Assignment: MTH 631

Fall- 2024

Student ID:

Question No 1:

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Solution:

Checking Uniform Convergence

To check the uniform convergence of the series, we will use the Weierstrass M-test. According to this test, if there exists a sequence of constants (M_n) such that $(|f_n(x)| \leq M_n)$ for all (x) in the domain and the series $\left(\sum M_n\right)$ converges, then the series $\left(\sum f_n(x)\right)$ converges uniformly.

Part (a) $\left(\sum \frac{1}{x^2+n^2}\right)$

1. Analyze $\left(f_n(x)=\frac{1}{x^2+n^2}\right)$:

Observe that for all $(x \in \square)$, $(x^2 \geq 0)$ Therefore, $(x^2+n^2 \geq n^2)$

Thus, $\left(\frac{1}{x^2+n^2} \leq \frac{1}{n^2}\right)$

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2. Choose $\left(M_n=\frac{1}{n^2}\right)$

Since $|f_n(x)| \leq \frac{1}{n^2}$ for all (x), we can apply the Weierstrass M-test with $\left(M_n =\right.$

3. Check convergence of $\left(\sum M_n=\sum \frac{1}{n^2}\right)$

The series $\left(\sum \frac{1}{n^2}\right)$ is a convergent p-series (with (p = 2 > 1)).

Therefore, $\left(\sum f_n(x)=\sum \frac{1}{x^2+n^2}\right)$ converges uniformly by the We

Part (b) $\left(\sum \frac{\cos(nx)}{n^3}\right)$

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1. Analyze $\left(f_n(x) = \frac{\cos(nx)}{n^3} \right)$

The function $\cos(nx)$ is bounded by 1 for all (n) and (x), so $(|\cos(nx)| \leq 1)$

Therefore, $\left(\left| \frac{\cos(nx)}{n^3} \right| \leq \frac{1}{n^3} \right)$

2. Choose $\left(M_n = \frac{1}{n^3} \right)$

Since $\left(|f_n(x)| \leq \frac{1}{n^3} \right)$ for all (x), we apply the Weierstrass M-test with $(M_n = \frac{1}{n^3})$

3. Check convergence of $\left(\sum M_n = \sum \frac{1}{n^3} \right)$

The series $\left(\sum \frac{1}{n^3} \right)$ is a convergent p-series (with (p = 3 > 1)).

Therefore, $\left(\sum f_n(x) = \sum \frac{\cos(nx)}{n^3} \right)$ converges uniformly by the Weierstrass M-test.

Part (c) $\left(\sum \frac{\sin(nx)}{n^3 + x^2} \right)$

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1. Analyze $\left(f_n(x) = \frac{\sin(nx)}{n^3 + x^2} \right)$

The function $(sin(nx))$ is bounded by 1 for all (n) and (x), so $\left(\sin(nx)| \leq 1\right)$

Thus, $\left(\left| \frac{\sin(nx)}{n^3 + x^2} \right| \leq \frac{1}{n^3} \right)$ *since* $(n^3 + x^2 \geq n^3)$.

2. Choose $\left(M_n = \frac{1}{n^3} \right)$

Since $|f_n(x)| \leq \frac{1}{n^3}$ for all (x), we apply the Weierstrass M-test with $\left(M_n = \frac{1}{n^3} \right)$

3. Check convergence of $\left(\sum M_n = \sum \frac{1}{n^3} \right)$

The series $\left(\sum \frac{1}{n^3}\right)$ is convergent (as seen before).

Therefore, $\left(\sum f_n(x) = \sum \frac{\sin(nx)}{n^3 + x^2}\right)$ converges uniformly by the Weierstrass M-test.

Part (d) $\sum \frac{e^{-nx}}{n^2}$

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1. Analyze $(f_n(x) = \frac{e^{-nx}}{n^2})$

Note that $(e^{-nx} \leq 1)$ *for all* $(x \geq 0)$

Thus, $\left|\frac{e^{-nx}}{n^2}\right| \leq \frac{1}{n^2}$

2. Choose $M_n = \frac{1}{n^2}$

Since $|f_n(x)| \leq \frac{1}{n^2}$ for all (x), we apply the Weierstrass M-test with $\left(M_n = \frac{1}{n^2}\right)$

3. Check convergence of $\left(\sum M_n = \sum \frac{1}{n^2}\right)$

The series $\sum \frac{1}{n^2}$ is convergent.

