VECTORS AND CLASSICAL MECHANICS (MTH622)

Highlighted HandMade Notes (Mid-Term)

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Scalar point Function:-

If to each point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$, then ϕ is called Scalar point function in R.

Scalar Field:-

* The set of all values of scalar point Function ϕ in R, together forms a Scalar Field.

* A Scalar Field is a Function define on Space whose value at each point is a scalar Quantity.

vector point Function;

If to each point (x, y, z) of a region R in Space there exist a unique vector $\vec{A}(x, y, z)$, then $\vec{A}$ is called a vector point Function in R.

vector Field;

A function of a Space whose value at each point is a vector Quantity is called vector field.

$\vec{A} = \bar{A}(x, y, z) = A_1(x, y, z) + A_2(x, y, z) + A_3(x, y, z$

Module # 03

The operator Del and Gradient of function

* The Del operator;

∇ is defined by;

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} \hat{k}$$

The del operator is not a particular operator but when we applied it on a Scalar point function, or a vector point function; this may be known as the gradient, the divergence, and the Curl.

Gradient : $\text{grad}(f) = \nabla\phi$

Curl :- $\text{Curl}(\vec{A}) = \nabla \times \vec{A}$

Divergence :- $\text{div}(\vec{A}) = \nabla . \vec{A}$

* Gradient Function;

Let $\phi(x, y, z)$ be a scalar point Function defined on a specific region on R and also differentiable on the same domain, Then we can apply del operator on ϕ in order to obtain gradient. written as $\nabla\phi$;

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k.$$

$\nabla\phi$ is defined as vector field.

$\nabla\phi = 0$ iff ϕ is Constant.

Module No. 4

properties of the Gradient

(i) $\nabla(C\phi) = C\nabla\phi$

(ii) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$

(iii) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(iv) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}, \quad \psi \neq 0$

Proof: $\nabla(C\phi) = C\nabla\phi$

$$= \frac{\partial C\phi}{\partial x} i + \frac{\partial(C\phi)}{\partial y} j + \frac{\partial(C\phi)}{\partial z} k$$

$$= C\frac{\partial\phi}{\partial x} i + C\frac{\partial\phi}{\partial y} j + C\frac{\partial\phi}{\partial z} k$$

$$= C\left(\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k\right)$$

$$= C\nabla\phi \Rightarrow R.H.S$$

(ii) $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

$$\nabla(\phi+\psi) = \frac{\partial(\phi+\psi)}{\partial x}i + \frac{\partial(\phi+\psi)}{\partial y}j + \frac{\partial(\phi+\psi)}{\partial z}k$$

$$= \left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right) + \left(\frac{\partial\psi}{\partial x}i + \frac{\partial\psi}{\partial y}j + \frac{\partial\psi}{\partial z}k\right)$$

$$= \nabla\phi + \nabla\psi = \text{R.H.S.}$$

(iii) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

$$\nabla(\phi\psi) = \frac{\partial(\phi\psi)}{\partial x}i + \frac{\partial(\phi\psi)}{\partial y}j + \frac{\partial(\phi\psi)}{\partial z}k$$

$$= \left(\phi\frac{\partial\psi}{\partial x}i + \psi\frac{\partial\phi}{\partial x}\right)i + \left(\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right)j + \left(\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right)k$$

$$= \phi\left(\frac{\partial\psi}{\partial x}i + \frac{\partial\psi}{\partial y}j + \frac{\partial\psi}{\partial z}k\right) + \psi\left(\frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k\right)$$

$$= \phi\nabla\psi + \psi\nabla\phi = \text{R.H.S.}$$

(iv) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi + \phi\nabla\psi}{\psi^2}$

$$\nabla\left(\frac{\phi}{\psi}\right) = \nabla\left(\phi\frac{1}{\psi}\right) = \phi\nabla\left(\frac{1}{\psi}\right) + \psi\nabla\left(\frac{1}{\phi}\right)$$

$$= \phi\left(\frac{-1}{\psi^2}\right)\nabla\psi + \psi\nabla\left(\frac{1}{\phi}\right)$$

$$= \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2} = \text{R.H.S}$$

Module # 05

Directional Derivative

The procedure to determine the derivative in a specific direction, other than the Coordinate axes (x, y, z) is called directional derivative.

In Particular, if we assume $\hat{a}$ has the direction of the positive x-axis, the $\hat{a} = \hat{i}$ then equation (1) will become $\nabla\phi \cdot \hat{i}$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\right) \cdot \hat{i}$$

$$\nabla\phi \cdot \hat{i} = \frac{\partial \phi}{\partial x}$$

Similarly, $\nabla\phi \cdot \hat{j} = \frac{\partial \phi}{\partial y}$, $\nabla\phi \cdot \hat{k} = \frac{\partial \phi}{\partial z}$

Module # 06

Theorem related Directional Derivative

The Maximum value of the directional derivative of $\phi(x, y, z)$ is equal to the Magnitude of $\phi(x, y, z)$, and Magnitude of $\phi(x, y, z)$ is equal to the Magnitude of $\nabla\phi$.

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{a}$$

$$= |\nabla \phi| |\hat{a}| \cos\theta$$

$-1 \leq \cos\theta \leq 1$, $\frac{\partial \phi}{\partial s}$ is Maximum when $\cos\theta = 1$, or $\theta = 0^\circ$

Directiond derivative $\frac{\partial \phi}{\partial s}$ is Zero, when $\theta = 90^\circ$

Module # 07

(Example of the Directional Derivative.)

Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Solve:-

Directional Derivative $= \nabla \phi \cdot \hat{a}$

$$\nabla \phi = \nabla (x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$\nabla \phi$ at $(1, -2, -1)$

$$= (2(1)(-2)(-1) + 4(-1)^2)\hat{i} + ((1)^2(-1))\hat{j} + ((1)^2(-2) + 8(1)(-1))\hat{k}$$

$$= 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector;

$$\hat{a} = \frac{\vec{a}}{|a|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

(v) Now

$$\nabla\phi \cdot \hat{a} = (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right)$$

$$= \left(\frac{16}{3} + \frac{1}{3} + \frac{20}{3}\right) = \frac{37}{3}$$

Since this is positive, ϕ is increasing in this direction.

Module # 08, 9, 10

Examples done

Module # 11

Geometrical Interpretation of Gradient

* Gradient of a scalar function represents a normal vector to the surface.

$\nabla\phi|_{(x,y,z)}$ represent the normal vector of the surface at (x, y, z).

Example:

if $\phi = x^2yz$

$$\nabla\phi = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

At $(1, 1, 1)$

$$\nabla\phi|_{(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$$

which is Normal to the surface $\phi = x^2yz$

Module # 12

(Theorem Related to Gradient)

Statement:

"$\nabla\phi$ is a vector perpendicular to the surface $\phi(x,y,z) = C$, where C is a Constant."

Module # 13

(Related Proble 1: Gradient)

(Q:1) If $\vec{A} = 2x^2 i - 3yz j + xz^2 k$ and $\phi = 2z - x^3 y$

Find:

(i) $\vec{A} \cdot \nabla\phi$

(ii) $\vec{A} \times \nabla\phi$

Sol:

$$\phi = 2z - x^3 y$$

$$\nabla\phi = -3x^2 y\, i - x^3 j + 2k$$

$\nabla\phi$ at $(1, -1, 1)$

$$\nabla\phi = 3i - j + 2k$$

Given vector $\vec{A} = 2x^2 i - 3yz j + xz^2 k$ at $(1, -1, 1)$

$$A = 2i + 3j + k$$

(i) $A \cdot \nabla\phi = (2i - 3j + k)(3i - j + 2k)$

$= 6 - 3 + 2 \Rightarrow \boxed{5}$ Answer

(5) $\vec{A} \times \nabla \phi = (2i + 3j + k) \times (3i - j + 2k)$

$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= (6 - (-1))\, i - (4 - 3)\, j + (-2 - 9)\, k$$

$$= 7i - j - 11k$$

Module # 14

Gradient

MCQ; $\nabla F(r) = \dfrac{f'(r)\vec{r}}{r}$

$$\nabla f(r) = \frac{\partial}{\partial x} f(r)\, i + \frac{\partial}{\partial y} f(r)\, j + \frac{\partial}{\partial z} f(r)\, k$$

$$= \frac{\partial f}{\partial x}\frac{\partial r}{\partial x} i + \frac{\partial f}{\partial y}\frac{\partial r}{\partial y} j + \frac{\partial f}{\partial z}\frac{\partial r}{\partial z} k$$

$$= \frac{df}{dr}\frac{\partial r}{\partial x} i + \frac{df}{dr}\frac{\partial r}{\partial y} j + \frac{df}{dr}\frac{\partial r}{\partial z} k$$

Since $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) \Rightarrow \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} 2y = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} 2z = \frac{z}{r}$$

$$= f'(r)\frac{x}{r}\hat{i} + f'(r)\frac{y}{r}\hat{j} + f'r\frac{z}{r}\hat{k}$$

$$= \frac{f'(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla f(r) = \frac{f'(r)\,\vec{r}}{r}$$

Hence proved.

