Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set S and if $\{F_n\}$ is a pointwise bounded and equicontinuous on S, then

Objective

1. $\{F_n\}$ is uniformly bounded on S,
2. $\{F_n\}$ contains a uniformly convergent subsequence.

Proof: Since $\{F_n\}$ is equicontinuous then by definition for every $\varepsilon > 0$, we have

$$|F_n(x) - F_n(y)| < \varepsilon, \qquad |x - y| < \delta.$$

From Analysis I, we know that S is compact then there are finitely many points $p_1, p_2, ...p_r$ in S such that to every $x \in S$ corresponds at least one p_1 such that $|x - p_1| < \delta$.

Since $\{F_n\}$ is pointwise bounded, there exists $M_i < \infty$ such that

$$|F_n(p_i)| < M_i, n \in \mathbb{N}.$$

If we take

$$M = \max\{M_1, ..., M_r\},$$

then $|F_n(x)| < M + \varepsilon$ for every $x \in S$. This proves the first part of the theorem.

Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set S and if $\{F_n\}$ is a pointwise bounded and equicontinuous on S, then

1. $\{F_n\}$ contains a uniformly convergent subsequence.

Proof: Let E be a countable dense subset of S. Then from previous theorem we have a subsequence $\{F_{n_i}(x)\}$ such that the subsequence $\{F_{n_i}(x)\}$ converges for every $x \in E$.

Fix the notation $F_{n_i}(x) = g_i$, we shall prove that $\{g_i\}$ converges uniformly on S.

Let $\varepsilon > 0$, and choose δ as before. Let $V(x, \delta)$ be the set of all $y \in S$ such that $|x - y| < \delta$.

$$f(x) = \sum_{k=0} c_k x^k, \qquad |x| < R,$$

the set $\{1, x, ..., x^n, ...\}$ is a base set of functions.

Fourier Series: A Fourier series expansion of a function is a representation of the function as a linear combination of sines and cosines, that is, the base set of the representation is

$$\{1, \cos nx, \sin nx\}_{n=1}^{\infty}.$$

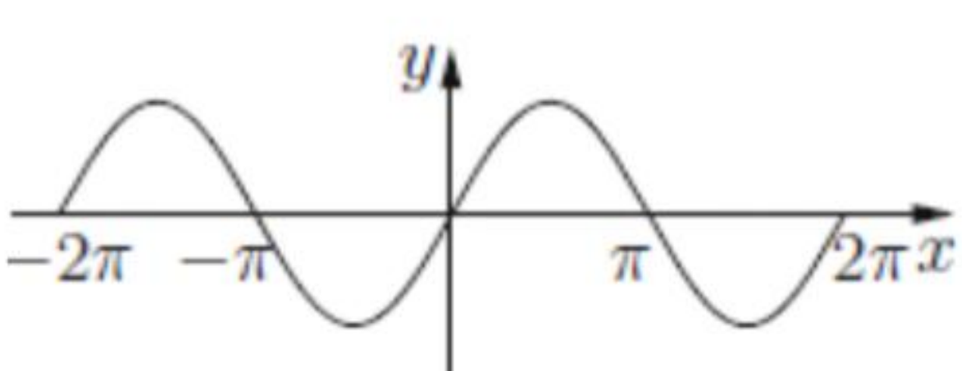
Objective - memorise equation

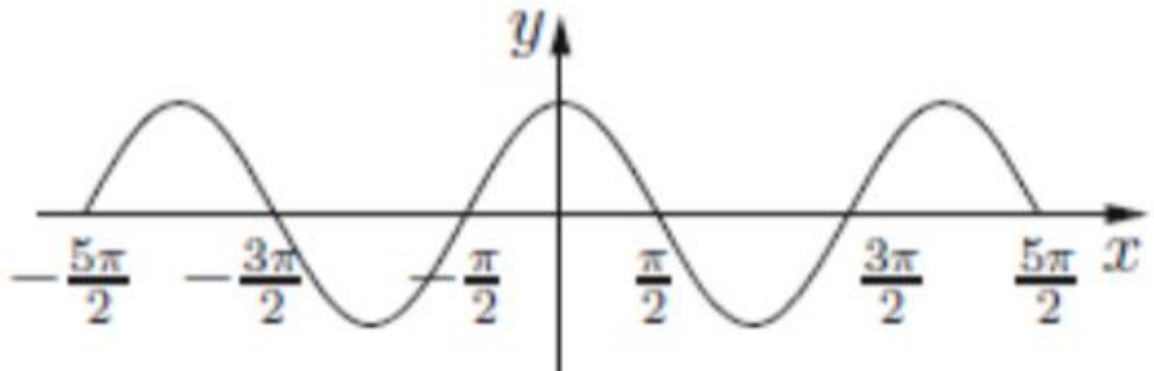
1.15.1 Periodic Functions

A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ is said to be periodic if there exists a nonzero real number ω such that

