Spring 2021 Papers Solution Solved By

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STA642

Q No. 1

$$f(x,y) = \begin{cases} 2 & , 0<x<y, 0<y<1 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) Find the marginal distributions:

(i) Marginal distribution of X:

$$f_x(x) = \int_x^1 2\,dy = 2[y]_x^1 = 2(1-x),$$

(ii) Marginal distribution of Y:

$$f_y(y) = \int_0^y 2\,dx = 2[x]_0^y = 2(y-0) = 2y$$

(b) Find the conditional distributions:

$$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{2}{2y} = \frac{1}{y}$$

and $$f(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

Q No. 2 Define Incomplete Gamma Function

Ans: In mathematics, the upper and lower incomplete gamma functions are types of special functions.

The upper incomplete gamma function is defined as:

$$\Gamma(\alpha,y) = \int_y^\infty t^{\alpha-1} e^{-t}\,dt$$

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The lower incomplete gamma function is defined as:

$$\gamma(\alpha, y) = \int_0^y t^{\alpha-1} e^{-t}\, dt$$

It is interesting to note that

$$\gamma(\alpha, y) + \Gamma(\alpha, y) = \Gamma(\alpha)$$

Q No. 3: In which conditions the poisson distribution is applicable?

Ans: Conditions of poisson distribution:

(i) An event can occur any number of times during a time period

(ii) Events occur independently.

Q No. 4 Write down the formula of cumulative distribution function of normal distribution.

Ans:

$$F(x) = \frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\delta\sqrt{2}}\right)\right] \quad for \quad -\infty < x < \infty$$

erf is the error function.

Q No. 5: The mean and variance of poisson distribution is 4. Find COV.

Given $\mu = \delta^2 = 4$

$$COV = \frac{\delta}{\mu} \times 100\% = \frac{\sqrt{4}}{4} \times 100\% = \frac{2}{4} \times 100\% = 50\%$$

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Q No. 6: Deriration of the mean of exponential distribution:

Sol: The PDF of exponential distribution

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0, \lambda > 0$$

$$Mean = E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx$$

$$= \left[-x e^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty$$

$$= -\left[x e^{-\lambda x}\right]_0^\infty - \left[\frac{1}{\lambda} e^{-\lambda x}\right]_0^\infty$$

$$= -[0-0] - \left[0 - \frac{1}{\lambda}(1)\right]$$

$$= 0 - \left[-\frac{1}{\lambda}\right] = \frac{1}{\lambda}$$

$$Mean = E(X) = \mu = \frac{1}{\lambda}$$

Derivation of variance of exponential distribution

$$E(X^2) = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty x^2 \lambda e^{-\lambda x} = \ldots = \frac{2}{\lambda^2}$$

$$E(X) = \frac{1}{\lambda}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$S.D(X) = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

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Q No. 7 General Concept of Ellipse

Ans: We may assume that one of the many diameters of the circle is being "elongated"

As soon as we "elongate" one of the diameters, we get a major axis as well as a minor axis.

Q No. 8 The concept of Contours

Ans: By the term "contour", we mean the set of points on the $X_1 X_2$ floor corresponding to which the ordinates are of equal height.

Q No. 9: Drive the mean of normal distribution.

Sol: $$f(x) = \frac{1}{\delta\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\delta}\right)^2}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\delta\sqrt{2\pi}} e^{-\left(\frac{x-\mu}{\sqrt{2}\delta}\right)^2}$$

$$\text{let} \quad z = \frac{x-\mu}{\delta} \Rightarrow x = \mu + z\delta \Rightarrow dx = \delta\, dz$$

$$\Rightarrow E(x) = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + z\delta)\, e^{-\frac{z^2}{2}} (\delta\, dz)$$

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$$E(x) = \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \underbrace{\frac{\delta}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2}\, dz}_{\text{odd function} = 0}$$

$$E(x) = \mu \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + 0$$

$$E(x) = \mu [1] + 0 = \mu \qquad \because \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz = 1$$

The area under a normal curve is unity (1).

μ is the mean of Normal Distribution.

Q.NO.10. What do you mean by statistically Independent variables?

Ans: Statistical models normally specify how one set of variables, called dependent variables, functionally depend on another set of variables, called independent variables.

OR Two random variables X and Y are independent, statistically independent if

$$f_{XY}(x,y) = f_X(x) f_Y(y).$$

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Q No. 11: Write down PDF of exponential distribution with $\theta = 1.5$.

Ans. PDF of exponential distribution.