Module # 15

Example done

Module # 16

Divergence of a vector point Function

Defi:

Divergence is a vector operator, when we applied this operator to the Quantity of a vector Field, it produce a Scalar Field.

Divergence of V, written as $\nabla \cdot V$ or div V;

$$\nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

(6) MCQ's;

* $\nabla \cdot \bar{V}$ is a scalar Quantity

* $\nabla \cdot \bar{V} \neq \bar{V} \cdot \nabla$

* If $\bar{V}$ is a Constant vector, then $\nabla \cdot \bar{V} = 0$

* If $\nabla \cdot \bar{V} = 0$ everywhere in some region of R, Then $\bar{V}$ is called "Solenoid vector point Function" in the Region.

(*) Example;

Determine the Constant a so that the vector $\bar{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is Solenoidal.

Sol;

$$\nabla \cdot V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left[(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right]$$

$$= \left[\frac{\partial (x+3y)}{\partial x} + \frac{\partial (y-2z)}{\partial y} + \frac{\partial (x+az)}{\partial z}\right]$$

$$= 1 + 1 + a \Rightarrow 2 + a = 0$$

$$\boxed{a = -2}$$

Subsitute $a = -2$ in given,

$$\bar{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x-2z)\hat{k}$$

$$\nabla \cdot \bar{V} = 1 + 1 - 2$$

$$\nabla \cdot \bar{V} = 2 - 2$$

$$\nabla \cdot V = 0$$

(So given $\bar{V}$ is Solenoidal.)

Module #17

Properties of Divergence

(i) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(ii) $\nabla (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \cdot \phi)$

Proof;

Let; $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

$\vec{B} = B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k}$

(i) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

L.H.S $= \nabla \cdot (\vec{A} + \vec{B}) = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

$$\nabla \cdot (\vec{A} + \vec{B}) = \frac{\partial}{\partial x}(A_1 + B_1)\hat{i} + \frac{\partial}{\partial y}(A_2 + B_2)\hat{j} + \frac{\partial}{\partial z}(A_3 + B_3)\hat{k}$$

$$= \left(\frac{\partial A_1}{\partial x}\hat{i} + \frac{\partial A_2}{\partial y}\hat{j} + \frac{\partial A_3}{\partial z}\hat{k}\right) + \left(\frac{\partial B_1}{\partial x}\hat{i} + \frac{\partial B_2}{\partial y}\hat{j} + \frac{\partial B_3}{\partial z}\hat{k}\right)$$

$= \nabla \cdot \vec{A} + \nabla \cdot \vec{B} = R.H.S$

(ii) $\nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \phi)$

L.H.S $= \nabla \cdot (\phi \vec{A})$

$\phi A = \phi A_1 \hat{i} + \phi A_2 \hat{j} + \phi A_3 \hat{k}$

$$(\nabla \cdot \phi \vec{A}) = \frac{\partial}{\partial x}(\phi A_1)\hat{i} + \frac{\partial}{\partial y}(\phi A_2)\hat{j} + \frac{\partial}{\partial z}(\phi A_3)\hat{k}$$

$$= \phi \frac{\partial A_1}{\partial x} + A_1 \frac{\partial \phi}{\partial x} + \phi \frac{\partial A_2}{\partial y} + A_2 \frac{\partial \phi}{\partial y} + \phi \frac{A_3}{\partial z}$$

$+ A_3 \partial \phi / \partial z$

(7)

~~$= \varphi(\nabla \cdot \vec{A}) + \left(\frac{\partial \varphi}{\partial x} i + \right.$~~

$$= \varphi\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) + \left(A_1 \frac{\partial \varphi}{\partial x} + A_2 \frac{\partial \varphi}{\partial y} + A_3 \frac{\partial \varphi}{\partial z}\right)$$

$$= \varphi(\nabla \cdot \vec{A}) + \left(\frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k\right)(A_1 i + A_2 j + A_3 k)$$

$$= \varphi(\nabla \cdot A) + (\nabla \cdot \varphi)(\vec{A})$$

$$= \varphi(\nabla \cdot \vec{A}) + (\nabla \cdot \varphi)(\vec{A})$$

Module # 18

Laplacian

"The Laplacian or laplace operator is a second-order differential operator given by the divergence of the gradient of a given defined over a space R."

$$\Delta \vec{U} = \nabla^2 \vec{U} = \nabla \cdot \nabla \vec{U}$$

MCQs:

In Cartesian coordinate system, the Laplacian is given by the sum of second order partial derivative of the function w.r.t each independent variable

Denoted by;

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

MCQs.

* It is a scalar operator.

* In two-dimention, the Laplace operator reduce to;

$$\nabla^2 = \frac{\partial^2}{\partial x^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

* Divergence of gradient of ϕ written as

$$\nabla \cdot \nabla \phi = \nabla^2 \phi$$

* $\nabla^2 \phi$ is called the Laplacian of ϕ

* The equation $\nabla^2 \phi = 0$ is called Laplace's equation.

* If a scalar function ϕ satisfies the Laplace equation $\nabla^2 \phi = 0$ in a Cartesian Region R, then ϕ is said to be a Harmonic Function in the Region R.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

* we can express the Laplace operator in polar coordinate (r, θ) notation.

* Two Dimensional Laplace operator can be expressed as;

$$= \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

(8)

Module # 19

Example of Divergence

(Q) If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, Find $\nabla.\vec{A}$ at the point $(1,-1,1)$

$$\nabla.\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k})$$

$$= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z}$$

$$= 2xz - 6y^2z^2 + xy^2$$

$\nabla.\vec{A}$ At $(1,-1,1)$

$$= 2(1)(1) - 6(-1)^2(1) + (1)(-1)^2$$

$$= 2 - 6 + 1 = -3$$

$\nabla.\vec{A}$ at $(1,-1,1)$ is -3.

Imp Modules For Subjective

Module # 20

Examples done

Module # 21

Examples done

Module # 22

Problem Related; Laplacian

Show that;

(1) $\nabla.(r^3\vec{r}) = 6r^3$

General Formula;

$$\nabla.[\vec{r}f(r)] = f(r)(\nabla.\vec{r}) + \vec{r}.(\nabla f(r))$$

Product Rule

$$= 3f(r) + \vec{r} \cdot \frac{f'(r)\,\vec{r}}{r}$$

$\because \nabla \cdot \vec{r} = 3$

$\therefore \nabla f(r) = \frac{f'r(\hat{r})}{r}$

General Formula;

$$\boxed{\nabla \cdot (f(r) \cdot \vec{r}) = (3+n) r^n}$$

(i) $\nabla \cdot (r^3 \vec{r}) = (3+n) r^n = (3+3) r^3$

$= 6r^3$

Module # 23

Curl of a vector point function

(MCQs)

* The infinitesimal rotation of a vector field is described by the curl.

Def;

If $\vec{V}(x,y,z)$ is a differentiable vector field in a certain region of space, then the curl or rotation of V, written $\nabla \times \vec{V}$.

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (V_1 i + V_2 j + V_3 k)$$

we can express this expression in Matrix form;

$$\nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

(MCQs)

* A vector field whose curl is zero is called irrotational.

