9 pop songs.

**THE PLAY-LENGTHS (IN MINUTES) OF THE SELECTED SONGS ARE LISTED IN THE FOLLOWING TABLE**

| Semi-Classic Music | Pop Music |
|---|---|
| 3.80 | 3.88 |
| 3.30 | 4.13 |
| 3.43 | 4.11 |
| 3.30 | 3.98 |
| 3.03 | 3.98 |
| 4.18 | 3.93 |
| 3.18 | 3.92 |
| 3.83 | 3.98 |
| 3.22 | 4.67 |
| 3.38 | |

Calculate a 99% confidence interval to estimate the difference in population means for these two types of recordings.

**SOLUTION:**

In this problem, we are dealing with a t-distribution with n1+n2 - 2 = 10 + 9 – 2 = 17 degrees of freedom. The table t-value for a 99% level of confidence and 17 degrees of freedom is t005.17 = 2.898.

**Calculations:**

| Semi-Classical Music | Pop Music |
|---|---|
| $n_1 = 10$ | $n_2 = 9$ |
| $\overline{X}_1 = 3.465$ | $\overline{X}_2 = 4.064$ |
| $S_1 = .0.3575$ | $S_2 = .0.2417$ |

Hence:

$$s_p = \sqrt{\frac{(.3575)^2(9)+(.2417)^2(8)}{10+9-2}}$$

$$= \sqrt{\frac{1.1503+0.4674}{17}}$$

$$= \sqrt{\frac{1.6177}{17}} \quad = \sqrt{0.0952}$$

$$= 0.31$$

The confidence interval is

$$(3.465-4.064)$$

$$\pm(2.898)(0.31)\sqrt{\frac{1}{10}+\frac{1}{9}} = -0.599 \pm 0.411$$

$$i.e.\ the\ C.I.\ is:$$

$$-1.010 \le \mu_1 - \mu_2 \le -.188$$

With 99% confidence, the record company executive can conclude that the true difference in population average length of play is between –1.01 minutes and –.188 minute. Zero is not in this interval, so she could conclude that there is a significant difference in the average length of play time between semi-classical music and pop music songs' recordings. Examination of the sample results indicates that pop music songs' recordings are longer. The result and conclusion obtained above can be used in the tactical and strategic planning for programming, marketing, and production of recordings.

**EXAMPLE**

From an area planted in one variety of guayule (a rubber producing plant), 54 plants were selected at random. Of these, 15 were off types and 12 were aberrant. Rubber percentages for these plants were:

| | |
|---|---|
| Offtypes | 6.21, 5.70, 6.04, 4.47, 5.22, 4.45, 4.84, 5.88, 5.82, 6.09, 6.06, 5.59, 6.74, 5.55 |
| Aberrant | 4.28, 7.71, 6.48, 7.71, 7.37, 7.20, 7.06, 6.40, 8.93, 5.91, 5.51, 6.36 |

Test the hypothesis that the mean rubber percentage of the Aberrants is at least 1 percent more than the mean rubber percentage of off types. Assume the populations of rubber percentages are approximately normal and have equal variances. Let subscript 1 stand for Aberrants, and let subscript 2 stand for off types. Then, we proceed as follows:

**i)** We formulate our null and alternative hypotheses as

H0 : $\mu$1 - $\mu$2 > 1,

and

H1 : $\mu$1 - $\mu$2 < 1

**ii)** We set the significance level at $\alpha = 0.05$.

**iii)** The test-statistic, if $H_0$ is true, is

$$t = \frac{\left(\overline{X}_1 - \overline{X}_2\right) - \left(\mu_1 - \mu_2\right)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

which has a Student's t-distribution with
$\nu = n_1 + n_2 - 2$, i.e. 25 degrees of freedom.
**iv)** Computations:
We have

$$\bar{x}_1 = \frac{\sum x_1}{n_1} = \frac{80.92}{12} = 6.74,$$

$$\bar{x}_2 = \frac{\sum x_2}{n_2} = \frac{84.25}{15} = 5.62,$$

And

$$\sum\left(x_1 - \bar{x}_1\right)^2 = \sum x_1^2 - \frac{\left(\sum x_1\right)^2}{n_1}$$

$$= 561.6402 - \frac{(80.92)^2}{12}$$

$$= 561.6402 - 545.6705$$

$$= 15.9697$$

$$\sum\left(x_2 - \bar{x}_2\right)^2 = \sum x_2^2 - \frac{\left(\sum x_2\right)^2}{n_2}$$

$$= 478.9779 - \frac{(84.25)^2}{15}$$

$$= 478.9779 - 473.2042$$

$$= 5.7737$$

$$Now\ s_p^2 = \frac{\sum\left(x_1 - \bar{x}_1\right)^2 + \sum\left(x_2 - \bar{x}_2\right)^2}{n_1 + n_2 - 2}$$

$$= \frac{5.9697 + 5.7737}{12 + 15 - 2}$$

$= 0.8697,$

so that

$$s_p = \sqrt{0.8697} = 0.93,$$

Hence, the computed value of our test statistic comes out to be

$$\therefore t = \frac{\left(6.74 - 5.62\right) - 1}{0.93\sqrt{\frac{1}{12} + \frac{1}{15}}} = \frac{0.12}{0.36} = 0.33$$

**v)** Critical Region:

Since this is a left-tailed test, therefore the critical region is given by

$t < -t0.05(25)$ i.e. $t < -1.708$

**vi)** Conclusion:

Since the computed value of t = 0.33 falls in the acceptance region, therefore we accept H0. We may conclude that the mean rubber percentage of the Aberrants is at least 1 percent more than the mean rubber percentage of Off types.

## T-DISTRIBUTION IN THE CASE OF PAIRED OBSERVATIONS

In testing hypotheses about two means, until now we have used independent samples, but there are many situations in which the two samples are not independent. This happen when the observation are found in pairs such that the two observations of a pair are related to each other. Pairing occurs either naturally or by design. Natural pairing occurs whenever measurement is taken on the same unit or individual at two different times. For example, suppose ten young recruits are given a strenuous physical training programme by the Army. Their weights are recorded before they begin and after they complete the training. The two observations obtained for each recruit i.e. the before-and-after measurement constitute natural pairing. The above is natural pairing.

**EXAMPLE:**
Ten young recruits were put through a strenuous physical training programme by the Army. Their weights were recorded before and after the training with the following results:

| Recruit | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Weight before | 125 | 195 | 160 | 171 | 140 | 201 | 170 | 176 | 195 | 139 |
| Weight after | 136 | 201 | 158 | 184 | 145 | 195 | 175 | 190 | 190 | 145 |

Using $\alpha = 0.05$, would you say that the programme affects the average weight of recruits?
Assume the distribution of weights before and after to be approximately normal. When the observations from two samples are paired, we find the difference between the two observations of each pair, and the test-statistic in this situation is:

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

$$= \frac{\bar{d} - 0}{s_d / \sqrt{n}}$$

$$= \frac{\bar{d}}{s_d / \sqrt{n}}$$

# LECTURE NO. 41

- Hypothesis-Testing regarding Two Population Means in the Case of Paired Observations (t-distribution)
- The Chi-square Distribution
- Hypothesis Testing and Interval Estimation Regarding a Population Variance (based on Chi-square Distribution)

In the last lecture, we began the discussion of hypothesis-testing regarding two population means in the case of paired observations. It was mentioned that, in many situations, pairing occurs naturally. Observations are also paired to eliminate effects in which there is no interest. For example, suppose we wish to test which of two types (A or B) of fertilizers is the better one. The two types of fertilizer are applied to a number of plots and the results are noted. Assuming that the two types are found significantly different, we may find that part of the difference may be due to the different types of soil or different weather conditions, etc. Thus the real difference between the fertilizers can be found only when the plots are paired according to the same types of soil or same weather conditions, etc.

We eliminate the undesirable sources of variation by taking the observations in pairs. This is pairing by design.

We illustrate the procedure of hypothesis-testing regarding the equality of two population means in the case of paired observations with the help of the same example that we quoted at the end of the last lecture:

**EXAMPLE**

Ten young recruits were put through a strenuous physical training programme by the Army. Their weights were recorded before and after the training with the following results:

| Recruit | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Weight before | 125 | 195 | 160 | 171 | 140 | 201 | 170 | 176 | 195 | 139 |
| Weight after | 136 | 201 | 158 | 184 | 145 | 195 | 175 | 190 | 190 | 145 |

Using $\alpha = 0.05$, would you say that the programme affects the average weight of recruits? Assume the distribution of weights before and after to be approximately normal.

**SOLUTION**

The pairing was natural here, since two observations are made on the same recruit at two different times. The sample consists of 10 recruits with two measurements on each. The test is carried out as below:

Hypothesis-Testing Procedure:

**i)** We state our null and alternative hypotheses as

$H0 : \mu d = 0$ and

$H1 : \mu d \neq 0$

**ii)** The significance level is set at $\alpha = 0.05$.

**iii)** The test statistic under H0 is

$$t = \frac{\bar{d}}{s_d / \sqrt{n}},$$

which has a t-distribution with n – 1 degrees of freedom.

**iv)** Computations:

| Recruit | Weight | | Difference, $d_i$ (after minus before) | $d_1^2$ |
|---|---|---|---|---|
| | Before | After | | |
| 1 | 125 | 136 | 11 | 121 |
| 2 | 195 | 201 | 6 | 36 |
| 3 | 160 | 158 | – 2 | 4 |
| 4 | 171 | 184 | 13 | 169 |
| 5 | 140 | 145 | 5 | 25 |
| 6 | 201 | 195 | 6 | 36 |
| 7 | 170 | 175 | – 6 | 25 |
| 8 | 176 | 190 | 5 | 196 |
| 9 | 195 | 190 | 14 | 25 |
| 10 | 139 | 145 | – 5 | 36 |
| $\sum$ | 1672 | 1719 | 6 | 673 |

$$\bar{d} = \frac{\sum d}{n} = \frac{47}{10} = 4.7,$$

$$s_{d^2} = \frac{\sum\left(d-\bar{d}\right)^2}{n-1} = \frac{1}{n-1}\left[\sum d^2 - \frac{\left(\sum d\right)^2}{n}\right]$$

$$= \frac{1}{9}\left[673 - \frac{(47)^2}{10}\right] = \frac{673-220.9}{9} = 50.23,$$

so that

$$s_d = \sqrt{50.23} = 7.09.$$

Hence, the computed value of our test-statistic comes out to be :

$$t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{4.7}{7.09/\sqrt{10}} = \frac{(4.7)(3.16)}{7.09} = 2.09.$$

**v)** The critical region is $|t| \geq t_{0.025(9)}$
$= 2.262.$

**vi)** Conclusion:

Since the calculated value of t = 2.09 does not fall in the critical region, so we accept H0 and may conclude that the data do not provide sufficient evidence to indicate that the programme affects average weight.

From the above example, it is clear that the hypothesis-testing procedure regarding the equality of means in the case of paired observations is very similar to the t-test that is applied for testing
H0 : μ = 0.(The only difference is that when we are testing H0 : μ = 0, our variable is X, whereas when we are testing H0 : μd=0, our variable is d.)

## HYPOTHESIS-TESTING PROCEDURE REGARDING TWO POPULATIONS MEANS IN THE CASE OF PAIRED OBSERVATIONS

When the observations from two samples are paired either naturally or by design, we find the difference between the two observations of each pair. Treating the differences as a random sample from a normal population with mean μd = μ1 - μ2 and unknown standard deviation σd, we perform a one-sample t-test on them. This is called a paired difference t-test or a paired t-test.

**Testing the hypothesis**

H0 : μ1 = μ2 against
HA : μ1 ≠ μ2 is equivalent to testing H0 : μd = 0 against
HA : μd ≠ 0.

Let d = x1 – x2 denote the difference between the two samples observations in a pair. Then the sample mean and standard deviation of the differences are

$$\bar{d} = \frac{\sum d}{n} \text{ and } s_d = \frac{\sum\left(d-\bar{d}\right)^2}{n-1}$$

where n represents the number of pairs.
Assuming that

**1)** $d_1, d_2, \ldots, d_n$ is a random sample of differences, and

**2)** the differences are normally distributed,

the test-statistic

$$t = \frac{\bar{d}-0}{s_d/\sqrt{n}} = \frac{\bar{d}}{s_d/\sqrt{n}}$$

follows a t-distribution with ν = n – 1 degrees of freedom. The rest of the procedure for testing the null hypothesis H0 : μd = 0 is the same

**EXAMPLE**

The following data give paired yields of two varieties of wheat.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Variety I | 45 | 32 | 58 | 57 | 60 | 38 | 47 | 51 | 42 | 38 |
| Variety II | 47 | 34 | 60 | 59 | 63 | 44 | 49 | 53 | 46 | 41 |

Each pair was planted in a different locality.

**a)** Test the hypothesis that, on the average, the yield of variety-1 is less than the mean yield of variety-2. State the assumptions necessary to conduct this test.

**b)** How can the experimenter make a Type-I error? What are the consequences of his doing so?

**c)** How can the experimenter make a Type-II error? What are the consequences of his doing so?

**d)** Give 90 per cent confidence limits for the difference in mean yield.

***Note:*** The pairing was by design here, as the yields are affected by many extraneous factors such as fertility of land, fertilizer applied, weather conditions and so forth.

**SOLUTION:**

**a)** In order to conduct this test, we make the following assumptions:

**ASSUMPTIONS**

- The differences in yields are a random sample from the population of differences,
- The population of differences is normally distributed.

i) We state our null and alternative hypotheses as

$$H_0 : \mu_d \geq 0 \ (\text{or} \ \ \mu_1 \geq \mu_2),$$

i.e. the mean yields are equal and

$$H_1 : \mu_d < 0 \ (\text{or} \ \ \mu_1 < \mu_2).$$

ii) We select the level of significance at $\alpha = 0.05$.

iii) The test statistic to be used is

$$t = \frac{\bar{d} - 0}{s_d / \sqrt{n}} = \frac{\bar{d}}{s_d / \sqrt{n}}$$

where $\bar{d} = \bar{x}_1 - \bar{x}_2$ and sd2 is the variance of the differences di.

If the populations are normal, this statistic, when H0 is true, has a Student's t-distribution with (n – 1) d. f.

**iv) Computations:**

Let X1i and X2i represent the yields of Variety I and Variety II respectively. Then the necessary computations are given below:

| $X_{1i}$ | $X_{2i}$ | $d_i = X_{1i} - X_{2i}$ | $d_i^2$ |
|---|---|---|---|
| 45 | 47 | –2 | 4 |
| 32 | 34 | –2 | 4 |
| 58 | 60 | –2 | 4 |
| 57 | 59 | –2 | 4 |
| 60 | 63 | –3 | 9 |
| 38 | 44 | –6 | 36 |
| 47 | 49 | –2 | 4 |
| 51 | 53 | –2 | 4 |
| 42 | 46 | –4 | 16 |
| 38 | 41 | –3 | 9 |
| Σ | — | –28 | 94 |

Now $\bar{d} = \frac{\sum d_i}{n} = \frac{-28}{10} = -2.8$, and

$$s_d^2 = \frac{1}{n-1}\left[\sum d_i^2 - \frac{\left(\sum d_i\right)^2}{n}\right] = \frac{1}{9}\left[94 - \frac{(-28)^2}{10}\right]$$

$= \frac{15.6}{9} = 1.7333$, so that $s_d = 1.32$

$$\therefore \quad t = \frac{\bar{d}}{s_d/\sqrt{n}} = \frac{-2.8}{1.32/\sqrt{10}} = \frac{(-2.8)(3.1623)}{1.32} = -6.71$$

v) As this is a one-tailed test therefore, the critical region is given by
$t < t_{0.05(9)} = -1.833$

**vi) Conclusion**

Since the calculated value of t = –6.71 falls in the critical region, we therefore reject H0. The data present sufficient evidence to conclude that the mean yield of variety-1 is less than the mean yield of variety-2.