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} \qquad x \geq 0, \theta > 0$$

$\theta = 1.5$

$$f(x;1.5) = \frac{1}{1.5} e^{-x/1.5} \qquad 0 < x < \infty$$

Q No. 12: Drive the CV of exponential distribution

Ans:

$$CV = \frac{\delta}{\mu}$$

$\delta = 1/\lambda$ and $\mu = \frac{1}{\lambda}$

$$CV = \frac{1/\lambda}{1/\lambda} = 1$$

$CV = 1$

Coefficient of variation of exponential distribution is always equal to 1.

Q No 13: Wride down MGF of Binomial distribution

Ans: MGF of Binomial distribution:

$$M(t) = [(1-p) + pe^t]^n \quad \forall \text{ value of } t.$$

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Q No 14: If $p = 0.2$, then find mean of the geomatric distribution.

Sol $p = 0.2$

$$\text{Mean} = E(x) = \mu = \frac{1}{p} = \frac{1}{0.2} = 5$$

Q No 15: Covariance of Multinomial distribution.

Ans: The covariance of multinomial distribution is

$$Cov(X_i, X_j) = -np_ip_j \qquad i \neq j$$

Q No 16: Theorem: If X and Y have a bivariate normal distribution with correlation coefficient ρ. Then X and Y are independent if and only if $\rho = 0$.

i.e $f_{XY}(x,y) = f_X(x) f_Y(y)$

proof:

$$f_{XY}(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}$$

Substituting $\rho = 0$ in the pdf of the bivariate normal distribution, we have:

$$f_{XY}(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma_x}\right)^2 + \left(\frac{y-\mu}{\sigma_y}\right)^2\right]}$$

Further simplifying, we have:

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$$f_{xy}(x,y) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\,\delta x\,\delta y}\, e^{\frac{-1}{2}\left(\frac{x-\mu(x)}{\delta x}\right)^2} \cdot e^{\frac{-1}{2}\left(\frac{y-\mu(y)}{\delta y}\right)^2}$$

$$f_{xy}(x,y) = \frac{1}{\sqrt{2\pi}\,\delta x}\, e^{\frac{-1}{2}\left(\frac{x-\mu(x)}{\delta x}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\,\delta y}\, e^{\frac{-1}{2}\left(\frac{y-\mu y}{\delta y}\right)^2}$$

$$f_{xy}(x,y) = f_x(x)\, f_y(y)$$

(Hence proved)

Q No 17: PDF of bivariate distribution. $N(\mu_x, \mu_y \delta_x^2, \delta_y^2, \rho)$

Ans: The probability density function of bivariate distribution (X, Y) is

$$f_{xy}(x,y) = \frac{1}{2\pi\,\delta_x\,\delta_y\,\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\delta_x}\right)^2 + \left(\frac{y-\mu_y}{\delta_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\delta_x}\right)\left(\frac{y-\mu_y}{\delta_y}\right)\right]}$$

where $-\infty < x, y < \infty$

and the parameters are such that

$-\infty < \mu_x, \mu_y < \infty$,

$\delta_x, \delta_y > 0$

$-1 < \rho < 1$

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Q No. 18: properties of Gamma Function.

Ans:

(i) If $\alpha = 1$, clearly $\Gamma(1) = \int_0^\infty e^{-y} dy = 1$

(ii) If $\alpha > 1$, an Integration by parts show that

$$\Gamma(\alpha) = (\alpha-1)\int_0^\infty y^{\alpha-2} e^{-y} dy$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

Accordingly,

If α is positive integer greater than 1.

$$\Gamma(\alpha) = (\alpha-1)(\alpha-2)\cdots(3)(2)(1)\Gamma(1)$$

$$= (\alpha-1)!(1)$$

$$\Gamma(\alpha) = (\alpha-1)!$$

Since $\Gamma(1) = 1$, this suggests we take

$$\Gamma(1) = 0! = 1$$

It is important to note that the value of a gamma function is always positive.

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Q No. 19 properties of Beta distribution

Ans:

(i) $\text{Mean} = \mu = E(X) = \frac{\alpha}{\alpha+\beta}$

(ii) $\text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(iii) $\text{Mode} = \frac{\alpha-1}{\alpha+\beta-2}$ $\alpha, \beta > 1$

Q No. 20 Example: If it is known that, on the average, two minor accidents occur at a busy crossing per day, what is the probability that, at the most two minor accidents will occur during an entire week?