* Identies of Curl;

* A gradient has zero curl;

$$\nabla \times \nabla \phi = 0$$

* A Curl has zero divergence;

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Note;

$$\nabla \cdot (\nabla \times \vec{V}) = (\nabla \times \nabla) \cdot \vec{V} = 0$$

Because $\nabla \times \nabla = 0$

Module # 24

(Properties of the Curl)

If $\vec{A}$ and $\vec{B}$ are differentiable vector ~~Field~~ Functions, and φ is differentiable scalar Functions of position (x, y, z), then

(i) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(ii) $\nabla \times (\varphi \vec{A}) = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}$

(iii) $\nabla \times (\nabla \varphi) = 0$

(iv) $\nabla \cdot (\nabla \times \vec{A}) = 0$

Module # 25

Examples of Curl

Q; If $\vec{A} = xz^3 \hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$.

Find

$\nabla \times \vec{A}$ at the point $(1, -1, 1)$

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(xz^3 \hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k})$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

(10)

$$= \left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z}\right)\hat{i} + \left(\frac{\partial(2yz^4)}{\partial x} + \frac{\partial(xz^3)}{\partial z}\right)\hat{j}$$

$$+ \left(\frac{\partial(-2x^2yz}{\partial x} - \frac{\partial(xz^3)}{\partial y}\right)\hat{k}$$

$$\nabla \times \vec{A} = (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

Now we calculate; $\nabla \times \vec{A}$ At $(1, -1, 1)$

$$= (2(1)^4 + (2)(1)^2(-1))\hat{i} + 3(1)(1)^2\hat{j} + 4(1)(-1)(1)$$

$$= (2-2)\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\boxed{\nabla \times \vec{A} = 3\hat{j} + 4\hat{k}}$$

Module # 26

Examples done

Module # 27

Example done

MCQs;

* $\vec{A} = \nabla\phi$ is called a Conservative vector Field and ϕ is called the Scalar potentional.

* If $\vec{A} = \nabla\phi$ then $\nabla \times \vec{A} = 0$

Module # 28 (Imp subjective)

Curl Example

Q1 if $v = \omega \times r$ prove $\omega = \frac{1}{2}\text{Curl } v$ where ω is a constant vector.

Solution;

$$\text{Curl } v = \nabla \times v = \nabla \times (\omega \times r)$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times \left[(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 z - \omega_1 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 z - \omega_1 x)\right)\hat{i} - \left(\frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y)\right)\hat{j} + \left(\frac{\partial}{\partial x}(\omega_3 z - \omega_1 x) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y)\right)\hat{k}$$

$$= (\omega_1 - \omega_1)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (+\omega_3 + \omega_3)\hat{k}$$

$$= 2\omega_1 \hat{i} + 2\omega_2 \hat{j} + 2\omega_3 \hat{k}$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$\Rightarrow \nabla \times (\omega \times r) = 2\vec{\omega}$$

$$\Rightarrow \nabla \times v = 2\vec{\omega}$$

$$\Rightarrow \vec{\omega} = \frac{1}{2}\nabla \times v \Rightarrow \vec{\omega} = \frac{1}{2}\text{Curl } v$$

(11) (MCQs) * If $\text{Curl } \vec{A} = 0$, then there would be no rotation and the field is called irrotational field.

* A field which is not irrotational is sometime called a vortex field.

Module # 29

Vector Identities

MCQs;

* $\nabla \cdot (\vec{A} \times \vec{r})$ if $\nabla \times \vec{A} = 0$
* $\nabla \times (\nabla \times \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

Module # 30

Line Integral

Imp objective

Def; An integral where the functions is evaluated along a curve is called line integral.

Symbolic Form of line integral;

$$\int_C \vec{A} \cdot d\vec{r}$$

$d\vec{r} = dx\,i + dy\,j + dz\,k$ is called the differential displacement vector.

MCQS

* The value of the line Integral is the sum of values of the field at all points on the curve.

* line Integral can be transformed into double Integral and Surface Integrals and vice versa.

* The Integrals that involves differential displacement vector $d\vec{r}$ are called line Integrals.

* line Integral can also be expressed as;

$$= \int_C (A_1 dx + A_2 dy + A_3 dz)$$

* line Integral $\int_C A \cdot dr$ is sometimes called a Scalar line Integral of vector Field $\vec{A}$.

* If C is a closed curve, the line Integral Around C is denoted by;

$$\oint_C \vec{A} \cdot d\vec{r} = \oint_C (A_1 dx + A_2 dy + A_3 dz)$$

* Simple closed curve;
a curve which does not intersect itself anywhere.

(12) Applications of line Integrals;

MCQS:

* If a vector field to be Integrated $\vec{A}$ is the force $\vec{F}$ on a particle move along C, then this line Integral show the work done by the Force.

* In Fluid Mechanics, if the vector field to be integrated $\vec{A}$ represents the velocity of some Fluid then the Integral is called the Circulation of $\vec{A}$ about C.

* Any integral which is to be calculated along a curve is called a line Integral.

Module #31

Other Forms and General Properties of line Integrals

* Line Integral using Scalar Function,

$$\int_c \varphi \, d\vec{r} = \int_c \varphi (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \hat{i}\int_c \varphi \, dx + \hat{j}\int_c \varphi \, dy + \hat{k}\int_c \varphi \, dz$$

MCQs;

* General Integral indicates the dot product ('.') between the given vector Field and the differential displacement vector.

* we can also express it as a cross product (x) of both the vectors, According to requirement

$$\int_C \vec{A} \times d\vec{r} = i\int_C (A_2 dz - A_3 dy) + \hat{j}\int A_3 dx - A_1 dz) + \hat{k}\int (A_1 dy - A_2 dx)$$

* General properties of line Integral;

(i) $\int_C K\vec{A} \cdot d\vec{r} = K\int_C \vec{A} \cdot d\vec{r}$

(ii) $\int_C (\vec{A} + \vec{B}) \cdot dr = \int_C \vec{A} \cdot d\vec{r} + \int_C \vec{B} \cdot d\vec{r}$

(iii) $\int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r}$

MCQs;

If the sense of orientation along C is reversed, the value of the integral is multiplied by '-1'.

(iv) If C is piece-wise smooth, Consisting of smooth curves $C_1, C_2 \dots C_n$;

$$\int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} + \dots \int_{C_n} \vec{A} \cdot d\vec{r}$$

Module #32

(13)

Example of line Integral

Q; If $\phi = 2xyz^2$, $F = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$.

a) Evaluate the integrals;

(i) $\int_C \phi \cdot d\vec{r}$

By subsituting the given values of x, y, z, we obtain;

$$\phi = 2xyz^2 = 2(t^2)(2t)(t^3)^2$$
$$= 4t^9$$

And $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow r = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$\Rightarrow d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$$

$$\int_C \phi\, d\vec{r} = i\int_C \phi\, dx + j\int_C \phi\, dy + k\int_C \phi\, dz$$

$$= \int_0^1 4t^9(2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$$

$$= \hat{i}\int_0^1 8t^{10}dt + \hat{j}\int_0^1 8t^9 dt + \hat{k}\int_0^1 12t^{11}dt$$

$$= i\left|\frac{8t^{11}}{11}\right|_0^1 + \hat{j}\left|\frac{8t^{10}}{10}\right|_0^1 + \hat{k}\left|\frac{12t^{12}}{12}\right|_0^1$$

$$= \left[\frac{8}{11}i\right] + \frac{8}{10}\hat{j} + \hat{k}$$

(ii) $\int_C F \times d\vec{r}$

$$F = xy\,\hat{i} + z\,\hat{j} + x^2\hat{k},$$

$$F = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$$

$$r = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

$$dr = (2t\,\hat{i} + 2\hat{j} + 3t^2\hat{k})\,dt$$

Now,

$$F \times dr = (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\,\hat{i} + 2\hat{j} + 3t^2\hat{k})$$

$$\begin{vmatrix} i & j & k \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = (-3t^5 - 2t^4)\,\hat{i} - (6t^5 - 2t^5)\,\hat{j} + (4t^3 + 2t^4)\hat{k}$$

$$\int_C \vec{F} \times d\vec{r} = i\int_0^1 (-3t^5 - 2t^4)\,dt - j\int_0^1 (6t^5 - 2t^5)\,dt + \hat{k}\int_0^1 (4t^3 + 2t^4)\,dt$$

$$= i\left| -\frac{3t^6}{6_2} - \frac{2t^5}{5} \right|_0^1 - j\left| \frac{6t^6}{6} - \frac{2t^6}{6} \right|_0^1 + \hat{k}\left| \frac{4t^4}{4} + \frac{2t^5}{5} \right|_0^1$$

$$= i\left[-\frac{(1)^6}{2} - \frac{2}{5} \right] - j\left[1 - \frac{2}{6_3} \right] + \hat{k}\left[1 + \frac{2}{5} \right]$$

$$= \left[-\frac{9}{10}\,i - \frac{2}{3}\,\hat{j} + \frac{7}{5}\,\hat{k} \right]$$

Module # 33

Example

love

Module # 34

Line Integral Dependent on Path

MCQS;

The value of line Integrals $\int_C \vec{A} \cdot d\vec{r}$ Generally depends not only on the end points P_1 & P_2 of the curve C, but also the geometric Shape of the path C.

