MTH 501 Linear Algebra

Lec #01

Introduction and overview

History
Named in Honor of Muhammad Ibn-e-Musa Alkhwarizmi.
The Sci of reduction & cancellation writen by Muhammad Ibn-e-Musa in 825.
Algebra is the branch of mathematics that uses to discribe relationship b/w things and that vary over time.
Defination
Algebra is a Part of math in which unknown qualities are found with the help of relations b/w unknown and known.

Algebraic Term:-

The basic unit of algebraic expression is a term. which is either a Product of a # & with one or more variable.

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Example

$4x$ is an algebraic term in which 4 is coefficient and x is said to be variable.

Algebraic Expression

An expression is a collection of numbers variables and +ve sign or -ve sign of operation

Example

$8x^2+9x-1$ is an algebraic express.

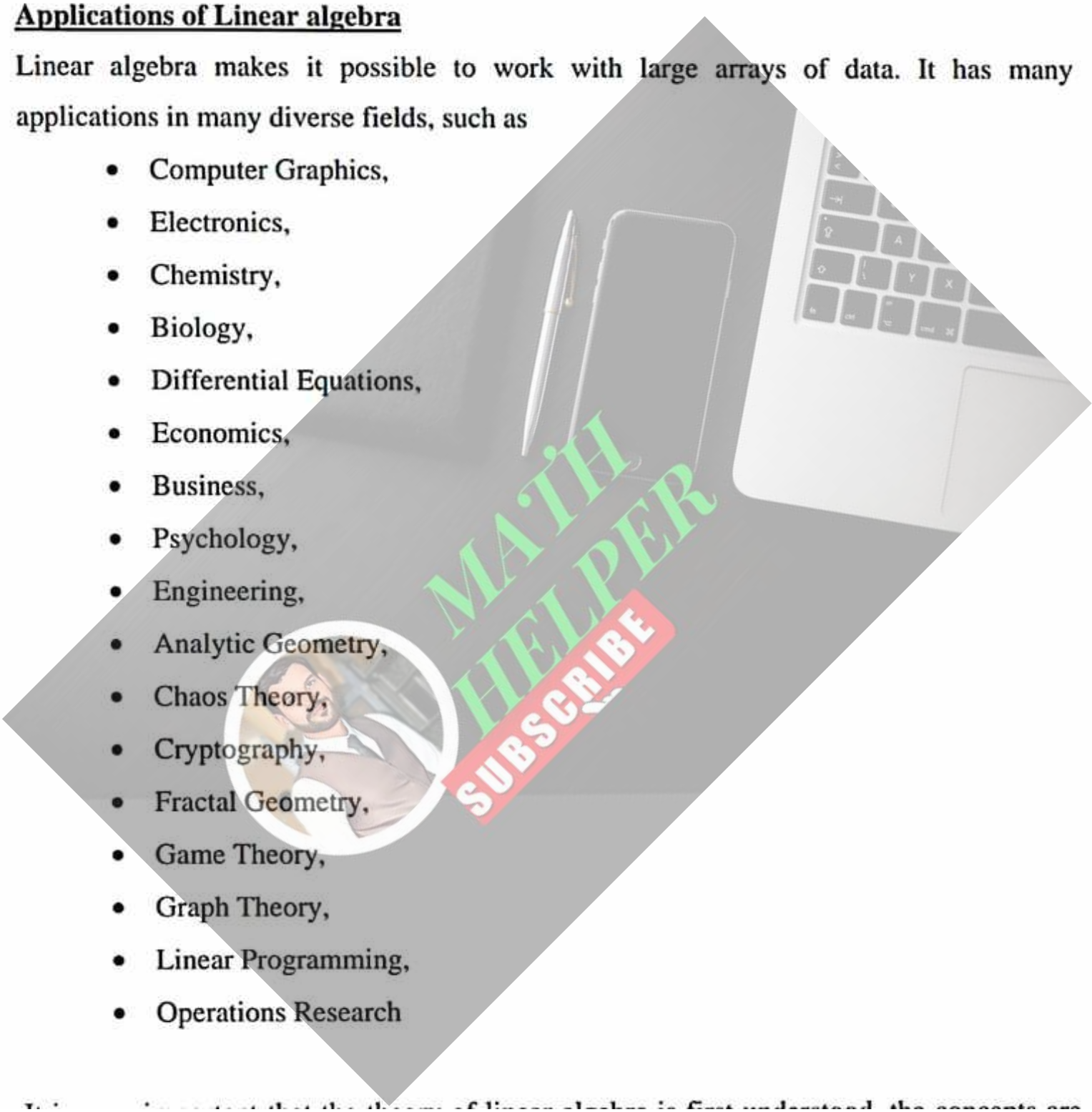


of trying to solve and analyze systems of linear equations.

Applications of Linear algebra

Linear algebra makes it possible to work with large arrays of data. It has many applications in many diverse fields, such as

- Computer Graphics,
- Electronics,
- Chemistry,
- Biology,
- Differential Equations,
- Economics,
- Business,
- Psychology,
- Engineering,
- Analytic Geometry,
- Chaos Theory,
- Cryptography,
- Fractal Geometry,
- Game Theory,
- Graph Theory,
- Linear Programming,
- Operations Research

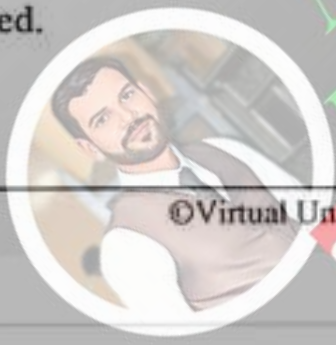


It is important that the theory of linear algebra is first understand, the concepts are

Objects of study in linear algebra

Linear algebra merits study at least because of its ubiquity in mathematics and its applications. The broadest range of applications is through the concept of vector spaces and their transformations. These are the central objects of study in linear algebra

1. The solutions of homogeneous systems of linear equations form paradigm examples of vector spaces. Of course they do not provide the only examples.
2. The vectors of physics, such as force, as the language suggests, also provide paradigmatic examples.
3. Binary code is another example of a vector space, a point of view that finds application in computer sciences.
4. Solutions to specific systems of differential equations also form vector spaces.
5. Statistics makes extensive use of linear algebra.
6. Signal processing makes use of linear algebra.
7. Vector spaces also appear in number theory in several places, including the study of field extensions.
8. Linear algebra is part of and motivates much abstract algebra. Vector spaces form the basis from which the important algebraic notion of module has been abstracted.



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9. Vector spaces appear in the study of differential geometry through the tangent bundle of a manifold.
10. Many mathematical models, especially discrete ones, use matrices to represent critical relationships and processes. This is especially true in engineering as well as in economics and other social sciences.

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Lec #2

Intro to Matrix

Matrix :-

A matrix is a collection of number or functions arranged into row and columns.

Denoted by capital letter A, B, C etc

The numbers or function are called elements. Elements denoted by Small letter a, b, c etc

Rows and Colums :-

Horizontal line of matrix called Row

Vertical Line of matrix called columns.

Order of matrix

The size of matrix is called order of matrix.

In short number of Rows and columns written as $r \times c$: r is row c is columns

Squar matrix

# of Rows = # of columns

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Equality of Matrix

(1) Same order

(2) Corresponding elements must be equal.

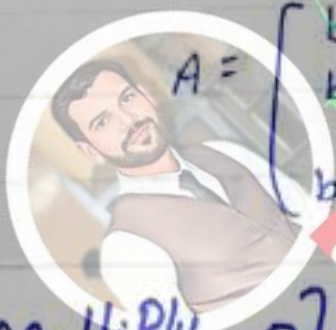
$$A = \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 7 & -8 \\ 9 & 3 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A = B$$

Column Matrix

A matrix having n rows and only one column

$$A = \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \end{bmatrix}$$



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Multiply of Matrix

Ex

$$5 \cdot \begin{bmatrix} 2 & -3 \\ 4 & -1 \\ 1/5 & 6 \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 20 & -5 \\ 1 & 30 \end{bmatrix}$$

we can write $KA = AK$

Addition of Matrices:-

only same order of matrix can be added by adding corresponding element.

Ex

$$A = \begin{bmatrix} 2 & 4 \\ 9 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 8 \\ 1 & 5 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2+6 & 4+8 \\ 9+1 & 7+5 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & 12 \\ 10 & 12 \end{bmatrix}$$

Difference of Matrix

only same order of matrix can subtract by subtracting corresponding elements.

Multiplication of Matrix

only multiple if the # of columnes of first matrix equal to number of Rows of 2nd matrix.

$$A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$$

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$$A = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 9 & -2 \\ 6 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4(9)+7(6) & 4(-2)+7(8) \\ 3(9)+5(6) & 3(-2)+5(8) \end{bmatrix}$$

$$= \begin{bmatrix} 36+42 & -8+56 \\ 27+30 & -6+40 \end{bmatrix}$$

$$AB = \begin{bmatrix} 78 & 48 \\ 57 & 34 \end{bmatrix}$$

Note

$$AB \neq BA$$

Matrix multiplication is commutative.

Multiplicative identity:-

This is commutative $I \cdot A = A \cdot I = A$

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A \cdot I = \begin{bmatrix} 2(1)+4(0) & 2(0)+4(1) \\ 5(1)+6(0) & 5(0)+6(1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} = A$$

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Zero or Null Matrix

All elements of matrix is zero...

Note

$$O + A = A + O = A$$

Associative Law:-

$$A(BC) = (AB)C$$

Singular Matrix

If determinant $= 0$ then Singular

If determinant $\neq 0$ then non Singular

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Minor of an element of a matrix

Find Minor of $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor of an element of matrix

Formula:- $C_{ij} = (-1)^{i+j} M_{ij}$

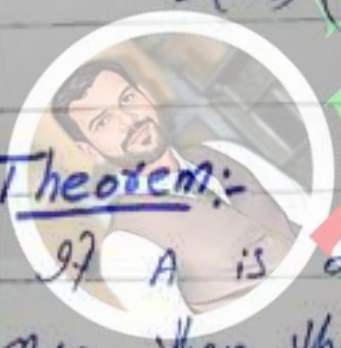
Example

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{bmatrix}$$ is a square matrix

Find cofactor of $3 \in A$ Denoted by C_{12}

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix}$$

$$= (-1)^3 (5 - 0) = -5$$



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Theorem:-

If A is a square matrix of order $n \times n$ then the matrix has a multiplicative inverse A^{-1} if and only if the matrix A is non singular.

