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1. In the form of interval, real line can be denoted by $(-\infty, +\infty)$
2. To each point on a co-ordinate line or real line, there is associated
   i. A real number
   ii. An integer
   iii. A natural number
   iv. A rational number.
3. How many real numbers lie in the interval [1, 5]
   i. 5
   ii. 3
   iii. 10
   iv. Infinite
4. All the three axes are positive in
   i. Third octant
   ii. First octant
   iii. Second octant
   iv. Eigth octant
5. Let w = f(x, y, z) such that $w = \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} + \frac{(z-4)^2}{c^2}$ where a, b and c are real numbers, not equal to zero. Is the function defined at origin? If yes, what is its value and if not, give the reason.

Solution:

Origin means (x, y, z) = (0, 0, 0). So putting these values in given equation

$$w = \frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} + \frac{(z-4)^2}{c^2}$$
$$= \frac{(0-1)^2}{a^2} + \frac{(0-2)^2}{b^2} + \frac{(0-4)^2}{c^2}$$
$$= \frac{1}{a^2} + \frac{4}{b^2} + \frac{16}{c^2}$$

Thus the given function is defined at origin and its value is

$\frac{1}{a^2}+\frac{4}{b^2}+\frac{16}{c^2}$.

6. Find the domain of the following functions.

i. $f(x\ ,\ y)=\sqrt{4-x^2-2y^2}$

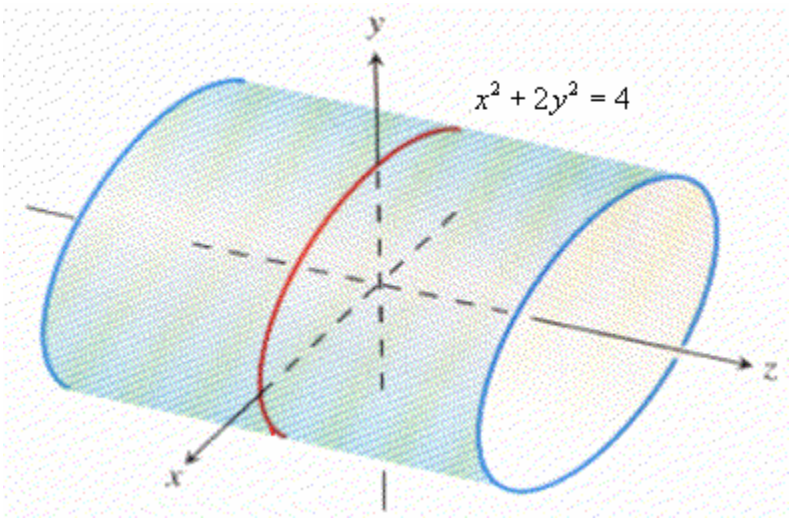
Domain of f consists of region in xy-plane where $4-x^2-2y^2\geq 0$

$4\geq x^2+2y^2$

$x^2+2y^2\leq 4$

Now $x^2+2y^2=4$ is an equation of surface of elliptic cylinder in three dimensional space which has infinite length along z-axis.

Thus domain of f is inside and on the surface of elliptic cylinder.



ii. $f(x,\ y)=\frac{1}{(x^2+y^2)^2}$

Domain of f is entire xy-plane except the origin because at origin function is not defined.

iii. $f(x,\ y)=x^2+y^3-xy+\cos(xy)$

Domain of f is entire xy-plane.

7. Which of the following statement is true? Correct the false statement.

i. A straight line is a special kind of curve.

TRUE

Generally, a curve is considered to be any one-dimensional collection of points. For your convenience, think a curve as a thread which we use in our daily life. The straight line is a special kind of curve.

ii. Plane is an example of surface.

TRUE

Surface is a two-dimensional geometric figure (a collection of points) in three-dimensional space. The simplest example is a plane—a flat surface. Some other common surfaces are spheres, cylinders, and cones.

iii. The intersection of two surfaces in three dimensional space gives a surface.