Module # 35

Independece of Path

Definition;

The line integral $\int_C \vec{V} \cdot d\vec{r}$ is said to Independent of the path C in a given region R, If the value of the line Integral $\int_{P_1}^{P_2} D \cdot d\vec{r}$ is the Same for all paths C Joining any two points P_1 and P_2 in R.

Theorem;

Prove that a necessary and sufficient Condition for $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ to be independent of the path Joining any two point P_1 & P_2 (MCQS; $\vec{V}$ to be Conservative) in a given region is that $\oint_C \vec{V} \cdot d\vec{r} = 0$ for all closed path C in the Region.

Proof;

Let C be any simple closed curve, and P_1 and P_2 be any two points on C.

Then since by the Supposition, the Integral is Independent of Path;

$$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} = \int_{P_1BP_2} \vec{V}\cdot d\vec{r}$$

Reversing the direction of Integration in the integral on the right, we have

$$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} = -\int_{P_1BP_2} \vec{V}\cdot d\vec{r}$$

$$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} + \int_{P_1BP_2} \vec{V}\cdot d\vec{r} = 0$$

$$\oint_C \vec{V}\cdot d\vec{r} = 0$$

Conversely, if $\oint_C \vec{V}\cdot d\vec{r} = 0$

then $$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} + \int_{P_1BP_2} \vec{V}\cdot d\vec{r} = 0$$

$$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} = -\int_{P_1BP_2} \vec{V}\cdot d\vec{r}$$

Again Changing the Direction;

$$\int_{P_1AP_2} \vec{V}\cdot d\vec{r} = \int_{P_1BP_2} \vec{V}\cdot d\vec{r}$$

Module # 36

Theorems on line Integrals

=> **Scalar Potential Function;**

A scalar potential function ϕ is a single-valued function for which there exists a continous vector field $\vec{V}$ in a simply connected region R that satisfies the Relation $\vec{V} = \nabla \vec{V}$.

Module #37

Example;

Q) If $\vec{A}(t) = t\hat{i} + t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\hat{k}$.

i) Evaluate;

$$\int_0^2 \vec{A}\cdot\vec{B}\,dt$$

$$\vec{A}\cdot\vec{B} = (t\hat{i} + t^2\hat{j} + (t-1)\hat{k})\cdot(2t^2\hat{i}) + 0\hat{j} + 6t\hat{k})$$

$$= 2t^3 + 6t(t-1)$$

$$\int_0^2 \vec{A}\cdot\vec{B}\,dt = \int_0^2 2t^3 + 6t(t-1)\,dt$$

$$= \int_0^2 2t^3 + \int_0^2 6t^2\,dt - \int_0^2 6t\,dt$$

$$= \left|\frac{2t^4}{2}\right|_0^2 + \left|\frac{6t^3}{3}\right|_0^2 - \left|\frac{6t^2}{2}\right|_0^2$$

$$= \left|\frac{t^4}{2} + 2t^3 - 3t^2\right|_0^2 = \boxed{12}$$

(ii) $\int_0^2 \vec{A} \times \vec{B}\, dt$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$$

$$= -6t^3 \hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4 \hat{k}$$

$$\int_0^2 \vec{A} \times \vec{B}\, dt = \int_0^2 (-6t^3 \hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4 \hat{k})\, dt$$

$$= -\hat{i}\int_0^2 6t^3\, dt - \hat{j}\int_0^2 (8t^2 - 2t^3)\, dt + \hat{k}\int_0^2 2t^4\, dt$$

$$= \left| -\hat{i}\,\frac{3}{2}t^4 - \hat{j}\left(\frac{8}{3}t^3 - \frac{t^4}{2}\right) + \hat{k}\,\frac{2}{5}t^5 \right|_0^2$$

$$= -\frac{3(16)}{2}\hat{i} - \hat{j}\,\frac{80}{6} + \frac{64}{5}\hat{k}$$

$$\int_0^2 \vec{A} \times \vec{B}\, dt = -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}.$$

Required Result.

Module #38

Example # Line Integrals

Q) If $F = 3xy\hat{i} - y^2\hat{j}$, evaluate work done by the force on the curve C in the xy plane, $y = 2x^2$, From (0,0) to (1,2)

Solution;

Work done;

$$\int_C \vec{F} \cdot d\vec{r}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

By Subsituting;

$$= \int_0^1 (3xy\hat{i} - y^2\hat{j}) - (dx\hat{i} + dy\hat{j})$$

$$= \int_0^1 (3xy\,dx - y^2\,dy)$$

$y = 2x^2$, $dy = 4x\,dx$

$$= \int_0^1 3x(2x^2)\,dx - (2x^2)^2 4x\,dx)$$

$$= \int_0^1 6x^3\,dx - \int_0^1 16x^5\,dx$$

$$= \left| \frac{3}{2}x^4 - \frac{8}{3}x^6 \right|_0^1$$

$$= \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}$$

Module #39

Example done

Module #40

Example done

Module #41

Surface Integral

MCQS:

Def: Integrals that involves the differential surface elements $\vec{ds}$ are called Surface Integrals.

$$\iint \vec{A} \cdot \vec{ds}$$

* $\hat{n}$ unit normal $\hat{n}^S$ to any point of the positive side of S is called a positive or outward drawn unit Normal.

* The Surface Integral of a vector field $\vec{A}$ is called the Flux of $\vec{A}$ through S.

* ϕ is a scalar point function

* If S is a closed surface, then the Surface Integrals may be written as;

$\oint \vec{A} \cdot d\vec{S}$, $\oint \vec{A} \times d\vec{S}$ OR $\oint \varphi \, d\vec{S}$

* Properties of Surface Integrals;

(i) $\iint_S K\vec{A} \cdot d\vec{S} = K \iint_S \vec{A} \cdot d\vec{S}$

(ii) $\iint_S (\vec{A} + \vec{B}) \cdot d\vec{S} = \iint_S \vec{A} \cdot d\vec{S} + \iint_S \vec{B} \cdot d\vec{S}$

(iii) $\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S}$

Module # 42

Evaluation of the Surface Integral

MCQs;

Theorem;

Let R be the projection of the Surface S on the xy-plane, then;

$$\iint_S \vec{A} \cdot \hat{n} \, ds = \iint_R \vec{A} \cdot \hat{n} \, \frac{dx\,dy}{|\hat{n} \cdot \hat{k}|}$$

Module #43

Example done

Module # 44

Example on Surface Integral

Q; If $\vec{F} = 2yi - zj + x^2k$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$

Evaluate; $\iint_S \vec{F} \cdot \hat{n}\, d\vec{S}$

Solution;

A vector normal to S is,

$$\nabla(8x - y^2) = 8\hat{i} - 2y\hat{j}$$

$$n = \frac{8i - 2yj}{\sqrt{(8)^2 + (2y)^2}} \Rightarrow \frac{8i - 2yj}{2\sqrt{16 + y^2}} \Rightarrow \frac{2(4i - yj)}{2(\sqrt{16 + y^2})}$$

$$\hat{n} \cdot \hat{i} = \frac{4}{\sqrt{16 + y^2}}$$

$$\vec{F} \cdot \hat{n} = (2yi - zj + x^2k)\left(\frac{4i}{\sqrt{16 + y^2}} + \frac{-yj}{\sqrt{16 + y^2}}\right)$$

$$= \frac{8y + 2y}{\sqrt{16 + y^2}}$$

(18) $\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \vec{A} \cdot \hat{n}\, \frac{dz\,dy}{|n \cdot i|}$

$$= \int_0^4 \int_0^6 \frac{8y + zy}{\sqrt{16+y^2}} \cdot \frac{\sqrt{16+y^2}}{4}\, dz\,dy$$

$$= \frac{1}{4} \int_0^4 \int_0^6 (8y + zy)\, dz\,dy \Rightarrow \frac{1}{4} \int_0^4 66y\, dy$$

$$= \left| \frac{33}{4} y^2 \right|_0^4 \Rightarrow \frac{3}{4}(4)^2 = 132$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = 132$$

Module 45, 46, 47 Examples

Module # 48

Volume Integral

MCQs;

A volume integral refers to an Integral over a 3-dimentional domain.