Then inverse of matrix A

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

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Lec # 03

System of Linear Equation

Linear eq.

Stright Line general eq.

$$y = mx + c$$

where m show slope of line

c is y intercept

R^2 (2 dimension) can be repersented by an eq. $a_1x + a_2y = b$ (a_1, a_2, b constant number not equal to 0.

in R^3 (3 dimensional space)

$$a_1x + a_2y + a_3z = b$$

A linear eq of n variables $x_1, x_2, \ldots, x_n$ can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \text{ (hyper plane in } R^n)$$

Homogeneous Linear eq.

If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ $\quad b = 0$

this is called homogeneous Linear eq.

NOTE In Linear eq does not involve any Product or S.q.rt of variable. & all variable occure only to the first power.

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Ex (1) $2x_1 + 3x_2 + 2 = x_3$ Linear eq

2) $x_2 = 2(\sqrt{5} + x_1) + 2x_3$ Linear eq.

↓ S.Q.R.T of number allow

$3x_1 - 2x_2 = x_1 x_2$ not Linear eq.

$x_2 = 4\sqrt{x_1} - 6$ " " "

System of Linear eq.

A finite set of Linear eq is called System of Linear eq or Linear system.

$$4x_1 - x_2 + 3x_3 = -1$$

The variables called unknown.

General system of Linear eq.

" " " " " m

eq and n-unknown

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Solution of system of Linear eq.

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Linear system with two unknow:-

when 2 lines intersect in R^2

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

There are three Posibilities

(1) The Lines may be Parallel no intersection no Solution

(2) The Lines intersect at one Point one solution

(3) The Lines may coincide there are infinity many points of intersection consequently Infinity many Sol.

Consistent and inconsistent system:-

consistant: A system have at least one Sol.

inconsistent: If system has no Sol.

Example

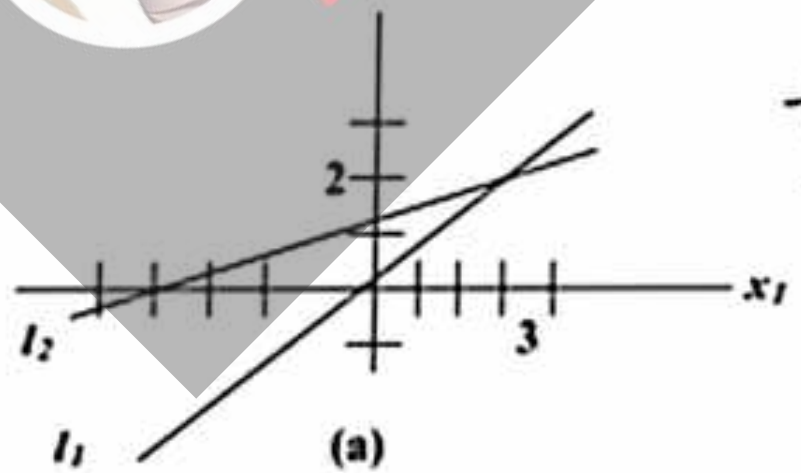
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Thus, a consistent linear system of two equations in two unknowns has either one solution or infinitely many solutions – there is no other possibility.

Example consider the system of linear equations in two variables
$x_1 - 2x_2 = -1,\ -x_1 + 3x_2 = 3$

Solve the equation simultaneously:

Adding both equations we get $x_2 = 2$. Put $x_2 = 2$ in any one of the above equation we get $x_1 = 3$. So the solution is the single point (3, 2). See the graph of this linear system



(a)

This system has exactly one solution

$x_1 - 2x_2 = -1$ — (1)

$+\quad -x_1 + 3x_2 = 3$ — (2)

$x_2 = 2$

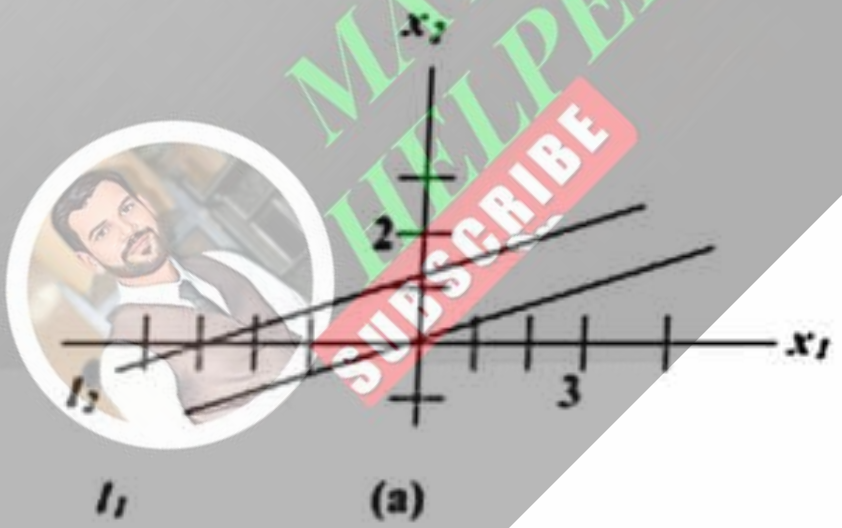
Put in (1)

$x_1 - 2(2) = -1$

$x_1 = 3$

(a) $x_1 - 2x_2 = -1$

$-x_1 + 2x_2 = 3$

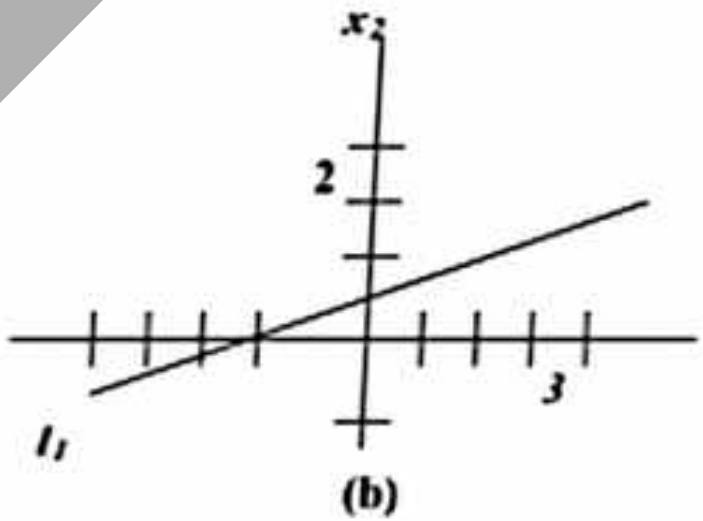


(a)

(a) No solution.

(b) $x_1 - 2x_2 = -1$

$-x_1 + 2x_2 = 1$



(b)

(b) Infinitely many solutions.

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Linear system with 3 unknowns:-

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Parametric Representation:-

Matrix Notation:-

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$A \qquad X = B$$

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An augmented matrix when we write coefficient matrix with an added column of constants from the right side denoted by A_b

Solve a linear system:-

(1) Successive elimination Method:-

Ex

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

Augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Sol To Perform Procedure of elimination

convert ↓

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To eliminate x_1 from third eq multiply eq 1 by 4

$$4x_1 - 8x_2 + 4x_3 = 0$$
$$+ \quad -4x_1 + 5x_2 + 9x_3 = -9$$
$$\overline{-3x_2 + 13x_3 = -9}$$

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So the result written as

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-3x_2 + 13x_3 = -9$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

Now Multiply eq 2 by $1/2$ to convert x_2 coefficient 1

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$-3x_2 + 13x_3 = -9$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

To eliminate x_2 from eq 3 add 3 time eq 2

$$3x_2 - 12x_3 = 12$$
$$-3x_2 + 13x_3 = -9$$
$$\overline{x_3 = 3}$$

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now to eliminate x_3 from eq 2 add 4 time eq 3

$$x_2 - 4x_3 = 4$$
$$+4x_3 = 12$$
$$\overline{x_2 = 16}$$

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$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 = 16$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Subtract eq 3 from eq 1