**b)** The experimenter can make a Type-I error by rejecting a true null hypothesis.
In this case, the Type-I error is make by rejecting the null hypothesis when the mean yield of variety-1 is actually not different from the mean yield of variety-2.
In so doing, the consequences would be that we will be saying that variety-2 is better than variety-1 although in reality they are equally good.

**c)** The experimenter can make a Type-II error by accepting of false null hypothesis.
In this case, the Type-II error is made by accepting the null hypothesis when in reality the mean yield of variety-1 is less than the mean yield of variety-2 and the consequence of committing this error would be a loss of potential increased yield by the use of variety-2.

**d)** The 90% confidence limits for the difference in means $\mu 1 - \mu 2$ in case of paired observations, are given by

$$\bar{d} \pm t_{\alpha/2,(n-1)} \cdot \frac{s_d}{\sqrt{n}}$$

Substituting the values, we get

$$-2.8 \pm 1.833 \ \frac{1.32}{\sqrt{10}}$$

or -2.8 + 0.765
or -3.565 to -2.035

Hence the 90% confidence limits for the difference in mean yields, $\mu 1 - \mu 2$, are (-3.6, -2.0) .

Until now, we have discussed statistical inference regarding population means based on the Z-statistic as well as the t-statistic.

Also, we have discussed inference regarding the population proportion based on the Z-statistic.

In certain situations, we would be interested in drawing conclusions about the variability that exists in the population values, and for this purpose, we would like to carry out estimation or hypothesis-testing regarding the population variance $\sigma 2$.

Statistical Inference regarding the population variance is based on the chi-square distribution.
We begin this topic by presenting the formal definition of the Chi-Square distribution and stating some of its main properties:

## THE CHI-SQUARE ($\chi 2$) DISTRIBUTION

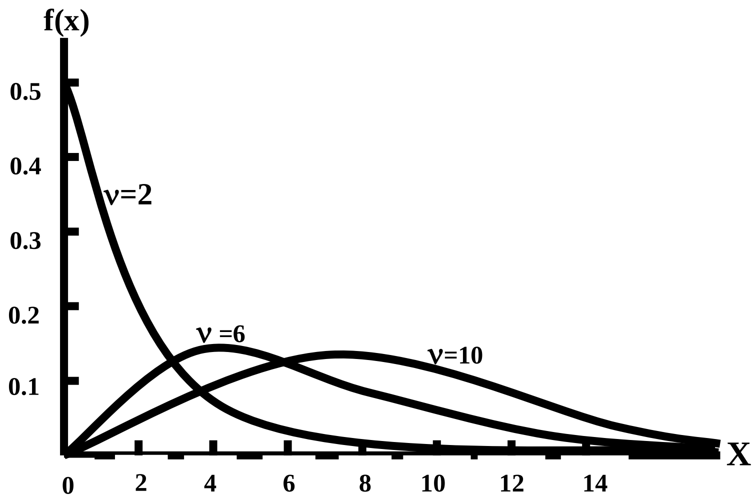
The mathematical equation of the Chi-Square distribution is as follows:

$$f(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}(x)^{(\nu/2)-1}.e^{-x/2}, \quad 0 < x < \infty$$

This distribution has only one parameter $\nu$, which is known as the degrees of freedom of the Chi-Square distribution.

## PROPERTIES OF THE CHI-SQUARE DISTRIBUTION

The Chi-Square ($\chi2$) distribution has the following properties:
**1.** It is a continuous distribution ranging from 0 to $+\infty$ .
The number of degrees of freedom determines the shape of the chi-square distribution. Thus there is a different chi-square distribution for each number of degrees of freedom. As such, it is a whole family of distributions.
**2.** The curve of a chi-square distribution is positively skewed.
The skewness decreases as $\nu$ increases.




**$\chi2$-distribution for various values of $\nu$**

As indicated by the above figures, the chi-square distribution tends to the normal distribution as the number of degrees of freedom approaches infinity.
**3.** The mean of a chi-square distribution is equal to $\nu$, the number of degrees of freedom.
**4.** Its variance is equal to $2\nu$.
**5.** The moments about the origin are given by

$$\mu'_1 = \nu$$
$$\mu'_2 = \nu(\nu+2)$$
$$\mu'_3 = \nu(\nu+2)(\nu+4)$$
$$\mu'_4 = \nu(\nu+2)(\nu+4)(\nu+6)$$

As such, the moment-ratios come out to be

$$\beta_1 = \frac{8}{\nu}$$
$$\beta_1 = 3 + \frac{12}{\nu}$$

Having discussed the basic definition and properties of the chi-square distribution, we begin the discussion of its role in interval estimation and hypothesis-testing. We begin with interval estimation regarding the variance of a normally distributed population:

**EXAMPLE:**

Suppose that an aptitude test carrying a total of 20 marks is devised, and administered on a large population of students, and, upon doing so, it was found that the marks of the students were normally distributed. A random sample of size n = 8 is drawn from this population, and the sample values are 9, 14, 10, 12, 7, 13, 11, 12.
Find the 90 percent confidence interval for the population variance $\sigma2$, representing the variability in the marks of the students.

**SOLUTION:**

The 90% confidence interval for σ2 is given by

$$\frac{\sum(Xi-\overline{X})^2}{\chi^2_{0.05(n-1)}} < \sigma^2 < \frac{\sum(X_i-\overline{X})^2}{\chi^2_{0.95(n-1)}}$$

The above formula is linked with the fact that if we keep 90% area under the chi-square distribution in the middle, then we will have 5% area on the left-hand-side, and 5% area on the right-hand-side, as shown below:

## χ2(N-1)-DISTRIBUTION

In order to apply the above formula, we first need to calculate the sample mean $\overline{X}$, which is

$$\overline{X} = \frac{\sum X}{n} = \frac{88}{8} = 11$$

Then, we obtain

$$\sum_{i=1}^{8}(X_i-\overline{X})^2 = (9-11)^2 + (14-11)^2 + \ldots + (12-11)^2 = 36$$

Next, we need to find :
1) the value of χ2 to the left of which the area under the chi-square distribution is 5%
2) the value of χ2 to the right of which the area under the chi-square distribution is 5%
For this purpose, we will need to consult the table of areas under the chi-square distribution.

## THE CHI-SQUARE TABLE

The entries in this table are values of x2α(ν), for which the area to their right under the chi-square distribution with ν degrees of freedom is equal to α.

**Upper Percentage Points of the Chi-square Distribution**

| ν \ α | 0.99 | 0.98 | 0.975 | 0.95 | 0.10 | 0.05 | 0.03 | 0.02 | 0.01 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.0002 | 0.001 | 0.001 | 0.004 | 2.71 | 3.84 | 5.02 | 5.41 | 6.64 |
| 2 | 0.020 | 0.040 | 0.051 | 0.103 | 4.61 | 5.99 | 7.38 | 7.82 | 9.21 |
| 3 | 0.115 | 0.185 | 0.216 | 0.352 | 6.25 | 7.82 | 9.35 | 9.84 | 11.34 |
| 4 | 0.297 | 0.429 | 0.484 | 0.711 | 7.78 | 9.49 | 11.14 | 11.67 | 13.28 |
| 5 | 0.554 | 0.752 | 0.831 | 1.145 | 9.24 | 11.07 | 12.83 | 13.39 | 15.09 |
| 6 | 0.87 | 1.13 | 1.24 | 1.64 | 10.64 | 12.59 | 14.45 | 15.03 | 16.81 |
| 7 | 1.24 | 1.56 | 1.69 | 2.17 | 12.02 | 14.07 | 16.01 | 16.62 | 18.48 |
| 8 | 1.65 | 2.03 | 2.18 | 2.73 | 13.36 | 15.51 | 17.54 | 18.17 | 20.09 |
| 9 | 2.09 | 2.53 | 2.70 | 3.32 | 14.68 | 16.92 | 19.02 | 19.68 | 21.67 |
| 10 | 2.56 | 3.06 | 3.25 | 3.94 | 15.99 | 18.31 | 20.48 | 21.16 | 23.21 |
| 11 | 3.05 | 3.61 | 3.82 | 4.58 | 17.28 | 19.68 | 21.92 | 22.62 | 24.72 |
| 12 | 3.57 | 4.18 | 4.40 | 5.23 | 18.55 | 21.03 | 23.34 | 24.05 | 26.22 |
| 13 | 4.11 | 4.76 | 5.01 | 5.89 | 19.81 | 22.36 | 24.74 | 25.47 | 27.69 |
| 14 | 4.66 | 5.37 | 5.63 | 6.57 | 21.06 | 23.68 | 26.12 | 26.87 | 29.14 |
| 15 | 5.23 | 5.98 | 6.26 | 7.26 | 22.31 | 25.00 | 27.49 | 28.26 | 30.58 |

Chi-Square Table (continued):

| | α | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **ν** | **0.99** | **0.98** | **0.975** | **0.95** | **0.10** | **0.05** | **0.025** | **0.02** | **0.01** |
| **16** | 5.81 | 6.61 | 6.91 | 7.96 | 23.54 | 26.30 | 28.84 | 29.63 | 32.00 |
| **17** | 6.41 | 7.26 | 7.56 | 8.67 | 24.77 | 27.59 | 30.19 | 31.00 | 33.41 |
| **18** | 7.02 | 7.91 | 8.23 | 9.39 | 25.99 | 28.87 | 31.53 | 32.35 | 34.81 |
| **19** | 7.63 | 8.57 | 8.91 | 10.12 | 27.20 | 30.14 | 32.85 | 33.69 | 36.19 |
| **20** | 8.26 | 9.24 | 9.59 | 10.85 | 28.41 | 31.41 | 34.17 | 35.02 | 37.57 |
| **21** | 8.90 | 9.92 | 10.28 | 11.59 | 29.62 | 32.67 | 35.48 | 36.34 | 38.93 |
| **22** | 9.54 | 10.60 | 10.98 | 12.34 | 30.81 | 33.92 | 36.78 | 37.66 | 40.29 |
| **23** | 10.20 | 11.29 | 11.69 | 13.09 | 32.01 | 35.17 | 38.08 | 38.97 | 41.64 |
| **24** | 10.86 | 11.99 | 12.40 | 13.85 | 33.00 | 36.42 | 39.36 | 40.27 | 42.92 |
| **25** | 11.52 | 12.70 | 13.12 | 14.61 | 34.38 | 37.65 | 40.65 | 41.57 | 44.31 |
| **26** | 12.20 | 13.41 | 13.84 | 15.38 | 35.56 | 38.88 | 41.92 | 42.86 | 45.64 |
| **27** | 12.88 | 14.12 | 14.57 | 16.15 | 36.74 | 40.11 | 43.19 | 44.14 | 46.96 |
| **28** | 13.56 | 14.85 | 15.31 | 16.93 | 37.92 | 41.34 | 44.46 | 45.42 | 48.28 |
| **29** | 14.26 | 15.57 | 16.05 | 17.71 | 39.09 | 42.56 | 45.72 | 46.69 | 49.59 |
| **30** | 14.95 | 16.31 | 16.79 | 18.49 | 40.26 | 43.77 | 46.98 | 47.96 | 50.89 |

From the χ2-table, we find that

$\chi 20.05\ (7) = 14.07$

and

$\chi 20.95\ (7) = 2.17$

Hence the 90 percent confidence interval for σ2 is

$$\frac{\sum\left(Xi-\overline{X}\right)^2}{\chi^2_{0.05,(7)}} < \sigma^2 < \frac{\sum\left(X_i-\overline{X}\right)^2}{\chi^2_{0.95,(7)}}$$

or $$\frac{36}{14.07} < \sigma^2 < \frac{36}{2.17}$$

or $$2.56 < \sigma^2 < 16.61$$

Thus the 90% confidence interval for $\sigma^2$ is (2.56, 16.61).

If we take the square root of the lower limit as well as the upper limit of the above confidence interval, we obtain (1.6, 4.1).

So, we may conclude that, on the basis of 90% confidence, we can say that the standard deviation σ of our population lies between 1.6 and 4.1 .We can obtain a confidence interval for σ by taking the square root of the end points of the interval for σ2, but experience has shown that σ cannot be estimated with much precision for small sample sizes.

The formula of the confidence interval for σ2 that we have applied in the above example is based on the fact that:

If $\overline{X}$ and S2 are the mean and variance (respectively) of a random sample X1, X2, …, Xn of size n drawn from a normal population with variance σ2, then the statistic

$$\frac{\sum\left(X_i-\overline{X}\right)^2}{\sigma^2} = \frac{nS^2}{\sigma^2} = \frac{\left(n-1\right)s^2}{\sigma^2}$$

follows a chi-square distribution with (n – 1) degrees of freedom.

Next, we consider hypothesis - testing regarding the population variance σ2 :

We illustrate this concept with the help of an example:

**EXAMPLE**

The variability in the tensile strength of a type of steel wire must be controlled carefully. A sample of the wire is subjected to test, and it is found that the sample variance is S2 = 31.5. The sample size was n = 16 observations.

Test the hypothesis that the population variance is 25 against the alternative that the variance is greater than 25. Use a 0.05 level of significance.

**SOLUTION**

**a)i)** We have to decide between the hypotheses

H0 : σ2 = 25, and
H1 : σ2 > 25

**ii)** The level of significance is $\alpha = 0.05$.

**iii)** The test statistic is $\chi^2 = \frac{nS^2}{\sigma_0^2}$, which under $H_0$, has a $\chi^2$-distribution with (n–1) degrees of freedom, assuming that the population is normal.

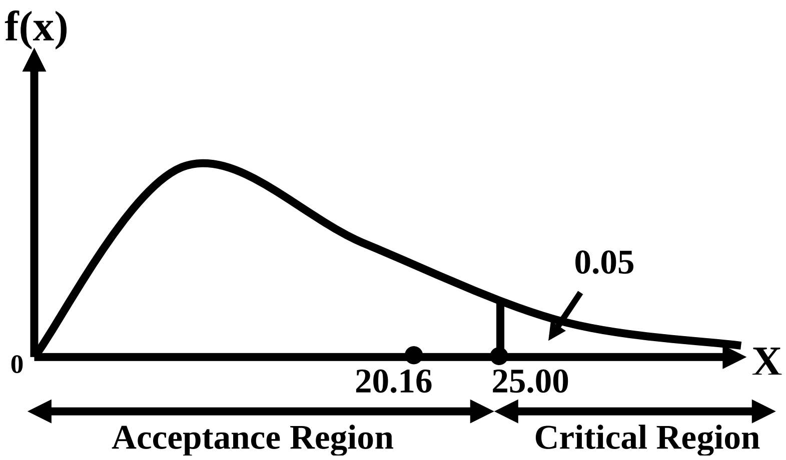
**iv)** We calculate the value of $\chi^2$ from the sample data as

$$\chi^2 = \frac{nS^2}{\sigma_0^2} = \frac{16(31.5)}{25} = 20.16 .$$

**v)** The critical region is $\chi 2 > \chi^2_{0.05,(15)} = 25.00$ (one tailed test)

**vi)** Conclusion.

Since the calculated value of χ2 falls in the acceptance region, so we accept our null Hypothesis, i.e. we have reasonable evidence to conclude that σ2 = 25.The Chi-Square Distribution with 15 degrees of Freedom:




The above example points to the following general procedure for testing a hypothesis regarding the population variance σ2: Suppose we desire to test a null hypothesis H0 that the variance σ2 of a normally distributed population has some specified value, say σ02. To do this, we need to draw a random sample $X_1, X_2, \ldots, X_n$ of size n from the normal population and compute the value of the sample variance $S^2$. If the null hypothesis $H_0 : \sigma^2 = \sigma^2_0$ is true, then the statistic $\chi^2 = \frac{nS^2}{\sigma_0^2}$ has a $\chi^2$-distribution with (n–1) degrees of freedom.