Def; Let $\vec{A}$ be a given vector point Function which is defined and Continuous in a Closed Region R. Then

$$\iiint_R \vec{A}\, d\vec{v}$$

* If we Have Scalar point Function $\varphi(x,y,z)$ Defined as Continous over Region R,

$$\iiint_R \varphi \, dV$$

* In Rectangular Coordinate System,

$dV = dx\,dy\,dz$ so the Volume Integral becomes;

$$\iiint_R \varphi \, dV = \iiint_R \varphi(x,y,z)\,dx\,dy\,dz$$

Which is Ordinary triple Integral of $\varphi(x,y,z)$ over region R.

* If $\varphi(x,y,z) = 1$;

$$\iiint_R dx\,dy\,dz = \iiint_R dV$$

* Volume Integral can be expressed also in Cylindrical and Spherical Coordinates.

(i) Volume Integral in ~~Cylindrical~~ Spherical Coordinate;

$$\iiint_R g(\rho,\varphi,z)\,\rho\,d\rho\,d\varphi\,dz$$

(ii) Volume Integral in ~~Cylindrical~~ Spherical Coordinate;

$$\iiint_R g(r,\theta,\varphi)\,r^2 \sin\varphi\,dr\,d\theta\,d\varphi$$

Module #50

Example Volume Integral

Q; Find the volume of the Region Common to the Intersecting cylinders $x^2+y^2=a^2$ & $x^2+z^2=a^2$

Solution;

Let M be the Required common Region,

Required volume = 8times volume of Region

$$\iiint_V M\,dV = 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= 8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy\,dx = 8\int_{x=0}^{a}(a^2-x^2)\,dx \Rightarrow 8\left(\frac{2a^3}{3}\right)$$

$$= \frac{16a^3}{3}$$

$$\because \int_0^a (a^2-x^2)\,dx \Rightarrow a^2\int_0^a 1\,dx - \int_0^a x^2\,dx$$

$$\Rightarrow a^2 x\Big|_0^a - \frac{x^3}{3}\Big|_0^a \Rightarrow a^2\cdot a - \frac{a^3}{3}$$

$$\Rightarrow \frac{3a^3-a^3}{3} \Rightarrow \frac{a^3(3-1)}{3}$$

$$= \left(\frac{2a^3}{3}\right] \Rightarrow \frac{16a^3}{3}$$

Module # 52

Divergence Theorem

MCQs;

* Divergence Theorem is also called Gauss's Divergence Theorem.

* It's used to derive equation governing the Flow of Fluids, Heat conduction wave propagation and electrical Field.

Statement;

It States that if R is the Region Bounded by a closed Surface S and $\vec{A}$ is a vector point Function with Continous First partial derivative;

$$\iint_S \vec{A} \cdot \hat{n}\, ds = \iiint_R \nabla \cdot \vec{A}\, dV$$

$\hat{n}$ is outward drawn unit Normal to S.

Module #53
Divergence Theorem In Rectangular form

MCQs:

* Cartesian Coordinate System (x, y, z) is also called Rectangular system.

* The Quantites $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of $\hat{n}$.

Hence; $\vec{A} \cdot \hat{n} = A_1 \cos\alpha + A_2 \cos\beta + A_3 \cos\gamma$

54 # Module
Skip

Module # 55
Example: Divergence Theorem

Evaluate; $\iiint_R \vec{F} \cdot \hat{n}\, dV$

$F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ 'S' is the Surface of the cube bounded S by,

$x = 0, 1, \quad y = 0, 1, \quad z = 0, 1$

By Divergence Theorem,

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iiint_S \nabla\cdot\vec{A}\,dv$$

$$\nabla\cdot\vec{A} = \nabla\cdot(4x^2\hat{i} + y^2\hat{j} + yz\hat{k})$$

$$\iiint \nabla\cdot\vec{A}\,dV = \iiint (4x - 2y + y)\,dV$$

$$= \iiint (4x - y)\,dV$$

$$= \int_0^1\int_0^1\int_0^1 (4x - y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 (2x^2 - y)\,dy\,dx$$

$$= \int_0^1\int_0^1 (2 - y)\,dy\,dx$$

$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)dx = \int_0^1 2 - \frac{1}{2}\,dx$$

$$\frac{3}{2}\int_0^1 dx = \frac{3}{2}(x)_0^1$$

$$= \frac{3}{2}$$

$$\boxed{\iiint_R F\cdot\hat{n}\,dv = \frac{3}{2}}$$

Module # 56

Theorem of Divergence Example;

Q; prove The Identity;

$$\iiint_V \nabla\varphi\, dv = \iint_S \varphi\, \hat{n}\, ds$$

Solution; Let $\vec{A} = \varphi\vec{C}$, $\vec{C}$ is constant

$$\iiint_V \nabla\cdot(\varphi\vec{C})\, dv = \iint_S \varphi\vec{C}\cdot\hat{n}\, ds$$

since; $\nabla\cdot(\varphi C) = (\nabla\varphi)\cdot\vec{C} = C\cdot\nabla\varphi$

$$\varphi C\cdot n = \vec{C}\cdot(\varphi\vec{A})$$

Substituting these values in above;

$$\iiint_V C\cdot\nabla\varphi\, dv = \iint_S \vec{C}\cdot(\varphi\hat{n})\, ds$$

$$\vec{C}\iiint_V \nabla\varphi\, dv = \vec{C}\iint_S \varphi\,\hat{n}\, ds \quad \because C \text{ is constant}$$

Hence prove.

Module #57

Example of Divergence Theorem

MCQs:

A fluid of density $\rho(x, y, z, t)$ moves with velocity $V(x, y, z, t)$. If ther is no sources or sinks,

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

wher $J = \rho V$

we can also write as:

$$\nabla \cdot \rho V + \frac{\partial \rho}{\partial t} = 0$$

* This ervation is called the Continuity equation, If ρ is a constant, the fluid is Incompressible and $\nabla \cdot V = 0$

$\nabla \cdot V = 0$ i.e v is is Solenoidial.

* ρ is the Charge Density.
* $J = \rho V$ is the current density.

(22) Imp

Module #58

Example #03 Divergence Theorem

(Q) Prove the Relation;

$$\iiint_V \nabla \times \vec{B}\, dV = \iint_S \hat{n} \times \vec{B}\, ds$$

where $\vec{B}$ is any vector Field;

Solution; Since the divergence theorem;

$$\iiint_V \nabla \cdot \vec{A}\, dV = \iint_S \vec{A} \cdot \hat{n}\, ds$$

Let; $\vec{A} = B \times C$

$$\iiint_V \nabla \cdot (\vec{B} \times \vec{C})\, dV = \iint_S (\vec{B} \times \vec{C}) \cdot \hat{n}\, ds$$

Since;

$$\nabla \cdot (B \times C) = \vec{C} \cdot (\nabla \times \vec{B})$$

$$(\vec{B} \times \vec{C}) \cdot \hat{n} = B \cdot (\vec{C} \cdot \hat{n}) = (\vec{C} \times \hat{n}) \cdot \vec{B} = \vec{C}(\hat{n} \times \vec{B})$$

Above Integral will become;

$$\iiint_V \vec{C} \cdot (\nabla \times B)\, dV = \iint_S \vec{C} \cdot (\hat{n} \times \vec{B})\, ds$$

$$C \iiint_V \nabla \times B\, dV = C \iint_S \hat{n} \times B\, ds$$

$\because C$ is constant

$$\iiint_V \nabla \times B\, dV = \iint_S \hat{n} \times B\, ds$$

Module #59

Stokes' Theorem

MCQS:-

Stoke's Theorems as the line Integral of the Tangential Component of a vector Function.