$$x_1 - 2x_2 + x_3 = 0$$
$$- \quad -x_3 = 3$$
$$\overline{x_1 - 2x_2 = -3}$$

$$x_1 - 2x_2 = -3$$
$$x_2 = 16$$
$$x_3 = 3$$

Now add eq 2 by 2 time in eq 1

$$x_1 - 2x_2 = -3$$
$$+ 2x_2 = 32$$
$$\overline{x_1 = 29}$$

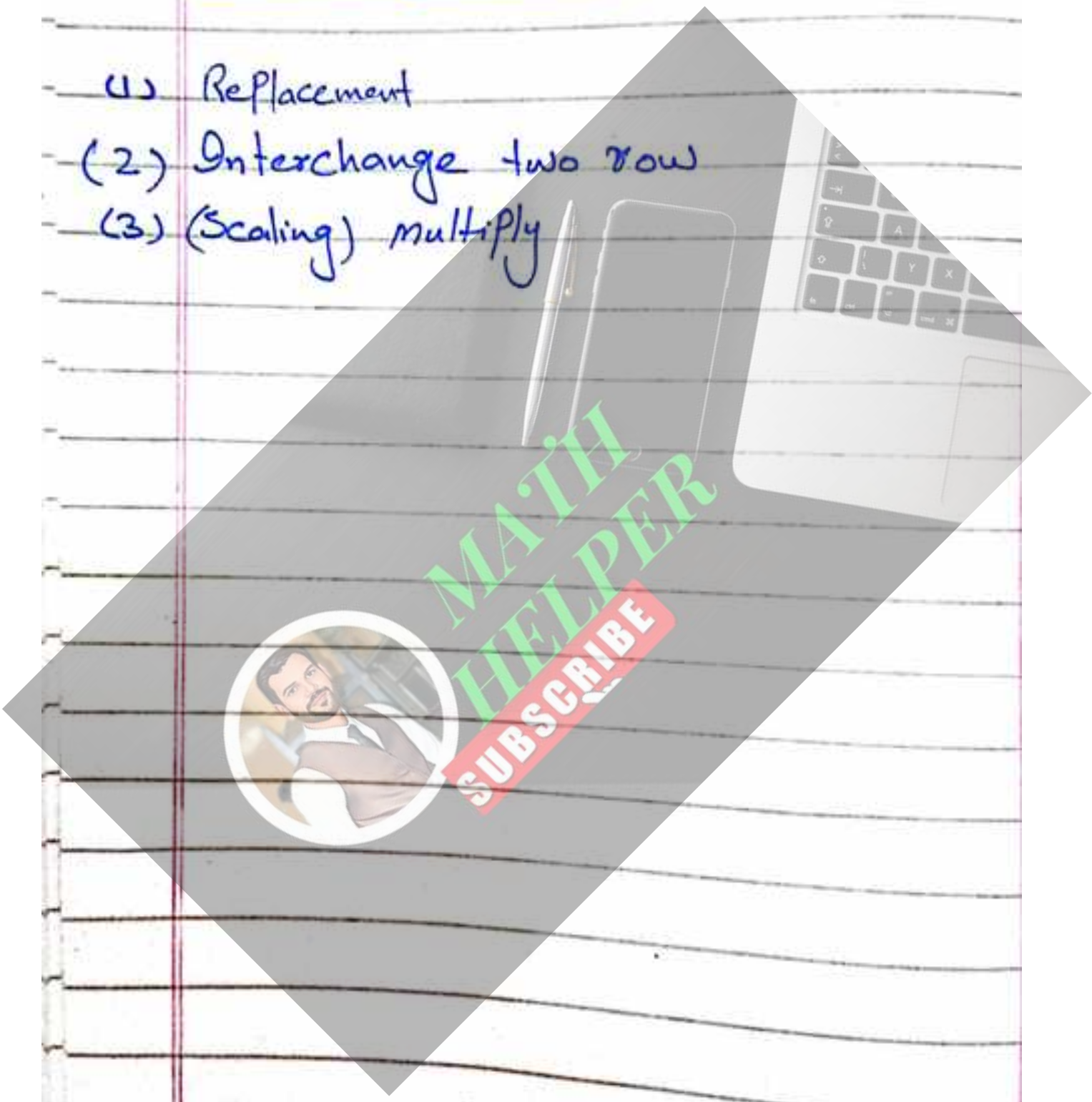
$$x_1 = 29$$
$$x_2 = 16$$
$$x_3 = 3$$

$$\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

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2 Elementry Row operation

(1) Replacement
(2) Interchange two row
(3) (Scaling) multiply



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Lecture # 04

Row Reduction & Echelon Forms

Echelon Form of a matrix

A rectangular matrix is in echelon form if it has the following three properties

(1) All nonzero rows are above any rows of all zeros.

(2) Each leading entry of a row is in column to the right of the leading entry of the row above it.

(3) All entries in a column below a leading entry are zero.

Reduced Echelon form of a matrix

If a matrix is in echelon form satisfy the following additional condition

(1) The leading entry in each nonzero row is 1.

(2) Each leading 1 is the only nonzero entry in its column.

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Ex. of echelon matrix

(1) $\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 6 & 5/2 \end{bmatrix}$

Ex. Reduced Echelon form of matrix:

(1) $\begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

NOTE A matrix may be row reduced into more than one matrix in echelon form, using different sequences of row operations. However the reduced echelon form obtained from matrix is unique.

Theorem:- Each matrix is row equivalent to one & only one reduced echelon form.

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Pivot Positions:-

A Pivot Position in a matrix A is a location in A that corresponds to a leading entry in an echelon form of A

Pivot Column:-

A Pivot Column is a Column of A that Contains a Pivot position.

Example

Apply elementry row operations to transform the following matrix first into echelon and then into reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

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Solution

$$= \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim (R_1 \leftrightarrow R_3)$$

$$= \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \sim (R_2 - R_1)$$

$$= \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & -2 & 4 \end{bmatrix} \sim \left(R_3 - \frac{3}{2} R_2\right)$$

⇓

This form is echelon form/Row echelon form of given matrix

Now Convert into Reduced echelon form

$$= \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 6 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} \sim \frac{1}{3} R_1 \\ \sim \frac{1}{2} R_2 \\ \sim -\frac{1}{2} R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim (R_1 + 3R_2)$$

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$$= \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 10 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim (R_1 - 5R_3)$$

$$= \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 10 \\ 0 & 1 & -2 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim (R_2 - R_3)$$

$\downarrow$

This is reduced echelon form of given matrix.

Solution of linear System :-

When the algorithm is applied to the augmented matrix of the system it gives solution set of linear system.

Example

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 5x_3 = 1$

$x_2 + x_3 = 4$

$0 = 0$ which mean x_3 is free

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Lecture #5

Vector Equation:

Column vector:-

A matrix with only one column is called column vector or simply a vector.

Ex $u=\begin{bmatrix}3 & -1\end{bmatrix}^T=\begin{bmatrix}3\\-1\end{bmatrix}$ $v=\begin{bmatrix}2 & 3 & 5\end{bmatrix}^T=\begin{bmatrix}2\\3\\5\end{bmatrix}$

Vectors in R^2

If R is the set of all real numbers then the set of all vectors with two entries is denoted by $R^2=R\times R$

Ex $u=\begin{bmatrix}3 & -1\end{bmatrix}^T=\begin{bmatrix}3\\-1\end{bmatrix}\in R^2$

Note The entries in vectors are assumed to be the elements of a set called a Field denoted by F.

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Algebra of vectors

Equality of vector in R^2

Two vectors in R^2 are equal if and only if their corresponding entries are equal

Ex

if $U=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $V=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2$

then $U=V$ iff $u_1=v_1 \wedge u_2=v_2$

NOTE In fact vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in R^2 are nothing but order pairs (x,y) of real number both representing the position of a point w.r.t origin.

Addition of vectors:-

Given to vectors u & v in R^2 their sum is the vector $u+v$ obtained by adding corresponding entries of vector u & v

Ex

$$U=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, V=\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2$$

$$U+V=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} \in R^2$$

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Scalar multiplication of a vector

Let u be a given vector and c is a number then scalar multiplication of vector obtained multiply each entry of u with c.

Notations

The scalar multiply denoted as cu
scalar light face and vector bold face

NOTE:

We can write a ~~matrix~~ vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form $(3, -1)$ but not $[3, -1]$

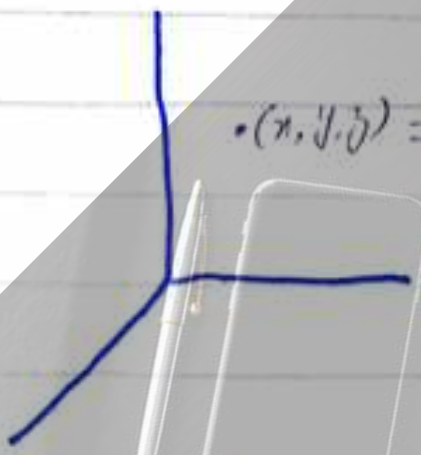
$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = (3, -1) \qquad \begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq [3 \quad -1]$$



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Geometric Descriptions of R^2

Vector in R^3



Algerbic Properties of R^n

i) $u + v = v + u$ commutative

$(u+v)+w = u+(v+w)$ Associative

$u + 0 = 0 + u = u$ Additive identity

$u + (-u) = (-u) + u = 0$ " Inverse

$c(u+v) = cu + cv$

$(c+d)u = cu + dv$

$d(du) = (cd)u$

$1u = u$

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Linear Combination:-

Let $v_1, v_2 \cdots v_p \in R^n$ and $c_1, c_2 \cdots c_p$ are scalars then

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

is called linear combi. of $v_1, v_2 \cdots v_p$ using weight $c_1, c_2 \cdots c_n$

Spanning set:-

If $v_1, \cdots v_p$ are in R^n then the set of all linear combi. of $v_1, \cdots v_p$ is denoted by $\text{Span}\{v_1, \cdots v_p\}$ is called the subset of R^n spanned

Vector eq of Line

$$x - x_0 = tv \qquad (-\infty < t < +\infty)$$

This is vector eq of line through x_0 & parallel to v

when the line passes through origin

$$x = tv \qquad (-\infty < t < +\infty)$$

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Parametric eq of a line in R^2

$$x = x_0 + at \;,\; y = y_0 + bt \quad (-\infty < t < +\infty)$$

Parametric eq of a line in R^3

$$x = x_0 + at \;,\; y = y_0 + bt \;,\; z = z_0 + ct \quad (-\infty < t < +\infty)$$

Example

(a) Find a vector eq & Parametric eq of the line in R^2 that passes through the origin & is // to the vector $v = (-2, 3)$

(b) R^3 that passes through the point $P_0(1, 2, -3)$ & // to the vector $v = (4, -5, 1)$

(c) Use vector eq obtained in Part 'b' to find two points on the line that are different from P_0.