Sol:

This is a poisson process with Occurance Rate $= \lambda = 2$ per day

Also Time one week $= 7$ days $\Rightarrow t = 7$

Hence $\lambda t = 2 \times 7 = 14$

$$P(X = x) = \begin{cases} \frac{e^{-\lambda t}(\lambda t)^x}{x!} = \frac{e^{-14}(14)^x}{x!}, & x = 0, 1, 2, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

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Now we require $X \leq 2$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{e^{-14}(14)^0}{0!} + \frac{e^{-14}(14)^1}{1!} + \frac{e^{-14}(14)^2}{2!}$$

$$= e^{-14} + 14e^{-14} + 98e^{-14}$$

$$= 113e^{-14}$$

$$= 0.0000\,9396$$

$$P(X \leq 2) = 9.396 \times 10^{-5}$$

We can say that the probability that at most 2 accidents will occur at a busy crossing in a week, this probability is almost 0, which is "Next to impossible".

Q No. 21: Write down the mean of the multinomial distribution,

Ans:

The mean of multinomial distribution is

$$E(X_i) = np_i$$

The variance of multinomial distribution is

$$Var(X_i) = np_i(1-p_i)$$

$$= np_i q_i$$

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Q No. 22: Find the first derivative of

$$K(t) = \mu t + \frac{\delta^2 t^2}{2}$$

Sol: Differentiate with respect to "t"

$$K'(t) = \frac{d}{dt}\left(\mu t + \frac{\delta^2 t^2}{2}\right)$$

$$= \mu \frac{d}{dt}(t) + \frac{\delta^2}{2}\frac{d}{dt}(t^2)$$

$$= \mu(1) + \frac{\delta^2}{2}(2t)$$

$$K'(t) = \mu + \delta^2 t$$

Q No. 23: prove the theorem Bivariate Normal Distribution.

Statement: If X and X have a bivariate normal distribution with correlation ρ. Then X and Y are independent if and if $\rho = 0$

proof: ~~If X and~~ The statemad iff means:

(a) If X and Y are independent, the $\rho = 0$

(b) If $\rho = 0$, then X and Y are independ

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(a) In general, if X and Y are independent random variables, then their correlation coefficient is 0.

As such, there is no need to prove the first part of statement.

(b) We know that

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_x}\right)\right]}$$

Substituting $\rho = 0$ in the pdf of the B.N.D we have

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$\Rightarrow f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$\Rightarrow f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)} \cdot \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$\Rightarrow f_{XY}(x,y) = f_X(x)\, f_Y(y) \quad \text{(Hence proved)}$$

X and Y are independent.

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Q No 24: Example:

If X is $b(n, 1/3)$, then $P(Y \geq 1) = 1 - (2/3)^n$

Find the smallest value of n that yields

$P(Y \geq 1) > 0.80$

Sol: let us first see how

$$P(Y \geq 1) = 1 - P(Y < 1)$$
$$= 1 - P(Y = 0)$$
$$= 1 - (2/3)^n$$

We know that

$$P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$P(Y = 0) = \binom{n}{0} p^0 (1-p)^{n-0}$$

$$P(Y = 0) = (1)(1)(1-p)^n = (1-p)^n$$

Here $p = 1/3$

$$P(Y = 0) = (1 - 1/3)^n = (2/3)^n$$

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Now we required to find the smallest value of n.

$$P(Y \geq 1) > 0.80$$

$$1 - (2/3)^n > 0.80$$

$$1 - 0.80 > (2/3)^n$$

$$0.20 > (2/3)^n$$

Taking "ln"

$$\ln(0.20) > \ln(2/3)^n$$

$$-1.6094 > n \ln(2/3)$$

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$\Rightarrow -1.6094 > -0.4055n$

$1.6094 < 0.4055n$

$\frac{1.6094}{0.4055} \lessgtr n$

$n > 3.97$

$n \cong 4$

Q No. 25: Write two properties of normal curve.

Ans: properties of normal curve are;

(i) The normal curve is asymptotic to the x-axis.

(ii) The ~~area~~ total area of normal curve is unity.

(iii) The maximum ordinate of the Normal Distribution at $x = \mu$ is equal to $\frac{1}{\sigma\sqrt{2\pi}}$.

Q No. 26: Write the formula of variance of the Hypergeomatric distribution.