Statement:-

It states that if S is open, two Sided Surface bounded by a simple Closed Curve C, then if $\vec{A}$ has Continuous First partial derivatives.

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \vec{n} \, ds$$

C is transvered in the positive direction

Module # 60

Stokes Theorem in Rectangular Form

MCQS.

Let $\vec{A} = A_1 i + A_2 \hat{j} + A_3 \hat{k}$ and $\hat{n} = n_1 i + n_2 \hat{j} + n_3 \hat{k}$ be the outward drawn unit normal to the Surface S.

If α, β, γ are the Angles which the unit Normal $\hat{n}$ Makes with the Positive directions of x, y and z.

(23) The Quantity $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosine of $\vec{n}$.

$n_1 = \cos\alpha$, $n_2 = \cos\beta$, $n_3 = \cos\gamma$

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

~~Hh~~ Module # 61 Skip

Module No. 62

Example of Stoke's Theorem

Prove that;

$$\oint_C d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \nabla) \times \vec{B}\, ds$$

Solution;

Stokes' theorem is;

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S ((\nabla \times \vec{A}) \cdot \hat{n})\, ds$$

Let $\vec{A} = \vec{B} \times \vec{C}$, C is constant.

$$\oint_C (\vec{B} \times \vec{C})\, d\vec{r} = \iint_S (\nabla \times (\vec{B} \times \vec{C})) \cdot \hat{n}\, ds$$

$$= \iint_S [(\vec{C} \cdot \nabla)\vec{B} - \vec{C}(\nabla \cdot \vec{B})] \cdot \hat{n}\, ds$$

$$= \iint_S (\vec{C} \cdot \nabla)\vec{B} \cdot \hat{n}\, ds - \iint_S [\vec{C}(\nabla \cdot B)] \cdot \hat{n}\, ds$$

$$= \iint_S \vec{C}\,[\nabla(\vec{B} \cdot \hat{n})]\, ds - \iint_S \vec{C} \cdot [\hat{n}(\nabla \cdot \vec{B})]\, ds$$

(ii)

$$= \vec{C} \cdot \iint_S [\nabla(\vec{B}\cdot\hat{n}) - \hat{n}(\nabla\cdot\vec{B})]\,ds$$

$$= \vec{C} \cdot \iint_S (\hat{n}\times\nabla)\times\vec{B}\,ds$$

C is constant.

$$\oint_C d\vec{r}\times\vec{B} = \iint_S (\hat{n}\times\nabla)\times\vec{B}\,ds$$

Hence prove.

Imp

Module #63

Related Theorems;

Stokes' Theorem

Statement;

Prove that a necessary and sufficient condition that $\int_C \vec{A}\cdot d\vec{r} = 0$ for every closed curve C is that $\nabla\times\vec{A} = 0$ identically.

Proof; **Sufficiently;** Suppose $\nabla\times\vec{A} = 0$.

Then by Stoke's theorem,

$$\oint_C \vec{A}\cdot d\vec{r} = \iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds$$

Since; $\nabla\times\vec{A} = 0$, so;

$$\iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds = 0$$

so, $\oint A\cdot d\vec{r} = 0$

Necessity;

Suppose $\oint_C \vec{A} \cdot d\vec{r} = 0$ Around Every Closed path, and $\nabla \times \vec{A} \neq 0$ at some point P. Let S be a Surface Contained in this Region whose Normal vector $\hat{n}$ at each point has the same direction

$$\nabla \times \vec{A} = \alpha \hat{n}$$

Stokes' theorem Become;

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds = \iint_S \alpha \hat{n} \cdot \hat{n} \, ds > 0$$

It Contradicts the hypothesis that $\oint_C \vec{A} \cdot d\vec{r} = 0$ and show that $\nabla \times \vec{A} = 0$

Hence proved.

Module # 64

Stoke's Theorem Example

Q; prove that;

$$\oint_C \varphi \, dr = \iint_S d\vec{s} \times \nabla \varphi$$

Solution;

By Stoke's Theorem;

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \hat{n} \, ds$$

Let $\vec{A} = \varphi \vec{C}$ where $\vec{C}$ is a constant non-zero -

$$\oint_C \varphi \vec{C} \cdot d\vec{r} = \iint_S \nabla \times (\varphi \vec{C}) \cdot \hat{n} \, ds$$

$$= \iint_S (\nabla \varphi \times \vec{C}) \cdot ds$$

Since $\nabla \times C = 0$

$$\oint_C \vec{C} \cdot \varphi \, d\vec{r} = \iint_S \nabla \varphi \cdot \vec{C} \times d\vec{s}$$

$$= \iint_S \vec{C} \times d\vec{s} \cdot \nabla \varphi$$

$$= \iint_S \vec{C} \cdot d\vec{s} \times \nabla \varphi$$

$$= C \cdot \iint_S d\vec{s} \times \nabla \varphi$$

C is constant;

$$\oint_C \vec{C} \cdot \varphi \, d\vec{r} = \iint_S d\vec{s} \times \nabla \varphi$$

Hence proved.

Module # 65

Skip

lengthy Example

Module # 66

Imp

Example of Stokes' Theorem.

Q; If $\oint \vec{E} \cdot d\vec{r} = -\frac{1}{c} \frac{\partial}{\partial t} \iint_S H \cdot ds$

Where S is any surface Bounded by the Curve C, Show that;

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

Solution;

As we know Stokes' theorm;

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} \, ds$$

by stokes' theorm;

$$\oint \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot \hat{n} \, ds$$

$$\iint_S (\nabla \times \vec{E}) \cdot d\vec{s} = -\iint_S \frac{1}{c} \frac{\partial \vec{H}}{\partial t} \cdot ds$$

$$\iint_S \left(\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{H}}{\partial t}\right) \cdot d\vec{s} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial H}{\partial t} = 0 \quad \text{OR}$$

$$\nabla \times E = -\frac{1}{c} \frac{\partial H}{\partial t}$$

proved.

MCQ + short

Module # 67

Simply & Multiply Connected Regions.

★ **Simply Connected Region;**

A Simply Closed curve is a Closed Curve which does not INtersects itself anywhere. ⇒ (R₀)

★ **Multiply Connected Region;**

A Region which is not simply Connected is called Multiply Connected.

OR

The Region which have holes are Called Multiply Connected.



Module # 68

SHRT + MCQs

GREEN'S THEOREMS IN THE PLANE

If R is simply-connected Region of the xy-plane bounded by a closed Curve C & if M and N are Continuous Functions of x and y having Continous derivatives in R, then.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Module # 69 lengthy example

skip

You can skip All lengthy Examples.

Module # 70

GREEN'S Theorem in the plane in vector Notation.

→ GREEN'S Theorem;

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

Now $M dx + N dy = (M\hat{i} + N\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$

$= \vec{A} \cdot d\vec{r}$

GREEN'S theorem in the plane in vector is written by;

$$\oint_C \vec{A} \cdot d\vec{r} = \iint (\nabla \times A) \cdot \vec{K}\, dR$$

→ Green's theorem's is sometime called stoke's theorem in the plane.

→ Green's theorem's in plane is a special case of stokes theorm.

Module #71

MCQs

→ Green theorem can be expressed in the plane vector notations which are also named the Tangential form or Normal form of Green's theorem.

→ The Tangential Form of Green theorem is also called First vector form of Green's theorem.

Module #72

MCQs

* The Generalization of Green's theorem as Gauss's divergence theorem is also called the second vector form of Green's theorem.