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Sol

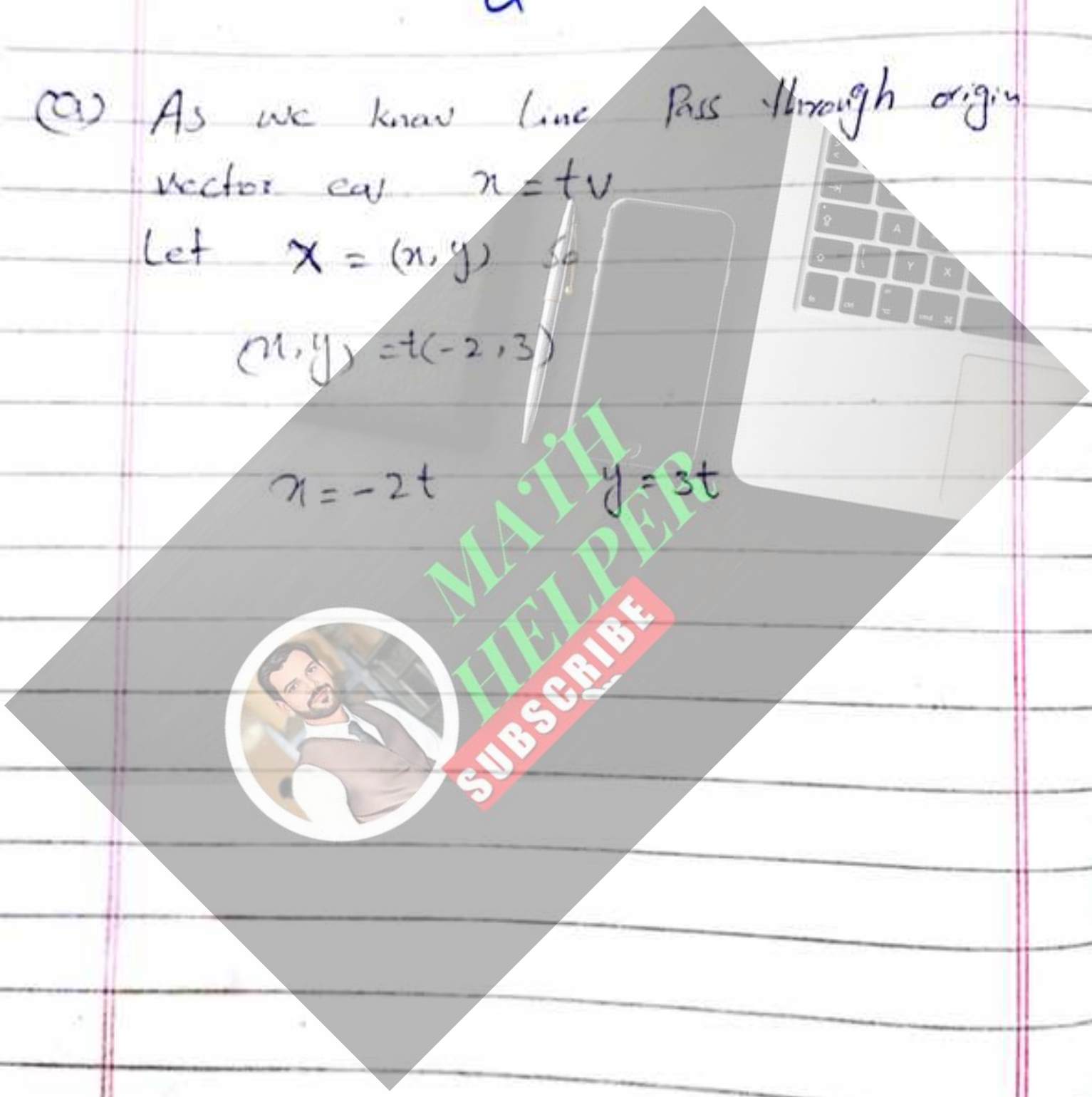
(a) As we know Line Pass through origin

vector eq $x = tv$

Let $x = (x, y)$ so

$(x, y) = t(-2, 3)$

$x = -2t \qquad y = 3t$



Vector eqn. of a Plane

$$r - r_0 = t_1 v_1 + t_2 v_2$$

If Plane passes through origin then eqn will be

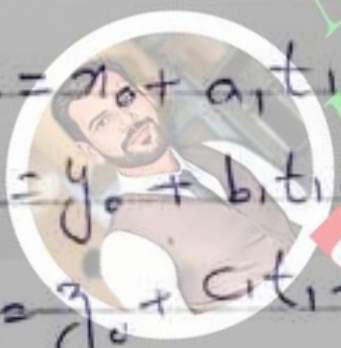
$$r = t_1 v_1 + t_2 v_2$$

Parametric eqn of Plane

$$x = x_0 + a_1 t_1 + a_2 t_2$$

$$y = y_0 + b_1 t_1 + b_2 t_2 \quad (-\infty < t_1 < +\infty)$$

$$z = z_0 + c_1 t_1 + c_2 t_2 \quad (-\infty < t_2 < +\infty)$$



MTH 501

Lecture #6

Matrix eq.

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector

Definition

If A is an $m \times n$ matrix with column $a_1, a_2 \cdots a_n$ and if x is in R^n then the product of A and x denoted by Ax is the linear combination of the column of A using the corresponding entries in x as weights that is

$$Ax = [a_1 a_2 \cdots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Note That Ax is defined only if the number of columns of A equals the number of entries in x.

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Example

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} =$$

$$= 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \end{bmatrix} + \begin{bmatrix} -7 \\ 21 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{Ans}$$

Ex For v_1, v_2, v_3 in R^m write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector

So! Place v_1, v_2 & v_3 in the columns of a matrix A & Place the weights $3, -5$ & 7 into vector x

$$3v_1 - 5v_2 + 7v_3 = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = Ax$$

How to write system of linear eq. as a vector eq involving a linear combination of vector

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system =

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1$$

is equivalent to

Linear combination form

$$= x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \rightarrow \text{matrix eq.}$$

$Ax = b$ is matrix eq

Theorem:-

If A is an $m \times n$ matrix with columns $a_1, a_2 \cdots a_n$ and if 'b' is in R^m the matrix eq. $Ax = b$ has the same solution set as the vector eq

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

Existance of Solutions:-

The eq. $Ax = b$ has a solution iff b is a linear combination of the columns of A.

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Ex: Which of the following are linear combination of

$$A=\begin{bmatrix}4 & 0\\ -2 & -2\end{bmatrix}, \quad B=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}, \quad C=\begin{bmatrix}0 & 2\\ 1 & 4\end{bmatrix}$$

(a) $\begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix}$ (b) $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}6 & 0\\ 3 & 8\end{bmatrix}$

Sol:

$$\begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix} = aA + bB + cC$$

$$\begin{bmatrix}6 & -8\\ -1 & -8\end{bmatrix} = a\begin{bmatrix}4 & 0\\ -2 & -2\end{bmatrix} + b\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix} + c\begin{bmatrix}0 & 2\\ 1 & 4\end{bmatrix}$$

$$4a+b=6 \quad \text{---(i)}$$

$$-b+2c=-8 \quad \text{---(ii)}$$

$$-2a+2b+c=-1 \quad \text{---(iii)}$$

$$-2a+3b+4c=-8 \quad \text{---(iv)}$$

So

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Subtracting eq 4 from 3

$$-b-3c=7$$

Subtracting eq 5 from 2

$$5c=-15$$

$$\boxed{c=-3}$$

From (2) $-b+2(-3)=-8$

$$\boxed{b=2}$$

From (3) $2a+2(2)-3=-1$

$$a=1$$

It means that $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$ is linear combination of A, B & C

Theorem:- Let A be an mxn matrix then the following statments are logically equivlent

(a) For each b in R^m the eq $Ax=b$ has a solution

(b) The columns of A span R^m

(c) A has Pivot position in every row.

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Ex In each part Determin wether the given vector span R^3

(a) $V_1 = (2, 2, 2)$, $V_2 = (0, 0, 3)$

$V_3 = (0, 1, 1)$

اگر یہ مطلب ہمارے پاس جو موجود ویکٹرز ہیں
مختلف constant کو استعمال کرتے
space پیدا کر سکیں تو
اس کو $\text{Span } R^3$ کہتے ہیں

Sol

we have to determin whether arbitary vector $b = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination $b = k_1v_1 + k_2v_2 + k_3v_3$ of the vectors v_1, v_2, v_3

Expressing this in terms of components given by

$$b_1, b_2, b_3 = k_1(2,2,2) + k_2(0,0,3) + k_3(0,1,1)$$

$$= (2k_1 + 0k_2 + 0k_3 , 2k_1 + 0k_2 + k_3 , 2k_1 + 3k_2 + k_3)$$

$$b_1 = 2k_1 + 0k_2 + 0k_3$$

$$b_2 = 2k_1 + 0k_2 + k$$

$$b_3 = 2k_1 + 3k_2 + k_3$$

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$ has a non zero determinent

$$\det(A) = -6 \neq 0$$

Therefor v_1, v_2, v_3 span R^3

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Lecture #07 MATH HELPER

1.2, 1.3

Solution sets of Linear Systems

Solution set:

The set of all possible solutions is called solution set.

Homogeneous Linear system:-

A system of linear eq. is said to be homogeneous if it can be written in the form of $Ax = 0$ where A is $m \times n$ matrix & 0 is the 0 vector in R^m.

Trivial Solution:-

If a homogenous eq $Ax = 0$ is trivial solution if the value of variables of the eq have zero sol.

If the value of unknowns other than zero the solution is non-trivial sol.

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Existence & uniqueness Theorem:-

The homogeneous eq $Ax=0$ has a nontrivial solution iff the eq has at least one free variable.

Example

Find Sol. Set

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$3x_1 + 2x_2 - 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Sol:-

$$A = \begin{bmatrix} 3 & 5 & -4 \\ 3 & 2 & -4 \\ 6 & 1 & -8 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

$$\begin{bmatrix} 3 & 5 & -4 \\ 3 & 2 & -4 \\ 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Augmented Matrix

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 3 & 2 & -4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

For sol. set Row Reduce to Reduced Echelon form.

$$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \quad -1R_1+R_2,\ -2R_1+R_3$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -3R_2+R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{1}{3}R_1,\ \frac{1}{3}R_2,\ \frac{5}{3}R_2+R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (-1)R_2$$

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$$x_1 - \tfrac{4}{3}x_3 = 0$$

$$x_2 = 0$$

$$0x_3 = 0$$

It is clear that x_3 is a free variable so $Ax = 0$ has nontrivial sol.

$$x_1 = \tfrac{4}{3}x_3 \,, \quad x_2 = 0 \text{ with } x_3 \text{ free}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \tfrac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 v$$

where $v = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$

This shows that every sol $Ax = 0$ in this case is scalar multiple of v.

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Geometric interpretation:-

NOTE:- A nontrivial sol. x can have some zero entries so long as not all of its entries are 0.

