MTH 501

Lec # 19

Cramer's Rule, Volume & Linear Transformation

Theorem 1 (Crammer's Rule)

Let A be an invertible $n \times n$ matrix. For any b in R^n the unique sol. x of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A} \qquad i = 1, 2, \ldots, n$$

NOTE matric level $x_1 = \frac{|A_{x_1}|}{|A|}$

Example

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$$

$$A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

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Example Use Cramer's Rule to Solve the System

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8$$

Sol

write system in matrix form $Ax = b$

$$\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ & } b = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ -5 & 4 \end{vmatrix} = 12 - 10 = 2$$

Crammer Rule

$$x_1 = \frac{|A_1(b)|}{|A|}, \quad x_2 = \frac{|A_2(b)|}{|A|}$$

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$$

$$|A_1(b)| = \begin{vmatrix} 6 & -2 \\ 8 & 4 \end{vmatrix} = 24 + 16 = 40$$

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$$A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$$|A_2(b)| = \begin{vmatrix} 3 & 6 \\ -5 & 8 \end{vmatrix} \Rightarrow 24 + 30 = 54$$

$$x_1 = \frac{|A_1(b)|}{|A|} = \frac{40}{2} = 20$$

$$x_2 = \frac{|A_2(b)|}{|A|} = \frac{54}{2} = 27$$

Example:- Consider the following system in which s is an unspacified Parameter Determin the value of s using Cramer Rule For which system has unique sol.

$$3s x_1 - 2x_2 = 4$$
$$-6x_1 + 5x_2 = 1$$

Sol

$$\begin{bmatrix} 3s & -2 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$A \quad x \quad B$

$$|A| = 3s^2 - 12 = 3(s^2 - 4) \Rightarrow 3(s-2)(s+2)$$

Since the system has a unique Sol

$\det A \neq 0$

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$3(s+2)(s-2) \neq 0$

$s^2 - 4 \neq 0$

$s \neq \pm 2$

For such an s the sol is (x_1, x_2)

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{4s+2}{3(s+2)(s-2)}, \quad s \neq \pm 2$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}, \quad s \neq \pm 2$$

Formula for A^{-1} :-

$$A^{-1} = \frac{1}{|A|} adj(A)$$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$|A| = 10 - 3 = 7$

$$Adj\ A = \begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}$$

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Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 5 \\ 6 & 4 & 3 \end{bmatrix}$

Adj A = ?

cofactor formula

$C_{ij} = (-1)^{i+j} M_{ij}$ → Minor

Minor

This is denoted by M

Let a 2×2 matrix and find ~~determinant~~ Adjoint by using above formula

Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$

cofactor of this matrix $\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

$C_{11} = (-1)^{1+1} M_{11}$

$C_{11} = M_{11}$

$M_{11} = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$

$M_{11} = |5| = 5$

$C_{11} = 5$

$C_{21} = (-1)^{1+2} M_{21} \Rightarrow -M_{21}$

$C_{21} = -3$

$C_{12} = (-1)^{1+2} M_{12} = (-1)(-1)$

$C_{12} = 1$

$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^t$

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$$C_{22} = (-1)^{2+2} M_{22}$$

$$= 1 M_{22} \Rightarrow 2$$

$$C_{22} = 2$$

$$\text{Adj } A_2 \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^t = \begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}^t$$

$$(\text{Adj } A)^t = \begin{bmatrix} 5 & -3 \\ 1 & 2 \end{bmatrix}$$

Example Find inverse $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$|A| = 2\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= 2(2-4) - 1(-2-1) + 3(4+1)$$

$$|A| = -4+3+15 = 14$$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^t$$

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$$C_{11} = (-1)^{1+1} M_{11}$$

$$C_{11} = M_{11}$$

$$M_{11} = \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} \Rightarrow 2 - 4 = -2$$

$$C_{11} = -2$$

$$C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3$$

$$C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14$$

$$C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7$$

$$C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4$$

$$C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$$

$$\text{Adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

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Determinants as Area or Volume:-

Theorem:-

If A is a 2×2 matrix the area of the Parallelogram determined by the columns of A is $|\det A|$ if A is a 3×3 the volume of the Parallelopiped determined by the columns of A is $|\det A|$.