# LECTURE NO. 42

Masters

- The F-Distribution
- Hypothesis Testing and Interval Estimation in order to Compare the Variances of Two Normal Populations (based on F-Distribution)

Before we describe you statistical inference based on the F-distribution, let us consolidate the idea of hypothesis-testing regarding the population variance with the help of an example:

**EXAMPLE**

The manager of a bottling plant is anxious to reduce the variability in net weight of fruit bottled. Over a long period, the standard deviation has been 15.2 gm. A new machine is introduced and the net weights (in grams) in 10 randomly selected bottles (all of the same nominal weight) are 987, 966, 955, 977, 981, 967, 975, 980, 953, 972. Would you report to the manager that the new machine has a better performance?

**SOLUTION**

**i)** We have to decide between the hypotheses

H0 : $\sigma$ = 15.2, i.e. the standard deviation is 15.2gm

H1 : $\sigma$ < 15.2 i.e. the standard deviation has been reduced.

**ii)** We choose the significance level at $\alpha = 0.05$.

**iii)** The test-statistic is

$$\chi^2 = \frac{nS^2}{\sigma_0^2} = \frac{\sum\left(X_i - \overline{X}\right)^2}{\sigma_0^2}$$

which under $H_0$, has a $\chi^2$ -distribution with (n – 1) degrees of freedom, assuming that the weights are normally distributed.

**iv)** Computations.

n = 10, $\sum$Xi = 9713, $\sum$X2i = 9435347

**v)** The critical region is $\chi$ 2 < $\chi$20.95 (9) = 3.32 (the lower 5% point)

NOW

$nS^2 = \sum(X_i - \overline{X})^2 = \sum X_i^2 - (\sum X_i)^2/n$

$= 9435347 - (9713)^2/10 = 1110.1$

$$\therefore \quad \chi^2 = \frac{1110.1}{(15.2)^2} = \frac{1110.1}{231.04} = 4.81$$

**vi)** Conclusion:

Since the calculated value of $\chi^2 = 4.81$ does not fall in the critical region, we therefore cannot reject the null hypothesis that the standard deviation is 15.2 gm and hence we would not report to the manager that the new machine has a better performance.

The above example points to the fact that, if we wish to test a null hypothesis H0 that the variance $\sigma$2 of a normally distributed population has some specified value, say $\sigma$02, then, (having drawn a random sample X1, X2, …, Xn of size n from the normal population), we will compute the value of the sample variance S2.

The mathematics underlying this hypothesis-testing procedure states that:

If the null hypothesis $H_0 : \sigma^2 = \sigma_0{}^2$ is true, then the statistic $\chi^2 = \dfrac{nS^2}{\sigma_0^2}$ has a $\chi^2$-distribution with (n–1) degrees of freedom.

A point to be noted is that, since the random variable X is distributed as chi-square, therefore we call it $\chi$2.

If we do so, our equation of the chi-square distribution can be written as

$$f\left(\chi^2\right) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\left(\chi^2\right)^{(\nu/2)-1}.e^{-\chi^2/2}, \; 0 < \chi^2 < \infty$$

It should be obvious that the standard deviation of the normal population will be tested in the same way as the population variance is tested.

Next, we begin the discussion of statistical inference regarding the ratio of two population variances.

As this particular inference is based on the F-distribution, therefore we begin with the discussion of the mathematical definition and the main properties of the F-distribution.

## THE F-DISTRIBUTION

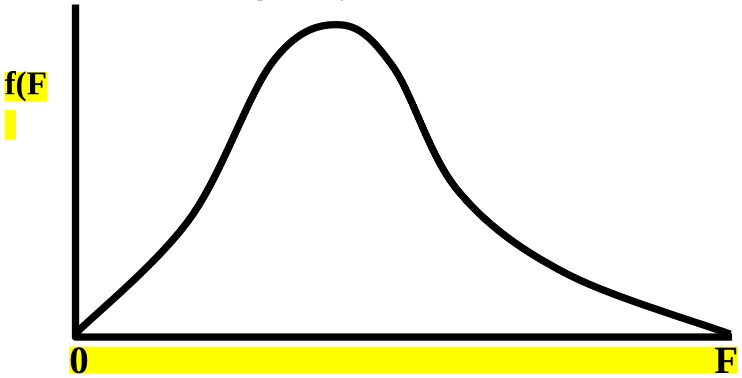
The mathematical equation of the F-distribution is as follows:

$$f(x) = \frac{\Gamma[(\nu_1+\nu_2)/2]\,(\nu_1/\nu_2)^{\nu_1/2}\;x^{(\nu_1/2)-1}}{\Gamma(\nu_1/2)\,\Gamma(\nu_2/2)\,[1+\nu_1 x/\nu_2]^{(\nu_1+\nu_2)/2}},\quad 0 < x < \infty$$

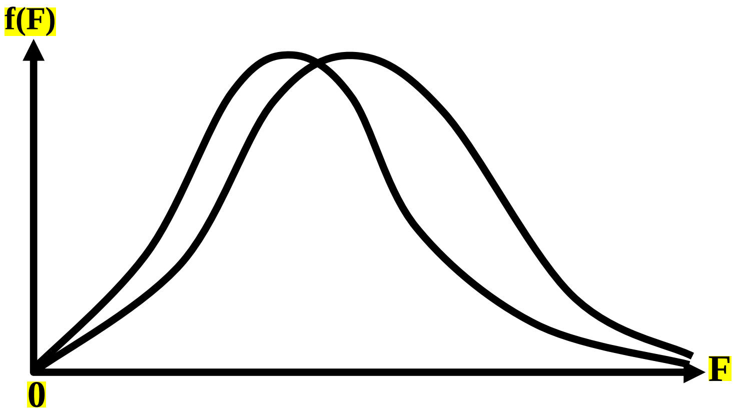
This distribution has two parameters ν1 and ν2, which are known as the degrees of freedom of the F-distribution.The F-distribution having the above equation have ν1 degrees of freedom in the numerator and ν2 degrees of freedom in the denominator. It is usually abbreviated as F(ν1, ν2).

## PROPERTIES OF F-DISTRIBUTION

**1.** The F-distribution is a continuous distribution ranging from zero to plus infinity.
**2**. The curve of the F-distribution is positively skewed.




But as the degrees of freedom ν1 and ν2 become large, the F-distribution approaches the normal distribution.




**3**. For ν2 > 2, the mean of the F-distribution is

$$\frac{\nu_2}{\nu_2 - 2}$$

which is greater than 1.
**4.**For ν2 > 4, the variance of the F-distribution is

$$\sigma^2 = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

**5.** The F-distribution for ν1 > 2, ν2 > 2 is unimodal, and the mode of the distribution with ν1 > 1 is at

$$\frac{\nu_2(\nu_1-2)}{\nu_1(\nu_2+2)}$$

which is always less than 1.
**6.** If F has an F-distribution with ν1 and ν2 degrees of freedom, then the reciprocal has an F-distribution with ν2 and ν1 degrees of freedom. Next, we consider the tables of the F-distribution. As the F-distribution involves two parameters,

ν1 and ν2, hence separate tables have been constructed for 5%, 2½ % and 1% right-tail areas respectively, as shown below:

**The F-table pertaining to 5% right-tail areas is as follows:**

# Upper 5 Percent Points of The F-Distribution i.e., $F_{0.05}$ ($\nu_1$, $\nu_2$)

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 161.4 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 238.9 | 243.9 | 249.0 | 254.3 |
| **2** | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.37 | 19.41 | 19.45 | 19.50 |
| **3** | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.84 | 8.74 | 8.64 | 8.53 |
| **4** | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.04 | 5.91 | 5.77 | 5.63 |
| **5** | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.82 | 4.68 | 4.53 | 4.36 |
| **6** | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.15 | 4.00 | 3.84 | 3.67 |
| **7** | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.73 | 3.57 | 3.41 | 3.23 |
| **8** | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.44 | 3.28 | 3.12 | 2.93 |
| **9** | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.23 | 3.07 | 2.90 | 2.71 |
| **10** | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.07 | 2.91 | 2.74 | 2.54 |
| **11** | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 2.95 | 2.79 | 2.61 | 2.40 |
| **12** | 4.75 | 3.88 | 3.49 | 3.26 | 3.11 | 3.00 | 2.85 | 2.69 | 2.50 | 2.30 |
| **13** | 4.67 | 3.80 | 3.41 | 3.18 | 3.03 | 2.92 | 2.77 | 2.60 | 2.42 | 2.21 |
| **14** | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.70 | 2.53 | 2.35 | 2.13 |
| **15** | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.64 | 2.48 | 2.29 | 2.07 |

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| **16** | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.59 | 2.42 | 2.24 | 2.01 |
| **17** | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.55 | 2.38 | 2.19 | 1.96 |
| **18** | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.51 | 2.34 | 2.15 | 1.92 |
| **19** | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.48 | 2.31 | 2.11 | 1.88 |
| **20** | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.45 | 2.28 | 2.08 | 1.84 |
| **21** | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.42 | 2.25 | 2.05 | 1.81 |
| **22** | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.40 | 2.23 | 2.03 | 1.78 |
| **23** | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.38 | 2.20 | 2.00 | 1.76 |
| **24** | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.36 | 2.18 | 1.98 | 1.73 |
| **25** | 4.24 | 3.38 | 2.99 | 2.76 | 2.60 | 2.49 | 2.34 | 2.16 | 1.96 | 1.71 |
| **26** | 4.22 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.32 | 2.15 | 1.95 | 1.69 |
| **27** | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.30 | 2.13 | 1.93 | 1.67 |
| **28** | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.44 | 2.29 | 2.12 | 1.91 | 1.65 |
| **29** | 4.18 | 3.33 | 2.93 | 2.70 | 2.54 | 2.43 | 2.28 | 2.10 | 1.90 | 1.64 |
| **30** | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.27 | 2.09 | 1.89 | 1.62 |
| **40** | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.18 | 2.00 | 1.79 | 1.51 |
| **60** | 4.00 | 3.15 | 2.76 | 2.52 | 2.37 | 2.25 | 2.10 | 1.92 | 1.70 | 1.39 |
| **120** | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.17 | 2.02 | 1.83 | 1.61 | 1.25 |
| **∞** | 3.84 | 2.99 | 2.60 | 2.37 | 2.21 | 2.10 | 1.94 | 1.73 | 1.52 | 1.00 |

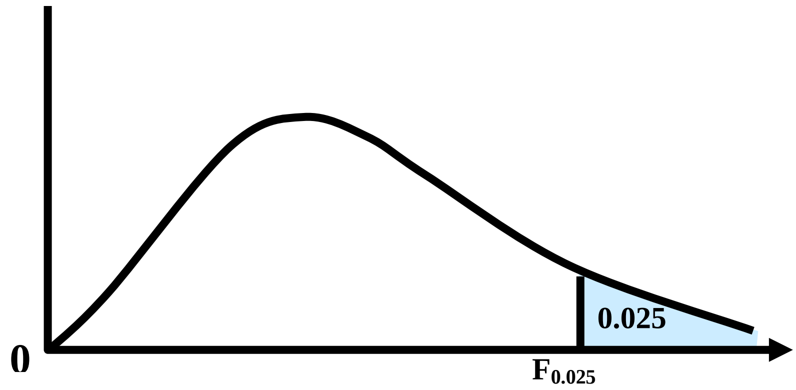
Similarly, the F-table pertaining to 2½% right-tail areas is as follows

**Upper 2.5 Percent Points of the F-Distribution i.e. $F_{0.025}(\nu_1, \nu_2)$**

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 647.8 | 799.5 | 864.2 | 899.6 | 921.8 | 937.1 | 956.7 | 976.7 | 997.2 | 1018 |
| 2 | 38.51 | 39.00 | 39.17 | 39.25 | 39.30 | 39.33 | 39.37 | 39.41 | 39.46 | 39.50 |
| 3 | 17.44 | 16.04 | 15.44 | 15.10 | 14.88 | 14.73 | 14.54 | 14.34 | 14.12 | 13.90 |
| 4 | 12.22 | 10.65 | 9.98 | 9.60 | 9.36 | 9.20 | 8.98 | 8.75 | 8.51 | 8.26 |
| 5 | 10.07 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.76 | 6.52 | 6.28 | 6.02 |
| 6 | 8.81 | 7.26 | 6.60 | 6.23 | 5.99 | 5.82 | 5.60 | 5.37 | 5.12 | 4.85 |
| 7 | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.90 | 4.67 | 4.42 | 4.14 |
| 8 | 7.57 | 6.06 | 5.42 | 5.05 | 4.82 | 4.65 | 4.43 | 4.20 | 3.95 | 3.67 |
| 9 | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.10 | 3.87 | 3.61 | 3.33 |
| 10 | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.85 | 3.62 | 3.37 | 3.08 |
| 11 | 6.72 | 5.26 | 4.63 | 4.28 | 4.04 | 3.88 | 3.66 | 3.43 | 3.17 | 2.88 |
| 12 | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.51 | 3.28 | 3.02 | 2.72 |
| 13 | 6.41 | 4.97 | 4.35 | 4.00 | 3.77 | 3.60 | 3.39 | 3.15 | 2.89 | 2.60 |
| 14 | 6.30 | 4.86 | 4.24 | 3.89 | 3.66 | 3.50 | 3.29 | 3.05 | 2.79 | 2.49 |
| 15 | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.20 | 2.96 | 2.70 | 2.40 |

**Upper 2.5 Percent Points of the F-Distribution i.e. $F_{0.025}(\nu_1, \nu_2)$ (Continued):**

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 6.12 | 4.69 | 4.08 | 3.73 | 3.50 | 3.34 | 3.12 | 2.89 | 2.63 | 2.32 |
| 17 | 6.04 | 4.62 | 4.01 | 3.66 | 3.44 | 3.28 | 3.06 | 2.82 | 2.56 | 2.25 |
| 18 | 5.98 | 4.56 | 3.95 | 3.61 | 3.38 | 3.22 | 3.01 | 2.77 | 2.50 | 2.19 |
| 19 | 5.92 | 4.51 | 3.90 | 3.56 | 3.33 | 3.17 | 2.96 | 2.72 | 2.45 | 2.13 |
| 20 | 5.87 | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 2.91 | 2.68 | 2.41 | 2.09 |
| 21 | 5.83 | 4.42 | 3.82 | 3.48 | 3.25 | 3.09 | 2.87 | 2.64 | 2.37 | 2.04 |
| 22 | 5.79 | 4.38 | 3.78 | 3.44 | 3.22 | 3.05 | 2.84 | 2.60 | 2.33 | 2.00 |
| 23 | 5.75 | 4.35 | 3.75 | 3.41 | 3.18 | 3.02 | 2.81 | 2.57 | 2.30 | 1.97 |
| 24 | 5.72 | 4.32 | 3.72 | 3.38 | 3.15 | 2.99 | 2.78 | 2.54 | 2.27 | 1.94 |
| 25 | 5.69 | 4.29 | 3.69 | 3.35 | 3.13 | 2.97 | 2.75 | 2.51 | 2.24 | 1.91 |
| 26 | 5.66 | 4.27 | 3.67 | 3.33 | 3.10 | 2.94 | 2.73 | 2.49 | 2.22 | 1.88 |
| 27 | 5.63 | 4.24 | 3.65 | 3.31 | 3.08 | 2.92 | 2.71 | 2.47 | 2.19 | 1.85 |
| 28 | 5.61 | 4.22 | 3.63 | 3.29 | 3.06 | 2.90 | 2.69 | 2.45 | 2.17 | 1.83 |
| 29 | 5.59 | 4.20 | 3.61 | 3.27 | 3.04 | 2.88 | 2.67 | 2.43 | 2.15 | 1.81 |
| 30 | 5.57 | 4.18 | 3.59 | 3.25 | 3.06 | 2.87 | 2.65 | 2.41 | 2.14 | 1.79 |
| 40 | 5.42 | 4.05 | 3.46 | 3.13 | 2.90 | 2.74 | 2.53 | 2.29 | 2.01 | 1.64 |
| 60 | 5.49 | 3.93 | 3.34 | 3.01 | 2.79 | 2.63 | 2.41 | 2.17 | 1.88 | 1.48 |
| 120 | 5.15 | 3.80 | 3.23 | 2.89 | 2.67 | 2.52 | 2.30 | 2.05 | 1.76 | 1.31 |
| ∞ | 5.02 | 3.69 | 3.12 | 2.79 | 2.57 | 2.41 | 2.19 | 1.94 | 1.64 | 1.00 |