Example

Solve the following system

$$10x_1 - 3x_2 - 2x_3 = 0 \quad \text{——(1)}$$

Sol

$$\frac{10x_1}{10} = \frac{3x_2}{10} + \frac{2x_3}{10}$$

$$x_1 = \frac{3}{10}x_2 + \frac{1}{5}x_3$$

As a vector general sol.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

u $\quad$ v

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This shows that every sol. of (4) one is a linear combination of the vector u, v show in (2) that is the Sol set is span{u, v}

Example #3

Find the sol. set of the following homogeneous system.

$$x_1 + 3x_2 + x_3 = 0$$
$$-4x_1 - 9x_2 + 2x_3 = 0$$
$$-3x_2 - 6x_3 = 0$$

Sol

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \quad 4R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{1}{2}R_2, (-3)R_2 + R_1$$

So $x_1 - 5x_3 = 0$

$x_2 + 2x_3 = 6$

$0 = 0$

x_3 is free variable

$x_1 = 5x_3$, $x_2 = -2x_3$ with x_3 is free variable

As a vector the general sol of $Ax = 0$ is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

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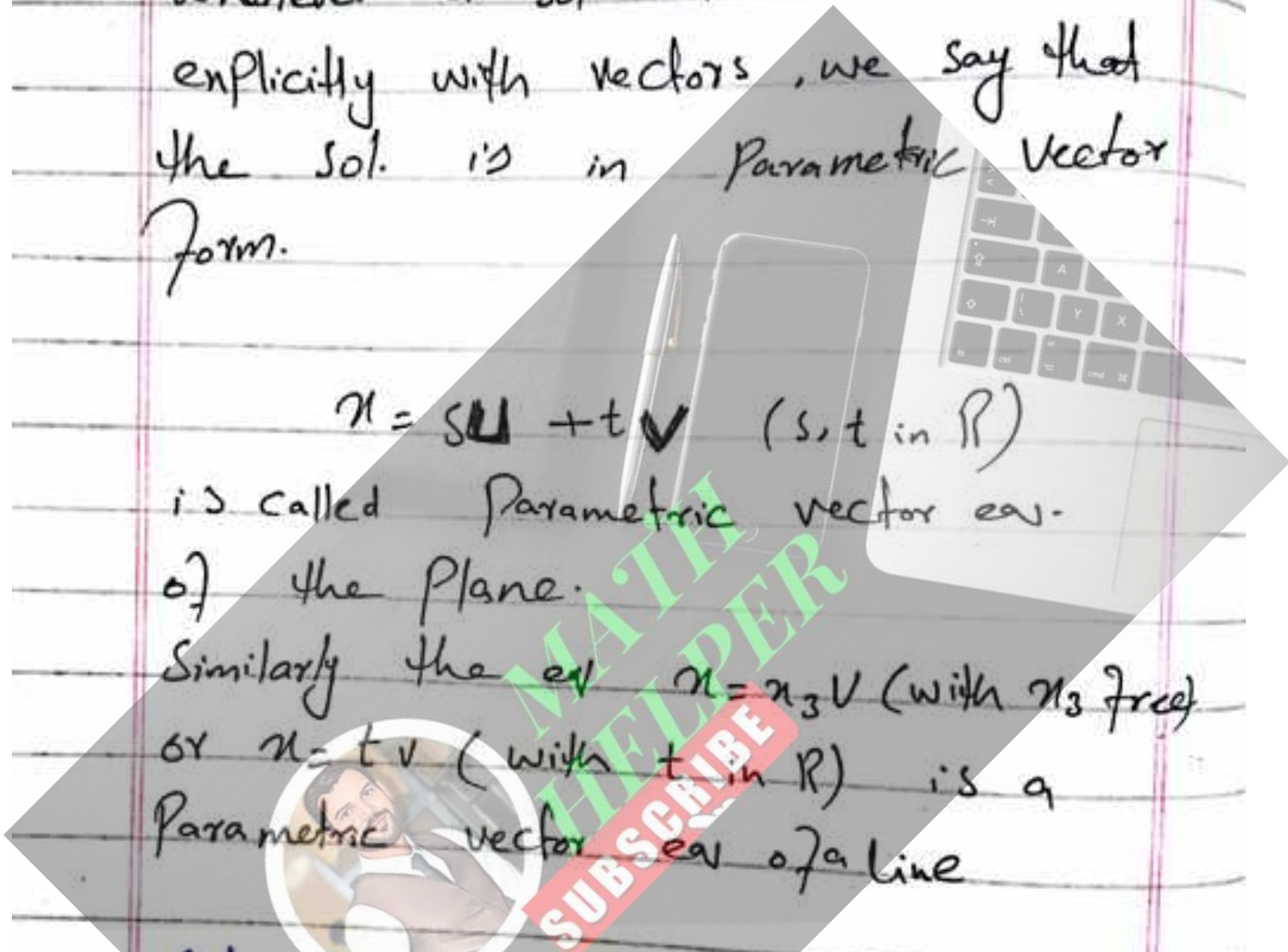
Parametric vector form of the sol

Whenever a sol set is described explicitly with vectors, we say that the sol. is in Parametric vector form.

$$x = su + tv \quad (s, t \text{ in } \mathbb{R})$$

is called Parametric vector eq. of the plane.

Similarly the eq $x = x_3 v$ (with x_3 free) or $x = tv$ (with t in $\mathbb{R}$) is a Parametric vector eq of a line



Sol. of nonhomogeneous system:-

Ex Describe all sol of $Ax = b$

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Sol

Augmented Matrix

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} \quad R_1+R_2, -2R_1+R_3$$

$$\sim \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad 3R_2+R_3, \frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -5R_2+R_1, \frac{1}{3}R_1$$

$$x_1 - \tfrac{4}{3}x_3 = -1$$

$$x_2 = 2$$

$$0 = 0$$

thus $x_1 = -1 + \frac{4}{3}x_3 \quad x_2 = 2$ & x_3 is free

as a vector the general sol of $Ax=b$ has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

$\downarrow$ P $\downarrow$ V

The eqn $x = P + x_3 V$ or, writing t as a general parameter

$$x = P + tV \ (t \text{ in } R)$$

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Theorem:-

Suppose the eqn. $Ax = b$ is consistent for some given b, and let P be a solution then the solution set of $Ax = b$ is the set of all vectors of form $w = p + v_h$ where v_h is any sol of the homogeneous eqn $Ax = 0$

Steps of writing a Sol. Set in a Parametric Vector Form

Step 1

Row Reduces the augmented matrix to reduced echelon form

Step 2

Express each basic variable in terms of any free variables appearing in an eq.

Step 3

Write a typical sol x as a vector whose entries depends on the free variables if any.

Step 4

Decompose x into a linear combination of vectors (with numeric entries) using the free variables as parameters.



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MTH 501

Lecture # 8

Linear Dependent

Definition:-

Set of vectors $\{v_1, v_2 \cdots v_p\}$ in R^n is said to be linearly independent if the vector eq $x_1v_1 + x_2v_2 + \cdots + x_pv_p = 0$ has only trivial solution.

The set of vectors $\{v_1, v_2 \cdots v_p\}$ is said to be linearly dependent if their weight exist not all zero such that

$$c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$$

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Ex Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ & } v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

(a) Determine the set of vector $\{v_1, v_2, v_3\}$ is linearly independent or not.

(b) If possible find a linear dependence relation among v_1, v_2, v_3

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Sol

(a) Row oper. on the associated augmented matrix

$$= \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \quad (-2)R_1 + R_2, (-3)R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 + R_1$$

Eq form

$$x_1 + 4x_2 + 2x_3 = 0$$

$$-3x_2 - 3x_3 = 0$$

$$0x_3 = 0$$

x_1 & x_2 are basic and x_3 is free variable as x_3 free variable and $x_3 = 0$ So the non trivial Sol. exist so by defination nontrivial Sol. shows Linearly dependent.

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(b) To find a linear dependence relation v_1, v_2, v_3 completly row reduce the augmented matrix & write the new system.

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} -\frac{1}{3}R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 4R_2$$

$$x_1 - 2x_3 = 0$$

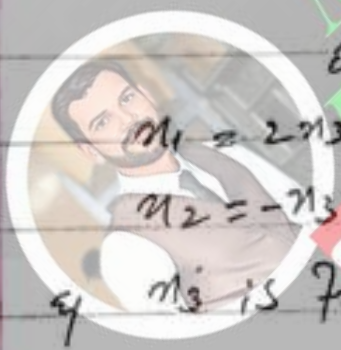
$$x_2 + x_3 = 0$$

$$0 = 0$$

$x_1 = 2x_3$

$x_2 = -x_3$

& x_3 is free



Choose any nonzero value of x_3

say $x_3 = 5$

then

$$x_1 = 2(5) = 10$$

$$x_2 = -(5) = -5$$

So general eq $x_1v_1 + x_2v_2 + x_3v_3 = 0$

So $10v_1 - 5v_2 + 5v_3 = 0$

This is one possible linear dependence relation among v_1, v_2 & v_3

Ex Check vectors are linearly dependent or linearly independent

$v_1 = (3, -1)$ $v_2 = (-2, 2)$

Sol

Consider two constant c_1 and c_2

Using relation

$$x_1 v_1 + x_2 v_2 = (0, 0)$$

replace x_1 as c_1 & x_2 as c_2

$$c_1 v_1 + c_2 v_2 = 0$$

$$c_1(3, -1) + c_2(-2, 2) = 0$$

$$(3c_1 - 2c_2, -c_1 + 2c_2) = 0$$

Now set each of the component equal to 0

$3c_1 - 2c_2 = 0$ — (i)

$-c_1 + 2c_2 = 0$ — (ii) Solving i & ii

$+$ ——————

$2c_1 = 0$

$c_1 = 0$ & $c_2 = 0$

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The trivial sol. is the only sol. So these two vectors are linearly independent.

Linear independence of matrix column

The column of a matrix are linearly independent iff the eq $Ax=0$ has only the trivial sol.

Ex Determine the column of

$$A=\begin{bmatrix}0 & 1 & 4\\ 1 & 2 & -1\\ 5 & 8 & 0\end{bmatrix}$$ are linearly independ

Sol

To study $Ax=0$, row reduce the augmented matrix

$$\begin{bmatrix}0 & 1 & 4 & 0\\ 1 & 2 & -1 & 0\\ 5 & 8 & 0 & 0\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 1 & 4 & 0\\ 5 & 8 & 0 & 0\end{bmatrix} \quad R_{12}$$

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$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & -2 & 5 & 0 \end{bmatrix} \quad (-5)R_1 + R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix} \quad 2(R_2) + R_3$$

at this point there are three basic variable no free variable so the eq $Ax=0$ has only trivial sol.
& the column of A are linearly independent.

Sets of one or two vectors :

A set containing only one vector (say, v) is linearly independent if and only if v is not zero vector.