Ex Calculate the area of the Parallelogram determined by the Points $(-2,-2), (0,3), (4,-1)$ & $(6,4)$

Sol:

Let $A(-2,-2), B(0,3), C(4,-1), D(6,4)$

Fixing one Point say $A(-2,-2)$ & find the adjecent lengths of Parallelogram which are given by the column vectors as follows

$$AB = \begin{bmatrix} 0-(-2) \\ 3-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$AC = \begin{bmatrix} 4-(-2) \\ -1-(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

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So the area of Parallelogram ABCD determined by above column vectors

$$\left|\det\begin{bmatrix}2 & 6\\ 5 & 1\end{bmatrix}\right| = |2-30| = |-28| = 28$$

Linear Transformation

Theorem

Let $T: R^2 \to R^2$ be the linear trans. determined by a 2×2 matrix A. if S is a parallelogram in R^2, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{Area of } S\}$$

If T is determined by a 3×3 matrix A and if S is a Parallelepiped in R^3 then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}$$

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Example

Let S be a Parallelogram determined by the vectors $b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & $b_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, Compute the area of image of S under the mapping $x \to Ax$

Sol

~~$|\det A| = \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$~~

$$|\det A| = \left| \det \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \right| = 2$$

$$S = \left| \det \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix} \right| = |1 - 15| = 14$$

The area of image of S under mapping $x \to Ax$

$$= |\det A| \cdot [\text{Area of } S] = 2 \cdot 14 = 28$$

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Lec # 20

Vector Spaces & SubSpaces

Defination:-

Let V be an arbitrary nonempty set of objects on which two operations are defined addition & Multiplication by scalars (Numbers). If the following axioms are satisfied by all objects u, v, w in V and all scalers k and l then we call V a vector space

Axioms of Vector Space

(1) **Closure Property:**

For any two vectors u & $v \in V$ implies $u+v \in V$.

(2) **Commutative Pro.**

For any two vectors u & $v \in V$ implies $u+v = v+u$

(3) **Associative Pro.**

For any Three vectors u, v & $w \in V$

$$u+(v+w) = (u+v)+w$$

(4) **Additive identity:**

For any vector $u \in V$ there exist a zero vector 0 such that $0+u = u+0 = u$

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(5) **Additive inverse :-**

For each vector $u \in V$ there exist a vector $-u$ in V such that

$$-u+u=0=u+(-u)$$

(6) **Scaler multiplication :-**

For any scalar k & a vector $u \in V$ implies $ku \in V$

(7) **Distributive Law :-**

For any scalar k if u & $v \in V$ then

$$k(u+v)=ku+kv$$

(8)

For scalars m, n and for any vector $u \in V$, $(m+n)u=mu+nu$

(9)

For scalars m, n and for any vector $u \in V$

$$m(nu)=(mn)u=n(mu)$$

(10)

For any vector $u \in V$ $1u=u$ where **1** is a multiplicative identity of real numbers.

NOTE from example 1

n Tuple R^n

اس کا مطلب ہے کہ ہم نے space R^n سے جو بھی ویکٹر
لینے ہیں ان کا کمپوننٹ n ہوگا components

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NOTE

The three most important special cases of R^n are R(Real num), R^2 (The vectors in the Plan) & R^3 (The vectors in 3-space)

Examples

Theorem

Let V be a vector space u a vector in V and K is a scalar then

(i) $0u = 0$ (ii) $k0 = 0$

(iii) $(-1)u = -u$ (iv) if $ku = 0$ then $k = 0$ or $u = 0$

Definition

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

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Theorem:-

If W is a set of one or more vectors from a vector space V then W is subspace of V if and only if the following conditions hold.

(a) if u & v are vectors in W, then u+v is ~~a~~ in W.

(b) If k is any scalar and u is any vector in W then ku is in W.

NOTE:-

Every vector space has at least two subspaces, itself and the subspace {0} consisting only of the zero vector. Thus the subspace {0} is called the zero subspace

NOTE:-

Let V is a vector space then every subset of V is not necessary a subspace of V.

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