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HIGHLIGHTED HANDOUT

MTH632

Complex Analysis and Differential Geometry

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MTH632- Complex Analysis and Differential Geometry

Topic:001

Complex Numbers: An Introduction

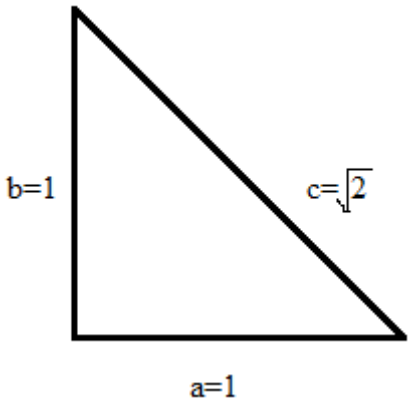
Natural Numbers: N = 1,2,3,4,5,6,7,……

Integers: Z = 0, $\pm 1, \pm 2, \pm 3, \pm 4$, ...

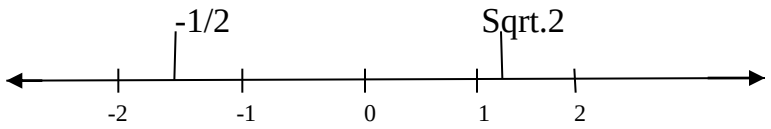
Rational Numbers: Q = $\left\{\frac{a}{b} : a, b \in Z, b \neq 0\right\}$

Irrational Numbers: Q'

Numbers that can not be written as $\frac{p}{q}$. The decimal expression is non-terminating and non periodic.



Real Numbers: R = Q U Q'



Numbers are useful: $6x + 1 = 13$

Do we need more numbers?

$X^2 + 2x + 2 = 0$, $x^2 + 1 = 0$

We need one more number : $\mathbf{i = \sqrt{-1}}$

Complex Numbers = $a + ib : a, b \in R$

" Cardino was the first to introduce complex numbers into algebra, but had misgivings about it"

"To divide 10 into two parts, the product of which is 40.

$5 + \sqrt{-5}$, $5 - \sqrt{-5}$

Rafael Bombelli authored I'Algebra(1572 and 1579) a set of three books.

"At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof".

L.Euler (1707-1783) introduced the notation $i = \sqrt{-1}$

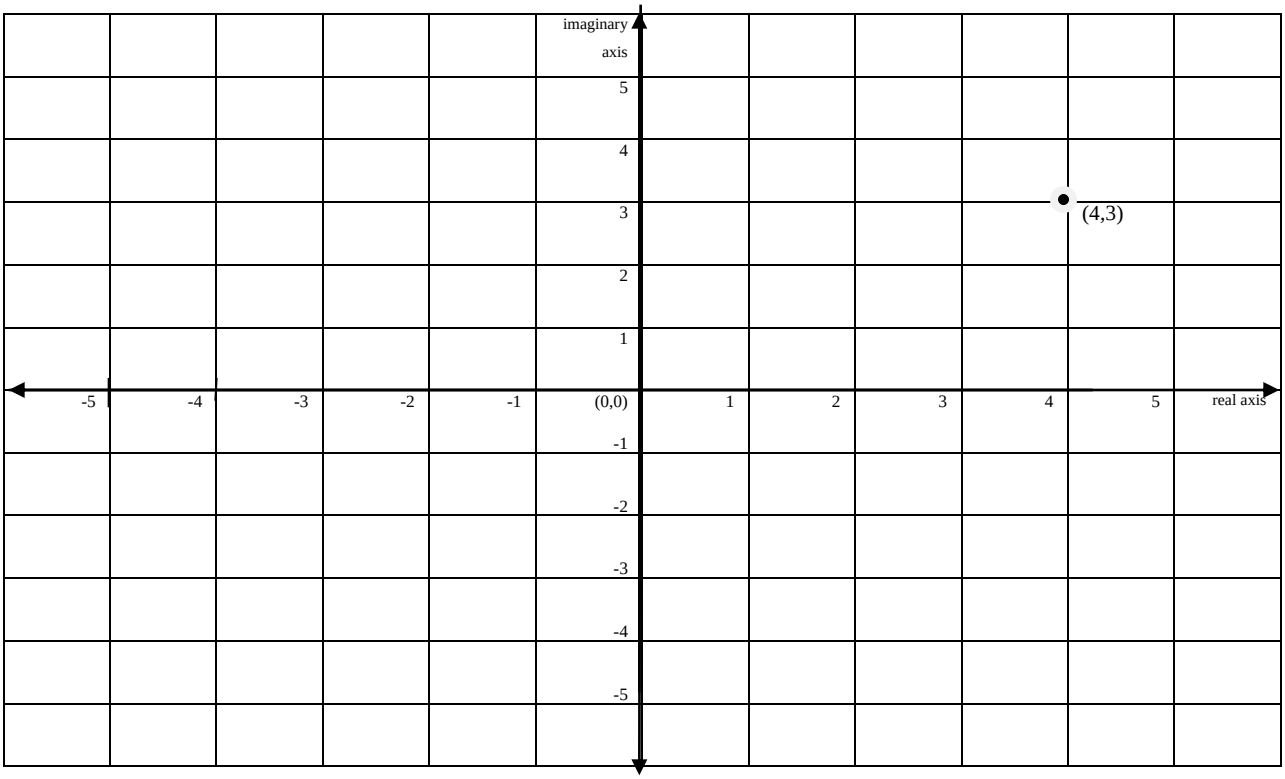
Topic:002

What are Complex Numbers?

A complex number z can be defined as an ordered pair z = (x,y) where x and y are real numbers.

Example: $z_1 = (2,1)$, $z_2 = (1,2)$ $z_1 \neq z_2$

Complex Numbers can interpreted as points in the Complex plane.



- When real numbers x are displayed as points (x,0) on the real axis the set of complex numbers includes the real numbers as a subset.

- Complex numbers of the form (0,y) correspond to points on the y-axis and are pure imaginary numbers when $y \neq 0$. The y-axis is then referred to as the imaginary axis.

For Complex Numbers z = (x,y), the real numbers x and y are known as the real and imaginary parts of z respectively; and we write: x = Re z , y = Im z

Equality of Complex Numbers:

Two complex Numbers z_1 and z_2 are equal whenever they have the same real parts and the same imaginary parts.

Thus the statement $z_1 = z_2$ means that z_1 and z_2 correspond to the same point in the complex, or z plane.

Topic:003

Computations in Complex Numbers

Addition of two Complex Numbers:

For:

$z_1 = (x_1 , y_1)$, $z_2 = (x_2 , y_2)$

The addition is defined as:

$z_1 + z_2 = (x_1 , y_1) + (x_2 , y_2)$

$z_1 + z_2 = (x_1 + x_2 , y_1 + y_2)$

Product of two Complex Numbers:

For:

$z_1 = (x_1 , y_1)$, $z_2 = (x_2 , y_2)$

The product is defined as:

$z_1 . z_2 = (x_1 , y_1) . (x_2 , y_2)$

$z_1 . z_2 = (x_1 x_2 - y_1 y_2 , y_1 x_2 + x_1 y_2)$

Example:

$z_1 = (1,2)$, $z_2 = (1,1)$

$z_1 . z_2 = (1.1 – 2.1 , 2.1 + 1.1)$

$z_1 . z_2 = (1 – 2 , 2 + 1)$

$z_1 . z_2 = (-1 , 3)$

The following definition of product (denoted by *) of Complex Numbers seems natural and very tempting

$(x_1 , y_1) * (x_2 , y_2) = (x_1x_2 , y_1y_2)$.

Both the above definition has many flaws, including the fact that we can’t find inverses with respect to this product.

Any Complex Number z = (x,y) can be written as:

$z = (x,y) = (x,0) + (0,y)$

Also we have $(0,1)(y,0) = (0,y)$. So,

$z = (x,y) = (x,0) + (0,1)(y,0)$.

The numbers of the form (x,0) are real numbers so can be simply denoted by x. Since, $(0,1)(0,1)=(-1,0)$ so we denoted the number (0,1) by i. Then,

$z = (x,y) = (x,0) + (0,1)(y,0) = x + iy$

The above representation makes the addition and multiplication of complex numbers simple.

Most of the time we use the following representation of complex numbers:

$z = x + iy$

The representation is due to Gauss.

In this representation, addition and multiplication becomes:

$$(x_1 + iy_1) + (x_2 + iy_2) = x_1 + iy_1 + x_2 + iy_2$$

$$= (x_1 + x_2) + i(y_1 + y_2)$$

$$(x_1 + iy_1) . (x_2 + iy_2) = x_1 x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

We have seen:

- Two ways to represent complex numbers.
- Addition of Complex Numbers.
- Product of Complex Numbers.

Topic:004

Properties of Addition of Complex Numbers

Although the ordered pair representation is the real foundational definition on which complex number system is built. But due to simplicity we prefer the following representation of complex numbers:

$$z = x + iy$$

The addition of complex numbers follow all the properties of real numbers (real numbers is a subset of complex numbers).

The Commutative Law: $z_1 + z_2 = z_2 + z_1$

The Associative Law: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

Due to Associative Laws we can write a sum $z_1 + z_2 + z_3$ without using paranthesis.

Additive Identity: There is unique complex number w such that z + w = w + z = z

Such number w is obviously (0,0) or 0 + i0 known as the additive identity and denoted as:

$$0 = (0,0) \text{ or } 0 = 0 + i0$$

Additive Inverse: For every complex number z = x + iy, there is a unique Ɛ such that:

$$z + Ɛ = 0$$

Obviously

$$Ɛ = -x - iy$$

Known as **additive inverse** and denoted as:

$$-z = -x - iy$$

Or in ordered pair notation:

$$-z = (-x , -y)$$

These properties of addition of complex numbers will be used in our algebraic manipulations.

Topic:005

Properties of product of Complex Numbers

We use the following representation of complex number

$$z = x + iy$$

This representation makes it easy to describe properties of multiplication of complex numbers.

The product of complex numbers follow all the properties of real numbers.

The Commutative Law: $z_1z_2 = z_2z_1$

The commutativity implies that: $iy = yi$

So we can write complex numbers in any of the following ways:

$$z = x + iy$$

Or

$$z = x + yi$$

The Associative Law: $(z_1z_2)z_3 = z_1(z_2z_3)$

Due to associative laws we can write a product $z_1z_2z_3$ without using paranthesis.

Multiplicative Identity: There is complex number w such that $zw = wz = z$. Such number w is obviously unique and is given by (1,0) or $1 + i0$, known as the multiplicative identity and denoted as:

$$1 = (1,0) \text{ or } 1 = 1 + i0$$

Multiplicative Inverse: For every complex number $z=x+iy$ other than $(0,0)$, there is a unique z^{-1} such that:

$$zz^{-1} = 1$$

In fact:

$$z^{-1} = \frac{1}{z} = \frac{1}{x+iy}$$

To write z^{-1} in standard form $x + iy$ we proceed as follows:

$$z^{-1} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x2-i2y2} = \frac{x-iy}{x2+y2}$$

Hence,

$$z^{-1} = \frac{x}{x2+y2} - i\frac{y}{x2+y2}$$

Division of Complex Numbers:

Using

$$z^{-1} = \frac{x}{x2+y2} - i\frac{y}{x2+y2}$$

We can define division of two complex numbers

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

as:

$$\frac{z1}{z2} = z_1 z_2^{-1}$$

$$\frac{z1}{z2} = (x_1 + iy_1)\left(\frac{x2}{x2+y2} - i\frac{y2}{x2+y2}\right)$$

($\left(\frac{x2}{x2+y2} - i\frac{y2}{x2+y2}\right)$ is the inverse of z_2 that could not be shown properly due to some typing issues)

$$\frac{z1}{z2} = \left(\frac{x1x2+y1y2}{(x2)2+(y2)2}, \frac{y1x2-x1y2}{(x2)2+(y2)2}\right)$$

For example:

If $z_1 = 4 + i$ and $z_2 = 2\text{-}3i$ then $\frac{z1}{z2}$ can be calculated using

$$\frac{z1}{z2} = \left(\frac{x1x2+y1y2}{(x2)2+(y2)2}, \frac{y1x2-x1y2}{(x2)2+(y2)2}\right)$$

We get:

$x_1 = 4$, $x_2 = 2$, $y_1 = 1$, $y_2 = -3$

$$\frac{z1}{z2} = \frac{5}{13} + i\frac{14}{13}$$

Distributive Law: The operation of multiplication is distributive over addition that is,

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

Distribtive law associates addition with multiplication.

Binomial Formula:

For any two complex numbers z_1 and z_2 , we have:

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^{\,k}\, z_2^{\,n-k}$$

For n = 1,2,3,…. Here,

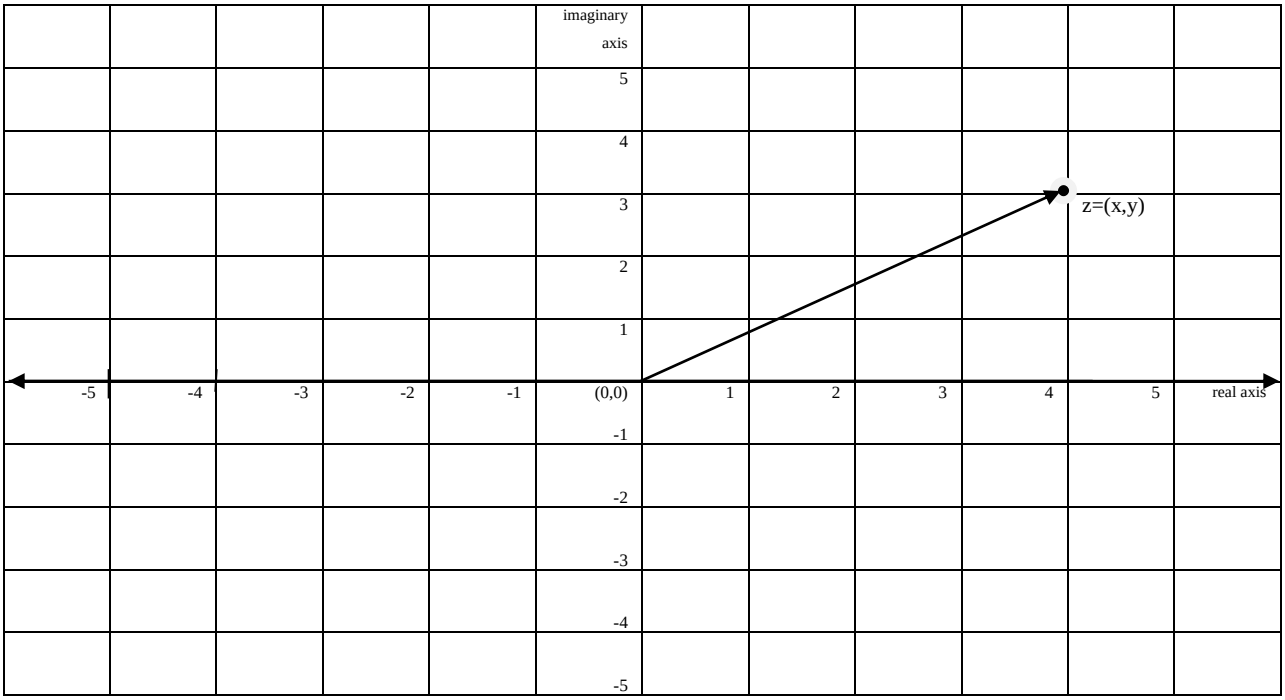
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

These properties of multiplication of complex numbers will be used in our algebraic manipulations.