Ex $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Sol

Notice that v_2 is the multiple of v_1

so $v_2 = 2v_1$

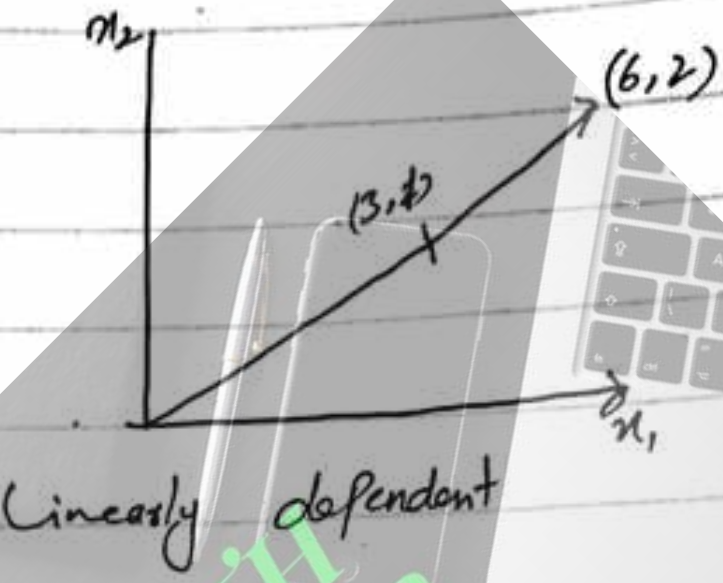
$v_2 - 2v_1 = 0$

$-2v_1 + v_2 = 0$

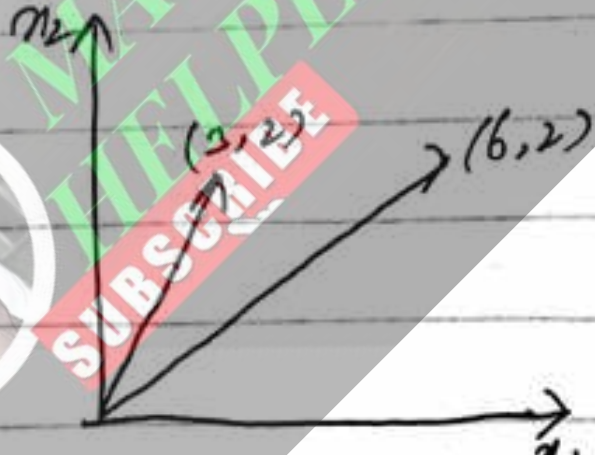
Which shows that $\{v_1, v_2\}$ is linearly dependent.

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NOTE A set of two vectors $\{v_1, v_2\}$ is linearly dependent iff one of the vector is multiple of the other



Linearly dependent



Linearly independent

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Theorem:- (Characterization of linearly dependent)

An indexed set $S=\{v_1, v_2 \cdots v_p\}$ of two or more vectors is linearly dependent iff one of the vectors in S is a linear combination of the others in fact if S is linearly dependent and $v \neq 0$

Theorem:-

If a set contains more vectors than there are entries in each vector then the set is linearly dependent. That is any set $\{v_1, v_2 \cdots v_p\}$ in R^n is linearly dependent if $p > n$

Theorem:- If a set $S=\{v_1, v_2 \cdots v_p\}$ in R^n contains the zero vector then the set is linearly dependent.

$$\begin{bmatrix}2\\3\\5\end{bmatrix} \cdot \begin{bmatrix}0\\0\\0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\8\end{bmatrix}$$

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Lec #10

The matrix of a linear transformation

Ex 1 The column of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Suppose T is a linear transformation from R^2 into R^3 such that

$$T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} \quad \& \quad T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

Find a formula for the image of an arbitrary x in R^2

So

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= x_1 e_1 + x_2 e_2$

T is linear transfo.

$$T(x) = x_1 T(e_1) + x_2 T(e_2)$$

$$T(x) = x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x + 0 \end{bmatrix}$$

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Theorem

Let $T: R^n \to R^m$ be a linear transformation. Then their exists a unique matrix A such that $T(x) = A(x)$ for all x in R^n

A matrix of $m \times n$ whose jth column is the vector $T(e_j)$ where e_j is the jth colum of the identify matrix R^n

$$A = [T(e_1) \cdots T(e_n)]$$

Example 2

For the standard dilation Transf. matrix A

$$T(x) = 3x \qquad x \in R^2$$

$$T(e_1) = 3e_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \& \quad T(e_2) = 3e_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

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Ex 3 Let $L: R^3 \to R^3$ is the linear operator defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix}$$

Find standard matrix representing L & verify $L(x) = A(x)$

So!

The standard matrix A show L is the 3×3 matrix whose cdu. are

$$[L(e_1) \cdot L(e_2) \cdot L(e_3)]$$

$$L(e_1) = L\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 0-0 \\ 1+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = col_1(A)$$

$$L(e_2) = L\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+1 \\ 1-0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = col_2(A)$$

$$L(e_3) = L\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0-1 \\ 0+1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = col_3(A$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y-z \\ x+z \end{bmatrix} = L(x)$$

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Ex A reflection w.r.t x-axis of a vector u in R^2 defined by the Linear operator

$$L(u) = L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Hence Standard matrix shows L is $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So $L(u) = A(u) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

Existence & uniqueness of the sol of $T(x) = b$



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Definition

A mapping $T: R^n \to R^m$ is said to be onto R^m if each b in R^m is the image of at least one x in R^n.

or

Equivalently T is onto R^m if for each b in R^m there exist at least one sol. of $T(x) = b$ "Does T map R^n onto R^m?" is an existance or.

The mapping T is not onto when there is some b in R^m such that the eqv. $T(x) = b$ has no sol.

Definition

A mapping $T: R^n \to R^m$ is said to be one-to-one (or 1:1) if each b in R^m is the image of at most one x in R^n

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MTH 501
Lec#11
Matrix operation

(i-j)th elements of a Matrix

Let A be a mxn matrix m and n are rows and column of matrix respectively. a_{ij} shows the ith row & jth column of the matrix

$$\text{Row } i \begin{bmatrix} a_{11} & a_{ij} & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{i1} & a_{ij} & a_{in} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{mj} & a_{mn} \end{bmatrix} = A$$



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Diagonal matrix

A square matrix whose non-diagonal entries are zero and minimum one entry in diagonal non zero

$$\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Null or zero matrix:-

A $m \times n$ matrix whose entries are all zero

Equal matrices:-

If they have same order or same number of rows and column and same corresponding entries.

Example

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x+1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

The matrix A & B equal if $x+1=5$ and $x=4$. $A \& B \neq C$ b/c different order

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Example

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$$

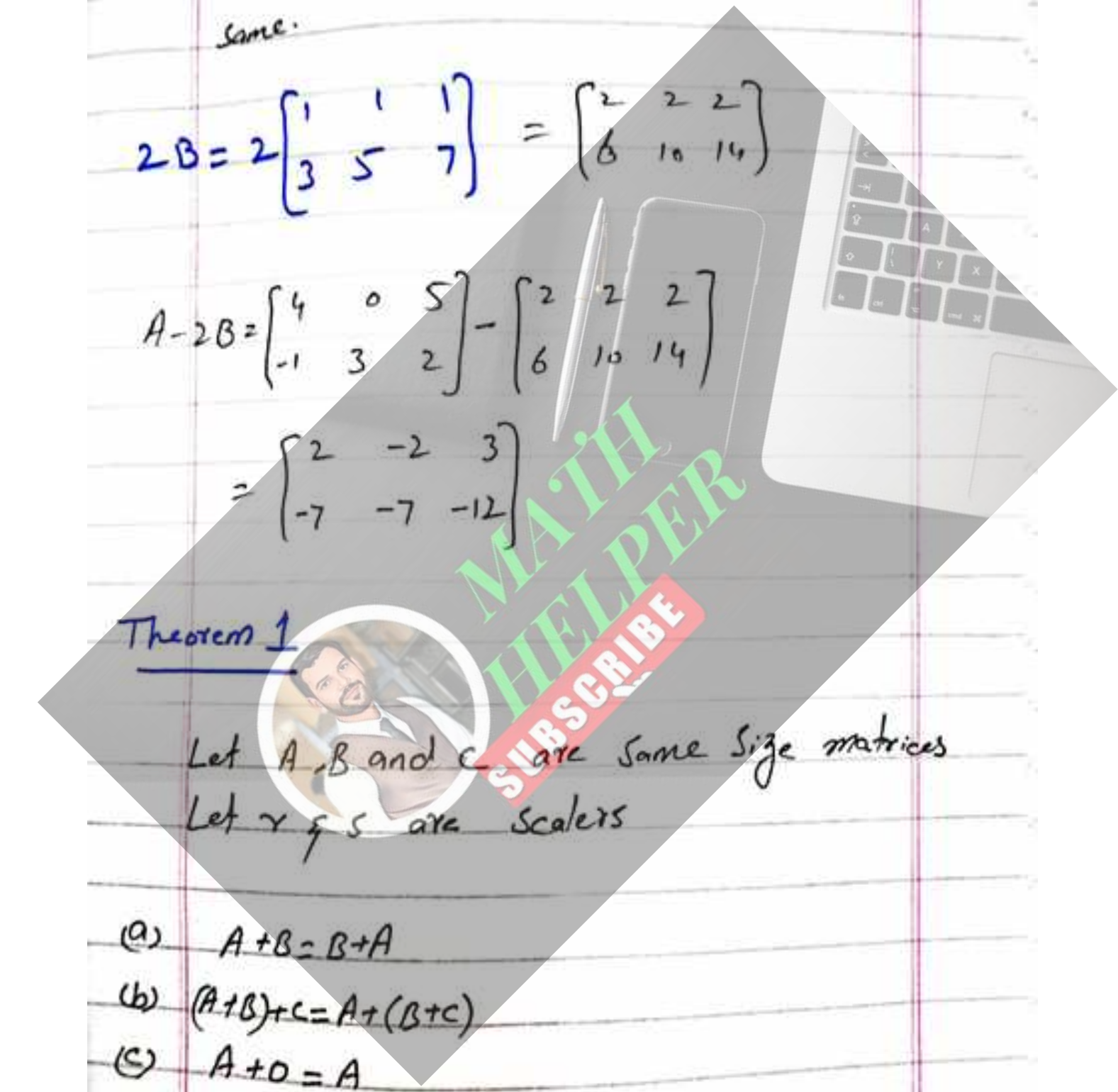
$$A+B = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}$$

NoTE For addition & subtraction both matrices order Should be same.

$$2B = 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}$$



Theorem 1

Let A, B and C are same Size matrices
Let r & s are scalers

(a) $A + B = B + A$

(b) $(A+B)+C = A+(B+C)$

(c) $A + 0 = A$

(d) $r(A+B) = rA + rB$

(e) $(r+s)A = rA + sA$

(f) $r(sA) = rs(A)$

Matrix Multiplication

If A is an $m \times n$ matrix & if B is an $n \times p$ with column $b_1 \cdots b_p$ then the Product AB is the $m \times p$ matrix whose columns are $Ab_1 \cdots Ab_p$

Example

Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and

$$B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

Sol

$$B = [b_1 \; b_2 \; b_3]$$

$$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}$$

$$AB = A[b_1 \; b_2 \; b_3]$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

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Example

(An undefined Product) Find BA

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix} \quad \& \quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

Sol

The # of cd. of B ≠ to # of row of A

So BA multiplication not possible.