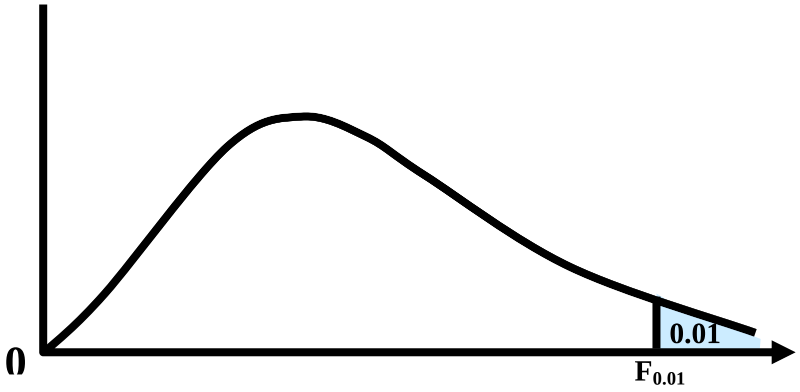

And, the F-table pertaining to 1% right-tail areas is as follows:

**Upper 1 Percent Points of the F-Distribution i.e. $F_{0.01}(\nu_1, \nu_2)$**

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4052 | 5000 | 5403 | 5625 | 5764 | 5859 | 5982 | 6106 | 6235 | 6366 |
| 2 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.37 | 99.42 | 99.46 | 99.50 |
| 3 | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.49 | 27.05 | 26.60 | 26.12 |
| 4 | 21.20 | 18.00 | 10.69 | 15.98 | 15.52 | 15.21 | 14.80 | 14.37 | 13.93 | 13.46 |
| 5 | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.29 | 9.89 | 9.47 | 9.02 |
| 6 | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.10 | 7.72 | 7.31 | 6.88 |
| 7 | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.84 | 6.47 | 6.07 | 5.65 |
| 8 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.03 | 5.67 | 5.28 | 4.86 |
| 9 | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.47 | 5.11 | 4.73 | 4.31 |
| 10 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.06 | 4.71 | 4.33 | 3.91 |
| 11 | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.74 | 4.40 | 4.02 | 3.61 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.50 | 4.16 | 3.78 | 3.36 |
| 13 | 9.07 | 6.70 | 5.74 | 5.20 | 4.86 | 4.62 | 4.30 | 3.96 | 3.59 | 3.17 |
| 14 | 8.86 | 6.51 | 5.56 | 5.03 | 4.69 | 4.46 | 4.14 | 3.80 | 3.43 | 3.00 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.00 | 3.67 | 3.29 | 2.87 |

**Upper 1 Percent Points of the F-Distribution i.e. $F_{0.01}(\nu_1, \nu_2)$ (continued)**

| $\nu_2$ \ $\nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 12 | 24 | ∞ |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 3.89 | 3.55 | 3.18 | 2.75 |
| 17 | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.79 | 3.45 | 3.08 | 2.65 |
| 18 | 8.28 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.71 | 3.37 | 3.03 | 2.57 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.63 | 3.30 | 2.92 | 2.49 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.56 | 3.23 | 2.86 | 2.42 |
| 21 | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.51 | 3.17 | 2.80 | 2.36 |
| 22 | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.45 | 3.12 | 2.75 | 2.31 |
| 23 | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.41 | 3.07 | 2.70 | 2.26 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.36 | 3.03 | 2.66 | 2.21 |
| 25 | 7.77 | 5.57 | 4.68 | 4.18 | 3.86 | 3.63 | 3.32 | 2.99 | 2.62 | 2.17 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.29 | 2.96 | 2.58 | 2.13 |
| 27 | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.26 | 2.93 | 2.55 | 2.10 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.23 | 2.90 | 2.52 | 2.06 |
| 29 | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.20 | 2.87 | 2.49 | 2.03 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.17 | 2.84 | 2.47 | 2.01 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 2.99 | 2.66 | 2.29 | 1.80 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.82 | 2.50 | 2.12 | 1.60 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.66 | 2.34 | 1.94 | 1.38 |
| ∞ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.51 | 2.18 | 1.79 | 1.00 |




Having discussed the basic definition and the main properties of the F-distribution, we now begin the discussion of the role of the F-distribution in statistical inference: First, we discuss interval estimation regarding the ratio of two population variances:

**CONFIDENCE INTERVAL FOR THE VARIANCE RATIO σ12/σ22**

Let two independent random samples of size n1 and n2 be taken from two normal population with variances σ12 and σ22 and let s12 and s22 be the unbiased estimators of σ12 and σ22.
Then, it can be mathematically proved that the quantity

$$F = \frac{s_1^2 / \sigma_1^2}{s_2^2 / \sigma_2^2}$$

has an F-distribution with (n1 – 1, n2 – 1) degrees of freedom.
The confidence interval for σ12/σ22 is given by

$$\left[\frac{s_1^2}{s_2^2}.\frac{1}{F_{\alpha/2}(n_1-1,\, n_2-1)},\ \frac{s_1^2}{s_2^2}.F_{\alpha/2}(n_2-1,\, n_1-1)\right]$$

We can also find a confidence interval for σ1/σ2 by taking the square root of the end points of the above interval.
We illustrate this concept with the help of the following example:

**EXAMPLE**

A random sample of 12 salt-water fish was taken, and the girth of the fish was measured. The standard deviation s1 came out to be 2.3 inches. Similarly, a random sample of 10 fresh-water fish was taken, and the girth of the fish was measured. The standard deviation of this sample i.e. s2 came out to be 1.5 inches. Find a 90% confidence interval for the ratio between the 2 population variances σ12/σ22. Assume that the populations of girth are normal.

**SOLUTION**

The 90% confidence interval for σ12/σ22 is given by

$$\left[\frac{s_1^2}{s_2^2}.\frac{1}{F_{0.05}(n_1-1,\, n_2-1)},\ \frac{s_1^2}{s_2^2}.F_{0.05}(n_2-1,\, n_1-1)\right]$$

Here s12 = (2.3)2 = 5.29,
s22 = (1.5)2 = 2.25,
n1– 1 = 12 – 1 = 11 and n2 – 1 = 10 – 1 = 9

Hence,
F0.05 (n1 – 1, n2 – 1) = F0.05 (11, 9) = 3.1
and
F0.05 (n2 – 1, n1 – 1) = F0.05 (9, 11) = 2.9

With reference to the F-table, it should be noted that if it is an abridged table and the F-values are not available for all possible pairs of degrees of freedom, then the required F-values are obtained by the method of interpolation. In this example, for the lower limit of our confidence interval, we need the value of F0.05(11, 9), but in the above table pertaining to 5% right-tail area, values are available for ν1 = 8 and ν1 = 12, but not for ν1 = 11. Hence, we can find the F-value corresponding to ν1 = 11 by the method of interpolation: The F-value corresponding to ν2 = 9 and ν1 = 8 is 3.23 whereas the F-value corresponding to ν2 = 9 and ν1 = 12 is 3.07.If we wish to find the F-value corresponding to ν2 = 9 and ν1 = 10, we can find the arithmetic mean of 3.23 and 3.07 which is 3.15.If we wish to find the F-value corresponding to ν2 = 9 and ν1 = 11, we can find the arithmetic mean of 3.15 and 3.07 which is 3.11, which, upon

rounding, is equal to 3.1.The above method of interpolation is based on the assumption that the F-values between any two successive F-values (printed in any row of the F-table) are equally spaced between the two given values.
If we do not wish to go through the rigorous procedure of interpolation, we can note that ν1 = 11 is close to ν1 = 12, and hence, we can consider that F-value which corresponds to ν1 = 12 (which in this case is 3.07 ~ 3.1 ----- exactly the same as what we obtained above (correct to one decimal place) by the method of interpolation). Going back to our example, the 90% confidence interval is

$$\left[\frac{5.29}{2.25}\cdot\left(\frac{1}{3.1}\right), \frac{5.29}{2.25}(2.9)\right]$$

or (0.76, 6.81).

Taking the square root of the end points (0.76, 6.81), we obtain the 90% confidence interval for σ1/σ2 as (0.87, 2.61).
Next, we discuss hypothesis - testing regarding the equality of two population variances: Suppose that we have two independent random samples of size n1 and n2 from two normal populations with variances σ12 and σ22, we wish to test the hypothesis that the two variances are equal. The main steps of the hypothesis - testing procedure are similar to the ones that we have been discussing earlier. We illustrate this concept with the help of an example:

**<u>EXAMPLE</u>**

In two series of hauls to determine the number of plankton organisms inhabiting the waters of a lake, the following results were found:
**Series I:** 80, 96, 102, 77, 97, 110, 99, 88, 103, 1089
**Series II:** 74, 122, 92, 81, 104, 92, 92
In series I, the hauls were made in succession at the same place. In series II, they were made in different parts scattered over the lake. Does there appear to be a greater variability between different places than between different times at the same place?

**<u>SOLUTION</u>**

If X denotes the number of plankton organisms per haul, then for each of the two series, X can be assumed to be normally distributed.
Hypothesis-testing Procedure:
**Step 1 :**
H0 : σ12 ≥ σ22 i.e. σ22 ≤ σ12
HA : σ12 < σ22 i.e. σ22 > σ12
**Step 2:** Level of significance: α = 0.05
**Step 3:** Test-statistic:
Since both the populations are normally distributed, hence, the statistic

$$F = \frac{s_2^{\,2}/\sigma_2^{\,2}}{s_1^{\,2}/\sigma_1^{\,2}}$$

will follow the F-distribution having (n2 - 1, n1 - 1) degrees of freedom.
**Step 4 :** Computations:

| $X_1$ | $X_1^2$ |
|---|---|
| 80 | 6400 |
| 96 | 9216 |
| 102 | 10404 |
| 77 | 5929 |
| 97 | 9409 |
| 110 | 12100 |
| 99 | 9801 |
| 88 | 7744 |
| 103 | 10609 |
| 108 | 11664 |
| 960 | 93276 |

| $X_2$ | $X_2^2$ |
|---|---|
| 74 | 5476 |
| 122 | 14884 |
| 92 | 8464 |
| 81 | 6561 |
| 104 | 10816 |
| 92 | 8464 |
| 92 | 8464 |
| 657 | 63129 |

Now

$$s_1^{\,2} = \frac{1}{n_1 - 1}\left[\sum X_1^{\,2} - \frac{\left(\sum X_1\right)^2}{n_1}\right]$$

and

$$s_2^{\,2} = \frac{1}{n_2 - 1}\left[\sum X_2^{\,2} - \frac{\left(\sum X_2\right)^2}{n_2}\right]$$

So $$s_1^{\,2} = \frac{1}{10-1}\left[9326 - \frac{(960)^2}{10}\right]$$

$$= \frac{1}{9}\left[93276 - 92160\right]$$

$$= \frac{1}{9}\left[1116\right] = 124$$

Similarly $$s_2^{\,2} = \frac{1}{n_2 - 1}\left[\sum X_2^{\,2} - \frac{\left(\sum X_2\right)^2}{n_2}\right]$$

$$= \frac{1}{7-1}\left[63129 - \frac{(657)^2}{7}\right]$$

$$= \frac{1}{6}\left[63129 - 61664.14\right]$$

$$= \frac{1}{6}\left[1464.86\right] = 244.14$$

Hence $$F = \frac{s_2^{\,2}}{s_1^{\,2}} = \frac{244.14}{124} = 1.97$$

**Step 5 :** Critical Region:

F > F0.05 (6, 9) = 3.37

**Step 6:** Conclusion:

Since 1.97 is less than 3.37, we do not reject H0; our data does not provide sufficient evidence to indicate that there is greater variability (in the number of plankton organisms per haul) between different places than between different times at the same place.

Let us consider another example:

**<u>EXAMPLE</u>**

Two methods of determining the moisture content of samples of canned corn have been proposed and both have been used to make determinations on proportions taken from each of 21 cans. Method I is easier to apply but appears to be more variable than Method II.

If the variability of Method I were not more than 25 per cent greater than that of Method II, then we would prefer Method I.

The sample results are as follows:

$$n_1 = n_2 = 21;\ \ \overline{X}_1 = 50;\ \overline{X}_2 = 53$$

$$\sum\left(X_1 - \overline{X}_1\right)^2 = 720; \qquad \sum\left(X_2 - \overline{X}_2\right)^2 = 340.$$

Based on the above sample results, which method would you recommend?

**<u>SOLUTION</u>**

In order to solve this problem, the first point to be noted is that, in this problem, our null and alternative hypotheses will be

H0: σ12 ≤ 1.25 σ22

and

H1: σ12 > 1.25 σ22.

Null and Alternative Hypotheses:

In this problem, we need to test
H0 : σ12 ≤ 1.25 σ22
against
H1 : σ12 > 1.25 σ22.
This is so, because 1.25 σ22 means 125% of σ22, and this means 25% greater than σ22. You are encouraged to work on this point on their own. The second point to be noted is that, in this problem, our test-statistic is not but is

$$F = \frac{s_1^{\ 2}}{1.25\ s_2^{\ 2}}.$$

Test Statistic:

$$\mathrm{F} = \frac{\mathrm{s}_1^{\ 2}}{1.25\ \mathrm{s}_2^{\ 2}}.$$

(Under the null hypothesis, s12 / 1.25 s22 has an F-distribution with ν1 = ν2 = 21-1 = 20 degrees of freedom.)
This is so because, (in accordance with the fact that has an F-distribution with (n1 – 1, n2 – 1) degrees of freedom), it can be shown that:

$$F = \frac{s_1^2 / \sigma_1^2}{s_2^2 / \sigma_2^2}$$

If we have
H0: σ12/ σ22 = k
then

$$F = \frac{s_1^{\ 2}}{s_2^{\ 2}} \cdot \frac{1}{k}$$

has an F-distribution with (n1 - 1, n2 - 1) degrees of freedom. (In this problem, k = 1.25.)You are encouraged to work on this problem also on their own, and to carry out the rest of the steps of the hypothesis-testing procedure (which are the usual ones), and to decide whether to accept or to reject the null hypothesis.

Masters

# LECTURE NO. 43

- Analysis of Variance
- Experimental Design

Earlier, we compared two-population means by using a two-sample t-test. However, we are often required to compare more than two population means simultaneously. We might be tempted to apply the two-sample t-test to all possible pairwise comparisons of means. For example, if we wish to compare 4 population means, there will be $\binom{4}{2} = 6$ separate pairs, and to test the null hypothesis that all four population means are equal, we would require six two-sample t-tests. Similarly, to test the null hypothesis that 10 population means are equal, we would need

$$\binom{10}{2} = 45$$

Separate two-sample t-tests. This procedure of running multiple two-sample t-tests for comparing means would obviously be tedious and time-consuming. Thus a series of two-sample t-tests is not an appropriate procedure to test the equality of several means simultaneously. Evidently, we require a simpler procedure for carrying out this kind of a test. One such procedure is the Analysis of Variance, introduced by Sir R.A. Fisher (1890-1962) in 1923:

## ANALYSIS OF VARIANCE (ANOVA)

It is a procedure which enables us to test the hypothesis of equality of several population means (i.e.
 $H_0 : \mu_1 = \mu_2 = \mu_3 = \ldots\ldots = \mu_k$
against
 $H_A$: not all the means are equal)
The concept of Analysis of Variance is closely related with the concept of Experimental Design:

## EXPERIMENTAL DESIGN

By an experimental design, we mean a plan used to collect the data relevant to the problem under study in such a way as to provide a basis for valid and objective inference about the stated problem. The plan usually includes:

- the selection of treatments whose effects are to be studied,
- the specification of the experimental layout, and
- the assignment of treatments to the experimental units.