Topic:006

Algebraic Operations and Geometric Representation of Complex Numbers

A Complex Number z = (x,y) = x + iy can be associated with the directed line segment, or vector, from origin to the point z=(x,y)



Recall:

For

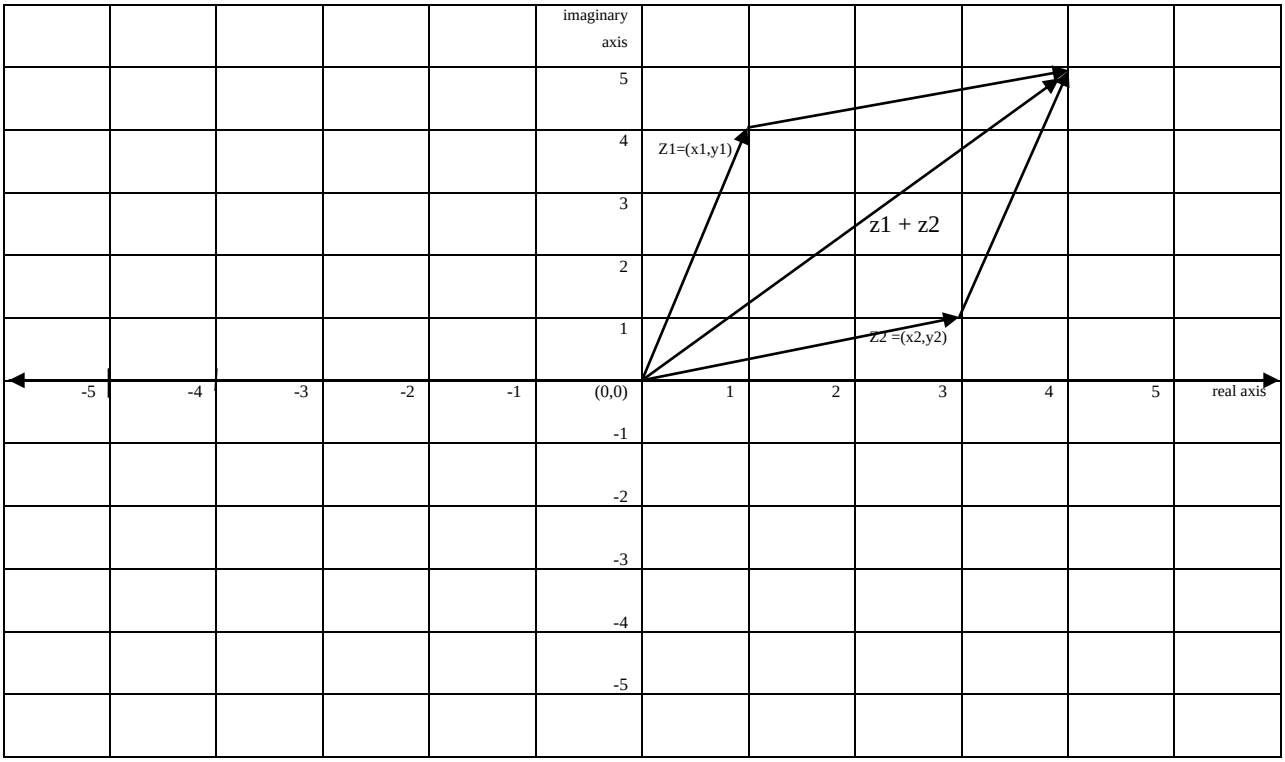
$z_1 = (x_1 , y_1)$, $z_2 = (x_2 , y_2)$

The addition is defined as:

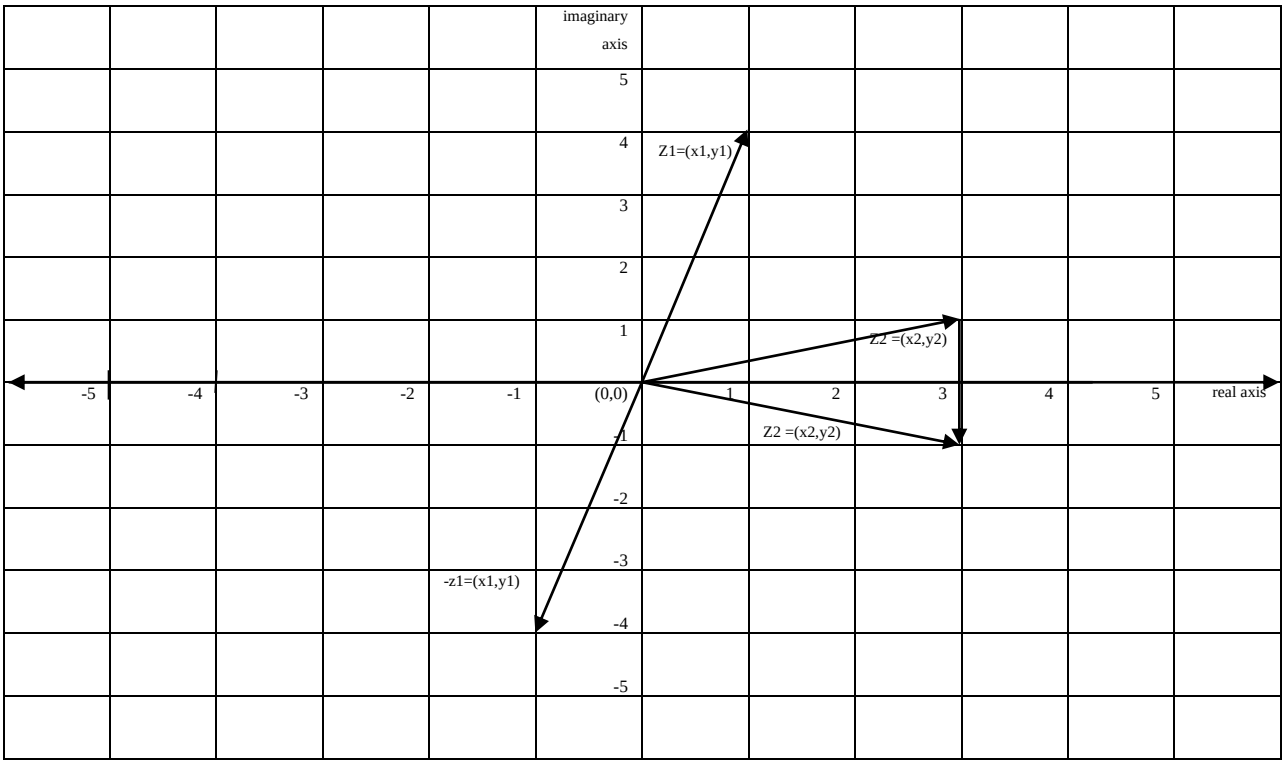
$z_1 + z_2 = (x_1 + x_2 , y_1 + y_2)$ OR

$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Geometrically, in Complex plane, $z_1 + z_2$ can be obtained using parallelogram law:



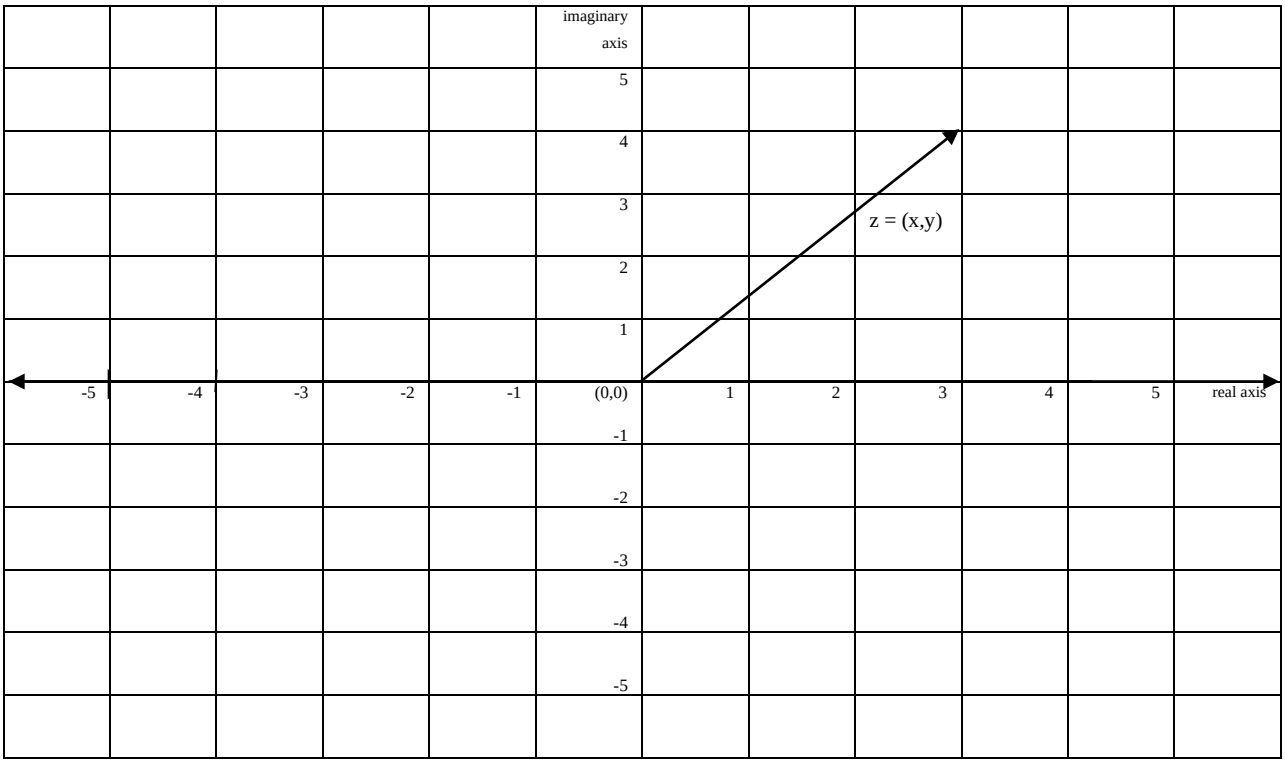
Similarly, we can calculate the difference $z_1 - z_2$ in the complex plane



Multiplication of two complex numbers is neither dot product nor vector product.

The vector interpretation helps us defining modulus or absolute value of a complex number.

Geometrically, the modulus of z = (x,y) = x + iy is distance between the point (x,y) and the origin.



The modulus or absolute value of a complex number z=(x,y)= x+iy is defined and denoted as:

$$|z| = \sqrt{x2 + y2}.$$

z = x+iy is real if and only if z = x + i0 , $|z| = \sqrt{x2} = |x|$

Example: $z = 2 + i3 = \sqrt{4 + 9} = \sqrt{13}$

Two Complex Numbers can be compared by finding their modulus.

We have seen:

- Geometry of algebraic operations

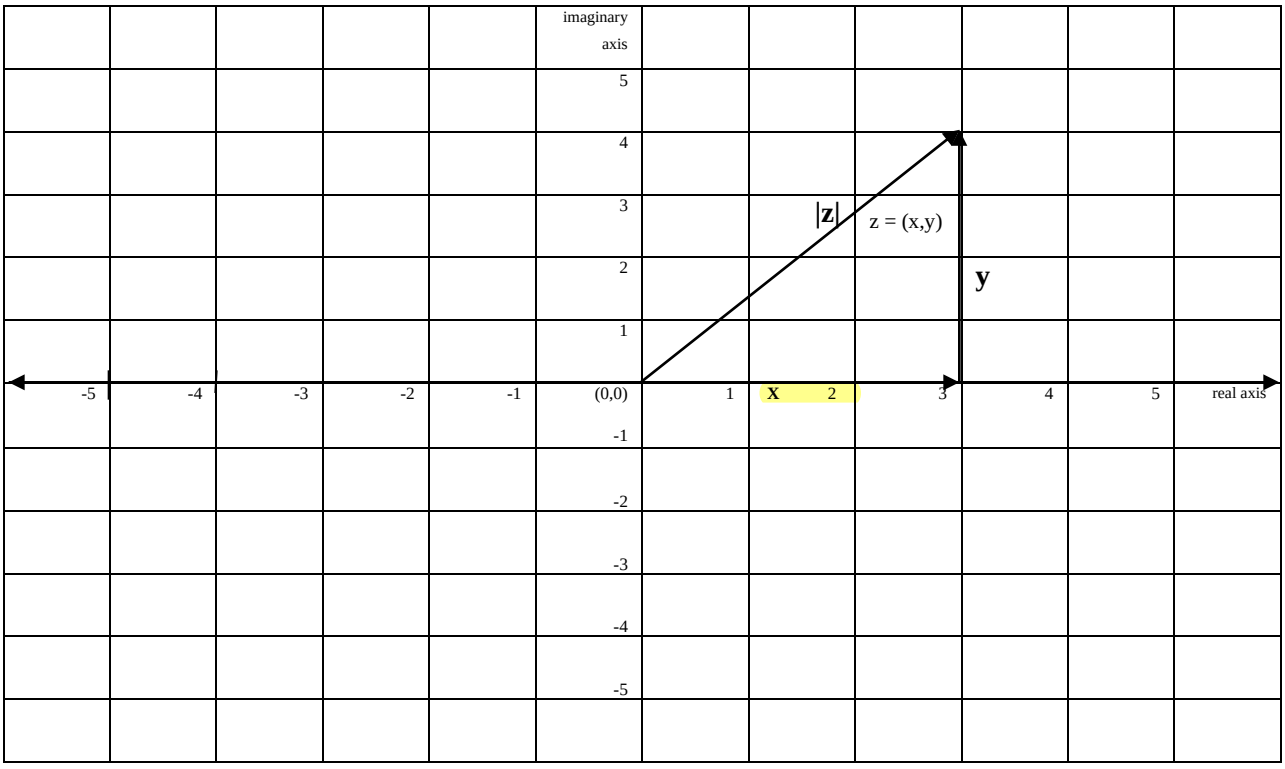
- Modulus of a Complex Number

Topic:007

Properties of Modulus

The modulus or absolute value of a complex number z=(x,y)= x+iy is defined and denoted as:

$$|z| = \sqrt{x2 + y2}.$$



x= Re(z) , y = Im(z)

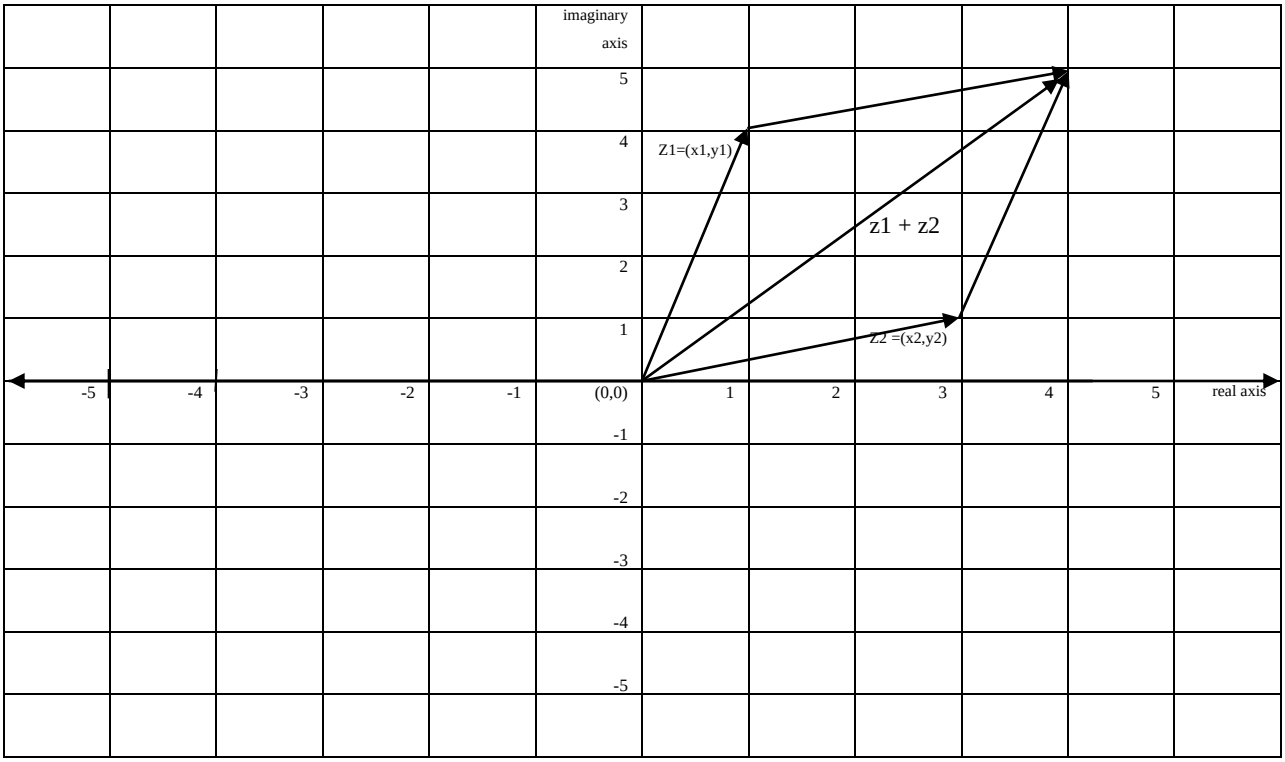
$Re(z) \leq |Re(z)| \leq |z|$

$Im(z) \leq |Im(z)| \leq |z|$

$|z|^2 = Re(z)^2 + Im(z)^2$

Triangle inequality:

$|z_1 + z_2| \leq |z_1| + |z_2|$



An important consequence of triangle inequality is:

$$|z_1 + z_2| \geq ||z_1| - |z_2||$$

Proof:

$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |- z_2|$ $\quad\quad\quad\quad$ $\therefore |z| = |-z|$

$\Rightarrow |z_1| - |z_2| \leq |z_1 + z_2|$

Interchanging z_1 and z_2

$|z_2| - |z_1| \leq |z_1 + z_2| \quad \rightarrow (1)$

$\Rightarrow -(|z_1| - |z_2|) \leq |z_1 + z_2|$

$\Rightarrow |z_1| - |z_2| \geq - |z_1 + z_2| \quad \rightarrow (2)$

$-|z_1 + z_2| \leq |z_1| - |z_2| \leq |z_1 + z_2| \rightarrow (3)$

Using triangle inequality:

$|z_1 + z_2| \leq |z_1| + |z_2|$

And the following inequality:

$|z_1 + z_2| \geq ||z_1| - |z_2||$

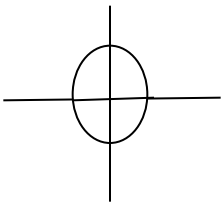
We get:

$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$

Also, $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

Example: Consider a complex number z such that $|z| = 1$.