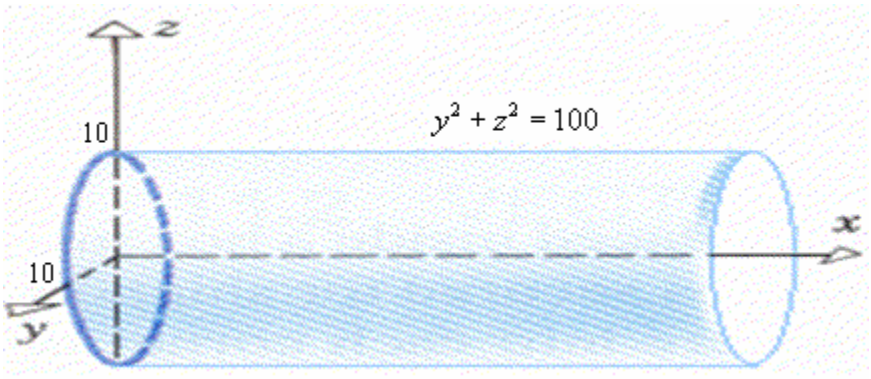
FALSE

Correct statement is "Intersection of two surfaces in three dimensional space gives a **curve**".

8. Identify the curve or surface in three dimensional space for each of the following equation.

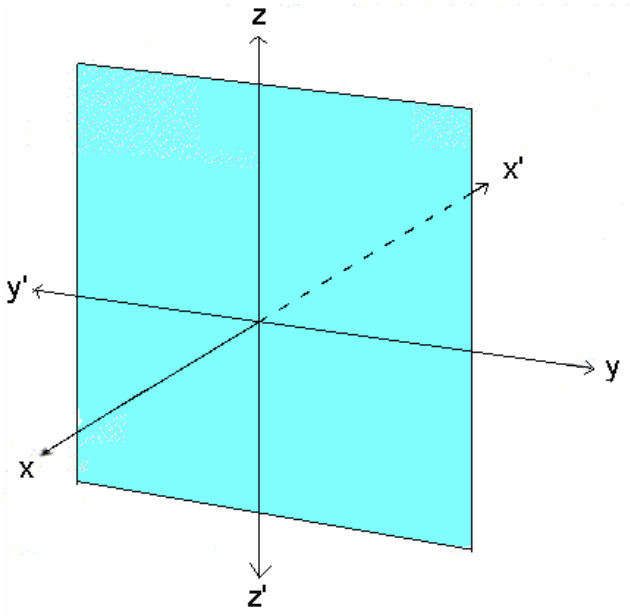
i. $y^2 + z^2 = 100$

This is the equation of surface of circular cylinder which has infinite length along x-axis.



ii. x = 0

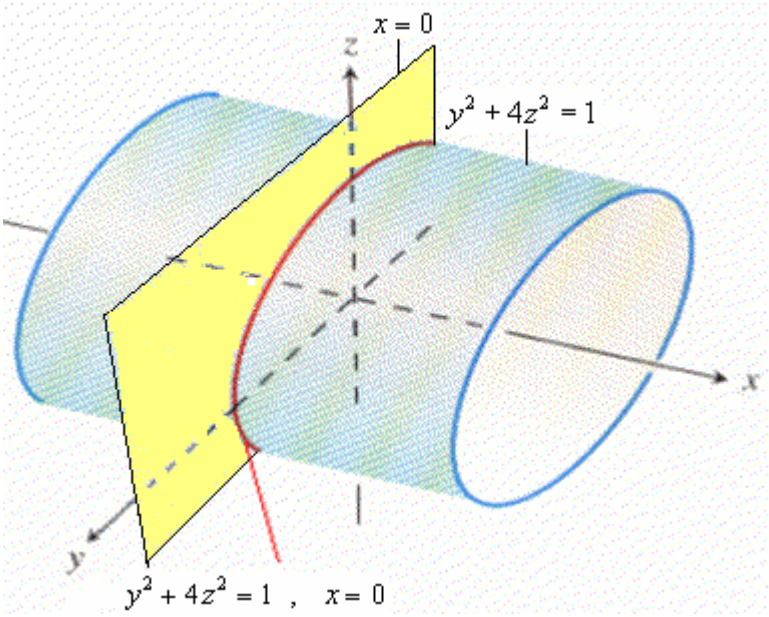
This is the equation of yz-plane.