Row column Rule for computing AB

Example

use dot Product rule to compute the individual entries in the Product of

$$A=\begin{bmatrix}1 & 2 & 4\\ 2 & 6 & 0\end{bmatrix} \quad \& \quad B=\begin{bmatrix}4 & 1 & 4 & 3\\ 0 & -1 & 3 & 1\\ 2 & 7 & 5 & 2\end{bmatrix}$$

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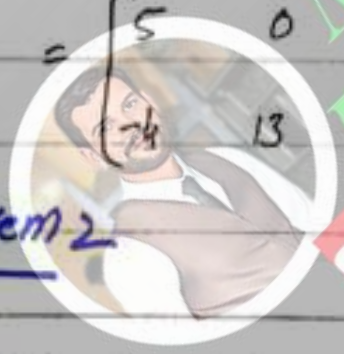
Finding specific entries in a matrix Product

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 6 \\ -2 & 3 & \end{bmatrix}$$

$$= \begin{bmatrix} \square\square & \square \\ \square\square & \square \end{bmatrix}$$

$$= \begin{bmatrix} 2\times1+3\times1 & 2\times3+(3\times2) & 2\times6+3\times3 \\ 1\times1+(-5)1 & 1\times3+(-5)(-2) & 1\times6+(-5)3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 21 \\ -14 & 13 & -9 \end{bmatrix}$$



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Theorem 2

(a) $A(BC) = (AB)C$ Associative

(b) $A(B+C) = AB+AC$ left distributive

(c) $(B+C)A = BA+CA$ right "

(d) $r(AB) = (rA)B = A(rB)$ for any scalar r

(e) $I_m A = A = A I_n$ identity

NOTE:-

(1) In general $AB \neq BA$

(2) Cancellation law do not hold in matrix multiplication $AB = AC$ then it is not true $B = C$

(3) If $AB = 0$ it does not mean $A = 0$ or $B = 0$

Power of matrix:-

Transpose of matrix

Interchange row into column or column into row

Theorem 3

Let A & B denote matrices whose size is appropriate for the following sum & product

(a) $(A^t)^t = A$ (b) $(A+B)^t = A^t + B^t$

(c) For any scaler γ $(\gamma A)^t = \gamma A^t$

(d) $(AB)^t = B^t A^t$

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MTH501

Lec #12

The inverse of a matrix

Inverse of Squar matrix

If A is an nxn matrix A matrix C of order nxn is called multiplicative inverse of A if $AC = CA = I$ where I is the nxn identity matrix

Invertible matrix

If the inverse of squar matrix exists it is called an invertible matrix.

Note If B is another inverse of A then we would have

$$B = BI = B(AC) = (BA)C = IC = C$$

when A is invertible its inverse is unique the inverse of A denoted by A^{-1}

$$A \cdot A^{-1} = I \quad \& \quad A^{-1}A = I$$

Note A matrix that is not invertible sometimes called Singular and invertible matrix Sometime called non singular matrix

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$|A| = 0$ Singular matrix

$|A| \neq 0$ non " "

Ex if $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ & $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$

Sol:

$$AC = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2(-7)+5(3) & 2(-5)+5(2) \\ -3(-7)-7(3) & (-3)(-5)+2(-7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$CA = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -14+15 & -35+35 \\ 6-6 & 15-14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus $C = A^{-1}$

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

if $ad - bc \neq 0$ then A is invertible $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

if $ad - bc = 0$ the A is not invertible

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Ex $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ Find A^{-1}=?

Sol

$$|A| = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} = 18 - 20 = -2$$

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\,\text{Adj}\,A$$

Theorem:-

(a) If A is invertible the A^{-1} is also invertible

$$(A^{-1})^{-1} = A$$

(b) If A & B are $n \times n$ invertible matrices then so is AB & the AB^{-1} is the product of A^{-1} and B^{-1} in the reverse order

$$(AB)^{-1} = B^{-1}A^{-1}$$

(c) If A is an invertible matrix then A^T and then inverse of A^T is the Transpose of A^{-1} that is

$$(A^T)^{-1} = (A^{-1})^T$$

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Generalize Point b

$$(A_1)(A_2)\cdots(A_n)^{-1} = A_n^{-1} \cdot A_{n-1}^{-1} \cdots A_2^{-1} \cdot A_1^{-1}$$

Theorem:-

If A is invertible & n is a non ~~Singular~~ -ve integer then

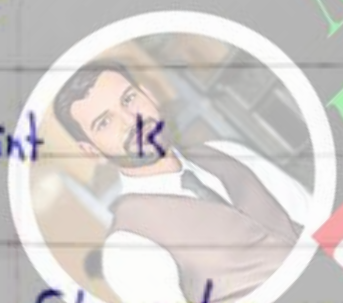
(a) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$

(b) kA is invertible for any non zero scalar k & $(kA)^{-1} = k^{-1}A^{-1}$

Point A^3 or $(A^{-1})^3$

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Point is



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Elementry matrices

Elementry Row operations

There are three types of Row oper.

* Interchange of any two rows
* Multiply any row by nonzero constant
* Adding a multiple of one row to other.

Elementry matrix

An elementry matrix is a matrix that results from applying a single elementry row oper. to an identity matrix

NOTE Elementry matrices are always square.

Theorem:-

If A is n×n identity matrix and if the elementry matrix E results by performing a certain row operation on the identity matrix then the product EA is the matrix that results when the same row oper. performed on A.

Theorem:-

An elementry matrix is invertible and the inverse is also an elemetry matrix.

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then $[A\ I]$ is row equivalent to $[I\ A^{-1}]$ otherwise A does not have an inverse

Example:-

Find the inverse of matrix

$$A=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 4 & -3 & 8\end{bmatrix}$$ if it exist

Sol:

$$[A\ I]=\left[\begin{array}{ccc|ccc}0 & 1 & 2 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{array}\right] R_{12}$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & -3 & -4 & 0 & -4 & 1\end{array}\right] -4R_1+R_3$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 2 & 3 & -4 & 1\end{array}\right] 3R_2+R_3$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 1 & 3/2 & -2 & 1/2\end{array}\right] -2R_3+R_2$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix} -3R_3 + R_3$$

Since

$A \sim I$ we conclude that A is invertible

and $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

It is good to check the final answer

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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MTH 501

Lec #13

Characterization of invertible Matrix

Solving Linear System by matrix inversion

Theorem

Let A be an $n \times n$ invertible matrix for any $b \in R^n$ then eq. $Ax = b$ — (i) has the unique Sol.

i.e $x = A^{-1}b$

Uniqueness:-

we assume that there is another Sol u. Indeed it is a Sol. of eq (i) So it must be $u = A^{-1}b$ it means $x = A^{-1}b = u$

this shows that $u = x$

Theorem:-

Let A & B be the square matrices such that $AB = I$ then A & B are invertible with $B = A^{-1}$ & $A = B^{-1}$

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Ex Solve the system of linear eq.

$$2x_1 + x_2 + x_3 = 1$$
$$5x_1 + x_2 + 3x_3 = 3$$
$$x_1 + 4x_3 = 6$$

Sol

The matrix form of system is

$$Ax = b$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

$$A \qquad x = b$$

Determinent of A

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}$$

$$= 2(4-0) - 1(20-3) + 1(0-1) = 8 - 17 - 1 = -10 \neq 0$$

So A is invertible

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Now apply inversion method

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 5 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 5 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$-5R_1 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -17 & 0 & 1 & -5 \\ 2 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$-2R_1 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -17 & 0 & 1 & -5 \\ 0 & 1 & -7 & 1 & 0 & -2 \end{array}\right]$$

$-R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -17 & 0 & 1 & -5 \\ 0 & 0 & 10 & 1 & -1 & 3 \end{array}\right]$$

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$\frac{1}{10} R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 0 & 1 \\ 0 & 1 & -17 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1/10 & -1/10 & 3/10 \end{array}\right]$$

$-4\left(\frac{3}{10}\right)$
$\frac{-12}{10} + 1$
$\frac{24}{5} + 3$
$\frac{-6}{5} + 1$

$-4R_3 + R_1$, $-17R_3 + R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/10 & 4/10 & -1/5 \\ 0 & 1 & 0 & 17/10 & -7/10 & 1/10 \\ 0 & 0 & 1 & 1/10 & -1/10 & 3/10 \end{array}\right]$$

Hence

$$A^{-1} = \begin{bmatrix} -2/5 & 2/5 & -1/5 \\ 17/10 & -7/10 & 1/10 \\ 1/10 & -1/10 & 3/10 \end{bmatrix}$$

As we know

$$Ax = B$$

$$x = A^{-1}B$$

B is given and we find A^{-1}

So

$$x = \begin{bmatrix} -2/5 & 2/5 & -1/5 \\ 17/10 & -7/10 & 1/10 \\ 1/10 & -1/10 & 3/10 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 1/5 \\ 8/5 \end{bmatrix}$$

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NOTE Matrix inversion method is only ~~applicable~~ when applicable

# of equations = # of unknowns

& fails if given matrix is not invertible

Ex

$$x_1 + 6x_2 + 4x_3 = 2$$
$$2x_1 + 4x_2 - x_3 = 3$$
$$-x_1 + 2x_2 + 5x_3 = 3$$

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$|A| = 1(20+2) - 6(10-1) + 4(4+4)$

$= 22 - 54 + 32$

$= 54 - 54 = 0$

So the inversion method fails

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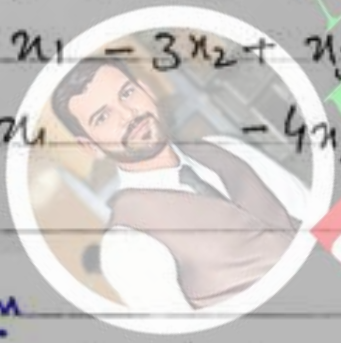
Sol for the System of a nxn Homogeneous Linear equations with an invertible coefficient matrix:-

Theorem:-

Let $Ax=0$ be a homogeneous Linear System of n equations and n unknowns then the coefficient matrix A is invertible iff this System has only a trivial Solution.