Now, If we want to calculate upper and lower bound



$|z-2|$ then using:

$||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

$|z-2| \leq |z+2| = 1 + 2 = 3$

Also: $|z-2| \geq ||z|-|2||$

$|z-2| \geq |1-2| = 1$

Hence, $1 \leq |z - 2| \leq 3$

We have seen:

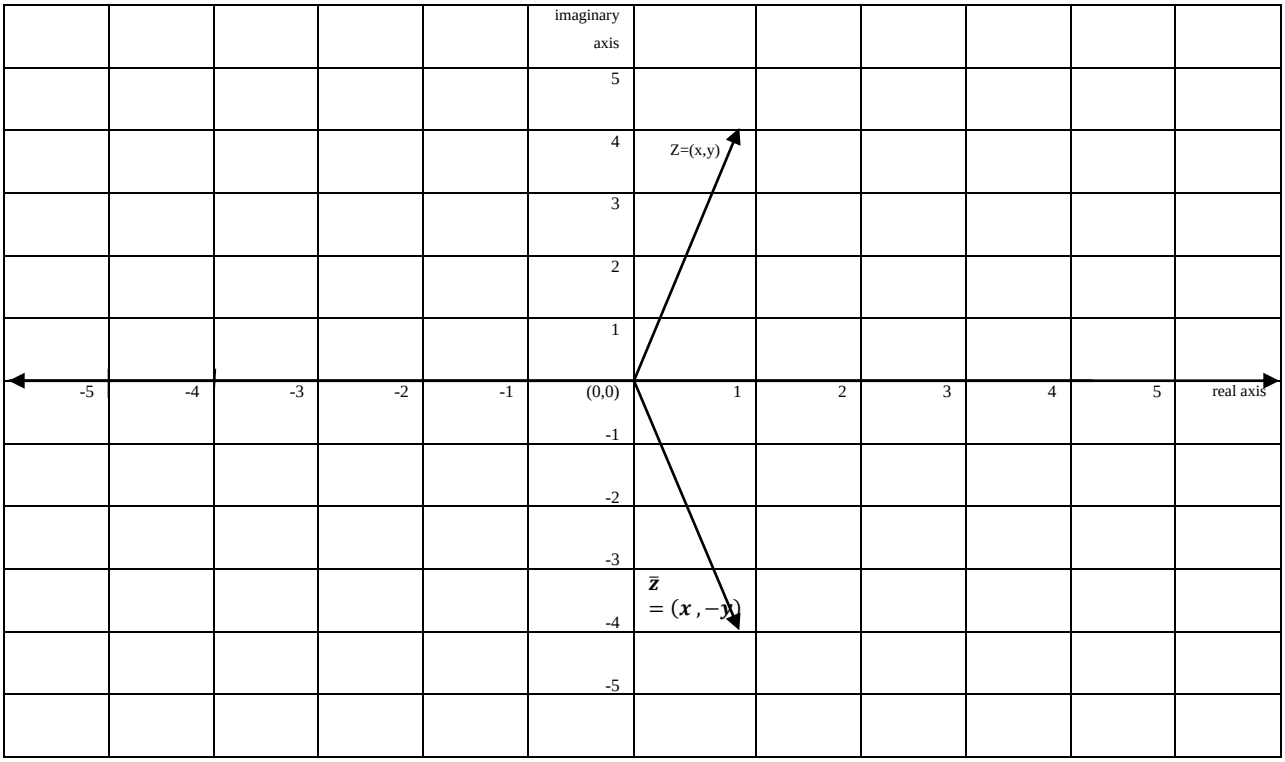
- Some Properties of modulus of Complex Numbers specially we have calculated upper and lower bound of modulus of Complex numbers.

Topic:008

Complex Conjugates

The complex conjugate or simply the comjugate of complex number z = x + iy is denoted by $\bar{z}$, is given by:

$$\bar{z} = (x\ , -y) = (x - iy)$$



Z = (x,y) , $\bar{z} = (x\ , -y)$, $|z| = |\bar{z}|$

Properties of Conjugate of Complex Numbers:

For two complex numbers $z_1=x_1 + iy_1$ and $z_2=x_2 + iy_2$

We have:**(1)** $\overline{z1 + z2} = \overline{z1} + \overline{z2}$

Proof: $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \Rightarrow z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \Rightarrow \overline{z1 + z2} = (x_1 + x_2) - i(y_1 + y_2)$

$\overline{z1 + z2} = (x_1 - iy_1) + (x_2 - iy_2) \Rightarrow \overline{z1 + z2} = \overline{z1} + \overline{z2}$

Similarly we can show the following properties:

(2) $\overline{z1 - z2} = \overline{z1} - \overline{z2}$ **(3)** $\overline{z1 z2} = \overline{z1}\ \overline{z2}$ **(4)** $\frac{\overline{z1}}{z2} = \frac{\overline{z1}}{\overline{z2}}$ **if** $z_2 \neq 0$

Complex conjugate , Re(z) , Im(z)

$z = (x + iy)$, $Re(z) = x$, $Im(z) = y$

$\bar{z} = (x - iy) \Rightarrow z + \bar{z} = 2x \Rightarrow x = \frac{z + \bar{z}}{2} \Rightarrow$ **$Re(\bar{z}) = \frac{z + \bar{z}}{2}$**

$z - \bar{z} = 2iy \Rightarrow y = \frac{z - \bar{z}}{2i} \Rightarrow$ **$Im(\bar{z}) = \frac{z - \bar{z}}{2i}$**

Similarly we have the following:

$Re(iz) = -Im(z)$ and $Im(iz) = Re(z)$

Complex Conjugate helps us in providing in properties of modulus.

First observe that: **$|z|^2 = z\bar{z}$**

$z = x + iy$, $|z| = \sqrt{x2 + y2}$, $|z|^2 = (x^2 + y^2)$

$z\bar{z} = (x+iy)(x-iy)$, $z\bar{z} = (x^2 - i^2y^2)$, $z\bar{z} = (x^2 + y^2)$

Also, For Complex Numbers z_1 and z_2 we have

$$|z_1 z_2| = |z_1|\,|z_2|$$

Proof: $|z_1 z_2|^2 = (z_1 z_2)\overline{(z1z2)} = (z_1 z_2)\,(\overline{z1}\ \ \overline{z2})$ $\quad\therefore |\mathbf{z}|^2 = \mathbf{z}\bar{\mathbf{z}}$

$= (z_1\overline{z1})\,(z_2\overline{z2}) = |z_1|^2\,|z_2|^2 \Longrightarrow |z_1 z_2| = |z_1|\,|z_2|$

On the same lines, we can prove:

$$\left|\frac{z1}{z2}\right| = \frac{|z1|}{|z2|}$$

Triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof:

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z1\ +\ z2)} = (z_1 + z_2)(\overline{z1} + \overline{z2})$

$= z_1\overline{z1} + z_1\overline{z2} + z_2\overline{z1} + z_2\overline{z2}$

$= |z_1|^2 + z_1\overline{z2} + \overline{z1\overline{z2}} + |z_2|^2$

$= |z_1|^2 + 2\mathrm{Re}(z_1\overline{z2}) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2$ $\quad\therefore \mathrm{Re}(z) \leq |z|\ ,\ |\overline{z2}| = |z_2|$

$= (|z_1| + |z_2|)^2$

$|z_1 + z_2| \leq |z_1| + |z_2|$ **Proved!**

Example: If a Complex number z satisfies $|z| < 2$ then

$|z^3 + 3z^2 - 2z + 1|$ has upper bound given by using triangle inequality and properties of modulus in the following way.

Solution: $|z^3 + 3z^2 - 2z + 1| \leq |z^3| + |3z^2| + |\text{-}2z| + |1|$

$= |z|^3 + 3|z|^2 + 2|z| + 1$ $\quad \therefore |z^3| = |z|^3 , |z_1 z_2| = |z_1|\,|z_2|$

$< 2^3 + 3(2)^2 + 2(2) + 1$

$= 8 + 12 + 4 + 1 = 25$

We have seen:

- Conjugate and its properties
- How conjugate is used to prove different properties of complex numbers.

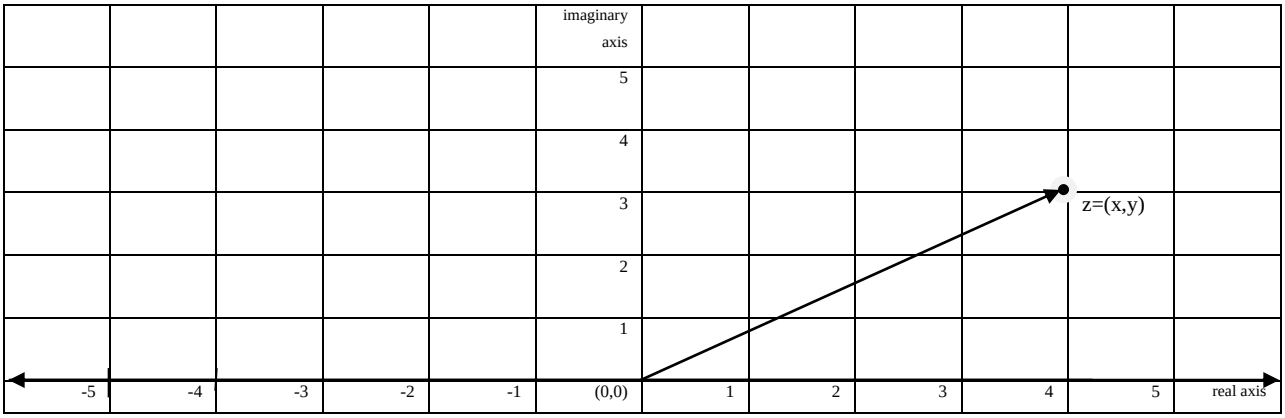
Topic:009

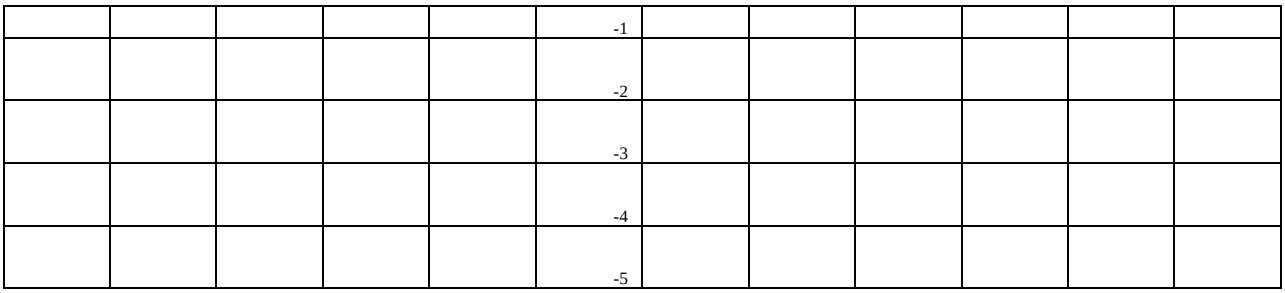
Exponential Form of Complex Numbers

We shall see that this way of writing down complex numbers will not only improve the understanding about the geometry of complex numbers but it will also help us in simplifying different operations on complex numbers like multiplication and division.

Recall:

A Complex Number z = (x,y) = x + iy can be associated with the directed line segment, or vector, from origin to the point z=(x,y)



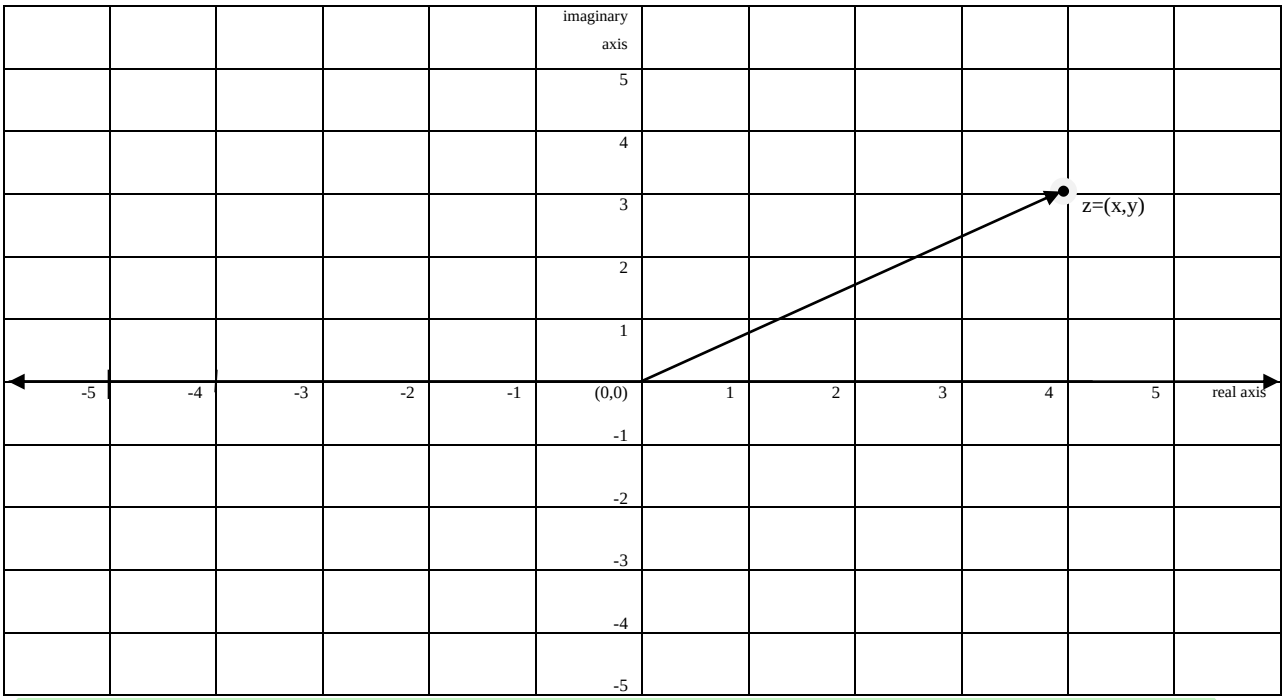


Now we know that vector can be completely specify by its direction and magnitude.

So, we can say that we can specify complex number by its magnitude and direction.

Let θ be the angle that the vector make with positive x-axis. Since, x = rcos θ and y = rsin θ So, the number z can be written in polar form as:

z = (rcos θ , rsin θ) = r(cos θ + i sin θ)



This is known as **polar representation of complex numbers** and the values **r and θ are called polar coordinates** of z.

If z=0 then coordinate θ is undefined. So whenever we talk about polar representation of complex numbers, we always assume that z is non-zero complex number.

The **real number r = |z| is not allowed to be negative** since it is length of the radius vector for z.

The direction which we calculate by an angle that this vector makes with positive x-axis, this can be positive or negative depending whether we are calculating counter-clockwise or clockwise. If we are calculating the angle counter-clockwise, then the angle is positive otherwise negative. The real number θ represents the angle, measured in radians, that z makes with positive x-axis.

Example: Let z = 1 + i

$|z| = \sqrt{1+1} = \sqrt{2}$, $\theta = \frac{\pi}{4}$

In Polar form: $z = \sqrt{2}\,((\cos\frac{\pi}{4}) + i \sin\frac{\pi}{4})$

For a complex number z = (x,y), the coordinate θ has infinite number of possible values. All these values differ by integral multiples of 2π.

So, the coordinate θ is any value for which the identities

$x = r\cos\theta$ and $y = r\sin\theta$ holds. Formally we have the following definition:

Definition (argument of z =(x,y)):

If $z \neq 0$, $\arg z = \{\theta : z = r(\cos\theta + i\sin\theta)\}$

If $\theta \in \arg z$ then we say that θ is an argument of z.

Definition (Principle value of arg z)

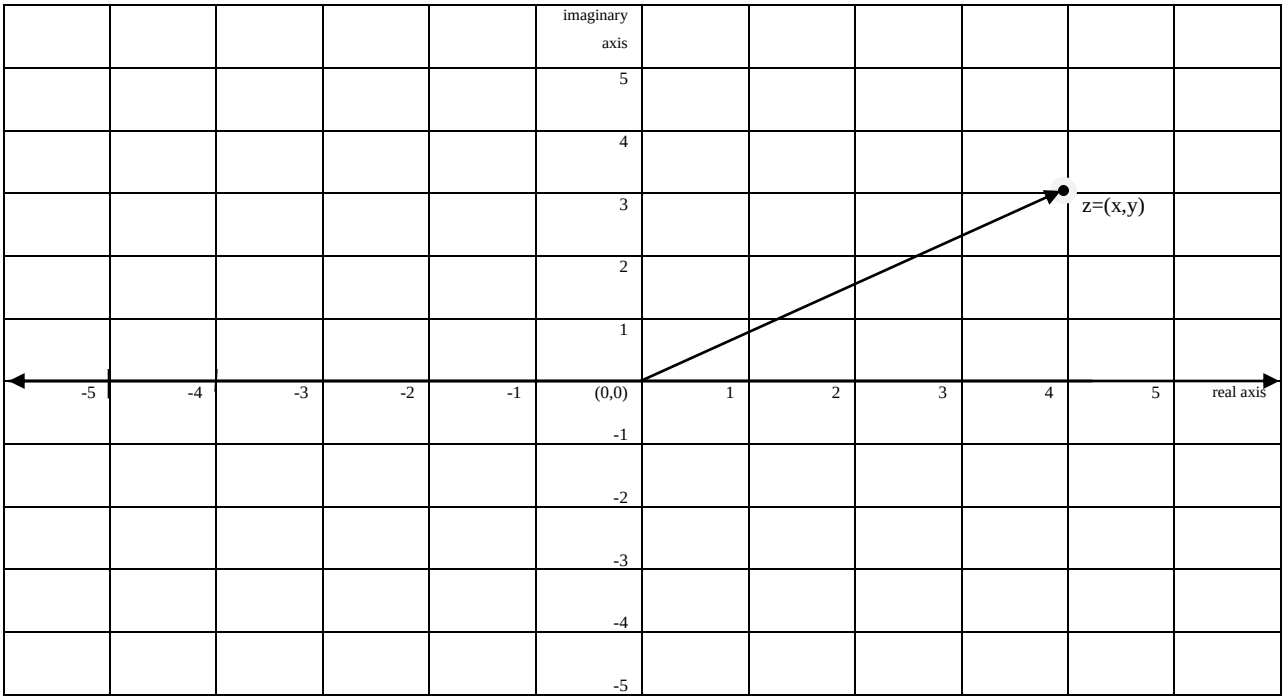
Let $z \neq 0$ be a complex number. Then principle value of argument of z, denoted by Arg z, is a unique value θ such that

$$z = r(\cos\theta + i\sin\theta) \text{ and } -\pi < \theta \leq \pi$$

Principle value of argument of z is also reffered as "the argument of z"

$$\textbf{arg z = Arg z + 2n}\pi\textbf{ , (n = 0, } \pm \textbf{1, } \pm \textbf{2 , } \pm \textbf{3 , ...)}$$

Calculation argument of z = (x,y):



Tanθ has period . Observe the quadrant of complex number.