iii. $y^2 + 4z^2 = 1$, $x = 0$

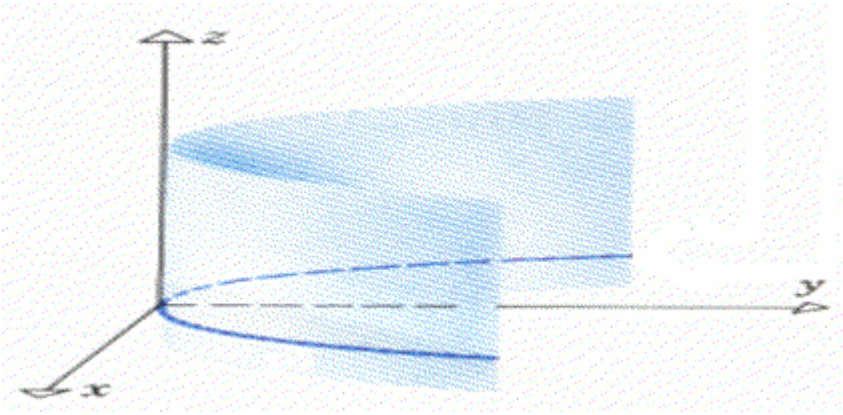
Since intersection of two surfaces is a curve in three dimensional space. So a curve in three dimensional space is represented by two equations representing the intersecting surfaces.

$y^2 + 4z^2 = 1$ is the equation of surface of an elliptic cylinder which has infinite length along x-axis and $x = 0$ is the equation of yz-plane. The intersection of these two surfaces is an elliptic curve. Thus these two equations shows elliptic curve in yz-plane.



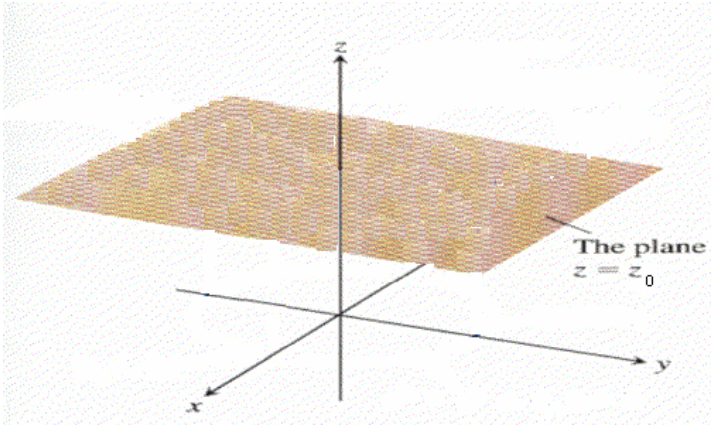
iv. $y = x^2$

This is the equation of surface of half cylinder in three dimensional space.



v. $z = z_0$, if (r, θ, z) represent spherical co-ordinates, and z_0 is any real number.

$z = z_0$ is a plane at point z_0 on z-axis. It is parallel to xy-plane.



9. Following are the co-ordinates of point in different co-ordinate systems. Complete the table.

Rectangular Co-ordinate (x, y, z)	**Spherical Co-ordinate** (ρ, θ, ϕ)	**Cylindrical Co-ordinate** (r, θ, z)
	$\left(\sqrt{3}, \frac{\pi}{3}, \frac{\pi}{2}\right)$	$r = \rho \sin\phi$ $r = \sqrt{3}\sin\frac{\pi}{2}$ $r = \sqrt{3}(1)$ $r = \sqrt{3}$ $\theta = \theta$ $\theta = \frac{\pi}{3}$ $z = \rho\cos\phi$ $z = \sqrt{3}\cos\frac{\pi}{2}$ $z = \sqrt{3}(0)$ $z = 0$ $\Rightarrow \left(\sqrt{3}, \frac{\pi}{3}, 0\right)$

$x = \rho \sin\phi \cos\theta$ $x = \sqrt{3} \sin\frac{\pi}{2} \cos\frac{\pi}{3}$ $x = \sqrt{3}(1)\left(\frac{1}{2}\right)$ $x = \frac{\sqrt{3}}{2}$ $y = \rho \sin\phi \sin\theta$ $y = \sqrt{3} \sin\frac{\pi}{2} \sin\frac{\pi}{3}$ $y = \sqrt{3}(1)\left(\frac{\sqrt{3}}{2}\right)$ $y = \frac{3}{2}$ $z = \rho \cos\phi$ $z = \sqrt{3} \cos\frac{\pi}{2}$ $z = \sqrt{3}(0)$ $z = 0$ $\Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{3}{2}, 0\right)$		
$x = r\cos\theta$ $x = (0)\cos\pi$ $x = (0)(-1)$ $x = 0$ $y = r\sin\theta$ $y = (0)\sin\pi$ $y = (0)(0)$ $y = 0$ $z = z$ $z = -\sqrt{2}$ $\Rightarrow \left(0, 0, -\sqrt{2}\right)$	$\rho = \sqrt{r^2 + z^2}$ $\rho = \sqrt{(0)^2 + (-\sqrt{2})^2}$ $\rho = \sqrt{2}$ $\theta = \theta$ $\theta = \pi$ $\tan\phi = \frac{r}{z}$ $\tan\phi = \frac{0}{-\sqrt{2}}$ $\tan\phi = 0$ $\phi = \tan^{-1} 0$ $\phi = 0$ $\Rightarrow \left(\sqrt{2}, \pi, 0\right)$	$\left(0,\ \pi,\ -\sqrt{2}\right)$