All these steps are accomplished before any experiment is performed. Experimental Design is a very vast area. In this course, we will be presenting only a very basic introduction of this area. There are two types of designs:

## SYSTEMATIC AND RANDOMIZED DESIGNS

In this course, we will be discussing only the randomized designs, and, in this regard, it should be noted that for the randomized designs, the analysis of the collected data is carried out through the technique known as Analysis of Variance.
Two of the very basic randomized designs are:

- The Completely Randomized (CR) Design,
- The Randomized Complete
- Block (RCB) Design

We will consider these one by one. We begin with the simplest design i.e. the Completely Randomized (CR) Design:

## THE COMPLETELY RANDOMIZED DESIGN (CR DESIGN)

A completely randomized (CR) design, which is the simplest type of the basic designs, may be defined as a design in which the treatments are assigned to experimental units completely at random, i.e. the randomization is done without any restrictions. This design is applicable in that situation where the entire experimental material is homogeneous (i.e. all the experimental units can be regarded as being similar to each other). We illustrate the concept of the Completely Randomized (CR) Design (pertaining to the case when each treatment is repeated equal number of times) with the help of the following example.

**EXAMPLE**
An experiment was conducted to compare the yields of three varieties of potatoes. Each variety as assigned at random to equal-size plots, four times. The yields were as follow:

| Variety | | |
|---|---|---|
| A | B | C |
| 23 | 18 | 16 |
| 26 | 28 | 25 |
| 20 | 17 | 12 |
| 17 | 21 | 14 |

Test the hypothesis that the three varieties of potatoes are not different in the yielding capabilities.

**SOLUTION**

The first thing to note is that this is an example of the Completely Randomized (CR) Design. We are assuming that all twelve of the plots (i.e. farms) available to us for this experiment are homogeneous (i.e. similar) with regard to the fertility of the soil, the weather conditions, etc., and hence, we are assigning the four varieties to the twelve plots totally at random. Now, in order to test the hypothesis that the mean yields of the three varieties of potato are equal, we carry out the six-step hypothesis-testing procedure, as given below:

Hypothesis-Testing Procedure:

**i)** $H_0 : \mu_A = \mu_B = \mu_C$
$H_A$ : Not all the three means are equal

**ii)** Level of Significance:
$\alpha = 0.05$

**iii)** Test Statistic:

$$F = \frac{MS\ Treatments}{MS\ Error}$$

which, if $H_0$ is true, has an F distribution with $\upsilon_1 = k-1 = 3 - 1 = 2$ and $\upsilon_2 = n-k = 12 - 3 = 9$ degree of freedom

**iv)** Computations:

The computation of the test statistic presented above involves quite a few steps, including the formation of what is known as the ANOVA Table.

First of all, let us consider what is meant by the ANOVA Table (i.e. the Analysis of Variance Table).

In the case of the Completely Randomized (CR) Design, the ANOVA Table is a table of the type given below:

**ANOVA TABLE IN THE CASE OF THE COMPLETELY RANDOMIZED (CR) DESIGN**

| Source of Variation | d.f. | Sum of Squares | Mean Square | F |
|---|---|---|---|---|
| Between treatments | k-1 | SST | MST | MST/MSE |
| Within treatments (Error) | n-k | SSE | MSE | -- |
| Total | n-1 | TSS | -- | -- |

Let us try to understand this table step by step:

The very first column is headed 'Source of Variation', and under this heading, we have three distinct sources of variation:

'Total' stands for the overall variation in the twelve values that we have in our data-set.

| Variety | | |
|---|---|---|
| A | B | C |
| 23 | 18 | 16 |
| 26 | 28 | 25 |
| 20 | 17 | 12 |
| 17 | 21 | 14 |

As you can see, the values in our data-set are 23, 26, 20, 17, 18, 28, and so on. Evidently, there is a variation in these values, and the term 'Total' in the lowest row of the ANOVA Table stands for this overall variation.

The term 'Variation between Treatments' stands for the variability that exists between the three varieties of potato that we have sown in the plots.

(In this example, the term 'treatments' stands for the three varieties of potato that we are trying to compare)

(The term ‘variation between treatments’ points to the fact that:
It is possible that the three varieties or, at least two of the varieties are significantly different from each other with regard to their yielding capabilities. This variability between the varieties can be measured by measuring the differences between the mean yields of the three varieties.)
The third source of variation is ‘variation within treatments’. This point to the fact that even if only one particular variety of potato is sown more than once, we do not get the same yield every time

| Variety | | |
|---|---|---|
| A | B | C |
| 23 | 18 | 16 |
| 26 | 28 | 25 |
| 20 | 17 | 12 |
| 17 | 21 | 14 |

In this example, variety A was sown four times, and the yields were 23, 26, 20, and 17 --- all different from one another! Similar is the case for variety B as well as variety C. The variability in the yields of variety A can be called ‘variation within variety A’.

Similarly, the variability in the yields of variety B can be called ‘variation within variety B’. Also, the variability in the yields of variety C can be called ‘variation within variety C’. We can say that the term ‘variability within treatments’ stands for the combined effect of the above-mentioned three variations. The ‘variation within treatments’ is also known as the ‘error variation’. This is so because we can argue that if we are sowing the same variety in four plots which are very similar to each other, then we should have obtained the same yield from each plot!

If it is not coming out to be the same every time, we can regard this as some kind of an ‘error’.
The second, third and fourth columns of the ANOVA Table are entitled ‘degrees of freedom’, ‘Sum of Squares’ and ‘Mean Square’.

**ANOVA TABLE IN THE CASE OF THE COMPLETELY RANDOMIZED (CR) DESIGN**

| Source of Variation | d.f. | Sum of Squares | Mean Square | F |
|---|---|---|---|---|
| Between treatments | k-1 | SST | MST | MST/MSE |
| Within treatments (Error) | n-k | SSE | MSE | -- |
| Total | n-1 | TSS | -- | -- |

The point to understand is that the sources of variation corresponding to treatments and error will be measured by computing quantities that are called Mean Squares, and ‘Mean Square’ can be defined as:

$$Mean\ Square = \frac{Sum\ of\ \ Squares}{Degrees\ of\ \ Freedom}$$

Corresponding to these two sources of variation, we have the following two equations:

$$1)\ \ 'MS\ Treatment' = \frac{'SS\ Treatment'}{d.f.}$$

AND

$$2)\ \ 'MS\ \ Error' = \frac{'SS\ \ Error'}{d.f.}$$

It has been mathematically proved that, with reference to Analysis of Variance pertaining to the Completely Randomized (CR) Design, the degrees of freedom corresponding to the Treatment Sum of Squares are k-1, and the degrees of freedom corresponding to the Error Sum of Squares are n-k. Hence, the above two equations can be written as:

$$1)\ \ 'MS\ Treatment' = \frac{'SS\ Treatment'}{k-1}$$

AND

$$2)\ \ 'MS\ \ Error' = \frac{'SS\ \ Error'}{n-k}$$

How do we compute the various sums of squares? The three sums of squares occurring in the third column of the above ANOVA Table are given by:

$$1)\ \ Total\ SS = TSS = \sum_i \sum_j X_{ij}^2 - C.F.$$

$$2)\ \ SS\ Treatment = SST = \frac{\sum_j T_{.j}^2}{r} - C.F.$$

where C.F. stands for 'Correction Factor', and is given by

$$C.F. = \frac{T_{..}^2}{n}$$

and r denotes the number of data-values per column (i.e. the number of rows). (It should be noted that this example pertains to that case of the Completely Randomized (CR) Design where each treatment is being repeated equal number of times, and the above formulae pertain to this particular situation. With reference to the CR Design, it should be noted that, in some situations, the various treatments are not repeated an equal number of times.

For example, with reference to the twelve plots (farms) that we have been considering above, we could have sown variety A in five of the plots, variety B in three plots, and variety C in four plots. Going back to the formulae of various sums of squares, the sum of squares for error is given by

$$3)\ \ SS\ \ Error = Total\ SS - SS\ Treatment$$

$$i.e.$$

$$SSE = TSS - SST$$

It is interesting to note that,
Total SS = SS Treatment + SS Error
In a similar way, we have the equation:
Total d.f. = d.f. for Treatment + d.f. for Error
It can be shown that the degrees of freedom pertaining to 'Total' are n - 1.
Now,

n-1 = (k-1) + (n-k)

i.e.
Total d.f. = d.f. for Treatment + d.f. for Error
The notations and terminology given in the above equations relate to the following table:

| | Variety | | | Total | $\sum_j X_{ij}^2$ |
|---|---|---|---|---|---|
| | A | B | C | | |
| | 23 (529) | 18 (324) | 16 (256) | -- | 1109 |
| | 26 (676) | 28 (784) | 25 (625) | -- | 2085 |
| | 20 (400) | 17 (289) | 12 (144) | -- | 833 |
| | 17 (289) | 21 (196) | 14 (196) | -- | 926 |
| $T_{.j}$ | 86 | 84 | 67 | 237 | 4953 |
| $T_{.j}^2$ | 7396 | 7056 | 4489 | 18941 | ↑ |
| $\sum_i X_{ij}^2$ | 1894 | 1838 | 1221 | 4953 | Check ← |

The entries in the body of the table i.e. 23, 26, 20, 17, and so on are the yields of the three varieties of potato that we had sown in the twelve farms. The entries written in brackets next to the above-mentioned data-values are the squares of those values.
For example:

529 is the square of 23,
676 is the square of 26,
400 is the square of 20,

and so on.

Adding all these squares, we obtain:

$$\sum_i \sum_j X_{ij}^2 = 4953$$

| | Variety | | | Total | $\sum_j X_{ij}^2$ |
|---|---|---|---|---|---|
| | A | B | C | | |
| | 23 (529) | 18 (324) | 16 (256) | -- | 1109 |
| | 26 (676) | 28 (784) | 25 (625) | -- | 2085 |
| | 20 (400) | 17 (289) | 12 (144) | -- | 833 |
| | 17 (289) | 21 (196) | 14 (196) | -- | 926 |
| $T_{.j}$ | 86 | 84 | 67 | 237 | 4953 |
| $T_{.j}^2$ | 7396 | 7056 | 4489 | 18941 | ↑ Check ← |
| $\sum_i X_{ij}^2$ | 1894 | 1838 | 1221 | 4953 | |

The notation T.j stands for the total of the jth column.(You must already be aware that, in general, the rows of a bivariate table are denoted by the letter 'i', whereas the columns of a bivariate table are denoted by the letter 'j'.

In other words, we talk about the 'ith row', and the 'jth column' of a bivariate table.)The 'dot' in the notation T.j indicates the fact that summation has been carried out over i (i.e. over the rows).

In this example, the total of the values in the first column is 86, the total of the values in the second column is 84, and the total of the values in the third column is 67.


| | Variety | | | Total | $\sum_j X_{ij}^2$ |
|---|---|---|---|---|---|
| | A | B | C | | |
| | 23 (529) | 18 (324) | 16 (256) | -- | 1109 |
| | 26 (676) | 28 (784) | 25 (625) | -- | 2085 |
| | 20 (400) | 17 (289) | 12 (144) | -- | 833 |
| | 17 (289) | 21 (196) | 14 (196) | -- | 926 |
| $T_{.j}$ | 86 | 84 | 67 | 237 | 4953 |
| $T_{.j}^2$ | 7396 | 7056 | 4489 | 18941 | ↑ Check ← |
| $\sum_i X_{ij}^2$ | 1894 | 1838 | 1221 | 4953 | |


Hence, ΣT.j is equal to 237.

ΣT.j is also denoted by T..

i.e.

T.. = ΣT.j The 'double dot' in the notation T.. indicates that summation has been carried out over i as well as over j.

The row below T.j is that of T.j2, and squaring the three values of T.j, we obtain the quantities 7396, 7056 and 4489.

Adding these, we obtain ΣT.j2 = 18941.

| | Variety | | | Total | $\sum_j X_{ij}^2$ |
|---|---|---|---|---|---|
| | A | B | C | | |
| | 23 (529) | 18 (324) | 16 (256) | -- | 1109 |
| | 26 (676) | 28 (784) | 25 (625) | -- | 2085 |
| | 20 (400) | 17 (289) | 12 (144) | -- | 833 |
| | 17 (289) | 21 (196) | 14 (196) | -- | 926 |
| $T._j$ | 86 | 84 | 67 | 237 | 4953 |
| $T._j^2$ | 7396 | 7056 | 4489 | 18941 | ↑<br>\| |
| $\sum_i X_{ij}^2$ | 1894 | 1838 | 1221 | 4953 | Check<br>← |

Now that we have obtained all the required quantities, we are ready to compute SS Total, SS Treatment, and SS Error:
We have

$$C.F. = \frac{T_{..}^2}{n} = \frac{(237)^2}{12} = 4680.75$$

Hence, the total sum of squares is given by

$$\begin{aligned} TSS \quad &= \sum_i \sum_j X_{ij}^2 - C.F. \\ &= 4953 - 4680.75 \\ &= 272.25 \end{aligned}$$

Also, we have

$$\begin{aligned} SS\ Treatment = SST &= \frac{\sum_j T._j^2}{r} - C.F. \\ &= \frac{18941}{4} - 4680.75 \\ &= 54.50 \end{aligned}$$

And, hence:
SS Error = SSE = TSS - SST
= 272.25 - 54.50 = 217.75

In this example, we have n = 12, and k = 3, hence:
n-1=11,
k- 1 = 2, and n - k = 9.
Substituting the above sums of squares and degree of freedom in the ANOVA table, we obtain:

**ANOVA TABLE**

| Source of Variation | d.f. | Sum of Squares | Mean Square | Computed F |
|---|---|---|---|---|
| Between treatments (i.e. Between varieties) | 2 | 54.50 | | |
| Error | 9 | 217.75 | | |
| Total | 11 | 272.25 | | |

Now, the mean squares for treatments and for error are very easily found by dividing the sums of squares by the corresponding degrees of freedom. Hence, we have

**ANOVA TABLE**

| Source of Variation | d.f. | Sum of Squares | Mean Square | Computed F |
|---|---|---|---|---|
| Between treatments (i.e. Between varieties) | 2 | 54.50 | 27.25 | |
| Error | 9 | 217.75 | 24.19 | |
| Total | 11 | 272.25 | -- | |

As indicated earlier, the test-statistic appropriate for testing the null hypothesis

H0 : μA = μB = μC

versus

HA : Not all the three means are equal is:

$$F = \frac{MS\ Treatments}{MS\ Error}$$

which, if H0 is true, has an F distribution with υ1 = k-1 = 3 – 1 = 2 and υ2 = n-k = 12 – 3 = 9 degree of freedom Hence, it is obvious that F will be found by dividing the first entry of the fourth column of our ANOVA Table by the second entry of the same column i.e.

$$F = \frac{\text{MS Treatment}}{\text{MS Error}} = \frac{27.25}{24.19} = 1.13$$

We insert this computed value of F in the last column of our ANOVA table, and thus obtain:

ANOVA TABLE

| Source of Variation | d.f. | Sum of Squares | Mean Square | Computed F |
|---|---|---|---|---|
| Between treatments (i.e. Between varieties) | 2 | 54.50 | 27.25 | 1.13 |
| Error | 9 | 217.75 | 24.19 | -- |
| Total | 11 | 272.25 | -- | -- |

The fifth step of the hypothesis - testing procedure is to determine the critical region. With reference to the Analysis of Variance procedure, it can be shown that it is appropriate to establish the critical region in such a way that our test is a right-tailed test. In other words, the critical region is given by:

Critical Region:

$F > F\alpha\ (k - 1, n - k)$

In this example:

The critical region is F > F0.05 (2,9) = 4.26

vi) Conclusion:

Since the computed value of F = 1.13 does not fall in the critical region, so we accept our null hypothesis and may conclude that, on the average, there is no difference among the yielding capabilities of the three varieties of potatoes.