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $x \neq 0$, then

$$\arg z \subset \arctan\left(\frac{y}{x}\right)$$

Based on these observations, we can state the following steps:

First Step: If $x \neq 0$, then calculate $\arctan\left(\frac{y}{x}\right)$

Second Step: Check the quadrant of z = (x,y)

Third Step: Add / Subtract multiples of π according to the quadrant in which z lies.

If we want to find the principal value of argument of z then we add an additional step.

Fourth Step: Add / Subtract multiples of 2π so that the angle θ satisfies the following:

$$-\pi < \theta \leq \pi$$

Example: Calculate Polar representation of $z = -1 - i$

$r = |z| = \sqrt{(-1)2 + (-1)2} = \sqrt{1+1} = \sqrt{2}$

$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = \arg z = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

$-\pi < \theta \leq \pi$ $\qquad$ $\text{Arg } z = \frac{5\pi}{4}$

Polar representation of z = $\sqrt{2}\left(\boldsymbol{cos}\frac{\boldsymbol{5\pi}}{\boldsymbol{4}} + \boldsymbol{isin}\frac{\boldsymbol{5\pi}}{\boldsymbol{4}}\right)$

If x = 0 then we can not use the following to find arg z

$$\tan\theta = \frac{y}{x}$$

If $z \neq 0$ and $x = 0$ then we use the following rule:

Arg $z = \frac{\pi}{2}$ if Im $z > 0$ and Arg $z = -\frac{\pi}{2}$ if Im $z < 0$

Example: Find Polar representation of $z = (0,5) = 5i$

$r = |z| = 5$, Arg $z = \frac{\pi}{2}$

Polar representation $z = 5(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

We have learnt:

- Polar Representation of Complex Numbers.

Topic 010:

Exponential Form of Complex Numbers (Part II)

Recall:

A Complex Number z can be written in Polar Form as:

$z = (r\cos\theta , r\sin\theta)$ or $z = r(\cos\theta + i\sin\theta)$

Euler Formula:

The symbol $e^{i\theta}$ or $\exp(i\theta)$ is defined by means of Euler formula as:

$$\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$$

Consider a complex number z in Polar Form:

$$\mathbf{z = r(cos\theta + isin\theta)}$$

The exponential form of z is given by:

$$\mathbf{z = r}\, e^{i\theta}$$

We will learn more about the exponential functions in our later discussions.

Example: The Polar representation of z = -1-i is given by:

$$z = \sqrt{2}(\cos\left(-\frac{3\pi}{4}\right) + isin\left(-\frac{3\pi}{4}\right))$$

$$e^{i\theta} = cos\theta + isin\theta$$

Exponential Form of z is:

$$\mathbf{z = r}\, e^{i\theta} \Longrightarrow \mathbf{z =} \sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)} \Longrightarrow \mathbf{z =} \sqrt{2}e^{-i\left(\frac{3\pi}{4}\right)}$$

$$\Longrightarrow z = \sqrt{2}\exp\left(i\left(-\frac{3\pi}{4}\right)\right)$$

We know that:

arg z = Arg z + 2n π , (n = 0,$\pm$1, $\pm$2, $\pm$3, ...)

So there are infinite exponential forms of z, given by:

$$z = \sqrt{2}\exp\left(i\left(-\frac{3\pi}{4} + 2n\pi\right)\right), \ n \in Z$$

Some Special Complex Numbers:

The case when r = 1

The Complex Numbers for which **r = 1** are **z = $e^{i\theta}$**

These Complex Numbers are located on a circle of radius 1.

For **θ =** , we get **z = $e^{i\pi}$ = cos π + $i$sinπ = -1**

So, we have **$e^{i\pi}$ + 1 = 0.** This is known as Euler's identity.

Euler's identity is most fascinating identity of Mathematics, since it involves many important mathematical symbols. **π** has been used in the works of Hippocrates in 400B.C. **i** is the main focus of this course. **e** is the base of natural logarithm.

The case r = R, where R is arbitrary.

The equation z = R$e^{i\theta}$ $0 \leq \theta \leq 2\pi$ is the parametric representation of the circle |z| = R.

More generally, $|z - z0|$ = R is a circle of radius R and center z_0, also written in the following form:

z = z_0 + R$e^{i\theta}$ $0 \leq \theta \leq 2\pi$. Here R and z_0 are fixed and θ is the parameter.

Example: Solve |$e^{i\theta}$ **- 1| = 2** geometrically.

|z-1| = 2 is a circle of radius 2 with center (1,0).

z = $e^{i\theta}$ is a circle of radius 1 with center (0,0).

So, the required point is the point of intersection of both circles.

Hence, |$e^{i\theta}$ **- 1| = 2** has solution **θ = π.**

We have seen:

- How to calculate exponential form of a complex number.
- How we can do simple calculations using this exponential form.

Topic 011:

Products and Powers in Exponential Form

By using Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

We get the following exponential form of complex numbers:

$$z = re^{i\theta}$$

Using Euler Formula and trigonometry we can verify the following:

$$e^{i\theta 1}e^{i\theta 2} = e^{i(\theta 1+\theta 2)}$$

Proof:

$e^{i\theta 1}e^{i\theta 2} = (\cos\theta 1 + i\sin\theta 1)\,(\cos\theta 2 + i\sin\theta 2)$

$= \cos\theta 1\cos\theta 2 + i\sin\theta 1\cos\theta 2 + i\sin\theta 2\cos\theta 1 + i^2\sin\theta 1\sin\theta 2$

$= (\cos\theta 1\cos\theta 2 - \sin\theta 1\sin\theta 2) + i(\sin\theta 1\cos\theta 2 + \sin\theta 2\cos\theta 1)$

$= \cos(\theta 1 + \theta 2) + i\sin(\theta 1 + \theta 2) = e^{i(\theta 1+\theta 2)}$

Product in exponential form:

Given: $z_1 = r_1 e^{i\theta 1}$, $z_2 = r_2 e^{i\theta 2}$

Then,

$z_1 z_2 = (r_1 e^{i\theta 1})(r_2 e^{i\theta 2}) = r_1 r_2 e^{i(\theta 1+\theta 2)}$

Example: $z_1 = 2\, e^{i\frac{\pi}{2}}$, $z_2 = 3\, e^{i\frac{\pi}{2}}$

$z_1 z_2 = (2)(3)\, e^{i\frac{\pi}{2}} e^{i\frac{\pi}{2}} = 6e^{i(\frac{\pi}{2}+\frac{\pi}{2})} = 6e^{i\pi}$

Division in Exponential form:

If $\mathbf{z = re^{i\theta}}$ then:

$z^{-1} = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1*e^{-i\theta}}{re^{i\theta}e^{-i\theta}} = \frac{1}{r}e^{-i\theta}\left(\frac{1}{e^{i(\theta-\theta)}}\right) = \frac{1}{r}e^{-i\theta}$

Hence, $z^{-1} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\left(\cos(-\theta) + isin(-\theta)\right).$

We know that:

$\cos(-\theta) = cos\theta$ and $sin(-\theta)$ = -sin (θ)

So, $z^{-1} = \frac{1}{r}\left(\cos(\theta) - isin(\theta)\right).$

If $z_1 = r_1 e^{i\theta 1}$ and $z_2 = r_2 e^{i\theta 2}$

Then $\frac{z1}{z2} = \frac{r1e^{i\theta 1}}{r2e^{i\theta 2}} = \frac{r1}{r2}e^{i(\theta 1-\theta 2)}$

Example: $z_1 = 2e^{i\frac{\pi}{3}}$, $z_2 = 4e^{i\frac{\pi}{2}}$

$\frac{z1}{z2} = \frac{2e^{i\frac{\pi}{3}}}{4e^{i\frac{\pi}{2}}} = \frac{1}{2}e^{i(\frac{\pi}{3}-\frac{\pi}{2})} = \frac{1}{2}e^{-i\frac{\pi}{6}}$

Powers in Exponential Form:

For Complex Number $z = re^{i\theta}$

$z^n = r^n e^{in\theta}$ for $n = 0, \pm1, \pm2, \pm3, \ldots..$

Proof: (By Mathematical Induction)

For n = 1:

$z^1 = r^1 e^{i1\theta} = z = re^{i\theta}$

For n = k:

$z^k = r^k e^{ik\theta}$

For n = k+1:

$z^{k+1} = r^{k+1} e^{i(k+1)\theta} \Longrightarrow z^{k+1} = (re^{i\theta})^{k+1}$

$z^{k+1} = (re^{i\theta})^k (re^{i\theta})^1 \Longrightarrow z^{k+1} = (r^k e^{ik\theta})(re^{i\theta}) \Longrightarrow z^{k+1} = r^{k+1} e^{i(k+1)\theta}$

For n = 0:

$z^0 = r^0 e^{i0\theta} = 1$

For negative numbers:

Let n is a negative integer. This implies n = -m where m is positive integer.

$z^n = z^{-m} = (re^{i\theta})^{-m} = (r^{-1}e^{-i\theta})^m = (r^{-1})^m e^{-i\theta(m)}$

$z^n = (r^{-1})^{-n} e^{-i\theta(-n)} = r^n e^{in\theta}$ **Proved!**

Example: $z = 7e^{i\frac{\pi}{4}}$ $z^{100} = 7^{100} e^{i\frac{\pi}{4}(100)}$ $z^{1000} = 7^{1000} e^{i\frac{\pi}{4}(1000)}$

Example: $(\sqrt{3} + i)^3$

First, we will convert this in exponential form:

$r = \sqrt{3+1} = \sqrt{4} = 2 \quad , \quad \theta = \frac{\pi}{6}$

$\sqrt{3} + i = re^{i\theta} = 2e^{i\frac{\pi}{6}} \Longrightarrow (\sqrt{3} + i)^3 = 2^3 e^{i\frac{3\pi}{6}} = 2^3 e^{i\frac{\pi}{2}}$

$(\sqrt{3} + i)^3 = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8i$

De Moivre's Formula:

The expression $\mathbf{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$

Where n = 0,$\pm$1, $\pm$2, ... is known as De Moivre's Formula.

Example: Show that $\mathbf{\cos 2\theta = \cos^2\theta - \sin^2\theta}$

and $\mathbf{\sin 2\theta = 2\sin\theta\cos\theta}$

Proof:

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + i^2\sin^2\theta + 2i\cos\theta\sin\theta$

$= (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$

By De Moivre's Formula:

$\cos 2\theta + i\sin 2\theta = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$

Hence, $\mathbf{\cos 2\theta = \cos^2\theta - \sin^2\theta}$ (by comparing real parts)

$\mathbf{\sin 2\theta = 2\sin\theta\cos\theta}$ (by comparing imaginary parts)

Exercise: Show that $\mathbf{\cos 5\theta = \cos^5\theta - \cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$.

We can use De Moivre's Formula to prove many trigonometric identities, including Lagrange's trigonometric identity:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{\sin\left[\frac{(2n+1)\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}$$

And:

$$\cos n\theta = \sum_{k=0}^{m} \binom{n}{2k} (-1)^k \cos^{n-2k}\theta \sin^{2k}\theta$$

We have seen:

- Product, division and powers in exponential form of complex numbers.
- De Moivre's Formula and its consequences

Topic:012

Roots of Complex Numbers

(Part I)

So far we have learnt basic operations of complex numbers. For example, How to add, how to multiply, how to divide complex numbers and some other elementary operations of complex numbers.

Now this topic is going to be our first non-trivial calculation about complex numbers where we are going to use those elementary but very important operations.

So, we are going to find roots of complex numbers. Off course we need to first learn what is root of complex numbers.

A number u is said to be the nth root of a complex number z if $u^n = z$.

We have **two problems** in finding root of complex numbers.

1. How many nth roots of a complex number z exist?

To answer this problem we are going to see very important concequence of fundamental theorem of algebra.

Corollary to the fundamental theorem of Algebra:

If p(z) is a polynomial of degree n (n>0) with complex coefficients, then equation p(z) = 0 has precisely (counting multiplicities) n solutions.

Recall Polynomial of degree n:

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots\ldots + a_1 z^1 + a_0$ where $a_i \in C$ i=0,1,2,,,n

Variable z is a complex variable.

Example: Let $p(z) = z^3 - (4+i)z^2 + (4+i4)z - 4i$

p(z) = 0 has following solutions:

$z^3 - (4+i)z^2 + (4+i4)z - 4i = 0$

$(z - 2)^2 (z - i) = 0$. Hence, $z_1 = 2$, $z_2 = 2$, $z_3 = i$

Now see how this corollary helps us.

To find nth root, find $u^n = z$ we can write: $u^n - z = 0$ where u is variable and z is given complex number. Then,

- There are precisely n numbers of nth roots of a complex number z_0.
- We denote the set of all nth roots of a complex number z_0 by $z_0^{1\backslash n}$.
- For positive real number r_0 the symbol $\sqrt[n]{r0}$ is reserved for one positive root.

How to find nth roots of a complex number z_0.

Equality of Complex Numbers in Exponential form.

Consider a complex number $z = re^{i\theta}$.

$z = re^{i\theta} \Longrightarrow z = re^{i(\theta+2\pi)} \Longrightarrow z = re^{i(\theta+2k\pi)}$

Given: $z_1 = r_1e^{i\theta 1}$ and $z_2 = r_2e^{i\theta 2}$

If $z_1 = z_2$ then $r_1 = r_2$ and

$\theta_1 = \theta_2 + 2k\pi$ $\quad k \in Z$. Also we know: $z^n = r^ne^{in\theta}$

We will use these observations in finding root of complex numbers.

Consider a complex number $z_0 = r_0e^{i\theta 0}$. Let $z = re^{i\theta}$ be nth root of z_0, that is, we have $z^n = z_0$

Now, $z^n = r^ne^{in\theta}$ and $z^n = z_0$

So, $r^ne^{in\theta} = r_0e^{i\theta 0}$

$\Longrightarrow r^n = r_0 \qquad \Longrightarrow r = \sqrt[n]{r0}$

And $n\theta = \theta_0 + 2k\pi \implies \theta = \frac{\theta 0}{n} + \frac{2k\pi}{n}$, $k \in Z$

So, $z = \sqrt[n]{r0} \exp\left(i\left(\frac{\theta 0}{n} + \frac{2k\pi}{n}\right)\right)$, $k \in Z$

Consider a complex number $z_0 = r_0 e^{i\theta 0}$. The nth root z is given by:

$z = \sqrt[n]{r0} \exp\left(i\left(\frac{\theta 0}{n} + \frac{2k\pi}{n}\right)\right)$, k = 0, $\pm 1, \pm 2$, ...

Example: Find square roots of 2i.

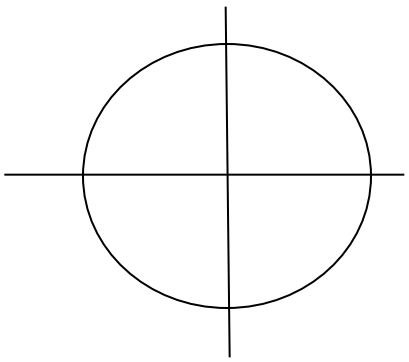
$2i = 2(\cos\frac{\pi}{2} + i\sin(\frac{\pi}{2}))$

$z_0 = 2e^{i\frac{\pi}{2}}$ then $z = \sqrt{2} \exp\left(i\left(\frac{\frac{\pi}{2}}{2} + \frac{2k\pi}{2}\right)\right)$, $k \in Z$

If k = 0 then z = 1 + i . If k = 1 , z = -1 – i.

If we take further values of k we will get same values again and again.