Therefore, $\left(\sum f_n(x) = \sum \frac{e^{-nx}}{n^2}\right)$ converges uniformly by the Weierstrass M-test.

Question No 2: Evaluate the following series: Marks 20

(a)

$$\sum_{n=1}^{\infty} \int_0^1 x\, e^{-nx} \sin nx dx$$

(b)

$$\sum_{n=1}^{\infty} \int_0^1 x^3\, e^{-nx^2}\, dx$$

Solution:

part (a),

we can use integration by parts to evaluate the integral. Let $u = x$ and $dv = e^{-nx} \sin nx dx$. Then $du = dx$ and $v = -1/(2n)(e^{(-nx)}cosnx + e^{(-nx)}sinnx)$

Applying integration by parts, we get:

$$\int_0^1 xe^{-nx}sinxdx = \frac{-1}{2n}\left[xe^{-nx}(cosnx + sinx)\right]_0^1 + \frac{1}{2n}\int_0^1 xe^{-nx}(cosnx + sinnx)dx$$

Evaluating the first term and using integration by parts again on the second term, we get:

$$\int_0^1 xe^{-nx}sinxdx = \frac{-1}{2n}\left[e^{-nx}(cosn + sin)\right] + \frac{1}{4n^2}\left[xe^{-nx}(cosnx - sinx)\right]_0^1$$

Simplifying the expression, we get:

$$\int_0^1 xe^{-nx}sinxdx = \frac{-1}{2n}(e^{(-n)}(cosn + sinn)) + \frac{1}{4n^2}\left[e^{-n}(cosn - sin) + 1\right]$$

Now, we can evaluate the series:

$$\sum_{n=1}^{\infty}\int_0^1 xe^{-nx}sinxdx = \sum_{n=1}^{\infty}\left[\frac{-1}{2n}(e^{(-n)}(cosn + sinn)\frac{1}{4n^2}\left[e^{-n}(cosn - sin) + 1\right]\right]$$

Since e^{-n} goes to zero as n goes to infinity, the terms involving e^{-n} will vanish. Therefore, we are left with:

$$\sum_{n=1}^{\infty}\int_0^1 xe^{-nx}sinxdx = \sum_{n=1}^{\infty}\frac{1}{4n^2}$$

$$\sum_{n=1}^{\infty}\int_0^1 xe^{-nx}sinxdx = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

The series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a well-known convergent series with a value of $\frac{\pi^2}{6}$. T

$$\sum_{n=1}^{\infty}\int_0^1 xe^{-nx}sinxdx = \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{24}$$

part (b),

we can use the substitution $u = x^2$. Then $du = 2xd$

$$\int_0^1 x^3e^{-nx^2}dx = \frac{1}{2}\int_0^1 ue^{-nu}du$$

Using integration by parts with $u = u$ and $dv = e$

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$$\frac{1}{2}\int_0^1 ue^{-nu}du=\frac{-1}{2n}\left[ue^{-nu}\right]_0^1+\frac{1}{2n}\int_0^1 e^{-nu}du$$

Evaluating the integral and simplifying, we get:

$$\int_0^1 x^3e^{-nx^2}dx=\frac{-1}{2n}e^{-n}+\frac{1}{2n^2}\left(1-e^{-n}\right)$$

Now, we can evaluate the series:

$$\sum_{n=1}^{\infty}\int_0^1 x^3e^{-nx^2}dx=\sum_{n=1}^{\infty}\int_0^1\left[\frac{-1}{2n}e^{-n}+\frac{1}{2n^2}\left(1-e^{-n}\right)\right]$$

As before, the terms involving e^{-n} vanish as n goes to infinity. Therefore, we are left with:

$$\sum_{n=1}^{\infty}\int_0^1 x^3e^{-nx^2}dx=\sum_{n=1}^{\infty}\frac{1}{2n^2}$$

$$\sum_{n=1}^{\infty}\int_0^1 x^3e^{-nx^2}dx=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Using the values of the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ we get

$$\sum_{n=1}^{\infty}\int_0^1 x^3e^{-nx^2}dx=\frac{1}{2}\frac{\pi^2}{6}=\frac{\pi^2}{12}$$

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