Ans: The variance of Hypergeomatric distribution is:

$$\text{Variance} = n \cdot \frac{K}{N} \cdot \frac{(N-K)}{N} \cdot \frac{N-n}{N-1}$$

or $\text{Variance} = npq\left(\frac{N-n}{N-1}\right)$, $p = \frac{K}{N}$, $q = \frac{N-K}{N}$

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Q No 27: Write PMF of Binomial distribution.

Ans: PMF of the Binomial distribution is

$$p(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0,1,2,\ldots n \\ 0 & \text{elsewhere.} \end{cases}$$

Q No 28: Derive Variance of Continuous Uniform distribution.

Ans: The PDF of C.U.D

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

$$Var(X) = E(X^2) - [E(X)]^2 \longrightarrow (1)$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{a}^{b} x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$E(X) = \frac{1}{2(b-a)} (b^2 - a^2) = \frac{a+b}{2}$$

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$$E(X^2) = \int_{\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{a}^{b} x^2 \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}\left[b^3-a^3\right]$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{a^2+ab+b^2}{3}$$

From (1)

$$Var(X) = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{2}$$

$$= \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-8ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$Var(X) = \frac{(a-b)^2}{12}$$

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Q No. 29: What is another name of rectangle distribution?

Ans: Uniform distribution is another name of rectangle distribution.

Q No. 30: MGF of Multinomial distribution.

Ans: The moment generating function of the multinomial distribution is

$$M_{\underline{X}}(t) = M_{\underline{x}}(t_1, t_2, \dots t_k)$$

$$= \left(\sum_{i=1}^{k} p_i e^{t_i}\right)^n$$

Q No. 31: MGF of Normal distribution:

Ans: The MGF of Normal distribution is

$$M(t) = e^{\mu t + \frac{1}{2}\delta^2 t^2} \quad \text{for } -\infty < t < \infty$$

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Q No. 32: Write CDF of continuous uniform distribution.

Ans: The CDF of the continuous uniform distribution is:

$$f(x) = \begin{cases} \frac{1}{b-a} & ; \ a \le x \le b \\ 0 & ; \ \text{elsewhere} \end{cases}$$

Q No. 33: Write the formula of variance in discrete uniform distribution.

Ans: The Variance of discrete uniform distribution is

$$Var(X) = \sigma^2 = \frac{n^2-1}{12}$$

Q No. 34: Write and derive the formula of mean in discrete uniform distribution.

Ans: The mean of discrete uniform distribution is

$$Mean = E(X) = \mu = \sum_{x=1}^{n} x\,p(x)$$

$$Mean = \mu = \sum_{x=1}^{n} x \cdot \frac{1}{n} = \frac{1}{n}(1+2+3+\cdots+n)$$

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$$Mean = \mu = \frac{1}{n} \times \frac{n(n+1)}{2}$$

$$Mean = \mu = \frac{n+1}{2}$$

Q No. 35: Derivation of variance of the discrete uniform distribution:

Ans:

$$Var(X) = E(X^2) - [E(X)]^2 \longrightarrow (1)$$

$$E(X^2) = \sum x^2 p(x) = \sum x^2 \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum x^2$$

$$E(X^2) = \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$E(X) = \sum x p(x) = \sum x \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum x$$

$$= \frac{1}{n} \cdot \frac{n(n+1)}{2}$$

$$E(X) = \frac{n+1}{2}$$

From (1) $$Var(X) = \frac{n(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{4n(n+1)(2n+1) - 6(n+1)^2}{24}$$

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$$Var(X) = \frac{4(2n^2+n+2n+1) - 6(n^2+2n+1)}{24}$$

$$= \frac{2[4n^2+2n+4n+2-3n^2-6n-3]}{24}$$

$$= \frac{2[n^2-1]}{24}$$

$$Var(x) = \frac{n^2-1}{12}$$

Q No: 36: A oil company conduets a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. The company is willing success twice(2) and is willing to try upto 5 times. What is the probability that the second succes comes on the fifth well drilled.

Sol:

We know that, the pmf of X is.

$$P(X=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \cdots$$

$p = 20\% = 0.20$ and $r = 2$

$$P(X=x) = \binom{x-1}{1} (0.20)^2 (1-0.20)^{x-2}, \quad x = 2, 3, 4, \cdots$$

$$P(X=x) = \binom{x-1}{1} (0.20)^2 (0.80)^{x-2}, \quad x = 2, 3, 4, \cdots$$

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We need to find $P(X=5)$

$$P(X=5) = \binom{5-1}{1} (0.2)^2 (0.8)^{5-2}$$

$$= \binom{4}{1} (0.2)^2 (0.8)^3$$

$$= 4 (0.04)(0.512)$$

$$P(X=5) = 0.08192$$

$$P(X=5) = 8.192\ \%$$

Q.No 37: Find coefficient of variation in poisson distribution.