Geometry of the roots:



We denote the distinct nth roots in the following way:

$$C_k = \sqrt[n]{r0}\ \exp\left(i\left(\frac{\theta 0}{n} + \frac{2k\pi}{n}\right)\right), \quad k = 0,1,2,,,,,n\text{-}1$$

When **k = 0 we denote by c_0**, when **k = 1 we denote by c_1** and so on. When we consider n = 2, in other words consider square roots of complex number, then these two complex numbers which are the square roots of given complex number lie at the opposite ends of a line segment. And off course if we change the complex number then these two complex numbers will be changed but will always at the opposite ends of a line. And similarly if we consider cube roots of any complex number, then they lie at three vertices of regular triangle. And when n = 4 then they are at the vertices of a square. And if we keep on increasing these values of n then we will get vertices of regular pentagon, vertices of hexagon and upto so on. So basically these roots of any given complex number off course lie on a circle of radius nth root of r_0 but there is much more about those roots that they are vertices of regular polygon.

We have seen:

- How to calculate roots of given complex number?
- We also discussed about geometry of roots.

Topic:013

Roots of Complex Numbers

(Part II)

Summary of the method:

Consider a complex number z_0. The following are steps for finding nth roots of z_0.

- Write z_0 in exponential form $z_0 = r_0 e^{i\theta}$
- Write z_0 in more general form:

  $z_0 = r_0 e^{i(\theta 0 + 2k\pi)}$, k = 0, ±1, ±2, ±3, ...
- The nth roots are:

  $z_0^{1/n} = \left[r0 e^{i(\theta 0 + 2k\pi)} \right]^{1/n}$

  $= \sqrt[n]{r0}\; e^{i\left(\frac{\theta 0}{n} + \frac{2\pi}{n}k\right)}$, k = 0,1,2,……,n-1

Example: Calculate nth roots of $z_0 = 1$.

Step 1: $z_0 = 1 = 1(\cos 0 + i\sin 0)$

$z_0 = e^{i0}$

Step 2: $z_0 = e^{i(0+2k\pi)}$, $k \in Z$

Step 3: $z_0^{1/n} = \sqrt[n]{1}\; e^{i\left(\frac{0}{n} + \frac{2\pi}{n}k\right)}$, k = 0,1,2,……,n-1

$z_0^{1/n} = e^{i\left(\frac{2\pi}{n}k\right)}$, k = 0,1,2,……,n-1

So, the nth roots of unity are:

$z_0^{1/n} = e^{i\left(\frac{2\pi}{n}k\right)}$, **k = 0,1,2,……,n-1**

If we denote: $\omega_n = e^{i\left(\frac{2\pi}{n}\right)}$ (k=1)

$\omega_n^2 = e^{i\left(\frac{2\pi}{n}2\right)}$ (k=2)

Similarly, ω_n^3 , ω_n^4 , ω_n^5 ,…..

Then the nth roots of unity become: 1 , ω_n , ω_n^2 , ω_n^3 , .., ω_n^{n-1}

Here ω_n is called the primitive nth root of unity.

Example: Find all square roots of unity.

1. $z_0 = 1 = e^{i(0)}$
2. $z_0 = e^{i(0+2k\pi)}$, k∈ Z
3. $z^{1/2} = e^{i\left(\frac{2k\pi}{2}\right)}$

$z^{1/2} = e^{i(k\pi)}$, k = 0 ,1

c_0 = 1 , c_1 = -1

Example: Find all cube roots of unity.

1. $z_0 = 1$, **2.** $z_0 = e^{i(0+2k\pi)}$, k∈ Z

3 $z^{1/3} = e^{i\left(\frac{2k\pi}{3}\right)}$ k = 0 , 1 , 2 , 3

c_0 = 1 , $c_1 = e^{i\left(\frac{2\pi}{3}\right)}$ = -1/2 + i $\sqrt{3}$/2

$c_2 = e^{i\left(\frac{4\pi}{3}\right)}$ = -1/2 - i $\sqrt{3}$/2

Geometry of cube roots of unity:

Geometry of nth roots of unity:

Relation between roots of complex number and roots of unity:

For a complex number $z_0 = r_0 e^{i\theta 0}$, the nth roots are:

$$C_k = \sqrt[n]{r0}\ \exp\left(i\left(\frac{\theta 0}{n} + \frac{2k\pi}{n}\right)\right), \quad k = 0,1,2,,,,,n-1$$

We have seen that the roots of any complex number z can not be calculated using this formula.

We can expand this formula in the following form:

$$c_k = \sqrt[n]{r0}\ \exp\left(\frac{i\theta 0}{n}\right)\exp\left(\frac{2\pi}{n}k\right)$$

$$\exp\left(\frac{2\pi}{n}k\right) = w_n^{\,k} = \text{roots of unity}$$

$$c_k = \sqrt[n]{r0}\ \exp\left(\frac{i\theta 0}{n}\right)\ w_n^{\,k} \quad , k=0,1,2,\ldots,n-1$$

We can say that whenever we have nth roots of given complex number then we can write them in the way we defined.

$$c_k = c_0\, w_n^{\,k}$$

So, if we know nth roots of unity and just one nth root of complex number z_0 , then we can calculate all of the nth roots of complex number.

In general, for a complex number $z_0 = r_0 e^{i\theta 0}$. If ζ is one of the nth roots of z_0 then the other nth roots are:

ζ , ζw_n , ζw_n^2 , ζw_n^3 , ….., ζw_n^{n-1}

Example: Find all values of $(2i)^{1/3}$.

Step 1: $z_0 = 2i = 2(\cos\frac{\pi}{2} + i\sin(\frac{\pi}{2}))$

$z_0 = 2e^{i(\frac{\pi}{2})}$

Step 2: $z_0 = 2e^{i(\frac{\pi}{2}+2k\pi)}$, $k \in Z$

Step 3: $z_0^{1/3} = \sqrt[3]{2}e^{i(\frac{\frac{\pi}{2}}{3}+\frac{2k\pi}{3})}$, $k \in Z$

$c_k = \sqrt[3]{2}e^{i(\frac{\pi}{6}+\frac{2k\pi}{3})}$, $k = 0,1,2$

$c_0 = \sqrt[3]{2}e^{i(\frac{\pi}{6})} = \sqrt[3]{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$c_1 = c_0 w_3$, $c_2 = c_0 w_3^2$

Since, we can calculate square roots, cube roots, and general nth roots of any given complex number. Now lets see quadratic formula.

Consider the equation:

$az^2 + bz + c = 0$

Then the quadratic formula states the following solutions of the above equation:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Find roots of the following equation:

$$z^2 + (2 + i)z + ¾ = 0$$

Here, a = 1 , b = (2+i) , c = ¾

Using Quadratic Formula:

$$z = \frac{-(2+i) \pm \sqrt{(2+i)^2 - 4(\frac{3}{4})}}{2}$$

$$z = \frac{-(2+i) \pm \sqrt{4i}}{2}$$

Discriminent = $\sqrt{4i} = \{\sqrt{2} + i\sqrt{2}\ , -\sqrt{2} - i\sqrt{2}\}$

Roots of given equation are:

$$\boldsymbol{z = \frac{-(2+i)+(\sqrt{2}+\sqrt{2}i)}{2}} \quad , \quad \boldsymbol{z = \frac{-(2+i)+(-\sqrt{2}-\sqrt{2}i)}{2}}$$

We have seen:

1) Roots of unity 2) Relation between roots of complex numbers and roots of unity 3) Quadratic formula.

Topic:014

Regions in the Complex Plane

(Part I)

In this discussion, we are going to study geometrical and topological properties of different regions in complex plane. So, in other words if we have region in a complex plane and we have a point then the question is how close the point p is with that region or if the point p lies inside the region, outside the region or at the boundary of that region. Now we have to write this easy idea in mathematical form.

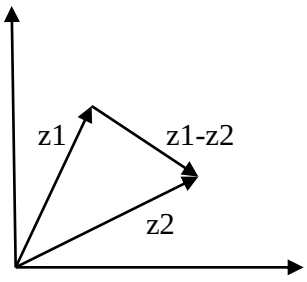
A circle in plane is a set of points whose distance from a fixed point say z_0 is constant c.

The point z_0 is called center and the constant c is called radius of the circle.

Notion of distance:

Given two points $z_1 = (x_1 , y_1)$ and $z_2 = (x_2 , y_2)$ the distance between them is given by:

$$|z_1 - z_2|$$



Now, lets use this distance to describe a circle in complex plane.

Equation of a circle of radius ε and center z_0 is given by:

$$|z - z_0| = \varepsilon$$

A closed disk with center z_0 and radius ε is defined by

$$|z - z_0| \leq \varepsilon$$

A open disk with center z_0 and radius ε is defined by

$$|z - z_0| < \varepsilon$$

ε Neighbourhood of a point z_0 is the following open disk:

$$|z - z_0| < \varepsilon$$

When value of ε is irrelevant then we just call it Neighbourhood of z_0.

Deleted Neighbourhood or Punctured Open Disk:

A deleted neighbourhood of z_0 contains all points in an ε neighbourhood of z_0 except the point z_0. Symbolically we have:

$$0 < |z - z_0| < \varepsilon$$

A point z_0 is **Interior point** of a set S whenever there is a neighbourhood of z_0 contained in S.

A point z_0 is **Exterior point** of a set S whenever there is a neighbourhood of z_0 not contained in S.

A point z_0 is a **Boundary Point** of a set S if it is neither an interior nor an exterior point of S.

Set of all boundary points of a set S is called boundary of S.

Example:

$$S_1: |z| \leq 1$$

This is a closed disk because we can write it in the following form:

$$S_1: |z - 0| \leq 1$$

Boundary of S_1 is circle of radius 1. i.e $|z| = 1$

In case of open disk:

$$S_2: |z| < 1$$

Boundary of S_2 is same as S_1 , circle of radius 1. i.e $|z| = 1$

In this discussion we learnt:

- Circles in the Complex Plane
- Neighbourhoods
- Interior , exterior and boundary points

Topic:015

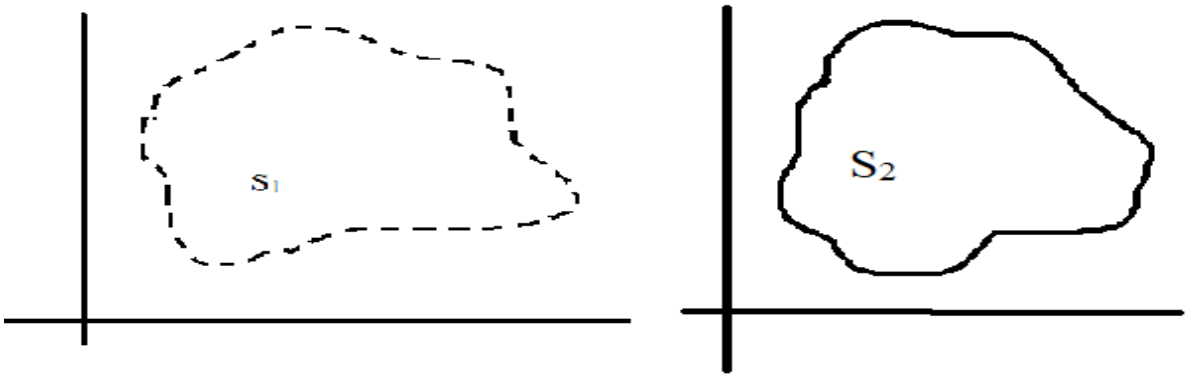
Regions in the Complex Plane (Part II)

We are going to continue our discussion of different regions in the complex plane and their topological and geometrical properties. And off course if a set is given what is the relationship between a set and a given point. Now this

discussion of different regions in the complex plane is going to help us in exploring different properties of complex functions. e.g their limit properties, their continuity, their differentiability and other properties.

Open Set:

A set S is called an open set if every point of S is an interior point of S.

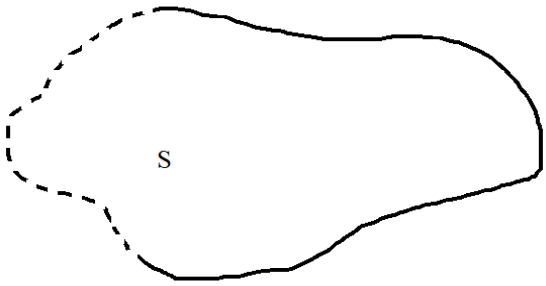


Closed Set:

A set S is called a closed set if it contains all its boundary points.

Accumulation Point / Limit Point:

A point z_0 is called an accumulation point of a set S if every deleted neighborhood of z_0 contains at least one point of S.



Closed Sets and Accumulation Points:

If a set is closed than it contains all its accumulation or limit points. We can use this condition to check whether a set is closed or not.

Closure of a set:

If a set is given it may or may not be closed but we can ask the following questions. Can we find the smallest closed set containing the given set? The answer is Yes. We can do that by finding the closure of that set.

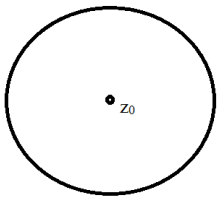
Closure of a set is the smallest closed set containing the set S.

And how to find the closure of a set? The procedure is very simple.

Closure of a set is the closed set consisting of all points in S together with boundary points.

Example: $S = \{z : |z| < 1\}$ is not closed because boundary is not included. Its Closure is: $S = \{z : |z| \leq 1\}$. This is the smallest closed set containing set S.

Some sets are neither open nor closed:



One of them is punctured disk.

This is not open because boundary points are not interior points and according to definition a set is open if every point of set is interior of set.

This is not closed because of the fact that the set is closed if it contains all of its accumulation or limit points. As z_0 is accumulation point but is not included in the set.

Similarly, we have examples of the sets that are both open as well as close.

Consider the entire complex plane. It is open because every point of this complex plane is an interior point.

Take any point and we can easily find a neighborhood of that point which is contained in the plane. So it is open.

Now let's see why it is not closed because there is no boundary. So, the condition is a set is closed if it contains all its boundary points. Since there is no boundary, so by definition, it is also closed.

There are many other examples of sets which are both open as well as closed.

In this topic, we learnt about:

- Open sets
- Closed Sets
- Closure of a Set

Topic:016

Regions in the Complex Plane (Part III)

To discuss further regions we need the concept of a curve.

Inituitively speaking, we can say that a curve is a thread on a flat plane or a string stretched in a flat plane.

In the precise way:

Curve:

A curve C is the range of a function given by:

$z(t) = (x(t) , y(t)) = x(t) + iy(t)$ where $t \in [a,b]$

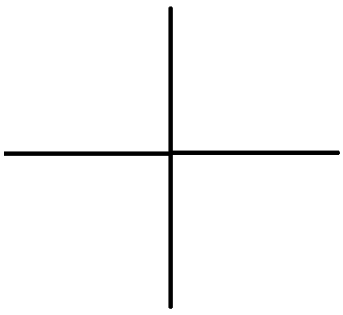
Where both x(t) and y(t) are continuous real valued functions.

We say that z(t) is parameterization of curve C.

Example 1: Consider a curve C_1 defines as:

$z_1(t) = 6\cos(t) + i\, 6\sin(t)$ where $t \in [0,2\pi]$

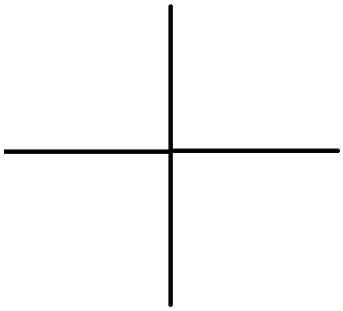
To get the points on C_1, we need to vary the value of t.



So we can say that a curve defined in this parametric way gives us more than a geometrical shape. Its geometrical shape is just a circle but it is also providing us the direction in which we are moving.

Example 2: Consider a curve C_2 defines as:

$z_2(t) = 6\cos(t) - i\,6\sin(t)$ where $t \in [0,2\pi]$

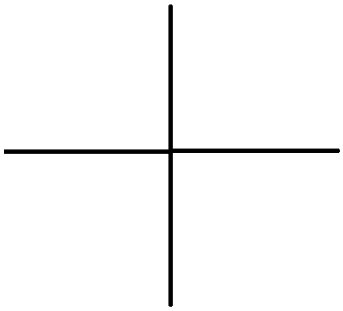


If a curve is given , when we vary the parameter t , we are travelling along the curve. As the parameter increases w move from initial point z(a) to the terminal point z(b) on the curve.

Closed Curve: If initial and terminal points are equal then the curve C is said to be closed curve.

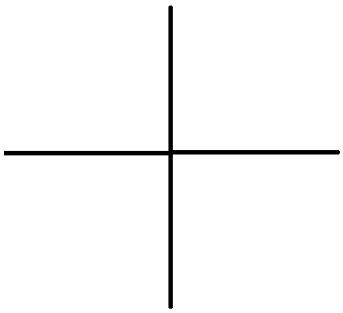
Example: The curve C_3 defines as:

$z(t) = 2t + i3t$ where $t \in [0,1]$ is not a closed curve.



Simple Curve: A curve is said to be simple if it does not cross itself. In other words the curve:

C: $z(t) = (x(t) , y(t))$, for $t \in [a,b]$ is simple provided $z(t)$ is one to one function or $z(t_1) = z(t_2) \Longrightarrow t_1 = t_2$



Examples:

A Simple closed curve C bounds an area A in Complex Plane.

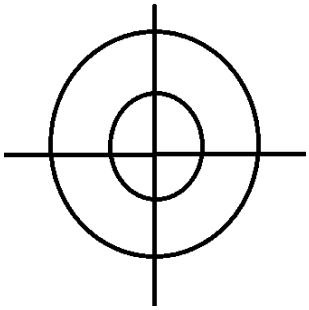
As the parameter increases we move along the curve. During this travel if the interior of A is on your left then the curve is said to be positively oriented.