10. Show that whether the limit of the following function exist or not at the origin.

$$f(x, y) = \frac{x^2 y^2}{x^4 + 3y^4}.$$

Solution:

Let us recall **Rule for Non-Existence of a Limit**
If in

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

we get two or more different values as we approach (a, b) along different paths, then $\lim_{(x,y)\to(a,b)} f(x, y)$

does not exist.

The paths along which (a, b) is approached may be straight lines or plane curves .

Two of the more common paths to check are the x and y-axis so let's try those.

If we approach $(0, 0)$ along the x-axis, their $y = 0$. This means that along the x-axis, we will plug-in $y = 0$ into the function and then take the limit as x approaches zero.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &= \lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^4 + 3y^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{x^2 (0)^2}{x^4 + 3(0)^4} \\ &= \lim_{(x,y)\to(0,0)} 0 \\ &= 0\end{aligned}$$

So, along the x-axis the function will approach zero as we move towards the origin.
Now, let's try the y-axis. Along this axis we have $x = 0$ and so the limit becomes,

$$\lim_{(x,y)\to(0,0)} f(x,\ y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(0)^2y^2}{(0)^4+3y^4} = \lim_{(x,y)\to(0,0)} 0 = 0$$

The same limit along two paths

Let's take another path. Move towards the origin along the path y = x.

To do this we will replace all the *y*'s with *x*'s and then let *x* approach zero.

$$\lim_{(x,y)\to(0,0)} f(x,\ y) \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4\quad 3y^4}$$

$$\lim_{(x,x)\to(0,0)} \frac{\left(x^2\right)\left(x^2\right)}{x^4\quad 3x^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4}{4x^4}$$

$$\lim_{(x,y)\to(0,0)} \frac{1}{4}$$

$$\frac{1}{4}$$

$f(r,\ s) = \ln\ (r^2+s^2-1)$

Solution:

The function is continuous in the region where $r^2+s^2-1>0$ because the natural logarithm (ln) is defined for non-zero positive real numbers only.

This implies $r^2+s^2>1$

Since $r^2+s^2=1$ is the equation of a circle, center at origin and unit radius.

Thus f is continuous in the region outside a unit circle center at origin.

12. Let **r** is a function and **s**, **t**, **u**, **v** are variables. Partial derivative is a derivative of a function **r** with respect to **u**

i. If **r** is a function of **s** and **s** is a function of **u**. FALSE

Say $r = f(s)$, $s = g(u)$ *then we have*

$$\frac{dr}{du} = \frac{dr}{ds}\frac{ds}{du}$$

ii. If **r** is a function of **s** and **s** is a function of two variables **u** and **v**. TRUE

Say $r \quad f(s)$, $s \quad g(u, v)$ *then we have*

$$\frac{r}{u} \quad \frac{dr}{ds}\frac{s}{u}$$

$$= \qquad = \qquad =$$

$$= \frac{\partial}{\partial} + \frac{\partial}{\partial}$$

iv. If **r** is a function of **s** and **t** and each of **s** and **t** is a function of both **u** and **v**. TRUE

Say $r = f(s, t)$, $s = g(u, v)$, $t = h(u, v)$ *then we have*

$$\frac{\partial r}{\partial u} = \frac{\partial r}{\partial s}\frac{\partial s}{\partial u} + \frac{\partial r}{\partial t}\frac{\partial t}{\partial u}$$

13. Let $f(x, y) = \dfrac{x^3 y - xy^3}{x^2 + y^2}$ and $f(0, 0) = 0$

Find $f_x(x, y)$ and $f_y(x, y)$.