In this course, we will not be discussing the details of the mathematical points underlying One-Way Analysis of Variance that is applicable in the case of the Completely Randomized (CR) Design. One important point that the students should note is that the ANOVA technique being presented here is valid under the following assumptions:

- The k populations (whose means are to be compared) are normally distributed;
- All k populations have equal variances i.e. $\sigma 12 = \sigma 22 = \ldots = \sigma k2$. (This property is called homoscedasticity)
- The k samples have been drawn randomly and independently from the respective populations.

Next, we begin the discussion of the Randomized Complete Block (RCB) Design:

**THE RANDOMIZED COMPLETE BLOCK DESIGN (RCB DESIGN)**

A randomized complete block (RCB) design is the one in which

- The experimental material (which is not homogeneous overall) is divided into groups or blocks in such a manner that the experimental units within a particular block are relatively homogeneous.
- Each block contains a complete set of treatments, i.e., it constitutes a replication of treatments.
- The treatments are allocated at random to the experimental units within each block, which means the randomization is restricted.(A new randomization is made for every block.)The object of this type of arrangement is to bring the variability of the experimental material under control.

  In simple words, the situation is as follows:

We have experimental material which is not homogeneous overall. For example, with reference to the example that we have been considering above, suppose that the plots which are closer to a canal are the most fertile ones, the ones a little further away are a little less fertile, and the ones still further away are the least fertile.

In such a situation, we divide the experimental material into groups or blocks which are relatively homogeneous. The randomized complete block design is perhaps the most widely used experimental design. Two-way analysis of variance is applicable in case of the randomized complete block (RCB) design.

We illustrate this concept with the help of an example:

**EXAMPLE**

In a feeding experiment of some animals, four types of rations were given to the animals that were in five groups of four each. The following results were obtained

| Groups | Rations | | | |
|---|---|---|---|---|
| | A | B | C | D |
| I | 32.3 | 33.3 | 30.8 | 29.3 |
| II | 34.0 | 33.0 | 34.3 | 26.0 |
| III | 34.3 | 36.3 | 35.3 | 29.8 |
| IV | 35.0 | 36.8 | 32.3 | 28.0 |
| V | 36.5 | 34.5 | 35.8 | 28.8 |

The values in the above table represent the gains in weights in pounds. Perform an analysis of variance and state your conclusions. In the next lecture, we will discuss this example in detail, and will analyze the given data to carry out the following test:

$H_0$ : $\mu A = \mu B = \mu C = \mu D$

$H_A$ : Not all the treatment-means are equal

Masters

# LECTURE NO. 44

- Randomized Complete Block Design
- The Least Significant Difference (LSD) Test
- Chi-Square Test of Goodness of Fit

At the end of the last lecture, we introduced the concept of the *Randomized Complete Block (RCB) Design*, and we picked up an example to illustrate the concept. In this lecture, we begin with a detailed discussion of the same example:

**EXAMPLE**

In a feeding experiment of some animals, four types of rations were given to the animals that were in five groups of four each. The following results were obtained:

| Groups | Rations | | | |
|---|---|---|---|---|
| | A | B | C | D |
| I | 32.3 | 33.3 | 30.8 | 29.3 |
| II | 34.0 | 33.0 | 34.3 | 26.0 |
| III | 34.3 | 36.3 | 35.3 | 29.8 |
| IV | 35.0 | 36.8 | 32.3 | 28.0 |
| V | 36.5 | 34.5 | 35.8 | 28.8 |

The values in the above table represent the gains in weights in pounds. Perform an analysis of variance and state your conclusions.

**SOLUTION**

Hypothesis-Testing Procedure:

**i a)** Our primary interest is in testing:
$H_0 : \mu_A = \mu_B = \mu_C = \mu_D$
$H_A$ : Not all the ration-means (*treatment-means*) are equal

**i b)** In addition, we can also test:
$H'_0 : \mu_I = \mu_{II} = \mu_{III} = \mu_{IV} = \mu_V$
$H'_A$ : Not all the group-means (*block-means*) are equal

**ii)** Level of significance
$\alpha = 0.05$

**iii a)** Test Statistic for testing
$H_0$ versus $H_A$:

$$F = \frac{MS\ Treatment}{MS\ Error}$$

which, if $H_0$ is true, has
an F-distribution with $v_1 = c-1 = 4-1 = 3$ and $v_2 = (r-1)(c-1) = (5-1)(4-1) = 4 \times 3 = 12$
degrees of freedom.

**iii b)** Test Statistic for testing
$H'_0$ versus $H'_A$:

$$F = \frac{\text{MS Block}}{\text{MS Error}}$$

which, if $H_0$ is true, has
an F-distribution with $v_1 = r-1 = 5-1 = 4$ and
$v_2 = (r-1)(c-1) = (5-1)(4-1) = 4 \times 3 = 12$ degrees of freedom.

Now, the given data leads to the following table:

**iv)** Computations:

| Groups | Ration | | | | $B_{i.}$ | $B_{i.}^2$ | $\sum_j X_{ij}^2$ |
|---|---|---|---|---|---|---|---|
| | A | B | C | D | | | |
| I | 32.3 (10.43.29) | 33.3 (1108.89) | 30.8 (948.64) | 29.3 (858.49) | 125.7 | 15800.49 | 3959.31 |
| II | 34.00 (1156.00) | 33.0 (1089.00) | 34.3 (1176.49) | 26.0 (676.00) | 127.3 | 16205.29 | 4097.49 |
| III | 34.3 (1176.49) | 36.3 (1317.69) | 35.3 (1246.09) | 29.8 (888.04) | 135.7 | 18414.49 | 4628.31 |
| IV | 35.0 (1225.00) | 36.8 (1354.24) | 32.3 (1043.29) | 28.0 (784.00) | 132.1 | 17450.41 | 4406.53 |
| V | 36.5 (1332.25) | 34.5 (1190.25) | 35.8 (1281.64) | 28.8 (829.44) | 135.6 | 18387.36 | 4633.58 |
| $T_{.j}$ | 172.1 | 173.9 | 168.5 | 141.9 | 656.4 | 86258.04 | 21725.22 |
| $T_{.j}^2$ | 29618.41 | 30241.21 | 28392.25 | 20135.61 | 108387.48 | | ↑ |
| $\sum_i X_{ij}^2$ | 5933.03 | 6060.07 | 5696.15 | 4035.97 | 21725.22 | ← | Check ←— |

Hence, we have Total SS $= \sum\sum X_{ij}^2 - \frac{T_{..}^2}{n}$

$$= 21725.22 - \frac{(656.4)^2}{20}$$
$$= 21725.22 - 21543.05$$
$$= 182.17$$

Treatment SS $= \frac{\sum_j T_{.j}^2}{r} - \frac{T_{..}^2}{n}$

$$= \frac{108387.48}{5} - \frac{(656.4)^2}{20}$$

= 21677.50 – 21543.05
= 134.45

Block SS $= \frac{\sum_i B_{i.}^2}{c} - \frac{T_{..}^2}{n}$

$$= \frac{86258.04}{4} - \frac{(656.4)^2}{20}$$
$$= 21564.51 - 21543.05$$

**= 21.46**

where c represents the number of observations per block (i.e. the number of *columns*)
And
Error SS= Total SS – (Treatment SS + Block SS)
= 182.17 – (134.45 + 21.46)
= 26.26
The degrees of freedom corresponding to the various sums of squares are as follows:

- Degrees of freedom for treatments: c - 1 (i.e. the number of treatments - 1)

- Degrees of freedom for blocks:
  r - 1 (i.e. the number of blocks - 1)
- Degrees of freedom for Total:
  rc - 1 (i.e. the total number of observations - 1)
- Degrees of freedom for error: degrees of freedom for Total minus degrees of freedom for treatments minus degrees of freedom for blocks

i.e. (rc-1) - (r-1) - (c-1)
= rc - r - c + 1
= (r-1) (c-1)

Hence the ANOVA-Table is:

**ANOVA-TABLE**

| Source of Variation | d.f. | Sum of Squares | Mean Square | F |
|---|---|---|---|---|
| Between Treatments (i.e. Between Rations) | 3 | 134.45 | 44.82 | $F_1 = 20.47$ |
| Between Blocks (i.e. Between Groups) | 4 | 21.46 | 5.36 | $F_2 = 2.45$ |
| Error | 12 | 26.26 | 2.19 | -- |
| Total | 19 | 182.17 | -- | -- |

**v a)** Critical Region for Testing H0 against HA is given by
F > F0.05 (3, 12) = 3.49

**v b)** Critical Region for Testing H′0 against H′A is given by
F > F0.05 (4, 12) = 3.26

**vi a) Conclusion Regarding Treatment Means**

Since our computed value F1 = 20.47 exceeds the critical value F0.05 (3, 12) = 3.49, therefore we *reject* the null hypothesis, and conclude that there *is* a difference among the means of *at least two* of the treatments (i.e. the mean weight-gains corresponding to at least two of the rations are different).

**vi b) Conclusion Regarding Block Means**

Since our computed value F2 = 2.45 does not exceed the critical value F0.05(4, 12) = 3.26, therefore we accept the null hypothesis regarding the equality of block means and thus conclude that blocking (i.e. the *grouping* of animals) was actually *not required* in this experiment. As far as the conclusion regarding the block means is concerned, this information can be used when designing a similar experiment in the future.

[If blocking is actually not required, then a future experiment can be designed according to the Completely Randomized design, thus retaining more degrees of freedom for Error. (The more degrees of freedom we have for Error, the better, because an estimate of the error variation based on a greater number of degrees of freedom implies an estimate based on a greater amount of information (which is obviously good).) ]

As far as the conclusion regarding the *treatment* means is concerned, the situation is as follows:

Now that we have concluded that there is a significant difference between the treatment means (i.e. we have concluded that the mean weight-gain is *not* the same for all four rations, then it is obvious that we would be interested in finding out, “Which of the four rations produces the greatest weight-gain?”

The answer to this question can be found by applying a technique known as the *Least Significant Difference (LSD) Test*.

**THE LEAST SIGNIFICANT DIFFERENCE (LSD) TEST**

According to this procedure, we compute the *smallest* difference that would be judged significant, and *compare* the absolute values of all differences of means with it. This smallest difference is called the least significant difference or LSD, and is given by:

*LEAST SIGNIFICANT DIFFERENCE (LSD):*

$$LSD = t_{\alpha/2,(\nu)}\sqrt{\frac{2(MSE)}{r}},$$

where MSE is the Mean Square for Error, r is the size of equal samples, and $t\alpha/2\ (\nu)$ is the value of t at $\alpha/2$ level taken against the error degrees of freedom ($\nu$).

The test-criterion that uses the least significant difference is called the LSD test.

Two sample-means are declared to have come from populations with significantly different means, when the absolute value of their difference *exceeds* the LSD.

It is customary to *arrange* the sample means in ascending order of magnitude, and to draw a *line* under any pair of adjacent means (or set of means) that are not significantly different.
The LSD test is applied *only* if the null hypotheses is rejected in the Analysis of Variance. We will not be going into the mathematical details of this procedure, but it is useful to note that this procedure can be regarded as an alternative way of conducting the t-test for the equality of two population means.
If we were to apply the usual two-sample t-test, we would have had to *repeat* this procedure quite a few times!
(The six possible tests are:
H0 : $\mu A = \mu B$
H0 : $\mu A = \mu C$
H0 : $\mu A = \mu D$
H0 : $\mu B = \mu C$
H0 : $\mu B = \mu D$
H0 : $\mu C = \mu D$ )
The LSD test is a procedure by which we can compare *all* the treatment means simultaneously.
We illustrate this procedure through the above example:
The Least Significant Difference is given by

$$\begin{aligned} LSD &= t_{\alpha/2,(\nu)}\sqrt{\frac{2(MSE)}{r}} \\ &= t_{0.025(12)}\sqrt{\frac{2(2.19)}{5}} \\ &= 2.179\sqrt{\frac{2(2.19)}{5}} \\ &= 2.179 \times 0.936 \\ &= 2.04. \end{aligned}$$

Going back to the given data:

| Groups | Rations | | | |
|---|---|---|---|---|
| | A | B | C | D |
| I | 32.3 | 33.3 | 30.8 | 29.3 |
| II | 34.0 | 33.0 | 34.3 | 26.0 |
| III | 34.3 | 36.3 | 35.3 | 29.8 |
| IV | 35.0 | 36.8 | 32.3 | 28.0 |
| V | 36.5 | 34.5 | 35.8 | 28.8 |
| Total | 172.1 | 173.9 | 168.5 | 141.9 |
| Mean | 34.42 | 34.78 | 33.70 | 28.38 |

We find that the four treatment means are:

$$\begin{aligned} \overline{X}_A &= 34.42 \\ \overline{X}_B &= 34.78 \\ \overline{X}_C &= 33.70 \\ \overline{X}_D &= 28.38 \end{aligned}$$

Arranging the above means in *ascending* order of magnitude, we obtain:

$$\begin{array}{cccc} \overline{X}_D & \overline{X}_C & \overline{X}_A & \overline{X}_B \\ 28.38 & 33.70 & 34.42 & 34.78 \end{array}$$

Drawing lines under pairs of adjacent means (or sets of means) that are not significantly different, we have:

$$\begin{array}{cccc} \overline{X}_D & \overline{X}_C & \overline{X}_A & \overline{X}_B \\ 28.38 & \underline{33.70} & \underline{34.42} & \underline{34.78} \end{array}$$

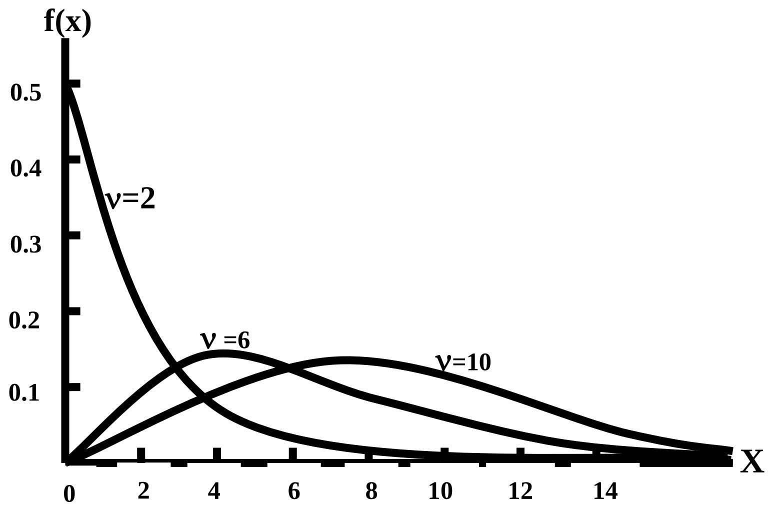
From the above, it is obvious that rations C, A and B are *not* significantly different from each other with regard to weight-gain. The only ration which *is* significantly different from the others is ration D.
Interestingly, ration D has the *poorest* performance with regard to weight-gain. As such, if our primary objective is to *increase* the weights of the animals under study, then we may recommend *any* of the other three rations i.e. A, B and C to the farmers (depending upon availability, price, etc.), but we must *not* recommend ration D.
Next, we will consider two important tests based on the
chi-square distribution. These are:

- The chi-square test of *goodness of fit*
- The chi-square test of *independence*

Before we begin the discussion of these tests, let us review the basic properties of the chi-square distribution:

**PROPERTIES OF THE CHI-SQUARE DISTRIBUTION**

The Chi-Square ($\chi 2$) distribution has the following properties:
**1.** It is a continuous distribution ranging from 0 to $+\infty$ .The number of degrees of freedom determines the *shape* of the chi-square distribution. (Thus, there is a different chi-square distribution for each number of degrees of freedom. As such, it is a whole *family* of distributions.)
**2.** The curve of a chi-square distribution is *positively skewed*. The skewness *decreases as* $\nu$ *increases*.