Now, how do we make sure that there is always going to be an area enclosed by a simple closed curve.

The Jordan Curve Theorem guarantees that a simple closed curve must enclose a region.

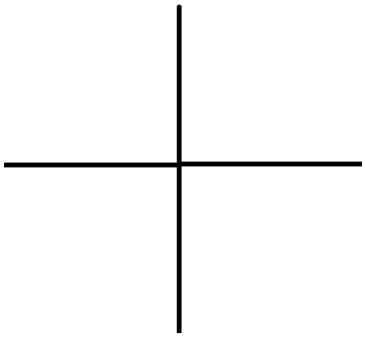
A set S is said to be **connected set** if for every pair of points z_1 , $z_2 \in S$ there is a curve C joining them which is entirely contained in S.

Example: The annulus A = {z:1<|z|<2} is connected.



A Connected Open Set is called a **Domain.**

Example: Check whether the half plane H={z: Re(z) > 0} is a domain or not.



Next Property of Region is **Boundedness.**

A set S is said to be bounded set if it can be completely contained in some closed disk, that is , there exist and $R > 0$ such that:

$$|z| \leq R \text{ for all } z \in S$$

A set which is not bounded is called **unbounded.**

Example:

The region $\{ z : 1 < Re(z) < 2 \}$ is not bounded.

Example:

The region $\{ z : |z| < 1 \}$ is bounded.

In this topic, we learnt about:

- Curves
- Closed Curves
- Simple Curves
- Orientation of Curves
- Connected Sets
- Domain
- Bounded Sets

Topic:017

Complex Functions

Part (I)

Before going into the complex part of the complex function, let's first try to understand **What is a function?**

There are many quantities around us that are related with each other. **For example:** If wind is blowing then the speed of the wind depends on the temperature and other quantities.

So, we can say that speed of wind is related to temperature. Similarly, if current is flowing in the circuit. Then the current I is related to voltage V across the circuit. Similarly there are many other factors.

Let's consider an example of temperature at different times of a day. So if we consider temperature at 9'O Clock in the morning, then the temperature will be different. If we consider it at 12 p.m it will be different and so on. So, we can say that the temperature is related with time. In fact, temperature depends on time. Time is basically an independent quantity in this example, but temperature depends on time. So, we can say that temperature is a function of time. And more precisely, when we take reading of temperature then it will give precisely one temperature value. And of course if this function is useful then we can get temperature value at each and every time. So we can

say that this temperature function is basically relating each and every value of time to unique value of temperature. So more precisely we can say that the temperature function T is associating each and every number between [0 , 24] to the temperature between [-20,50].

A temperature function T provides us temperature during different times of a day. In this case, we have A = [0,24] and B=[-20,50] and the rule T relates a time with a value of temperature.

So we can say that temperature is a kind of machine that takes input as one quantity and gives output as another quantity. These input and output quantities are related inside the machine.

More Precisely:

A function from set A to set B is a rule that assigns each value of A with a unique value of B.

Two rules or conditions are:

1. We should get output for each and every input from the set A.
2. The output should be unique.

As we are talking about complex numbers. Then why we need complex functions? As we know complex numbers are imaginary numbers and complex functions can be said as imaginary functions.

Are these numbers are really imaginary or realities are also involved?

Consider the **motion of a pendulum.**

If we leave the bob from its center then it starts moving back and forth. As the time moves on the amplitude of bob going to be decreased. There are many factors for this. One of them is air friction. Now if we want to completely understand this situation then we need to model this situation. In other words, we want to write down the quantities, relation between the quantities and conditions. If we want to do that and if we want to use real numbers and real valued functions then it's going to be very complicated model. If we want to simplify the situation, then we need complex valued functions.

Next example is basically **fluid dynamics**. If w are considering fluid in two dimensions with low viscous number, then we need complex valued functions and since fluid is very very applicable phenomenon around us. For example blood flow in our veins.

So, fluid dynamics is a very real life oriented thing. So, complex valued functions help us in understanding these phenomena. We can say that these complex valued functions are very very related to our real life.

Our next example is about circuit theory problems. So, if we want to solve the circuit then we need complex valued functions.

Complex Function:

A Complex valued function f of a complex variable z is a rule that assigns to each value z in a set D one and only one complex value w. Here z is independent and w is dependent variable.

We write w = f(z) and cal w **the image of z under *f***. Here the set D is the **domain of *f*** and the set of all images {w=f(z):z∈ D} is called **the range of *f*.**

In some context complex valued functions are also known as **mappings** or **transformations.**

Example: A simple example of a complex valued function $f:C\rightarrow C$ is given by the rule $w = f(z) = z^2$.

If we take:

$f(1+i) = (1+i)^2 = 1+i^2+2i = 1-1+2i = 2i$

A function has **two parts**:

1. Domain (what are the objects that we are going to relate)
2. Range/Rule (how we are going to relate those objects)

In this lecture, we learnt about why we need complex functions.

Topic:018

Complex Functions

Part (II)

Now, if we want to do calculations with complex functions for example if we want to know what are the images under some

function of a complex number or if we want to explore further properties like continuity, limit, derivatives and other properties of complex numbers then we need to have some very precise descriptions of these complex functions.

Real and imaginary parts of Complex functions

(In Cartesian Coordinate)

A complex number z can be expressed by its real and imaginary parts.

So, when we apply function on a complex number $x + iy$ then we get another complex number say $u + iv$. So we can say that:

A complex function w = $f(z)$ can be written as:

w = $f(z) = f(x + iy) = u + iv$ *here u and v depends on x and y*

So we can write

w = $f(z) = u(x , y) + iv(x , y)$

Here *u and v are real valued functions and inputs are x and y.*

Example: Express the function

$f(z) = \bar{z}\, Re(z) + z^2 + Im(z)$ in the form $f(z) = u(x , y) + iv(x , y)$.

Solution:

$f(z) = (x - iy)\, x + (x + iy)^2 + y$

$f(z) = (x - iy)\, x + (x^2 - y^2 + 2ixy) + y$

$f(z) = x^2 - iyx + x^2 - y^2 + 2ixy + y$

$f(z) = (2x^2 - y^2 + y) + i(xy)$

So,

$u(x , y) = (2x^2 - y^2 + y)$ *and* $v(x , y) = xy$

Conversely, Two real valued functions $u(x , y)$ *and* $v(x , y)$ determine a complex valued function.

We can use the formulas:

$x = \frac{z+\bar{z}}{2}$ *and* $y = \frac{z-\bar{z}}{2i}$

to find a formula for f involving variables z and $\bar{z}$.

Example: Express $f(z) = 4x^2 + i4y^2$ by a formula involving the variables z and $\bar{z}$.

Solution: Calculation reveals that:

$f(z) = 4\left(\frac{z+\bar{z}}{2}\right)^2 + i4\left(\frac{z-\bar{z}}{2i}\right)^2$

$f(z) = z^2+2z\bar{z}+\bar{z}^2 - i(z^2 - 2z\bar{z}+\bar{z}^2)$

$f(z) = z^2 (1 - i) + z\bar{z} (2 + 2i) + \bar{z}^2 (1 - i)$

Sometimes input of a function is a complex number but output is a real number, that is , $v(x,y) = 0$.

The following example shows one such real valued function.

Example: $f: C \rightarrow R$ defined by

$f(x+iy) = \sqrt{x2 + y2}$ *Or* $f(z) = |z|$

Real and Imaginary parts of a Complex Function

(In Polar Corrdinates)

Using $z = re^{i\theta}$ in the expression of the complex function $w = f(z)$ gives us the polar representation:

$$f(z) = f(r\,e^{i\theta}) = u(r\,,\theta) + iv(r\,,\theta)$$

where u *and* v are real functions of real variables r *and* θ.

Example: Express $f(z) = z^2$ in imaginary and real parts using Polar form.

Solution: For Polar Form, we use $z = re^{i\theta}$.

$$f(z) = z^2 \Longrightarrow f(re^{i\theta}) = (re^{i\theta})^2 = r^2e^{i2\theta} = r^2(cos2\,\theta + isin2\,\theta)$$

$$f(re^{i\theta}) = r^2cos2\,\theta + ir^2sin2\,\theta$$

$$u(r\,,\theta) = r^2cos2\,\theta \quad , iv(r\,,\theta) = ir^2sin2\,\theta$$

Example: Express $f(z) = z^2 + z + 1$ in imaginary and real parts using both Cartesian and Polar Form.

Solution: For Cartesian Form we use $f(z) = x + iy$

$$f(z) = z^2 + z + 1 \Longrightarrow f(x + iy) = (x + iy)^2 + (x + iy) + 1$$

$$f(x + iy) = x^2 - y^2 + 2ixy + x + iy + 1$$

$$f(x + iy) = (x^2 - y^2 + x + 1) + i(2xy + y)$$

So that:

$$u(x\,,y) = x^2 - y^2 + x + 1 \quad , \quad v(x\,,y) = 2xy + y$$

For Polar Form we use $z = re^{i\theta}$.

$f(z) = z^2 + z + 1 \Longrightarrow f(re^{i\theta}) = (re^{i\theta})^2 + re^{i\theta} + 1$

$f(re^{i\theta}) = r^2\cos 2\theta + ir^2\sin 2\theta + r\cos\theta + ir\sin\theta + 1$

$f(re^{i\theta}) = (r^2\cos 2\theta + r\cos\theta + 1) + i(r^2\sin 2\theta + r\sin\theta)$

$So,\ u(r,\theta) = r^2\cos 2\theta + r\cos\theta + 1\ ,\quad v(r,\theta) = r^2\sin 2\theta + r\sin\theta$

In Cartesian Coordinates:

$f(z) = (x^2 - y^2 + x + 1) + i(2xy + y)$

In Polar Coordinates:

$f(z) = (r^2\cos 2\theta + r\cos\theta + 1) + i(r^2\sin 2\theta + r\sin\theta)$

Remember: *Function is the same but if we change the coordinates , Real and Imaginary components will be changed.*

Now let's consider some very special complex valued functions which are going to be a part of our further discussion in Complex Analysis. One of the famous example of such functions is **Polynomial Functions.**

A polynomial of degree n, over complex numbers, is an expression of the form:

$P_n(z) = a_0 + a_1 z + \ldots\ldots + a_{n-1}z^{n-1} + a_n z^n$

where a_i are complex numbers. n is the degree.

Domain of polynomial function is the whole complex plane.

Linear Polynomial Functions:

One of special types of polynomial functions of degree n = 1

f(z) = az + b *where a , b are complex numbers , called linera polynomial function or just linear transformation.*

These polynomials have very interesting geometrical properties which we will see in further discussion.

Examples:

f(z) = 1 + z + z^2 is a polynomial function.

f(z) = 1 + z + z^2+ z^3 + z^4+….. is Not a Polynomial because Polynomial is always a finite sum of terms. In fact, it is a series.

f(z) = $\frac{1}{z}$ = z^{-1} *is Not a polynomial. Because powers should be non negative.*

Rational Functions:

A rational function is an expression of the following form:

$\frac{P(z)}{Q(z)}$, where P(z) and Q(z) are polynomials

(possibly of different degrees).

Linera Fractional Transformations:

A transformation of the form:

T(z) = $\frac{az+b}{cz+d}$, ad – bc ≠ 0 where a , b, c ,d are complex numbers.

Sometimes we are given an expression which is not a function by definition but we can construct many functions from this expression and of course we can use those functions to discuss the properties of that given expression.

Example: Consider

$$f(z) = z^{1/2} = \pm \sqrt{r}\, exp\left[i\frac{\theta}{2}\right]$$

where $r = |z|$ *and* θ *is the principal argument of z.*

This expression is not a function because input is complex numbers but the output is not unique. It will given two different roots as output.

But we can construct two single valued functions from here as:

$$f_1(z) = z^{1/2} = \sqrt{r}\, exp\left[i\frac{\theta}{2}\right], \quad f_2(z) = z^{1/2} = -\sqrt{r}\, exp\left[i\frac{\theta}{2}\right]$$

These two are complex valued functions because these give unique output of complex input.

To explore properties of f(z) we instead consider $f_1(z)$ *and* $f_2(z)$.

These kinds of multi valued functions appear many times in complex analysis. One of the trick to deal with such functions is dividing them into many functions.

In this part we studied how to write down complex valued function in real and imaginary parts and also discussed some complex valued functions.

Topic:019

Mappings

Now we want to see if we get geometrical representation of these complex valued functions. Now why we want to get geometrical representation? Because, we want to observe properties of these complex valued functions using geometry of them and on the other hand it will giveus some very beautiful pictures.

How we can get geometrical representation of the functions?

For that Recall calculus of functions of one variable.

Properties of real valued function $y = f(x)$ of a real variable are often exhibited by the graph of the function.

From the curves of the function we observe many many properties. For example, we can see that the function is increasing, function is smooth so we can say that function is differentiable etc. So there are many properties that we can observe by just looking at the graph of the function.

Now the question is: Can we have similar geometrical description of the function for complex valued functions?

For Complex Functions $w = f(z)$ no such convenient graphical representation of the function f is available.

Here domain z is a point in complex plane in R^2. And w is the range in R^2.

So following the same procedure, as we did for real valued functions of one variable, we take domain axis (x-axis) and range axis (y-axis). In this case, the domain is R^2 and the range is R^2. If we follow the same procedure then the graph of these functions will be in R^4 which is very difficult task.

To overcome this problem, we represent z in one plane, known as *z-plane*, and its image w is represented in a second plane, known as *w-plane*.

When we consider a function using such correspondence between two complex planes, *z-plane* and *w-plane* then we call complex functions as **"Complex Mappings"** or simply **"mapping"**.

Given a Complex Function $\boldsymbol{f(z) = u(x,y)+iv(x,y)}$ with domain D. The equations $\boldsymbol{u = u(x,y)}$ *and* $\boldsymbol{v = v(x,y)}$ describe a transformation (or mappings) from a set D in the *z-plane* onto the range R in the *w-plane.*

Example: Consider the following complex valued function

$w = f(z) = z + 1$ with domain $D = \{z: |z| \leq 1\}$.

Solution: $f(1) = 2$, $f(i) = i+1$, $f(-1) = 0$, $f(-i) = 1-i$

We can say that from this geometrical representation of the function, we can say that this function is translating the points of the domain to the right one unit.

Example: The image of the set $A = \left\{z : |z| \leq \frac{1}{2}\right\}$ is under $f(z)=z+1$ is the set $B = \left\{z : |z-1| \leq \frac{1}{2}\right\}$.

Example: Consider the following complex valued function

$w = f(z) = z + 1$ with domain $D = \{z: |z| \leq 1\}$.

Sometimes, it is useful to represent both domain and range in the same plane.

In general:

We know that a function is a procedure that takes z as an input and gives us w as an output. So, it's a kind of box that processes this z and gives us output as w.

Now the question is can we find reverse of this procedure? In general the answer is No. But fortunately there are some conditions on the function under which the inverse procedure of this process exists. To observe those conditions consider some examples.

Example: If $w = f(z) = iz$, the inverse image of the point -1 is the single point i, because $f(i) = i(i) = -1$, and i is the only point that maps to -1.

We can say that the inverse image of -1 is i. So in this case when we try to find reverse procedure of this function $f(z)$, then it is possible. Because inverse image of -1 is i. And if this happens for all the points in the range , the inverse image of each point is unique. Then it is possible to find the inverse of this function.

Example: If $w = f(z) = z^2$, the inverse image of the point -1 is the set $\{i , -i\}$.

In this case reverse is not possible.

If we have a function f from domain D to range R then its inverse image is a function from R to D only if f is one-to-one.

One to One: *f is one to one if we suppose that* $f(z_1) = f(z_2)$ *then from this assumption deduce that* $z_1 = z_2$.

If f is not one to one then the inverse image of a point is not unique.

Example: The function $f(z) = iz$ is one-to-one.

Example: The function $f(z) = z^2$ *is not one-to-one.*

If w=f(z) maps the set A one-to-one and onto the set B, then for each w in B there exists exactly one point z in A such that w=f(z)

If the function is one to one and onto, then we have inverse function.

Then we can solve the equation w = f(z) by solving for z as a function of w.

That is ***the inverse function*** $z = f^{-1}(w)$ *can be found, and the following equations hold*

$f^{-1}(f(z)) = z$ ***for all*** $z \in A$ *and* $f(f^{-1}(w)) = w$ ***for all*** $w \in B$

Example: The function $w = f(z) = iz$ for any complex number z find $f^{-1}(w)$.

Solution: Observe that *f* is one to one and onto the entire complex plane.

From here, $w = iz$ $z = w/i = -iw$

We have $f^{-1}(w) = -iw$ *for all complex numbers w.*

Sometimes we use same variable for inverse $f^{-1}(z) = -iz$.

Also the following equations hold:

$f^{-1}(f(z)) = z$ ***for all*** $z \in A$ *and* $f(f^{-1}(w)) = w$ ***for all*** $w \in B$

Different properties of mappings can be noticed by observing images of carefully chosen subsets of its domain. These subsets are usually described by equations or inequalities. In this situation inverse functions help us in describing images of subsets.