Ans: Mean and Variance of poisson distribution is μ.

$$E(X) = \mu \quad , \quad Var(X) = \mu$$

$$COV = \frac{S.D}{Mean} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{\mu}}$$

Q.No 38: What is PDF of normal random variable X or standard normal distribution of X.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \quad -\infty < x < \infty$$

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Q No. 39: The names of 5 men and 5 women are written on slips of paper and placed in a hat. Four names are drawn. What is probability that 2 are men and 2 are women?

Sol:

Let X denote the number of men.

Then, $N = 5+5 = 10$ names to be drawn from.

$K = 5$ and $n = 4$

and here the possible values of x are

0, 1, 2, 3, 4, i.e n.

Hence the hypergeomatric distribation is

$$h(x; N, n, K) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$h(x; 10, 4, 5) = \frac{\binom{5}{x}\binom{10-5}{4-x}}{\binom{10}{4}}$$

and the required probability i.e $P(X=2)$ is

$$h(2; 10, 4, 5) = \frac{\binom{5}{2}\binom{5}{4-2}}{\binom{10}{4}} = \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}}$$

$$= \frac{(10)(10)}{210} = \frac{10}{21} \quad \text{Ans}$$

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Q No.40. The probability that a planted radish seed germinates is 0.80. A gardner plants nine seeds.

(i) What is the average number of seeds.

(ii) What is the standard daviation of X.

(iii) What is the probability that, out of nine seeds, at least 8 will germinate?

Sol:

Let X denote the numbey of radish seeds that succesfully germinate.

(i) We know that

the mean of X is np

Mean $= \mu = np$

$n = 9$ and $p = 0.80$

Mean $= \mu = 9 \times 0.80 = 7.2$ seeds

(ii) We know that the variance of X is $np(1-p)$

Variance$(X) = np(1-p)$

$= 9(0.80)(1-0.80)$

$= 9(0.80)(0.20)$

$Var(X) = 1.44$

$S.D(X) = \sqrt{1.44} = 1.20$

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(iii) We know that

$$p(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$n = 9, \quad p = 0.80$$

$$p(X=x) = \binom{9}{x} (0.80)^x (0.20)^{9-x}$$

$$p(\text{at last } 8) = p(X \geq 8) = p(X=8) + p(X=9)$$

$$= \binom{9}{8} (0.80)^8 (0.20)^{9-8} + \binom{9}{9} (0.80)^9 (0.20)^{9-9}$$

$$= (9)(0.80)^8 (0.20) + (1)(0.80)^9 (1)$$

$$= 9 \times 0.1678 \times 0.20 + 0.1342$$

$$= 0.3020 + 0.1342$$

$$= 0.4362$$

$$= 43.62\ \%$$

Q No. 41: Let the joint pdf of X_1 and X_2 be

$$f(x_1, x_2) = \begin{cases} x_1 + x_2 & 0 < x_1, x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Show that X_1 and X_2 dependent.

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Sol: Here the marginal probability density function are obtained as follows:

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2$$

$$= \int_0^1 (x_1 + x_2)\, dx_2$$

$$= \left[x_1 x_2 + \frac{x_2^2}{2}\right]_0^1$$

$$= (x_1)(1) - (x_1)(0) + \frac{(1)^2}{2} - \frac{(0)^2}{2}$$

$$= x_1 - 0 + \frac{1}{2} - 0$$

$$\therefore \quad f(x_1) = x_1 + \frac{1}{2}, \quad 0 < x_1 < 1$$

Now $$f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1$$

$$= \int_0^1 (x_1 + x_2)\, dx_1$$

$$= \left[\frac{x_1^2}{2} + x_1 x_2\right]_0^1$$

$$= \frac{(1)^2}{2} - \frac{(0)^2}{2} + (1)(x_2) - (0)(x_2)$$

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$$f(x_2) = \frac{1}{2} + x_2 \quad , \quad 0 < x_2 < 1$$

Now $$f(x_1)\, f(x_2) = \left(x_1 + \frac{1}{2}\right)\left(x_2 + \frac{1}{2}\right)$$

$$f(x_1) f(x_2) = x_1 x_2 + \frac{1}{2} x_1 + \frac{1}{2} x_2 + \frac{1}{4}$$

$$f(x_1) f(x_2) \neq f(x_1, x_2)$$

So the random variables X_1 and X_2 are dependent (i.e. not independent).