Module #73

GREEN'S 1st Identity

If φ and ψ are scalar point Functions with Continous second order derivatives in a region R bounded by a closed Surface S, then;

$$\iiint_S [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV = \iint_S (\varphi \nabla \psi) \cdot d\vec{S}$$

PROOF

Divergence theorem;

$$\iint_S \vec{A} \cdot \hat{n}\, ds = \iiint_R \nabla \cdot \vec{A}\, dV$$

let; $\vec{A} = \varphi \nabla \psi$

$$\iiint_S (\varphi \nabla \psi) \cdot \hat{n}\, ds = \iint_S (\varphi \nabla \psi)\, d\vec{s} = \iiint_R \nabla \cdot (\varphi \nabla \psi)\, dV$$

Put;

$$\nabla \cdot \varphi \nabla \psi = \nabla \varphi \cdot \nabla \psi + \varphi (\nabla \cdot \nabla \psi)$$
$$= \varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)$$

Equation become;

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)]\, dV = \iint_S (\varphi \nabla \psi)\, d\vec{s}$$

proved.

MCQS

$$\nabla \psi \cdot \hat{n} = \frac{\partial \psi}{\partial n}, \quad \nabla \varphi \cdot \hat{n} = \frac{\partial \varphi}{\partial n}$$

$$\nabla \psi \cdot d\vec{s} = \nabla \psi \cdot \hat{n}\, ds = \frac{\partial \psi}{\partial n} ds$$

$$\nabla \varphi \cdot d\vec{s} = \nabla \varphi \cdot \hat{n}\, ds = \frac{\partial \varphi}{\partial n} ds$$

1st GREEN Identity can also be written as;

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)]\, dV = \iint_S \varphi \frac{\partial \psi}{\partial n} ds$$

Module # 74

GREEN'S 2nd Identity.

If φ and ψ are scalar point functions with continuous second order derivates in a Region R bounded by a closed surface S, then,

$$\iiint_R (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi) dV = \iint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\bar{s}$$

PROOF;

Greens 1st Identity;

$$\iiint_R [(\varphi \nabla^2 \psi + \nabla \varphi \nabla \psi)] dV = \iint_S (\varphi \nabla \psi) \cdot d\bar{s}' \rightarrow (1)$$

Just Interchang φ by ψ

$$\iiint_R [(\psi \nabla^2 \varphi + \nabla \psi \nabla \varphi)] dV = \iint_S (\psi \nabla \varphi \cdot d\bar{s}' \rightarrow (2)$$

Subtract eq (2) from (1)

$$\iiint_R [(\varphi \nabla^2 \psi - \psi \nabla^2 \varphi)] dV = \iint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\bar{s}$$

is called the second order Identity or symmetrical theorem.

Alternative form of Green's 2nd Identity;

$$\iiint_R [\varphi \nabla^2 \psi - \psi \nabla^2 \varphi] dV = \iint_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n}\right) ds$$

Module # 75

Example; Related Green's Theorems.

Imp Done

Evaluate; $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2 dy$

along the path $x^4 - 6xy^3 = 4y^2$

Solution; Direct evaluation is difficult;

By comparing it with Green's Theorem,

$$\oint_C M dx + N dy = \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2 dy$$

Here; $M = 10x^2 - 2xy^3$ and $N = -3x^2y^2$

$$\frac{\partial M}{\partial y} = -6x^2y^2$$

$$\frac{\partial N}{\partial x} = -6x^2y^2$$

The Integral is independent of the path.

- we can use any Path;

$$\int_{x=0}^{2} 10x^2\,dx = \frac{10}{5}x^5 = 2(32) = 64$$

$$\int_{y=0}^{1} -12y^2\,dy = -12\left(\frac{y^3}{3}\right) = -4(1) = -4$$

Then the value of the line Integral;

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy = 64 - 4 = \underline{60}$$

MCQs;

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, Integral is Independent of the path.

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, Integral is Dependent of the path.

~~Module # 76~~

MCQs;

$\oint M\,dx + N\,dy = 0$

$M\,dx + N\,dy = 0$ Around every closed curve C in a simply-connected region if;

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

everywhere in the Region.

Module # 77

Theorem;

Show that the Area bounded by a simple closed curve C is given by;

$$\frac{1}{2}\oint_C x\,dy - y\,dx.$$

Proof;

Greens Theorem;

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

put $M = -y$ and $N = x$ in theorem,

$$\oint_C -y\,dx + x\,dy = \iint_R \left(\frac{\partial (x)}{\partial x} - \frac{\partial (-y)}{\partial y}\right) dx\,dy$$

$$\oint_C x\,dy - y\,dx = 2 \iint_R dx\,dy$$

$$= 2A$$

$$\text{Area} = A = \frac{1}{2}\oint x\,dy - y\,dx$$

Example;

Find the Area of the ellipse

$x = a\cos\theta$, $y = b\sin\theta$

Solution;

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$$= \frac{1}{2}\oint_0^{2\pi} (a\cos\theta)(b\cos\theta)d\theta - (b\sin\theta)(-a\sin\theta)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2\theta\,d\theta + ab\sin^2\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)d\theta$$

$$= ab\cdot\frac{1}{2}\int_0^{2\pi} d\theta = ab\pi.$$

Module # 78

Go through example From Handouts if you want otherwise you can Skip it.

(INTRODUCTION TO Classical Mechanics.)

Imp Short + MCQs

Module # 79

Mechanics; Scienc of Motion

* Studies the state of rest and Motion and the law governing rest, equilibrium and Motion.

Division of Mechanics;

* Classical Mechanics
* Quantam Mechanics.

Classical Mechanics:-

* Deals with the Macroscopic object Macroscopic objects (which can be seen by naked eye)

* Two broad division;
  * Non-Relativistic
  * Relativistic Mechanics.

* Non-Relativistic Mechanics based on the law of Newton's is Concerned with bodies moving at Speeds and velocity negligibly small as compared to the Speed of light $C = 3 \times 10^5$ per Kilometer per second.

* Relativistic Mechanics is dispensable for partiles and bodies moving with Speed Comparable to the Speed of light.

Quantam Mechanics;

Mechanics of Microscopic object.

Microscopic objects (can't be seen by naked eye).

Division of Classical Mechanics:-

① Mechanics of particles and Rigid bodies:- based on Newton's law,

② Mechanics of Fluid; also based on Newton's law.

* Machnics of elastic Solids;

Deals with the behaviour of solids when the undergo deformation under Forces.

Module No. 80

Introduction to the Basic of Newton's law.

Newton's three law of Motion are Formulated in term of underlying concepts; the notations of Space, time, Mass and Force.

Module No. 81

Introduction to Rectangular Components of velocity & Acceleration

Rectangular Components;

The process of Spliting a vector into various parts or components is called as Resolution of vector.

If we spit a vector in a Rectangular plane OXY, Such components are called Rectangular Components.

* Component along x-axis called Horizontal Component of vector.

* Component along y-axis is called the vertical Component of a vector

(31) **Position vector;**

* In Retangular Coordinates, the position vector is simply;

$$r = x\hat{i} + y\hat{j}$$

The components of the position vector of a moving particle are functions of the time, namely,

$$x = x(t), \quad y = y(t)$$

* **Rectangular Components; Velocity;**

If the vector is the position vector r of a moving particle and the parameter is the time t, the derivative of r w.r.t 't' is called the velocity.

$$v = \frac{dr}{dt} = i\dot{x} + j\dot{y} + k\dot{z}$$

* The Magnitude of the velocity is called the Speed.

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

* The direction of v is tangent to the path of Motion.

✱ Rectangular Components: Acceleration:

Acceleration represents the Rate of change in the velocity of a particle.

$$a = \frac{dv}{dt} = \frac{d^2r}{dt^2} = a_x i + a_y j + a_z k$$

In Retangular Components,

$$a = \dot{V}_x t + \dot{V}_y j + \dot{V}_z k$$

$$a = i\ddot{x} + j\ddot{y} + k\ddot{z}$$

* Acceleration is a vector Quantity.
* Componets of Acceleration are the second derivative of the positional Coordinates of a Moving particle.
* The Direction of 'a' is usually not tangent to the path of the particle.