Example: Show that the function $f(z) = -z$ maps the line

$y = -x+1$ to the line $v = -u - 1$. *In this case subset of domain is* $A = \{x+iy : y = -x+1\}$. *Let* $B = f(A)$, *the defining conditions for B can be found as follows:*

$w = f(z) = u + iv \in B \Leftrightarrow f^{-1}(w) \in A \implies -w \in A$

$\implies -u-iv \in A \implies (-u, -v) \in A \implies -v = u+1 \implies v = -u-1$

If we change this set A, we get other information about the behavior of this function.

We have seen geometrical depiction of complex valued function, using this we can observe different properties of functions.

Topic:020

Mappings : Linear Transformation

Part (I)

So far we have discussed what are complex valued functions and what are mappings associated to them.

Mapping is a geometrical way of visualizing a given complex valued function and of course it helps us in understanding the behavior of the function.

From now on we are going to choose particular values of complex valued functions and discuss the mapping associated to them.

Our first example is Linear Transformations. These are very simple but very important complex valued functions and they have very interesting geometrical interpretations as well.

This is the first part of our discussion on linear transformation.

Linear Transformation: *A complex valued function of the form*

$L(z) = Az + B$ where A , B are fixed complex numbers and z is a complex variable. It is a special form of Polynomial.

If we change the values of A and B we will get particular examples of linear transformations.

Now, we are going to choose particular cases in which we choose particular values of A and B and we will see what are

the mappings associated to them.

Our first example is the case of ***Translation.***

In this case, we choose $A = 1$, $B = a + ib$

$T(z) = z + B \Longrightarrow T(z) = (x + iy)+(a+ib) \Longrightarrow T(z) = (x+a)+i(y+b)$

Geometrical Interpretation of this function:

Inverse Translation: *Translation is the one to one and onto function on the entire complex plane.*

In this case if we assume:

$w = T(z) = z + B \Rightarrow z = w - B \Rightarrow z = f^{-1}(w) = w - B$

$\Rightarrow f^{-1}(z) = z - B$

That's why they are inverses of each other.

The inverse mapping is given by $z = T^{-1}(w) = w - B$

$= u - a + i(v-b)$

And shows that *T* is one to one mapping from the *z-plane* onto the *w-plane*.

Rotation:

Our next particular case of linear transformation is Rotation. In case of rotation we take B = 0 and we choose its

complex number *A* such that $|A| = 1$. Linear transformation in this case will become:

$R(z) = Az$. If we want to understand its geometrical meaning more efficiently then we use in this case Polar Co-ordinates.

Let α be a fixed real number. Then

$A = e^{i\alpha}$, *if* $z = re^{i\theta}$ *then*

$R(z) = re^{i\alpha}e^{i\theta}$ $\qquad$ $R(z) = re^{i(\alpha+\theta)}$

When rotation is applied to a complex number, modulus of the complex number remains unchanged, the argument (angle) is changed. i.e the number will be rotated about the origin.

$w = R(z) = re^{i\alpha}e^{i\theta}$ $\qquad$ $z = e^{-i\alpha}w$

If $w = \rho e^{i\varphi}$ *then* $z = f^{-1}(w) = \rho e^{i(\varphi-\alpha)}$

Now as we can see that if we transform using this rotation then we are rotating by an angle α and if we use the inverse transformation, we just subtract angle α from the argument. In other words, we are rotating in opposite direction of α. We can easily observe the geometrically that it is inverse of the rotation.

Precisely:

If we use polar coordinates $w = \rho e^{i\varphi}$ in the w plane, then the inverse mapping is given by:

$z = R^{-1}(w) = we^{-i\alpha} = \rho e^{i\varphi}\, e^{-i\alpha} = \rho e^{i(\varphi-\alpha)}$

This shows that the rotation R is one to one mapping of the z-plane onto the w-plane.

Magnification:

Let k > 0 be a fixed positive real number. Then the trasformation

$w = S(z) = k\,z = k\,x + i\,k\,y$ is called Magnification.

It has following cases:

1. If k < 1, then it reduces the distance between points by the factor k.
2. If k > 1, it has the effect of stretching the distance between points by the factor k.
3. If k = 1 , there will be no change.

If A = $ke^{i\alpha}$ and B = a + ib where k > 0 is a positive real number then the transformation w = L(z) = Az + B is a one-to-one mapping of the z-plane onto the w-plane and is called a **Linear Transformation.**

So, L(z) = $ke^{i\alpha}$ z + B

We have observed three different cases, when we choose particular case for A and B, we have translation , magnification and rotation. And from this transformation, we observe that k is responsible for magnification, α is responsible for rotation and B is responsible for translation.

In terms of composition of a rotation, a magnification and a translation it can be written as a effect of rotating the plane

through an angle given by $\alpha = \text{Arg } A$ followed by magnification by the factor $k = |A|$ followed by a translation by the vector

$B = a + ib$

Hence, a linear transformation is a compodition of:

1. Translation
2. Rotation
3. Magnification

Inverse Transformation: L is a one to one mapping from the z-plane onto the w-plane. So we can find inverse.

$w = L(z) = Az + B \implies Az = w - B$

$z = \frac{w}{A} - \frac{B}{A} \implies z = f^{-1}(w) = \frac{w}{A} - \frac{B}{A} \implies L^{-1}(z) = \frac{z}{A} - \frac{B}{A}$

And of course, we can see that:

$L(L^{-1}(z)) = z$ and $L^{-1}(L(z)) = z$

These are the inverses of each other.

Geometrically, If we apply L then we will be translating, rotating and magnifying it. And if we apply inverse transformation, then we will be reversing whole process.

The inverse mapping is given by $z = L^{-1}(w)$

$z = \frac{1}{A} w - \frac{B}{A}$

We discussed geometrical meaning of linear transformation.

Topic:021

Mappings : Linear Transformation

Part (II)

Linear Transformation: A Complex valued function of the form $L(z) = Az + B$ where A B are complex numbers.\

Hence a linear transformation is a composition of:

1. Translation
2. Rotation
3. Magnification

In this part we are going to construct some examples of linear transformation with given data.

Example: The ellipse centered at the origin with a horizontal major axis of four units and vertical minor axis of two units can be represented by the parametric equation:

$$s(t) = 2cost + isint = (2cost\ ,\ sint) \quad for\ 0 \leq t \leq 2\pi$$

Now if we want to construct linear transformation that rotate the ellipse

1. By an angle of $\frac{\pi}{3}$ radians and
2. Shift the center of ellipse 3 units to the right and 1 unit up.

Solution: We generate a parametric equation r(t) given by:

$r(t) = e^{i\frac{\pi}{3}} s(t) + (3 + i)$

on simplifying by using Euler's formula:

$r(t) = (2cost + isint)\,(cos\frac{\pi}{3} + isin\frac{\pi}{3}) + (3 + i)$

$r(t) = 2cost\ cos\frac{\pi}{3} + i2cost\ sin\frac{\pi}{3} + isint\ cos\frac{\pi}{3} + i^2 sint\ sin\frac{\pi}{3}$

$r(t) = (2cost\ cos\frac{\pi}{3} - sint\ sin\frac{\pi}{3}) + i(2cost\ sin\frac{\pi}{3} + sint\ cos\frac{\pi}{3}) + (3+i)$

$r(t) = (cost - \sqrt{3}/2\ sint + 3) + i(\sqrt{3}cost + ½\ sint + 1)$

$r(t) = (cost - \sqrt{3}/2\ sint + 3\ ,\ \sqrt{3}cost + ½\ sint + 1)$

Now this is the linear transformation which rotates the ellipse by angle $\frac{\pi}{3}$ *and also translate it by (3 + i).*

Example: *Let* $f(z) = (3+4i)z - 2 + i.$

1. Find the image of the disk $|z - 1| < 1.$

Solution: *Let A be the disk of the form*

$A = \{z: |z - 1| < 1\}$ *with center is 1 and radius of this open disk is 1.*

We have to construct inverse of this linear transformation. As we know linear transformation is always one to one and onto on

entire complex plane. So we can construct its inverse.

The inverse transformation is: $z = \frac{w+2-i}{3+4i}$.

This inverse will help us in finding image of this disk in w-plane.

As w=f(z) then the inverse is $f^{-1}(w) = z \in A$.

Let's apply the definition of $f^{-1}(w)$*:*

$\Leftrightarrow \frac{w+2-i}{3+4i} \in A$

Now we know very well defined description of this set. An element $z \in A$ *if* $|z - 1| < 1$. *And here we are claiming that* $\frac{w+2-i}{3+4i} \in A$. *It means:*

$\left| \frac{w+2-i}{3+4i} - 1 \right| < 1.$

$\left| \frac{w+2-i}{3+4i} - 1 - 0i \right| |3 + 4i| < 1 \; |3+4i|.$

On simplifying we get:

$|w + 2 - i + ((-1 - 0i)(3+4i))| < 5$

$|w + 2 - i - 1 - 4i| < 5$

$|w - 1 - 5i| < 5 \quad \Longrightarrow |w - (1+5i)| < 5$

Example: *Let $f(z) = (3 + 4i)z - 2 + i$.*

2. Find the image of the half plane $Im(z) > 1$

Solution: *Write $w = f(z)$ in Cartesian form as:*

$z = x + iy$

$f(z) = w = u+iv = (3 + 4i)(x + iy) - 2 + i$

$u+iv = 3x + 3iy + 4ix - 4y - 2 + i$

$u+iv = 3x - 4y - 2 + i(4x + 3y + 1)$

Here, $u = 3x - 4y - 2$ *and* $v = 4x + 3y + 1$

Now, if we want to find image of given linear transformation then we have to find y as given condition is $Im(z) > 1$.

We have: $v = 4x + 3y + 1$ *which implies* $y = \frac{v-4x-1}{3}$

As given condition is $Im(z) > 1$ means that $y > 1$. So,

$$\frac{v - 4x - 1}{3} > 1$$

$$v - 4x - 1 > 3$$

$$v > 4 + 4x$$

We want to calculate an expression , inequality or equations in terms of u and v that defines the image of the given half plane in w-plane.

As we obtained the expression in v and x, so it is not

our required expression. To find our required expression, we need to replace x in terms of u and v.

Now taking out the value of x from the value of u, we get:

$u = 3x - 4y - 2 \implies x = \frac{u+4y+2}{3}$

in this expression, we replace y by its value in terms of u and v.

$x = \frac{u+4(\frac{v-4x-1}{3})+2}{3}$

On solving, we get

$x = \frac{u+v-2}{19}$

Now, putting value of x in v > 4 + 4x

$v > 4 + 4(\frac{u+v-2}{19})$

$v > \frac{4u+68}{3}$

Hence, the image of half plane Im(z) > 1 is $v > \frac{4u+68}{3}$.

This inequality can also be written as:

$v - \frac{4u}{3} - \frac{68}{3} > 0$

If we replace inequality by equality, then

$v - \frac{4u}{3} - \frac{68}{3} = 0$

Similarly, we can calculate other examples.

Topic: 022

The Mappings; $w = z^n$ and $w = z^{1/n}$

(Part I) (n = 2)

We are going to study some other kinds of complex valued functions, which are nth power and nth root functions. These functions are basically special cases of a broader class of complex valued functions, which are power functions.

A real function of the form $f(x) = x^a$ where a is the real constant and x is real variable, is called **Power Function**.

These functions are not just simple to handle geometrically and algebraically but also have some interesting geometrical and algebraic properties. That's why these functions are studied frequently.

Complex power function is a function of the form $f(z)=z^{\alpha}$ *where* α *is the complex constant.*

If we consider α *to be in complex number that involve I, then these are very complicated cases and will discuss later.*

Now we are focusing only when α *is a positive integer of form n and 1/n.*

In this discussion, we choose special number i.e n = 2. We will study square and square root functions and will also discuss different properties of these functions.

The mapping f(z) = z^2

To study these functions in simplified way, we should choose exponential form of complex number , z should not be 0.

By replacing z = $re^{i\theta}$, we get

w = f(z) = $r^2e^{i2\theta}$

If w = $\rho e^{i\varphi}$ then f(z) is given as:

$\rho = r^2$ and $\varphi = 2\theta$

This tells us about the behaviour of this mapping z^2. For example this tells us about the modulus.

Observe that if r > 1, then in the w-plane after applying this mapping z^2, it will be further away from the origin.

If r < 1, then after applying this mapping z^2, it will get closer to the origin.

***Example:** If the domain of the definition D for f(z) = z^2 is restricted to be the region*

A = $\{re^{i\theta}: 0 < r \leq 1$ and $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}\}$

Then the image of A is

B = $\{\rho e^{i\varphi}: 0 < \rho \leq 1$ and $-\pi < \varphi \leq \pi\}$

The inverse image of f, which we denote by g is then

$z = g(w) = w^{1/2} = \rho^{1/2} e^{i\frac{\varphi}{2}}$ *where* $w \in B$. *That is*

$z = g(w) = w^{1/2} = \rho^{1/2} e^{i\frac{Arg(w)}{2}}$, $w \neq 0$.

Definition (Principal Square Root)

The function $g(w) = w^{1/2} = \rho^{1/2} e^{i\frac{Arg(w)}{2}}$, $w \neq 0$

is called principal square root function.

(Remember when we calculate the square root of a function, there are two outputs, but here it is a single valued function. So what is the difference between these two arguments? Over there we use the fact that θ *is allowed to be any argument, so* θ *could be replaced by* $\theta + 2k\pi$ *, when we varied k then we get different square roots of complex number. We are not allow to add or subtract multiples of* 2π. *So it is the argument of complex number, that's why it is a single valued function, not a multi-valued function and the output is always going to be one number. We can also observe that what will happen when we apply this principle square root function.*

The modulus will be square rooted and the argument will be halfed.

Off course, we can calculate the composition.

It can be checked that f and g satisfy equations

g(f(z)) = z and f(g(w)) = w

then we say that they are inverses of each other.

Consider the mapping $f(z) = z^2$

We see images in w-plane of some special lines (parallel to real axis and parallel to imaginary axis) in the z-plane.

The mapping $w = z^2$ *can be thought of as the transformation*

$u = x^2 - y^2 \quad , \quad v = 2xy$

from the xy-plane into uv-plane.

Image of vertical line:

The vertical line x = a is mapped onto the set of points given by the equations:

$u = a^2 - y^2 \quad , \quad v = 2ay$

if $a \neq 0$ *then* $y = \frac{v}{2a}$, *putting the value of y in u we get*

$u = a^2 - (\frac{v}{2a})^2 = a^2 - v^2/4a^2 \qquad (1)$

$u - a^2 = -\frac{1}{4a2}(v^2)$

This is just a translation along u-axis which represents a

parabola with a vertex at a^2, oriented horizontally and opening to the left.

If $a > 0$, the set $\{(u,v): u = a^2 - y^2 \quad , v = 2ay\}$ has $v > 0$ whenever $y > 0$, so the part of line $x = a$ lying above the x-axis is mapped onto the top half of parabola.

Image of Horizontal line:

The horizontal line $y = b$ is mapped onto the parabola given by the equations:

$u = x^2 - b^2 \quad , \quad v = 2xb$

if $b \neq 0$ then $x = \frac{v}{2b}$, we get

$u = -b^2 + v^2/4b^2 \qquad (2)$

$u + b^2 = \frac{1}{4b2}v^2$

which represents a parabola with vertex at $-b^2$, oriented horizontally and opening to the right.

If $b > 0$, the part of the line $y = b$ to the right of y-axis is mapped to the top half of the parabola because the set $\{(u,v): u = x^2 – b^2, v = 2bx\}$ has $v > 0$ when $x > 0$.

We see that the quadrant 1 is mapped onto quadrant 1 and quadrant 2 by $w = z^2$, so the rectangle $0 < x < a$, $0 < y < b$ is mapped onto the region bounded by the top halves of the parabolas.

The vertices 0 , a ,a + ib and ib of the rectangle are mapped onto the four points 0 , a^2 , $a^2 – b^2 + i2ab$ and $–b^2$ respectively.

The vertical line $x = 0, y \neq 0$ is mapped to the set $\{(-y^2,0): y \neq 0\}$

Which is the set of negative real numbers.

Similarly, the horizontal line $y = 0$, $x \neq 0$ is mapped to the set $\{(x^2, 0): x \neq 0\}$ which is the set of positive real numbers.

In this session we started our discussion on very special power functions which are complex valued functions of the form z^n and $z^{1/n}$, we choose n = 2 in this session.

Topic: 023

The Mappings; $w = z^n$ and $w = z^{1/n}$

(Part II)

We are going to choose some other curves in the z-plane and study the behavior of these curves after applying mapping z^2.

These curves are basically hyperbolas.

Each branch of hyperbola $x^2 - y^2 = c_1$ $(c_1 > 0)$

is mapped in one to one manner onto the vertical line $x = c_1$.

From $x^2 - y^2 = c_1$ we get

$x = \pm\sqrt{c_1 + y^2}$

For finding v, we put the value of x in $v = 2xy$

$v = \pm 2y\sqrt{c_1 + y^2}$

Then the image of right hand side expressed in the form of u and v parametrically as:

$u = c_1$ $\quad v = 2y\sqrt{c_1 + y^2}$ $\quad (-\infty < y < \infty)$

The parametric representation for the image of left-branch of the hyperbola expressed as:

$u = c_1$ $\quad v = -2y\sqrt{c_1 + y^2}$ $\quad (-\infty < y < \infty)$

Each branch of hyperbola $2xy = c_2$ $(c_2 > 0)$ is transformed into the horizontal line $v = c_2$.