**χ2-distribution for various values of ν**

As indicated by the above figures, the chi-square distribution tends to the normal distribution as the number of degrees of freedom approaches infinity. Having reviewed the basic properties of the chi-square distribution; we begin the discussion of the Chi-Square Test of Goodness of Fit:

**CHI-SQUARE TEST OF GOODNESS OF FIT**

The chi-square test of goodness-of-fit is a test of hypothesis concerned with the comparison of observed frequencies of a sample, and the corresponding expected frequencies based on a theoretical distribution. We illustrate this concept with the help of the same example that we considered in Lecture No. 28 --- the one pertaining to the fitting of a binomial distribution to real data:

**EXAMPLE:**

The following data has been obtained by tossing a *LOADED* die 5 times, and noting the number of times that we obtained a *six*. Fit a binomial distribution to this data.

| No. of Sixes (x) | 0 | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|---|
| Frequency (f) | 12 | 56 | 74 | 39 | 18 | 1 | 200 |

**SOLUTION**

To fit a binomial distribution, we need to find n and p.
Here n = 5, the largest x-value.
To find p, we use the relationship $\bar{x} = np$.
We have:

| No. of Sixes (x) | 0 | 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|---|---|
| Frequency (f) | 12 | 56 | 74 | 39 | 18 | 1 | 200 |
| fx | 0 | 56 | 148 | 117 | 72 | 5 | 398 |

Therefore:

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{0 + 56 + 148 + 117 + 72 + 5}{200}$$

$$= \frac{398}{200} = 1.99$$

Using the relationship $\bar{x} = np$,
we obtain

$p = 1.99$ or $p = 0.398$.

Letting the random variable X represent the number of sixes, the above calculations yield the fitted binomial distribution as

$$b(x;5,\ 0.398) = \binom{5}{x}(0.398)^x(0.602)^{5-x}$$

Hence the *probabilities* and *expected frequencies* are calculated as below:

| No. of Sixes (x) | Probability f(x) | Expected frequency |
|---|---|---|
| **0** | $\binom{5}{0}q^5 = (0.602)^5$ **= 0.07907** | **15.8** |
| **1** | $\binom{5}{1}q^5 p = 5.(0.602)^4(0.398)$ **= 0.26136** | **52.5** |
| **2** | $\binom{5}{2}q^3 p^2 = 10.(0.602)^3(0.398)^2$ **= 0.34559** | **69.1** |
| **3** | $\binom{5}{3}q^2 p^3 = 10.(0.602)(0.398)^3$ **= 0.22847** | **45.7** |
| **4** | $\binom{5}{4}qp^4 = (0.602)(0.398)^4$ **= 0.07553** | **15.1** |
| **5** | $\binom{5}{5}p^5 = (0.398)^5$ **= 0.00998** | **2.0** |
| **Total** | **= 1.00000** | **200.0** |

Comparing the observed frequencies with the expected frequencies, we obtain:

| No. of Sixes $x$ | Observed Frequency $o_i$ | Expected Frequency $e_i$ |
|---|---|---|
| 0 | 12 | 15.8 |
| 1 | 56 | 52.5 |
| 2 | 74 | 69.1 |
| 3 | 39 | 45.7 |
| 4 | 18 | 15.1 |
| 5 | 1 | 2.0 |
| Total | 200 | 200.0 |

The above table seems to indicate that there is not much discrepancy between the observed and the expected frequencies. Hence, in Lecture No.28, we concluded that it was a reasonably *good* fit. But, it was indicated that *proper* comparison of the expected frequencies with the observed frequencies can be accomplished by applying the *chi-square test of goodness of fit*. The Chi-Square Test of Goodness of Fit enables us to determine in a *mathematical* manner whether or not the theoretical distribution fits the observed distribution reasonably well.
The procedure of the chi-square of goodness of fit is very *similar* to the *general* hypothesis-testing procedure:

**HYPOTHESIS-TESTING PROCEDURE**

**Step-1:**

$H_0$ : The fit is good
$H_A$ : The fit is not good

**Step-2:**

Level of Significance: $\alpha = 0.05$

**Step-3:** Test-Statistic:

$$\chi^2 = \sum_i \frac{(o_i - e_i)^2}{e_i}$$

which, if H0 is true, follows the chi-square distribution having k - 1 - r degrees of freedom(where k = No. of x-values (*after* having carried out the necessary mergers), and r = number of parameters that we estimate from the sample data).

**Step-4:** Computations:

| No. of Sixes $x$ | Observed Frequency $o_i$ | Expected Frequency $e_i$ | $o_i - e_i$ | $(o_i - e_i)^2$ | $(o_i - e_i)^2/e_i$ |
|---|---|---|---|---|---|
| 0 | 12 | 15.8 | -3.8 | 14.44 | 0.91 |
| 1 | 56 | 52.5 | 3.5 | 12.25 | 0.23 |
| 2 | 74 | 69.1 | 4.9 | 24.01 | 0.35 |
| 3 | 39 | 45.7 | -6.7 | 44.89 | 0.98 |
| 4<br>5 | $\left.\begin{matrix}18\\1\end{matrix}\right\}$ **19** | $\left.\begin{matrix}15.1\\2.0\end{matrix}\right\}$ **17.1** | 1.9 | 3.61 | 0.21 |
| Total | 200 | 200.0 | | | 2.69 |

**IMPORTANT NOTE**

In the above table, the category x = 4 has been merged with the category x = 5 because of the fact that the expected frequency corresponding to x = 5 was less than 5.

[It is one of the basic requirements of the chi-square test of goodness of fit that the expected frequency of any x-value (or any combination of x-values) should not be less than 5.]Since we have combined the category x = 4 with the category x = 5, hence k = 5. Also, since we have estimated one parameter of the binomial distribution (i.e. p) from the sample data, hence r = 1. (The other parameter n is already known.)
As such, our statistic follows the chi-distribution having k - 1 - r = 5 - 1 - 1 = 3 degrees of freedom. Going back to the above calculations, the computed value of our test-statistic comes out to be $\chi^2 = 2.69$.
**Step-5:** Critical Region:
Since $\alpha = 0.05$, hence, from the Chi-Square Tables, it is evident that the critical region is:$\chi^2 \geq \chi^2_{0.05}(3) = 7.82$
**Step-6:**
Conclusion:

Since the computed value of $\chi^2$ i.e. 2.69 is less than the critical value 7.82, hence we accept $H_0$ and conclude that the fit is good.(In other words, with only 5% risk of committing Type-I error, we conclude that the distribution of our random variable X can be regarded as a binomial distribution with n = 5 and p = 0.398.)

Masters

## LECTURE NO. 45

- Chi-Square Test of Goodness of Fit (in continuation of the last lecture)
- Chi-Square Test of Independence
- The Concept of Degrees of Freedom
- p-value
- Relationship Between Confidence; Interval and Tests of Hypothesis

An Overview of the Science of Statistics in Today's World (including Latest Definition of Statistics)
The students will recall that, towards the end of the last lecture, we discussed the chi-square test of goodness of fit. We applied the test to the example where we had fitted a binomial distribution to real data, and, since the computed value of our test statistic turned out to be insignificant, therefore we concluded that the fit was good.
Let us consider another example:

**EXAMPLE**

The platform manager of an airline's terminal ticket counter wants to determine whether customer arrivals can be modelled by using a Poisson distribution. The manager is especially interested in late-night traffic.
Accordingly, data for the time period of interest have been collected, as follows:

| Number of Arrivals Per Minute | Frequency |
|---|---|
| 0 | 84 |
| 1 | 114 |
| 2 | 70 |
| 3 | 60 |
| 4 | 32 |
| 5 | 16 |
| 6 | 15 |
| 7 | 4 |
| 8 | 5 |
| | 400 |

Is the distribution Poisson?

**SOLUTION:**

First of all, we fit a Poisson distribution to the given data Because a mean is not specified, it must be estimated from the sample data. The mean of the frequency distribution can be found by using the formula

$$\bar{x} = \frac{\sum fx}{n}$$

where n = Σf.
Thus we have the following calculations:

| Number of Arrivals x | Frequency f | fx |
|---|---|---|
| 0 | 84 | 0 |
| 1 | 114 | 114 |
| 2 | 70 | 140 |
| 3 | 60 | 180 |
| 4 | 32 | 128 |
| 5 | 16 | 80 |
| 6 | 15 | 90 |
| 7 | 4 | 28 |
| 8 | 5 | 40 |
| | 400 | 800 |

*Hence* :

$$Mean = \bar{x} = \frac{\sum fx}{n} = \frac{800}{400} = 2$$

Replacing μ by , the formula for the Poisson probabilities is

Hence, we obtain: $f(x) = \frac{e^{-\bar{x}}\bar{x}^x}{x!} = \frac{e^{-2}2^x}{x!}$

| Number of Customer Arrivals | Observed Frequencies | Poisson Probabilities f(x) | Expected Frequencies 400 f(x) |
|---|---|---|---|
| 0 | 84 | 0.1353 | 54.12 |
| 1 | 114 | 0.2707 | 108.28 |
| 2 | 70 | 0.2707 | 108.28 |
| 3 | 60 | 0.1804 | 72.16 |
| 4 | 32 | 0.0902 | 36.08 |
| 5 | 16 | 0.0361 | 14.44 |
| 6 | 15 | 0.0120 | 4.80 |
| 7 | 4 | 0.0034 | 1.36 |
| 8 | 5 | 0.0009 | 0.36 |
| 9 or more | 0 | 0.0002 | 0.08 |
| | 400 | 1 | 400 |

Next, we apply the chi-square test of goodness of fit according to the following procedure:

**HYPOTHESIS-TESTING PROCEDURE**

**Step-1:**
H0 : Arrivals are Poisson-distributed.
H1 : The distribution is not Poisson.

**Step-2:**
Level of Significance: $\alpha = 0.05$

**Step-3:** Test-Statistic: $\chi^2 = \sum_i \frac{(o_i - e_i)^2}{e_i}$

which, if H0 is true, follows the chi-square distribution having k - 1 - r degrees of freedom; (where k = No. of x-values (after having carried out the necessary mergers), and r = number of parameters that we estimate from the sample data)
**Step-4:**Computations:

The necessary calculations are shown in the following table:

| Number of Customer Arrivals | Observed Frequency $o_i$ | | Expected Frequency $e_i$ | | (0 – e) | (0-e)$^2$ | (0-e)$^2$/e |
|---|---|---|---|---|---|---|---|
| 0 | 84 | | 54.12 | | 29.88 | 892.81 | 16.50 |
| 1 | 114 | | 108.28 | | 5.72 | 32.72 | 0.30 |
| 2 | 70 | | 108.28 | | -38.28 | 1465.36 | 13.53 |
| 3 | 60 | | 72.16 | | -12.16 | 147.87 | 2.05 |
| 4 | 32 | | 36.08 | | -4.08 | 16.65 | 0.46 |
| 5 | 16 | | 14.44 | | 1.56 | 2.43 | 0.17 |
| 6 | 15 | | 4.80 | | | | |
| 7 | 4 | 24 | 1.36 | 6.60 | 17.40 | 302.76 | 45.87 |
| 8 | 5 | | 0.36 | | | | |
| 9 or more | 0 | | 0.08 | | | | |
| | 400 | | 400 | | | | $\chi^2$=78.88 |

With reference to the above, it should be noted that, since some of the expected frequencies are less than the required minimum of 5, it became necessary to combine some of those classes. Combination is best accomplished working from the bottom up.

In order that we obtain a number greater than 5, the last four expected frequencies had to be combined. Hence, the effective number of categories becomes 7.

**Step-5:**

Determination of the Critical Region:

Since the effective number of categories becomes 7

Therefore k = 7.

Also, since the one lone parameter of the Poisson distribution has been estimated from the sample data, hence r = 1.

Hence: Our statistic follows the chi-square distribution having

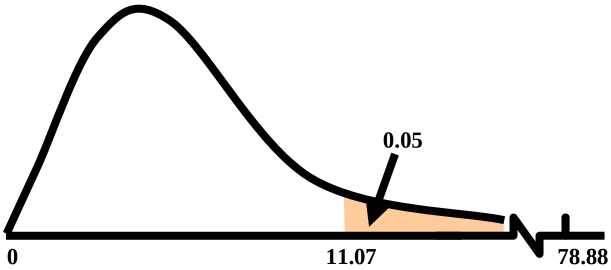
k - 1 - r = 7 - 1 - 1 = 5

degrees of freedom.

The critical region is given by

$\chi^2 \geq \chi^2_{0.05}(5) = 11.07$

CRITICAL REGION:




**Step-6:**

Conclusion:

Since the computed value of our test statistic i.e. 78.88 is much larger than the critical value 11.07, therefore, we reject H0 and conclude that the distribution is probably not a Poisson distribution with parameter 2.

(With only 5% risk of committing Type-1 error, we conclude that the fit is not good.)

In fact, the computed value of our test statistic i.e. 78.88 is so large that it is possible that if we had set the level of significance at 1%, even then it would have exceeded the critical value. The students are encouraged to check this up themselves. If the computed value does fall in the critical region corresponding to 1% level of significance, then our result is highly significant

**RATIONALE OF THE CHI-SQUARE TEST OF GOODNESS OF FIT**

It is clear that $\chi^2 = \sum_i \frac{(o_i - e_i)^2}{e_i}$ will be a small quantity when all the oi's are close to the corresponding ei's. (In fact, if the observed frequencies are exactly equal to the expected ones, then $\chi^2$ will be exactly equal to zero.)

The $\chi^2$ - statistic will become larger when the differences between the oi's and ei have become larger. Thus, $\chi^2$ measure the amount of deviation (or discrepancy) between the observed and the expected results.

**ASSUMPTIONS OF THE CHI-SQUARE TEST OF GOODNESS OF FIT**

While applying the chi-square test of goodness of fit, certain requirements must be satisfied, three of which are as follows:

**1.** The total number of observations (i.e. the sample size) should be at least 50.

**2.** The expected number ei in any of the categories should not be less than 5. (So, when the expected frequency ei in any category is less than 5, we may combine this category with one or more of the other categories to get $e_i \geq 5$.)

**3.** The observations in the sample or the frequencies of the categories should be independent.

Next, we begin the discussion of the Chi-Square Test of Independence:

In this regard, it is interesting to note that, (since the formula of chi-square in this particular situation is very similar to the formula that we have just discussed), therefore, the chi-square test of independence can also be regarded as a kind of chi-square test of goodness of fit. We illustrate this concept with the help of an example:

**EXAMPLE**

A random sample of 250 men and 250 women were polled as to their desire concerning the ownership of personal computers. The following data resulted:

| | Men | Women | Total |
|---|---|---|---|
| Want PC | 120 | 80 | 200 |
| Don't Want PC | 130 | 170 | 300 |
| Total | 250 | 250 | 500 |

Test the hypothesis that desire to own a personal computer is independent of sex at the 0.05 level of significance.

**SOLUTION**

**i)** H0 : The two variables of classification (i.e. gender and desire for PC) are independent, and
H1 : The two variables of classification are not independent.

**ii)** The significance level is set at $\alpha = 0.05$.

**iii)** The test-statistic to be used is $\chi^2 = \sum_i \sum_j \left(o_{ij} - e_{ij}\right)^2 / e_{ij}$

This statistic, if H0 is true, has an approximate chi-square distribution with (r - 1) (c - 1) = (2 - 1) (2 - 1) = 1 degrees of freedom.

**iv)** Computations:

In order to determine the value of $\chi^2$, we carry out the following computations:

The first step is to compute the expected frequencies. The expected frequency of any cell is obtained by multiplying the marginal total to the right of that cell by the marginal total directly below that cell, and dividing this product by the grand total.