Module # 82

Intro. To Tangential and Normal Components of VA

Introduction;

A vector at a point on a curve can be decomposed uniquely as a sum of two components

- Tangential component => Tangent to curve
- Normal component => Normal to curve

* Tangential velocity is measured at any point tangent to a circular path.
* Normal component of velocity is zero.

velocity; $V = VT$

Components of Acceleration

$$a = T\frac{dv}{dt} + \frac{v^2}{t}N$$

Tangential component; $T\frac{dv}{dt}$

Normal component ; $\frac{v^2}{t}N$

Module # 83

Dnp Example From Handouts

Module # 84

Curvature and Radius of Curvature

Curvature; Curve From being Straight as in the case of a line.

MCQs; ID $\bar{T}$ is a unit tangent vector to a Space curve C, then $\frac{dT}{ds}$ is Normal to T. $\Rightarrow R = 1/k$ is called the Radius of Curvature.

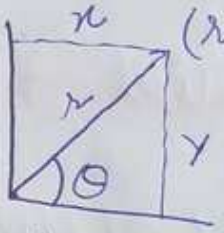
Module # 85

Introduction to Radial and Transverse Components of velocity and Acceleration

- **Polar Coordinate:**

Coordinate in form of (r, θ) are called polar coordinate and Framework is called polar Coordinate System.

- **Position vector:-**



The position of the Coordinate System can be written as the product of the Radial distance r by a unit Radial vector e_r:

$$r = r e_r$$

- The Acceleration vector in Plane polar Coordinates;

$$\underset{\sim}{a} = (\ddot{r} - r\dot{\theta}^2)\underset{\sim}{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underset{\sim}{e}_\theta$$

- Radial component of the Acceleration;

$$a_r = \ddot{r} - r\dot{\theta}^2$$

* Transverse component of Acceleration;

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

Module # 86

Example From Handouts

Module #87

Reference System or Inertial Frame.

* Frame work:
A framework that is used for the observation and mathematical description of Physical phenomena.

* Initial Frames:
Reference in which a body remains at rest or moves with constant linear velocity unless acted upon by forces.

* Any frame of reference that moves with constant velocity relative to an Initial Frame. These are also called Inertial Frame.

* Mechanics is based on Newton's laws of Motion.

* Newton's laws are not valid for every Frame of Reference.

* Newton's laws only Hold in restricted Class of Frames so called the Inertial Frame.

Module # 86

Example From Handouts

Module #87

Reference System or Inertial Frame.

* Frame work:
A framework that is used for the observation and Mathematical description of Physical phenomena.

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* Any frame of reference that moves with Constant velocity relative to an Initial Frame These are also called Inertial Frame.

* Mechanics is based on Newton's laws of Motion.

* Newton's laws are not valid for every Frame of Reference.

* Newton's laws only Hold in restricted Class of Frames So called the Inertial Frame.

Module #88

Examples

From Handouts

Koi ak part aa SKta hai.

Module #89

Newton's First and 2nd law;

Newton's 1st law of Motion;

* "A object at ~~w~~rest will Remain at rest or Moving with Constant Speed until an External Force applied on it."

* If the net Force acting on a body is Zero then the velocity of object is said to be Constant.

* velocity of an object is vector Quantity.

Newton's 2nd law of Motion:-

For any particle of Mass m, the net Force F on the particle of mass m times the particle's Acceleration.

$$F = ma$$

$$\boxed{P = mv} \text{ (Momentum)}$$

Module # 90

The Third law and Law of Conservation of Momentum.

=) **Newton's 3rd law;**

To every action there is an equal and opposite Reaction.

=) law of Conservation of Momentum;

=> **Def of linear Momentum :-**

If a body of a mass m is Moving with velocity, then the Momentum of that body is equal to the product of Mass and velocity of the specific body.

$$\vec{P} = m\vec{v}$$

It is vector Quantity.

SI unit is $Kgms^{-1}$.

=> **law of Conservation of Momentum:-**

If the sum of the external Forces on a system is Zero, the total Momentum of the System does not Change

$$\sum_i m_i v_i = Mv \text{ Constant.}$$

Momentum is alway Conserved.

Module #91

Validity of Newton's Law.

=) Newton's laws are not valid universally after the quantam Mechanics and theory of Relativity.

=) Initially Newton's laws were considered as Fundamental.

=) But Now quantum laws are Fundamental and Newto's law can be observed as derived from the quantum world.

=) Newton's First two laws are always valid In classical phenomena.

Module # 92, 93

Examples of Newton's laws

FROM Handouts

Module # 94

Introduction to Energy :- Kinetic Energy

Energy:- The ability to do work by the object.

=) Energy neither can be created nor be destroyed, but can be converted from one form to another.

=) SI unit of Energy is the Joule;

=) 1 Joule :- can be defined as the Energy transferred to an object by the work done of Moving, it a distance of 1 Meter against a force of 1 Newton.

=) Kinetic Energy; Energy stored in a body due to its Motion.

$$K.E = T = \frac{1}{2} mv^2.$$

=) Measuring unit of Kinetic energy is the Joule.

=) It is denoted by T.

=) Kinetic energy increase with the square of the velocity.

=) If a p denotes Momentum of the object & m is the mass then K.E in from of Momentum as;

$$T = \frac{p^2}{m}$$

Module # 95

Introduction to Work - Theorem

=) **Work Done:-** Product of Force and the displacement in the direction of applied Force.

$$W = Fd$$

=) **SI unit :-** Joule (J)

=) **Joule :-** Work done by a Force of one Newton through a displacement of one Meter.

$$dW = \vec{F} \cdot d$$

For Integration;

$$W = \int_{P_1}^{P_2} \vec{F} \cdot dr = \int_{r_1}^{r_2} \vec{F} \cdot dr$$

=) **Theorem;**

(Prove that) **WORK done** $= \int_{t_1}^{t_2} \vec{F} \cdot dr = \frac{1}{2} m \vec{v}_2^{\,2} - \frac{1}{2} m \vec{v}_1^{\,2}$

Proof;

$$L.H.S = \int_{t_1}^{t_2} F \cdot dr = \int_{t_1}^{t_2} \vec{F} \cdot \frac{dr}{dt} dt$$

$$= \int_{t_1}^{t_2} F \cdot v \, dt \qquad \because \frac{dr}{dt} = \vec{v}$$

$$= \int_{t_1}^{t_2} m \vec{a} \cdot \vec{v} \, dt$$

$$= m\int_{t_1}^{t_2} \frac{d\vec{v}}{dt}\cdot\vec{v}\,dt = m\int_{t_1}^{t_2} \vec{v}\cdot d\vec{v}$$

$$= \frac{1}{2}m\int_{t_1}^{t_2} d(\vec{v}\cdot\vec{v}) = \frac{1}{2}m|\vec{v}^2|_{t_1}^{t_2}$$

$$\text{Work done} = \frac{1}{2}mv_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Proved.

Module # 96

Examples of Work done

EXAMPLE #01;

Find the work done in moving an object along a vector $r = 8\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied Force is $\vec{F} = 2\hat{i} - \hat{j} - \hat{k}$.

Solution:-

$$W = F\cdot r$$
$$= (2\hat{i} - \hat{j} - \hat{k})\cdot(8\hat{i} + 2\hat{j} - 5\hat{k})$$
$$= 16 - 2 + 5 = 19 \text{ Joule.}$$

Example #02:-

Find the work done by the Force of a body of mass $m = 10^4$ kg with Initial velocity $V_1 = 1.5\times10^3 ms^{-1}$ to $V_2 = 3.0\times10^3 ms^{-1}$ at any time t.

Solution; Work done $= \frac{1}{2}m\vec{V}_2^2 - \frac{1}{2}mV_1^2$

$$= \frac{1}{2}m(V_2^2 - \vec{V}_1^2)$$

$$= \frac{1}{2}(10^4)[(3.0\times10^3)^2 - (1.5\times10^3)^2] = 3.38\times10^{10} kgms^{-1}$$
$$= 3.38\times10^{10} J$$

THE END

MTH622 Handmad Mid-Term Highlighted Notes.

Notes prepared by :- Fatima
(@Queen OF KING)

Remember me In your Beautiful Prayers

Best Of luck To All Of You.