Q No. 42: Write down the PDF of joint normal distribution of X_1 and X_2.

Sol: The PDF of joint normal distribution of X_1 and X_2 is.

$$f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]}$$

for $-\infty < x_1 < \infty$ and $-\infty < x_2 < \infty$

where $-\infty < \mu_1, \mu_2 < \infty$, $\sigma_1 > 0, \sigma_2 > 0$

and $-1 \leq \rho \leq 1$

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Q NB.43: In Bivariate Distribution; What is meant by the "X_1, X_2 Floor"?

Ans: Consider the three-dimensional space for example the room in which you are sitting right now.

* Now focus on the floor of the room.
* One "edge" of the floor will be called the X_1 axis.
* The other "edge" of the floor will be called the X_2 axis.

And as such, the floor will be called the "X_1X_2 floor" (actually it is the X_1X_2-plane)

Q No.44: Drive MGF of the exponential distribution:

Sol: $$M_0(t) = E[e^{tx}] = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x} dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty = \lambda \left[\frac{-e^{-(\lambda-t)x}}{\lambda - t} \right]_0^\infty$$

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Now if $t < \lambda$ then $\lambda - t > 0$, we have

$$M_0(t) = \lambda\left[\frac{-e^{-\infty}}{\lambda - t} - \frac{-e^{0}}{\lambda - t}\right] = \lambda\left[\frac{0}{\lambda - t} + \frac{1}{\lambda - t}\right]$$

$$M_0(t) = \frac{\lambda}{\lambda - t} \quad \text{for} \quad t < \lambda$$

Q No. 45: Define poisson process

Ans: The poisson process is one of the most widely used counting processes.

Whenever a particular event of interest occurs randomly over a time scale, we say that we are dealing with a poisson process.

Q No. 46: Write down the PDF of normal distribution.

Ans. A random variable X is said to be a normal distribution if its PDF is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\delta} e^{\frac{-1}{2}\left(\frac{x-\mu}{\delta}\right)^2} \qquad -\infty < x < \infty$$

where μ is the mean and $\delta > 0$ is the standard daviation of the distribution $\in N(\mu, \delta^2)$.

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Q No. 47: A statistical class takes two exams Exam 1 (Mid term Exam) and Exam 2 (Final term Exam), follow a bivariate normal distribution with parameters: $\mu_x = 70$, $\mu_y = 60$

$\delta_x = 10$, $\delta_y = 15$

$f = 0.6$

(1) What is the probability that the student scores over 75 on Exam 1 (Mid term Exam)?

Sol: mean $= \mu_x = 70$

Variance $= \delta_x = 10$

i.e. $X \sim N(70, 10)$

Hence the probability that a randomly selected student scores over 75 in Exam 1 is given by

$$P(\text{over } 75) = P(X > 75) = P\left(Z > \frac{x - \mu_x}{\delta_x}\right)$$

$$= P\left(Z > \frac{75 - 70}{10}\right) = P(Z > 0.5)$$

$$= 1 - P(Z < 0.5)$$

$$= 1 - 0.6915$$

$$= 0.3085$$

$P(\text{over } 75) = 30.85\% \cong 31\%$

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Q No.48: The duration of long distance telephone calls is found to be exponentially distributed with a mean of 3 minutes. What is the probability that a call will last
i) more than 3 minutes.
(ii) more than 5 minutes.

Sol: Let X be the exponential r.v with parameter θ. then mean $= \theta = 3$

The pdf can be written as

$$f(x) = \int_3^{\infty} \left(\frac{1}{\theta}\right) e^{-x/\theta} dx$$

(i) $p(\text{more than } 3) = p(X > 3) = \int_3^{\infty} \left(\frac{1}{3}\right) e^{-x/3} dx$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_3^{\infty} = -e^{-\infty} + e^{-3/3}$$

$$p(X > 3) = 0 + e^{-1} = e^{-1} = 0.3679$$

(ii) $p(\text{more than } 5) = p(X > 5) = \int_5^{\infty} \left(\frac{1}{3}\right) e^{-x/3} dx$

$$p(X > 5) = \frac{1}{3}\left[\frac{e^{x/3}}{-1/3}\right]_5^{\infty} = -e^{-\infty} + e^{-5/3}$$

$$p(X > 5) = 0 + e^{-5/3} = e^{-1.7} = 0.1827$$

Solved By Mahar Afaq Safdar Muhammadi

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