In this example,

$$e_{11} = \frac{(200)(250)}{500} = 100\ ,$$

$$e_{12} = \frac{(200)(250)}{500} = 100,$$

$$e_{21} = \frac{(300)(250)}{500} = 150\ ,$$

*and*

$$e_{22} = \frac{(300)(250)}{500} = 150.$$

Hence, we have:

Expected Frequencies:

| | Men | Women | Total |
|---|---|---|---|
| Want PC | 100 | 100 | 200 |
| Don't Want PC | 150 | 150 | 300 |
| Total | 250 | 250 | 500 |

Next, we construct the columns of oij - eij, (oij - eij)2 and (oij - eij)2 eij , as shown below:

| Observed Frequency $o_{ij}$ | Expected Frequency $e_{ii}$ | $o_{ij} - e_{ij}$ | $(o_{ij} - e_{ij})^2$ | $(o_{ij} - e_{ij})^2/e_{ii}$ |
|---|---|---|---|---|
| 120 | 100 | 20 | 400 | 4.00 |
| 130 | 150 | -20 | 400 | 2.67 |
| 80 | 100 | -20 | 400 | 4.00 |
| 170 | 150 | 20 | 400 | 2.67 |
| | | | | $\chi^2 = 13.33$ |

Hence, the computed value of our test-statistic comes out to be $\chi^2 = 13.33.$

**v)** Critical Region:

$\chi^2 \geq \chi^2_{0.05(1)} = 3.84$

**vi)**

Conclusion:

Since 13.33 is bigger than 3.84, we reject H0 and conclude that desire to own a personal computer set and sex are associated. Now that we have concluded that gender and desire for PC are associated, the natural question is, "Which gender is it where the proportion of persons wanting a PC is higher?" We have:

| | Men | Women | Total |
|---|---|---|---|
| Want PC | 120 | 80 | 200 |
| Don't Want PC | 130 | 170 | 300 |
| Total | 250 | 250 | 500 |

A close look at the given data indicates clearly that the proportion of persons who are desirous of owning a personal computer is higher among men than among women.
And, (since our test statistic has come out to be significant), therefore we can say that the proportion of men wanting a PC is significantly higher than the proportion of women wanting to own a PC.
Let us consider another example:

**EXAMPLE**

A national survey was conducted in a country to obtain information regarding the smoking patterns of the adults males by marital status. A random sample of 1772 citizens, 18 years old and over, yielded the following data :

| MARITAL STATUS | SMOKING PATTERN | | | Total |
|---|---|---|---|---|
| | Total Abstinence | Only at times | Regular Smoker | |
| **Single** | 67 | 213 | 74 | **354** |
| **Married** | 411 | 63 | 129 | **1173** |
| **Widowed** | 85 | 51 | 7 | **143** |
| **Divorced** | 27 | 60 | 15 | **102** |
| **Total** | **590** | **957** | **225** | **1772** |

Use this data to decide whether there is an association between marital status and smoking patterns. The students are encouraged to work on this problem on their own, and to decide for themselves whether to accept or reject the null hypothesis.(In this problem, the null and the alternative hypotheses will be:

H0: Marital status and smoking patterns are statistically independent.

HA : Marital status and smoking patterns are not statistically independent.)

This brings us to the end of the series of topics that were to be discussed in some detail for this course on Statistics and Probability. For the remaining part of today's lecture, we will be discussing some interesting and important concepts. First and foremost, let us consider the concept of

**DEGREES OF FREEDOM**

As you will recall, when discussing the t-distribution, the chi-square distribution, and the F-distribution, it was conveyed to you that the parameters that exists in the equations of those distributions are known as degrees of freedom. But the question is, 'Why these parameters are called degrees of freedom?' Let us try to obtain an answer to this question by considering the following:

Consider the two-dimensional plane, and consider a straight line segment in the plane. If one edge of the line segment is fixed at some point (x0, y0), the line segment can be rotated in the plane such that the fixed edge stays in its place. In other words, we can say that the line segment is free to move in the plane with one restriction. Hence, if we fix one end-point of the line segment, then we are left with one degree of freedom for its movement. Next, consider the case when we fix both end-points of the line segment in the plane. In this case, both degrees of freedom are lost, and therefore the line can no longer move in the plane. But, if we view the above situation with reference to the three-dimensional space --- the one that we live in --- we note that the whole plane (containing the fixed line segment) can move in three dimensions, and hence, we have one degree of freedom for its movement. Let us try to understand this concept in another way: Suppose we have a sample of size n = 6, and suppose that the sum of the sample values is 20. That is, we have the following situation: Our Sample:

| Sr. No. | Value |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| Total | 20 |

Now, the point is that, given this total of 20, if we choose the first 5 values freely, we are not free to choose the sixth value. Hence, one degree of freedom is lost. This point can also be explained in the following alternative way. Given that the sum of the six values is 20, if we have knowledge of the first five values, but the sixth value is missing, then we can re-generate the sixth value. This can also be expressed as follows.

If there are six observations and you find their sum; next, you throw away one of the six observations; then, you can re-generate that observation (because of the fact that you have already computed the sum).
Since, the number of values that can be re-generated is one, hence, the degrees of freedom are n minus one.
(The one which can be re-generated is not the one that we can choose freely.)
Going back to sampling distributions such as the t-distribution, the chi-square distribution and the F-distribution, 'degrees of freedom' can be defined as the number of observations in the sample minus the number of population parameters that are estimated from the sample data (from those observations). For example, in lecture number 39, we noted that the statistic follows the t-distribution having n-1 degrees of freedom.

$$t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

Here n denotes the number of observations in our sample, and since we are estimating one population parameter i.e. $\sigma$ from the sample data, hence the number of degrees of freedom is n-1.
Similarly, referring to lecture number 42, the students will recall that it was stated that the statistic $\frac{s_1^2}{s_2^2}$

Follows the F-distribution having (n1-1, n2-1) degrees of freedom
Here n1 denotes the number of observations in the first sample, and since we are estimating one parameter of the first population i.e. $\sigma$12 from the sample data, hence the number of degrees of freedom for the numerator of our statistic is n1 minus one. Similarly, n2 denotes the number of observations in the second sample, and since we are estimating one parameter of the second population i.e. $\sigma$22 from the sample data, hence the number of degrees of freedom for the denominator of our statistic is n2 minus one. In addition, in today's lecture, you learnt that the statistic

$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\frac{\left(0_{ij} - e_{ij}\right)^2}{e_{ij}},$$

follows the chi-square distribution having (r-1)(c-1) degrees of freedom. Let us try to understand this point: Consider the 2 $\times$ 2 contingency table ,similar to the one that we had in the example regarding the desire for ownership of a personal computer. In this regard, suppose that we have two variables of classification, A and B, and the situation is as follows:

| | $A_1$ | $A_2$ | Total |
|---|---|---|---|
| $B_1$ | | | 200 |
| $B_2$ | | | 300 |
| Total | 250 | 250 | 500 |

The point is that, given the marginal totals and the grand total, if we choose the frequencies of the first cell of the first row freely, we are not free to choose the frequency of the second cell of the first row. Also, given the frequency of the above-mentioned first cell, we are not even free to choose the frequency of the second cell of the first column.

Not only this, it is interesting to note that, given the above, we are not even free to choose the frequency of the second cell of the second row or the second column !Hence, given the marginal and grand totals, we have only degree of freedom (i.e. 1 = 1 $\times$ 1 = (2-1)(2-1) degrees of freedom).A similar situation holds in the case of a 2 x 3 contingency table. The students are encouraged to work on this point on their own, and to realize for themselves that, in the case of a 2 x 3 contingency table, there exist (2 - 1) ( 3 - 1) = 2 degrees of freedom . Next, let us consider the concept of p-value:

You will recall that, with reference to the concept of hypothesis-testing, we compared the computed value of our test statistic with a critical value. For example, in case of a right-tailed test, we rejected the null hypothesis if our computed value exceeded the critical value, and we accepted the null hypothesis if our computed value turned out to be smaller than the critical A hypothesis can also be tested by means of what is known as the p-value.

**P-VALUE**

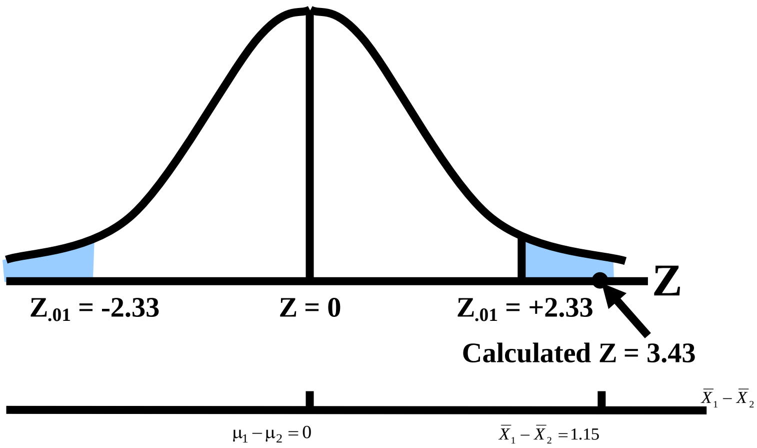
The probability of observing a sample value as extreme as, or more extreme than, the value observed, given that the null hypothesis is true. We illustrate this concept with the help of the example concerning the hourly wages of computer analysts and registered nurses that we discussed in an earlier lecture. The students will recall that the example was as follows:

**EXAMPLE**

A survey conducted by a market-research organization five years ago showed that the estimated hourly wage for temporary computer analysts was essentially the same as the hourly wage for registered nurses. This year, a random sample of 32 temporary computer analysts from across the country is taken. The analysts are contacted by telephone and asked what rates they are currently able to obtain in the market-place. A similar random sample of 34 registered nurses is taken. The resulting wage figures are listed in the following table.

| Computer Analysts | | | Registered Nurses | | |
|---|---|---|---|---|---|
| $ 24.10 | $25.00 | $24.25 | $20.75 | $23.30 | $22.75 |
| 23.75 | 22.70 | 21.75 | 23.80 | 24.00 | 23.00 |
| 24.25 | 21.30 | 22.00 | 22.00 | 21.75 | 21.25 |
| 22.00 | 22.55 | 18.00 | 21.85 | 21.50 | 20.00 |
| 23.50 | 23.25 | 23.50 | 24.16 | 20.40 | 21.75 |
| 22.80 | 22.10 | 22.70 | 21.10 | 23.25 | 20.50 |
| 24.00 | 24.25 | 21.50 | 23.75 | 19.50 | 22.60 |
| 23.85 | 23.50 | 23.80 | 22.50 | 21.75 | 21.70 |
| 24.20 | 22.75 | 25.60 | 25.00 | 20.80 | 20.75 |
| 22.90 | 23.80 | 24.10 | 22.70 | 20.25 | 22.50 |
| 23.20 | | | 23.25 | 22.45 | |
| 23.55 | | | 21.90 | 19.10 | |

Conduct a hypothesis test at the 2% level of significance to determine whether the hourly wages of the computer analysts are still the same as those of registered nurses. In order to carry out this test, the Null and Alternative Hypotheses were set up as follows:
Null and Alternative Hypotheses:
H0 : $\mu1 - \mu2 = 0$
HA : $\mu1 - \mu2 \neq 0$
(Two-tailed test)
The computed value of our test statistic came out to be 3.43, whereas, at the 5% level of significance, the critical value was 2.33, hence, we rejected H0.

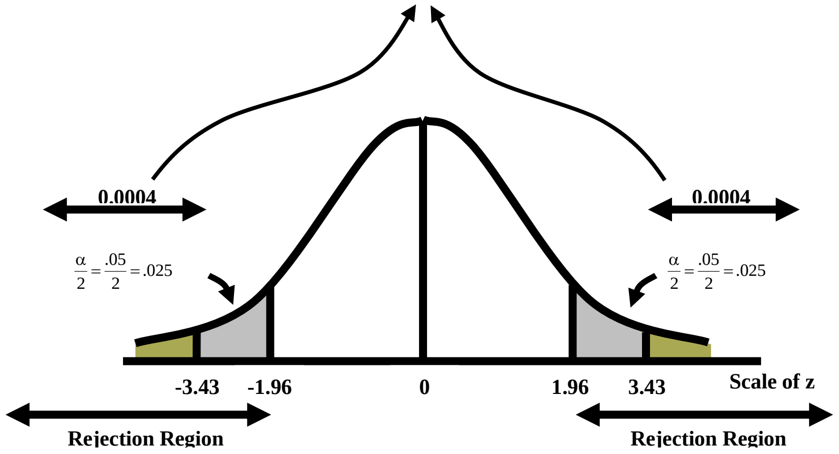



Hence, we concluded that there was a significant difference between the average hourly wage of a temporary computer analyst and the average hourly wage of a temporary registered nurse. This conclusion could also have been reached by using the

**P-VALUE METHOD**

**I.** Looking up the probability of $Z > 3.43$ in the area table of the standard normal distribution yields an area of .5000 – .4996 = .0004.
**II.** To compute the p-value, we need to be concerned with the region less than –3.43 as well as the region greater than 3.43 (because the rejection region is in both tails).

**p-value = 0.0004+0.0004 = 0.0008**




The p-value is 0.0004 + 0.0004 = 0.0008. Since this value is very small, it means that the result that we have obtained in this example is highly improbable if, in fact, the null hypothesis is true. Hence, with such a small p-value, we decide to reject the null hypothesis.
The above example shows that: The p-value is a property of the data, and it indicates "how improbable" the obtained result really is. A simple rule is that if our p-value is less than the level of significance$\alpha$, then we should reject H0, whereas if our p-value is greater than the level of significance$\alpha$, then we should accept H0. (In the above example, $\alpha$ = 0.02 whereas the p-value is equal to 0.0008, hence we reject H0.)

## RELATIONSHIP BETWEEN CONFIDENCE INTERVAL AND TESTS OF HYPOTHESIS

Some of the students may already have an idea that there exists some kind of a relationship between the confidence interval for a population parameter $\theta$ and a test of hypothesis about$\theta$. (After all: When deriving the confidence interval for $\mu$, the area that was kept in the middle of the sampling distribution of $\bar{X}$ was equal to 1-$\alpha$ so that the area in each of the right and left tails was equal to $\alpha/2$. And, when testing the hypothesis H0 : $\mu = \mu 0$ versus HA : $\mu \neq \mu 0$ at level of significance $\alpha$, the area in each of the right and left tails was again equal to $\alpha/2$.)Hence, consider the following proposition: Let [L, U] be a 100(1 - $\alpha$)% confidence interval for the parameter $\theta$.Then we will accept the null hypothesis H0 : $\theta = \theta 0$ against H1 : $\theta \neq \theta 0$ at a level of significance $\alpha$ if $\theta 0$ falls inside the confidence interval, but if $\theta 0$ falls outside the interval [L, U], we will reject H0.In the language of hypothesis testing, the (1 - $\alpha$) 100% confidence interval is known as the acceptance region and the region outside the confidence interval is called the rejection or critical region. The critical values are the end points of the confidence interval. The students are encouraged to work on this point on their own. As we approach the end of this course, we present an Overview of the Science of Statistics in Today's World: Statistics is a vast discipline! In this course, we have discussed the very basic and fundamental concepts of statistics and probability. But, there are numerous other topics that could have been discussed if we had the time. We could have talked about the Latin Square Design, we could have considered Inference Regarding Regression and Correlation Coefficients, we could have discussed Non-Parametric Statistics, and so on, and so forth.
The students are encouraged to study some of these concepts on their own --- as and when time permits --- in order to develop a better understanding and appreciation of the importance of the science of Statistics. In this course, numerous examples were discussed and many numerical problems were presented.
The solutions of these problems were presented in detail, and the various steps were worked out. In doing so, the purpose was to develop in the students a better understanding of the core concepts of the various techniques that were applied. But, it is interesting and useful to note that, a lot many of these numerical problems can be solved within seconds by using the wide variety of statistical packages that are available. These include **SPSS, SAS, Statistica, Statgraph, Minitab, Stata, S-Plus, etc.** (The students are welcome to try out some of these packages on their own.)Towards the end of this course, we present one of the latest definitions of Statistics:

## LATEST STATISTICAL DEFINITION

Statistics is a science of decision making for governing the state affairs. It collects analyzes, manages, monitors, interprets, evaluates and validates information. Statistics is Information Science and Information Science is Statistics. It is an applicable science as its tools are applied to all sciences including humanities and social sciences.



- THE END -