Ex

$$2x_1 + x_2 + 3x_3 = 0$$
$$x_1 - 3x_2 + x_3 = 0$$
$$x_1 \quad\quad - 4x_3 = 0$$



Theorem

Let A be a square $n \times n$ matrix the the following statments are equivalent

(1) A is an invertible matrix

(2) A is row equivalent to the $n \times n$ matrix

(3) A has n Pivot Positions.

(4) The eq. $Ax=0$ has only the trivial sol.

(5) The column of A form a linearly independent set.

(6) The linear transformation $x \to Ax$ is one-to-one

(7) The eq $Ax=b$ has at least one sol for each b in R^n

(8) The columns of A Span R^n

(9) The Linear transformation $x \to Ax$ maps R^n onto R^n.

(10) There is a $n \times n$ matrix C such that $CA=I$

(11) " " " " " D such that $AD=I$

Example

Show that the matrix

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$ is invertible by using invertible matrix theorem.

Sol By Row equivalent

$$\sim -R_1 + R_2$$

$$= \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

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$\sim -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9 \\ 0 & 0 & -7 \end{bmatrix}$$

It shows that A matrix has three pivot positions and hence, is invertible

Solving multiple linear system with a common coefficient matrix

This technique is used to solving a sequence of linear systems.

$Ax_1 = b_1$, $Ax_2 = b_2$ ---: $Ax_k = b_k$

Take the inverse of coefficient matrix A

So

$x_1 = A^{-1}b_1$, $x_2 = A^{-1}b_2$ --- $x_k = A^{-1}b_k$

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Ex Let L be the linear transformation from $R^2 \to P_2$ defined by

$$T(x,y) = xyt + (x+y)t^2$$

Find the matrix representing T w.r.t to the standard bases

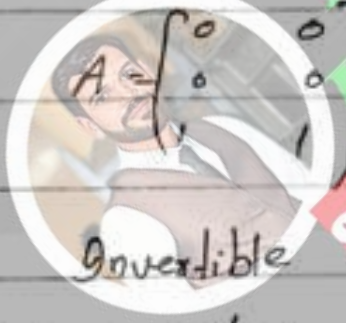
Sol

Sol

Let $A = \{(1,0),(0,1)\}$ be the basis of R^2

$$T(1,0) = (1)(0)t + (1+0)t^2$$
$$= t^2 = (0,0,1)$$
$$T(0,1) = t^2 = 0,0,1$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



NOTE Invertible matrix theorem applies only Squar matrix.

Invertible Linear transformation

A Transformat $T: R^n \to R^n$ be a Linear transformation and A be a standard matrix for T then T is invertible iff A is invertible

In that case Linear transformation S given by $S(x) = A^{-1}x$ is a unique function satisfying (1) & (2)

$S(T(x)) = x \ \forall \ x \in R^n$ ——— (1)

$T(S(x)) = x \ \forall \ x \in R^n$ ——— (2)

NOTE If the inverse of linear transformation exist then it is unique.

Proposition

Case 1 If $m < n$ then the system $Ax = y$ has either no solution or infinitely many sol.

If $m = n$ then the system $Ax = y$ has a unique sol. iff $Rank(A) = n$

If $m > n$ then the transformation $y = Ax$ is a non invertible b/c we can find vector y in R^m such that $Ax = y$ is inconsistent

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MTH501

Lecture # 14
Partitioned Matrices

A block matrix or a Partitioned matrix is a Partition of a matrix into rectangular Smaller matrices called blocks

General Partitioning of a matrix:-

A matrix can be Partitioned (subdivided) into sub matrices also called blocks in various ways by inserting lines b/w row & columns.

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Example

$$P = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 3 & 3 & 4 & 4 \\ 3 & 3 & 4 & 4 \end{pmatrix}$$

$$P_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P_{12} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad P_{21} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$P_{22} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

Example

The matrix $A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]_{3\times 6}$

can also be written as 2x3 Partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Note To know that in how many ways to block up an ordinary matrix A.

Example

Let a general matrix A of 5x3 order

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix}$$

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Addition of Blocked Matrices

If matrices A & B are the same size & are same partitioned in exactly the same way then it is natural to make the same partition of the ordinary matrix sum A+B in this case each block of A+B is the matrix sum of coressponding blocks of A & B

Theorem If A is mxn & B is nxp then

$$AB = Col_1(A)\,Row_1(B) + Col_2(A)\,Row_2(B) + \cdots + Col_n(A)\begin{bmatrix} Row_1(B) \\ Row_2(B) \\ \vdots \\ Row_n(B) \end{bmatrix}$$

$$= Col_1(A)\,Row_1(B) + Col_2(A)\,Row_2(B) + \cdots Col_n(A)\,Row_n(B)$$

Example

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ & } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}$$

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Similarly Subtraction of blocked matrices is defined

Note Multiplication of a Partitioned matrix by a scaler is also computed block by block

Multiplication of Partitioned Matrices

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{23} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

if the size of the blocks confirm for the required operation.

Example Find the block multiplication of the follow Partitioned matrices

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 0 & 2 \\ -1 & -2 & 3 & 1 & 1 \\ \hline 2 & 1 & -2 & 1 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{cc} 2 & 1 \\ 3 & 0 \\ -5 & 1 \\ \hline 0 & 8 \\ 0 & 2 \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

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Toeplitz matrix

A matrix in which each descending diagonal from left to right is constant is called a toeplitz Matrix or diagonal-constant matrix

$$A = \begin{bmatrix} a & 1 & 2 & 3 \\ 4 & a & 1 & 2 \\ 5 & 4 & a & 1 \\ 6 & 5 & 4 & a \end{bmatrix}$$ is toeplitz matrix

Inverses of Partitioned Matrices

Block Diagonal matrices

A Partition matrix A is said to be block diagonal if the main diagonal are square and all other position matrices are zero.

$$A = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & D_K \end{bmatrix}$$

where the matrices $D_1, D_2 \cdots D_K$ are square

It can be shown that the matrix A is invertible iff each matrix on diagonal is invertible.

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$$A^{-1} = \begin{bmatrix} D_1^{-1} & 0 & \cdots & 0 \\ 0 & D_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & D_k^{-1} \end{bmatrix}$$

Ex1

$$A = \left[\begin{array}{cc|cc|c} 8 & -7 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 5 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

Block upper triangular matrices

A Partitioned Squar matrix A is said to be ---- if the matrices on main diagonal are Square and all matrices below the main diagonals are Zero

NOTE Block lower Triangular matrix is Similar

Example upper Triangular matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$ where order of A_{11} & A_{22} $p \times p$ & $q \times q$ respectively Find A^{-1}

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Ex Find A^{-1}

$$A = \left[\begin{array}{cc|cc} 1 & 9 & -5 & 0 \\ 3 & 3 & 3 & -2 \\ \hline 0 & 0 & 7 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right]$$

$$A_{11} = \begin{bmatrix} 1 & 9 \\ 3 & 3 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -5 & 0 \\ 3 & -2 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 7 & 0 \\ 3 & 1 \end{bmatrix}$$

$$A_{11}^{-1} = \frac{\text{Adj } A}{|A|} = \frac{\begin{bmatrix} 3 & -9 \\ -3 & 1 \end{bmatrix}}{-24}$$

$$= \begin{bmatrix} -1/8 & 9/24 \\ 1/8 & -1/24 \end{bmatrix}$$

$$A_{22}^{-1} = \begin{bmatrix} 1/7 & 0 \\ -3/7 & 1 \end{bmatrix}$$

$$A_{11}^{-1} \cdot A_{12} \cdot A_{22}^{-1} = \begin{bmatrix} -1/8 & 9/24 \\ 1/8 & -1/24 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1/7 & 0 \\ -3/7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4/7 & 3/4 \\ 1/4 & -1/2 \end{bmatrix}$$

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MTH 501

Lec # 15

Matrix Factorization

Definition :-

A factorization of a matrix as a Product of two or more matrices is called Matrix factorization

Uses :-

It use several important Comp. Programs.

LU factorization or LU Decomposition

LU factorization is a matrix decomposition in which writes a matrix as the Product of Lower & Upper Triangular matrix. This decomposition is use to Solve system of linear equation or Calculated the def determinent.

Remarks :- * If A is a square matrix order n Then the order of L & U matrix also order n

* In general not every square matrix

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A has an LU Decomposition, not is an LU decomposition unique if it exist.

Theorem:-

If a square matrix A can be reduced to row echelon form with no row interchanges then A has an LU decomposition.

Procedure for finding an LU decomposition

Step 1:- Reduce matrix A to row echelon form U without using row interchange

Step 2:- In each position along the main diagonal of L place the reciprocal of the multiplier that introducing the leading 1 in that position in U

Step 3:- In each position below the main diagonal of L, place the -ve of the multiplier used to introduce the zero in that position in U

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Step 4:- Form the decomposition $A = LU$

Example

Find LU decomposition of

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$$

Sol

By using Crout's method

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{---(1)}$$

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21} & L_{21}U_{12}+L_{22} & L_{21}U_{13}+L_{22}U_{23} \\ L_{31} & L_{31}U_{12}+L_{32} & L_{31}U_{13}+L_{32}U_{23}+L_{33} \end{bmatrix}$$

$\boxed{L_{11} = 6}$

$\boxed{L_{21} = 9}$

$\boxed{L_{31} = 3}$

$L_{11}U_{12} = -2$

$6U_{12} = -2 \Rightarrow -\frac{1}{3}$

$\boxed{U_{12} = -\frac{1}{3}}$

$L_{11}U_{13} = 0$

$6U_{13} = 0 \Rightarrow \boxed{U_{13} = 0}$

$L_{21}U_{12} + L_{22} = -1$

$9(-\frac{1}{3}) + L_{22} = -1$

$\boxed{L_{22} = 2}$

$L_{21}U_{13} + L_{22}U_{23}$

$9(0) + 2U_{23} = 1$

$\boxed{U_{23} = \frac{1}{2}}$

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$L_{31}U_{12} + L_{23} = 7$

$3(-\frac{1}{3}) + L_{23} = 7$

$\boxed{L_{23} = 8}$

$L_{31}U_{13} + L_{32}U_{23} + L_{33} = 5$

$3(0) + 8(\frac{1}{2}) + L_{33} = 5$

$L_{33} = 5 - 4$

$\boxed{L_{33} = 1}$

Now put the obtained value in matrix form

$$\begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

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Solving Linear System by LU Factorization

First write system in

$$Ax = b$$

$$A = LU$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$LUx = b$$

Let $Ux = y$ — ①

$$Ly = b$$ — ②