Solution:

$$f(x, y) = \frac{x^3 y - xy^3}{x^2 + y^2}$$

$$f_x(x, y) = \frac{(3x^2 y - y^3)(x^2 + y^2) - (x^3 y - xy^3)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{3x^4 y + 3x^2 y^3 - x^2 y^3 - y^5 - 2x^4 y + 2x^2 y^3}{(x^2 + y^2)^2}$$

$$= \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{(x^3 - 3xy^2)(x^2 + y^2) - (x^3 y - xy^3)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x^5 + x^3 y^2 - 3x^3 y^2 - 3xy^4 - 2x^3 y^2 + 2xy^4}{(x^2 + y^2)^2}$$

$$= \frac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2}$$

14. Show whether the function $z(x, y) = \sin\left(y - \frac{x}{2}\right)$ satisfy Laplace's Equation or not?

Solution:

As we know for a function z(x, y) ,Laplace Equation is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

So for $z(x, y) = \sin\left(y - \frac{x}{2}\right)$

$$\frac{\partial z}{\partial x} = \cos\left(y - \frac{x}{2}\right) \quad \left(\frac{-1}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin\left(y - \frac{x}{2}\right) \quad \left(\frac{1}{4}\right)$$

$$\frac{\partial z}{\partial y} = \cos\left(y - \frac{x}{2}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = -\sin\left(y - \frac{x}{2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\left(\frac{1}{4}\right)\sin\left(y - \frac{x}{2}\right) - \sin\left(y - \frac{x}{2}\right)$$

$$= -\frac{5}{4}\sin\left(y - \frac{x}{2}\right) \neq 0$$

Since for given function z, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \neq 0$ so the given function does not satisfy Laplace Equation.

15. For a function $f(x, y) = y^2 x^4 e^x + 2$,

$$\frac{\partial^5 f}{\partial y^3 \partial x^2} = 0$$

Show the calculation steps.

Solution:

$f(x, y) = y^2 x^4 e^x + 2$

$\frac{\partial^5 f}{\partial y^3 \partial x^2}$ means differentiating f with respect to x, two times and then with respect to y three times. If we differentiate this function in this order it will be difficult because due to the presence of $x^4 e^x$, number of terms increase after each differentiation.

But if we first differentiate this function three times with respect to y and then two times with respect to x, this fifth derivative can be calculated in few steps.

As we know that this function and all of its partial derivatives are defined and continuous everywhere, so we use Euler's Theorem for mixed derivatives by which

$$\frac{\partial^5 f}{\partial y^3 \partial x^2} = \frac{\partial^5 f}{\partial x^2 \partial y^3}$$

So $f(x, y) = y^2 x^4 e^x + 2$

$$\frac{\partial f}{\partial y} = 2yx^4 e^x$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^4 e^x$$

$$\frac{\partial^3 f}{\partial y^3} = 0$$

$$\frac{\partial^4 f}{\partial x \partial y^3} = 0$$

$$\frac{\partial^5 f}{\partial x^2 \partial y^3} = 0$$

Thus

$$\frac{\partial^5 f}{\partial y^3 \partial x^2} = 0$$

16. Let $w = \frac{x}{y}$ *where* $x = e^t$, $y = \ln t$. Find the derivative of w with respect to t.

Solution:

Since w is a function of two variables x and y , and each x and y are function of one variable t. So by chain rule

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \quad \text{----------(1)}$$

Since $w = \frac{x}{y}$, $x = e^t$, $y = \ln t$ so

$$\frac{dx}{dt} = e^t \text{ and } \frac{dy}{dt} = \frac{1}{t}$$

$$\frac{\partial w}{\partial x} = \frac{1}{y} \text{ and } \frac{\partial w}{\partial y} = \frac{-x}{y^2}$$

Put values of x *and* y

$$\frac{\partial w}{\partial x} = \frac{1}{\ln t} \text{ and } \frac{\partial w}{\partial y} = \frac{-e^t}{(\ln t)^2}$$

Put values in equation (1)

$$\frac{dw}{dt} = \frac{1}{\ln t}\left(e^t\right) + \frac{-e^t}{(\ln t)^2}\left(\frac{1}{t}\right)$$

$$= \frac{te^t \ln t - e^t}{t\,(\ln t)^2}$$

$$= \frac{e^t(t\ln t - 1)}{t\,(\ln t)^2}$$

Observe that $\frac{dw}{dt}$ is a function of t only.

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