From v = 2xy we get v = c_2.

Using $y = \frac{c_2}{2x}$ in u = $x^2 - y^2$, we get u = $x^2 - \frac{c_2^2}{4x^2}$ $\quad 0 < x < \infty$

The image of the lower branch has the parametric representation:

u = $\frac{c_2^2}{4y^2} - y^2$ $\quad$ v = c_2 $\quad$ $-\infty < y < 0$

In general:

Geometry of the mapping w = $z^{\frac{1}{2}}$

If z = $re^{i\theta}$ then $f(z) = z^{\frac{1}{2}}$ becomes

$w = f(z) = |z|^{\frac{1}{2}} e^{i\frac{Arg(z)}{2}} = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ $\quad$ for z = $re^{i\theta} \neq 0$

if $w = \rho e^{i\varphi}$ *then image under f is given by*

$\rho = r^{\frac{1}{2}}$, $\varphi = \frac{\theta}{2}$

Geometry of the mapping $w = z^{\frac{1}{2}}$

Example: If the domain of definition D for $f(z) = z^{\frac{1}{2}}$ is restricted to be the region

$B = \{re^{i\theta} : 0 < r \leq 1 \; and \; -\pi < \theta \leq \pi\}$

Then the image of A under the mapping $f(z) = z^{\frac{1}{2}}$ is

$A = \{\rho e^{i\varphi} : 0 < \rho \leq 1 \; and \; -\frac{\pi}{2} < \varphi \leq \frac{\pi}{2}\}$.

Example: We have seen that hyperbolas are mapped onto lines under z^2. So the map $f(z) = z^{\frac{1}{2}}$ maps lines on hyperbolas. (exercise)

Example: Similarly, the parabolas are mapped onto lines under $z^{\frac{1}{2}}$ since lines are mapped onto parabolas under z^2. (exercise)

Topic: 024

The Mappings; $w = z^n$ and $w = z^{1/n}$

(Part III)

For any positive integer n, the function $f(z) = z^n$, $z = r\,e^{i\theta} \neq 0$, can be expressed as $w = f(z) = z^n = r^n e^{in\theta}$.

If $w = \rho e^{i\varphi}$ in the w-plane then the mapping $f(z) = z^n$ can be given by following the equations:

$$\rho = r^n \quad , \quad \varphi = n\theta$$

Example: Consider the mapping $f(z) = z^3$. The mapping $w = z^3$ increases the argument of a point by a factor of 3, the ray nearest x-axis in the first quadrant.

If we restrict the domain of $f(z) = z^n$ to the region

$$A = \{re^{i\theta} : 0 < r \leq 1 \text{ and } -\frac{\pi}{n} \leq \theta \leq \frac{\pi}{n}\}$$

The image of A under the mapping f(z) is given by:

$$B = \{\rho e^{i\varphi} : 0 < \rho \leq 1 \text{ and } -\pi < \varphi \leq \pi\}$$

The inverse mapping of f(z) denoted by g(z) is then

$g(w) = w^{\frac{1}{n}} = \rho^{\frac{1}{n}} e^{i\frac{\varphi}{n}}$, $w \neq 0$ *where* $w \in B$. *That is:*

$z = g(w) = w^{\frac{1}{n}} = |w|^{\frac{1}{n}} e^{i\frac{Arg(w)}{n}}$, $w \neq 0$

Definition (Principal nth root)

The function $g(w) = w^{\frac{1}{n}} = |w|^{\frac{1}{n}} e^{i\frac{Arg(w)}{n}}$, $w \neq 0$ is called the principal nth root function.

Geometry:

Example: Find image of the following region under $f(z) = z^4$

$$A = \{re^{i\theta} : r > 2 \text{ and } \frac{\pi}{4} < \theta < \frac{\pi}{3}\}$$

Solution: Using $z = re^{i\theta}$ in $w = f(z) = z^4$, we get $w = r^4 e^{i4\theta}$

if we use polar coordinates $w = \rho e^{i\varphi}$ *in w-plane then*

$\rho = r^4 > 16 \implies \rho > 16$

The image of $\frac{\pi}{4} < \theta < \frac{\pi}{3}$ *by applying* $f(z)=z^4$

becomes $\frac{4\pi}{4} < 4\theta < \frac{4\pi}{3} \implies \pi < \varphi < \frac{4\pi}{3}$

Example: Find the image of the sector $r > 0$, $-\pi < \theta < \frac{2\pi}{3}$ under the mapping $z^{\frac{1}{4}}$.

Solution: Using $z = re^{i\theta}$ in $w = f(z) = z^{\frac{1}{4}}$, we get $w = r^{\frac{1}{4}} e^{i\frac{1}{4}\theta}$

if we use polar coordinates $w = \rho e^{i\varphi}$ *in w-plane then*

$$r > 0$$

$$\Rightarrow r^{\frac{1}{4}} > 0$$

$$\Rightarrow \rho > 0$$

and the image of:

$$-\pi < \theta < \frac{2\pi}{3}$$

$$\Rightarrow -\frac{\pi}{4} < \frac{\theta}{4} < \frac{2\pi}{12}$$

$$\Rightarrow -\frac{\pi}{4} < \varphi < \frac{\pi}{6}$$

So, the image of mapping f(z) = $z^{\frac{1}{4}}$ *is the set:*

$$\{\, \rho e^{i\varphi} : \ \rho > 0 \ and \ \ -\frac{\pi}{4} < \varphi < \frac{\pi}{6}\}.$$

In this lecture, we have discussed a special kind of power functions which are nth power functions and nth root functions and we studied about their geometry. In other words we studied what will happen if we consider regions in the w-plane, what will happen after applying these mappings on those regions. So they have many interesting geometrical properties.

Topic: 025

Limits (Part I)

We are going to start our discussion on the first nontrivial concept about complex valued functions. This is the concept of

Limit. This is one of the very important concepts when we study functions. It tells us about the behavior of function at a point. First we see its concept in real valued functions.

Consider a real valued function $y = f(x)$ of real variable x. we can calculate its graph in R^2.

We say that the limit $\lim_{x \to x_0} f(x) = L$ if the functional values are getting closer and closer to this number L when we approaches x_0.

$$\lim_{x \to x_0} f(x) = L$$

Means that values $f(x)$ of the function f can be made **arbitrary close** to the real number L if values of x are choosen **sufficiently close** to, but not equal to, the real number x_0.

So, the concept of closeness, mathematically, is related to distance.

On the real line, the points x and x_0 are close if $|x - x_0|$ is a small positive number.

Similarly the points $f(x)$ and L are close if $|f(x) - L|$ is a small positive number.

Precisely:

"$f(x)$ can be made arbitrary close to the L" for any $\varepsilon > 0$, x can be choosen in that way so that

$$|f(x) - L| < \varepsilon$$

*"*sufficiently close to, but not equal to x_0*"*

There exist $\delta > 0$ such that if x is with in distance δ of x_0 and $x \neq x_0$, then

$$|f(x) - L| < \varepsilon$$

For every $\varepsilon > 0$ there exist a $\delta > 0$ such that if:

$$0 < |x - x_0| < \delta$$

Then, $$|f(x) - L| < \varepsilon$$

The complex limit is same than that of real limit except the notion "close"in the complex plane.

Distance between z_1 and z_2 is written as $|z_1 - z_2|$ and is a real number.

We can define limit as:

The limit of f as $z \to z_0$ exists and is equal to L if:

"For every $\varepsilon > 0$ there exist a $\delta > 0$ such that $|f(x) - L| < \varepsilon$

whenever $0 < |x - x_0| < \delta$"

This fact is denoted as: $\lim_{z \to z_0} f(z) = L$

There are many ways to approach z_0 in complex plane. We can choose any path in the neighborhood of z_0 to reach z_0.

If $\lim_{z \to z_0} f(z) = L$ we want that *f(z)* approaches the same complex number L along every possible curve through z_0.

Criteria for checking the non-existence of limit:

If f approaches two complex numbers $L_1 \neq L_2$ for two different curves or paths through z_0 , then $\lim_{z \to z_0} f(z)$ does not exist.

Example: Show that the limit $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

Solution: First we calculate the limit when z approaches to 0 along the real axis:

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x + i0}{x - i0} = \lim_{x \to 0} = 1 = 1$$

If z approaches to 0 along the imaginary axis:

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{y \to 0} \frac{0 + iy}{0 - iy} = \lim_{y \to 0} = -1 = -1$$

Hence, $\lim_{z \to 0} \frac{z}{\bar{z}}$ does not exist.

This criteria cannot be used to prove the existence of limits.

Because there are infinitely many ways to approach the number z_0. If you want to use this criteria you should be able to prove that along each and every path approaching z_0 , the functional

values are approaching to this number L. and if you can do that for those infinitely and uncountable many paths then the limit exists. Off course this is very impractical situation.

So, how do we check the limit exists , we have the following definition:

To show existence of limit

$$\lim_{z \to z_0} f(z) = L$$

We need to show the following:

"For every $\varepsilon > 0$ there exist a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$"

This is also known as ***$\varepsilon - \delta$ definition.***

Because it is very customary to choose this distance to be ε which is the distance of functional value and the distance on the domain or in z-plane is taken to be δ. So, that's why this definition is also known as $\varepsilon - \delta$ definition.

Example: Show that

$$\lim_{z \to z_0} \bar{z} = \bar{z}_0$$

Solution: Using definition ,take $\varepsilon > 0$, we need to find $\delta > 0$.

$\varepsilon > 0$ we need to prove $|f(z) - \bar{z}_0| < \varepsilon$

$\Leftrightarrow |\bar{z} - \bar{z}_0| < \varepsilon \quad \Leftrightarrow |\overline{z - z_o}| < \varepsilon \Leftrightarrow |z - z_o| < \varepsilon$

If $\delta = \varepsilon \implies |z - z_o| < \delta = \varepsilon \implies |f(z) - \bar{z}_0| < \varepsilon$

Hence, $|f(z) - \bar{z}_0| < \varepsilon$ *whenever* $|z - z_o| < \delta$.

Topic: 026

Limits (Part II)

So far we have learnt why we need concept of limits to understand the behavior of complex valued functions and we are also using simple observation we also agreed on a precise mathematical definition of limit of a complex valued function.

The limit of f as $z \to z_0$ *exists and is equal to L if:*

"For every $\varepsilon > 0$ *there exist a* $\delta > 0$ *such that* $|f(z) - L| < \varepsilon$

whenever $0 < |z - z_0| < \delta$*"*

This fact is denoted as: $\lim_{z \to z_0} f(z) = L$

The definition of limits of complex valued function does not provide a convenient way to calculate limits.

Our next aim is to provide a practical method for computing complex limits. We want to use the connection between complex limits and real limits to simplify computation of complex limits.

A complex function is of the form:

$f(z) = f(x+iy) = u(x,y)+iv(x,y)$

we find a relation between the complex limit of f(z) and the real limits of the multivariable real function u(x,y) and v(x,y).

Main Motivation

Theorem:

Let f(z) = u(x,y) + iv(x,y) be a complex function that is defined in some neighborhood of z_0 except perhaps at $z_0=x_0+iy_0$. Then

$$\lim_{z\to z_0} f(z)=u_0+iv_0$$

Iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

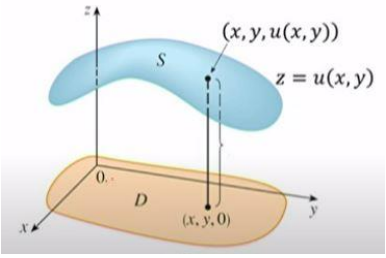
Limit of real valued functions of two variables

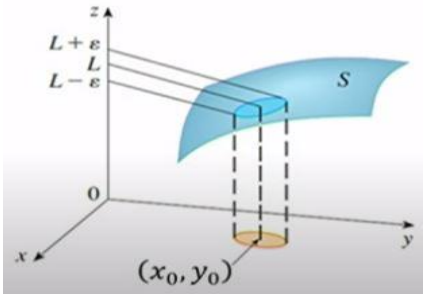
What does the following mean:

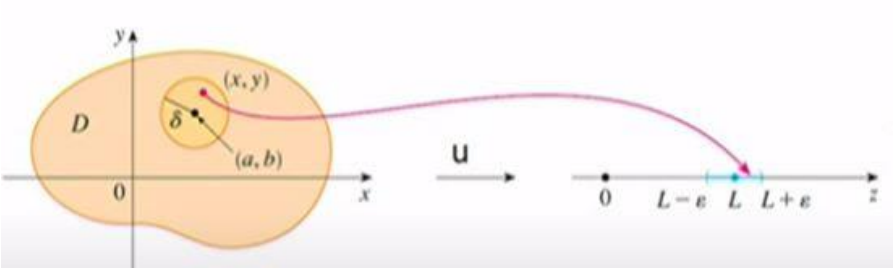
$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

The value of $u(x,y)$ can be made to get as close as we want to the value u_0 by taking (x,y) to be sufficiently close to (x_0,y_0).

To understand it geometrically, we know that the function $u(x,y)$ can be associated to a surface in $3D$.







What does the following mean:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

$|u(x,y) - u_0|$ can be made arbitrarily small by making distance between (x,y) and (x_0,y_0) small.

Here the point (x,y) is in the xy plane and the distance between (x,y) and (x_0,y_0) can be calculated using distance formula $\sqrt{(x-x_0)^2+(y-y_0)^2}$.

Definition:

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$$

means the following:

for each $\varepsilon > 0$, there exist $\delta > 0$ such that $|u(x,y) - u_0| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$.

Example: If $u(x,y) = \frac{2x^3}{(x^2+y^2)}$ then show that

$$\lim_{(x,y)\to(0,0)} u(x,y) = 0.$$

Solution: We will use polar coordinates:

$x = rcos\theta \quad , \quad y = rsin\theta$

$$u(rcos\theta, rsin\theta) = \frac{2r^3cos^3\theta}{(r^2cos^2\theta + r^2sin^2\theta)} = \frac{2r^3cos^3\theta}{r^2(cos^2\theta + sin^2\theta)}$$

$$= 2rcos^3\theta$$

We want to check $|u(x, y) - 0|$.

In polar coordinates:

$|2rcos^3\theta| = 2r|cos^3\theta| \leq 2r \qquad \because \ |cos^3\theta| \leq 1$

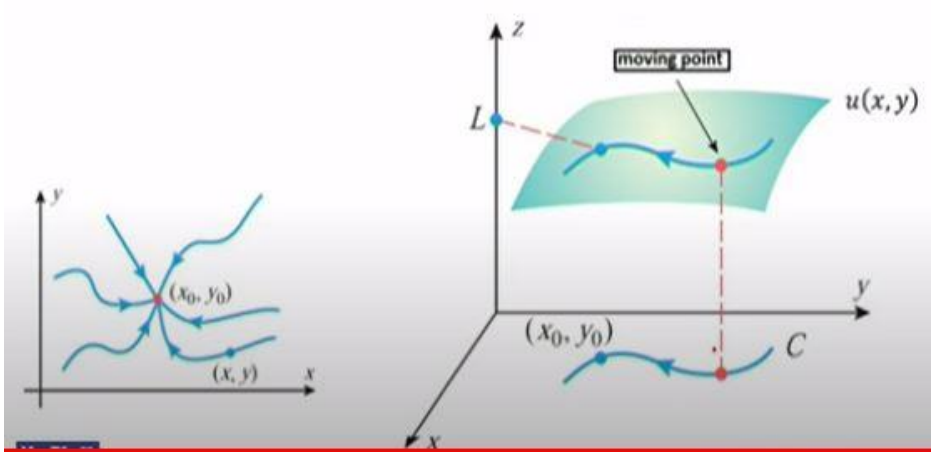
We have to show that $2r < \varepsilon \Longrightarrow r < \frac{\varepsilon}{2}$.

So, we have to find $\delta > 0$ such that $|u(x, y) - 0| < 2r < \varepsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$

where $\sqrt{(x-0)^2 + (y-0)^2} = r$

We choose $\delta = \frac{\varepsilon}{2} \Longrightarrow r < \frac{\varepsilon}{2} \Longrightarrow 2r < \varepsilon \Longrightarrow |u(x, y) - 0| < \varepsilon$

The value u_0 of the limit must not depend on how (x, y) approaches (x_0, y_0). So $u(x, y)$ must approach the value u_0 when (x, y) approaches (x_0, y_0) along any curve that ends at the point (x_0, y_0).



If we can find two curves L_1 and L_2 that end at $(x_0\ , y_0)$ along which $u(x,y)$ approaches two distinct values u_1 and u_2 then $u(x,y)$ does not have a limit as (x,y) approaches $(x_0\ , y_0)$.

Example: Show that the function $u(x,y) = \frac{xy}{x^2+y^2}$ does not have a limit as (x,y) approaches (0,0).

Solution:

Along y = x:

$$\lim_{(x,y)\to(0,0)} \frac{x(x)}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

Along y = -x:

$$\lim_{(x,y)\to(0,0)} \frac{x(-x)}{x^2+x^2} = \lim_{(x,y)\to(0,0)} \frac{-x^2}{2x^2} = -\frac{1}{2}$$

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Limits (Part III)