$$f(x) = f(x + \omega), \qquad x \in \Omega.$$

The simplest examples of periodic functions from $\mathbb{R}$ into $\mathbb{R}$ include the well known sine and cosine functions, since for each $k \in \mathbb{Z}\setminus\{0\}$





1.16.1 Fourier Series of Even and Odd Functions

Even and odd functions possess certain simple but useful properties:

- The product of two even (or odd) functions is an even function.
- The sum of two even (or odd) functions is an even (or odd) function.
- The product of an even and an odd function is an odd function.
- For a Riemann integrable function f defined on $[-c, c]$ $(c > 0)$, it is evident that

$$\int_{-c}^{c} f(x)dx = 2\int_{0}^{c} f(x)dx, \qquad \text{if } f \text{ is even}$$

$$\int_{-c}^{c} f(x)dx = 0, \qquad \text{if } f \text{ is odd}$$

Fourier series of even function: Suppose that $f(x)$ is a periodic function of

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Back Handouts-RealAnalysis-II

The *inner product* $\mathbf{X} \cdot \mathbf{Y}$ of $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{Y} = (y_1, y_2, \ldots, y_n)$ is

$$\mathbf{X} \cdot \mathbf{Y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

2.2 Schwarz's Inequality

Lemma: If $\mathbf{X}$ and $\mathbf{Y}$ are any two vectors in $\mathbb{R}^n$, then

$$|\mathbf{X} \cdot \mathbf{Y}| \leq |\mathbf{X}|\,|\mathbf{Y}|, \tag{2.3}$$

with equality if and only if one of the vectors is a scalar multiple of the other.

Proof: Suppose that $\mathbf{Y} \neq \mathbf{0}$ and t is any real number. Then

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} (x_i - t y_i)^2 \\ &= \sum_{i=1}^{n} x_i^2 - 2t \textstyle\sum_{i=1}^{n} x_i y_i + t^2 \sum_{i=1}^{n} y_i^2 \\ &= |\mathbf{X}|^2 - 2(\mathbf{X} \cdot \mathbf{Y})t + t^2 |\mathbf{Y}|^2. \end{aligned} \tag{2.4}$$

Describe and prove it- subjective - 5 marks

The last expression is a second-degree polynomial p in t. From the quadratic formula, the zeros of p are

$$t = \frac{(\mathbf{X} \cdot \mathbf{Y}) \pm \sqrt{(\mathbf{X} \cdot \mathbf{Y})^2 - |\mathbf{X}|^2 |\mathbf{Y}|^2}}{|\mathbf{Y}|^2}.$$

Hence,

$$(\mathbf{X} \cdot \mathbf{Y})^2 \leq |\mathbf{X}|^2 |\mathbf{Y}|^2. \tag{2.5}$$

because if not, then p would have two distinct real zeros and therefore be negative between them, contradicting the inequality (2.4).

Proof:

$$(\mathbf{X} \cdot \mathbf{Y})^2 \leq |\mathbf{X}|^2 |\mathbf{Y}|^2, \tag{2.6}$$

Taking square roots in (2.6) yields (2.3) if $\mathbf{Y} \neq \mathbf{0}$. If $\mathbf{X} = t\mathbf{Y}$, then $|\mathbf{X} \cdot \mathbf{Y}| = |\mathbf{X}||\mathbf{Y}| = |t||\mathbf{Y}|^2$ (verify), so equality holds in (2.3).

Conversely, if equality holds in (2.3), then p has the real zero $t_0 = (\mathbf{X} \cdot \mathbf{Y})/|\mathbf{Y}|^2$, and

$$\sum_{i=1}^{n} (x_i - t_0 y_i)^2 = 0$$

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Back Handouts-RealAnalysis-II

Corollary: If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are in $\mathbb{R}^n$, then

$$|\mathbf{X}-\mathbf{Z}| \leq |\mathbf{X}-\mathbf{Y}| + |\mathbf{Y}-\mathbf{Z}|.$$

Proof: Write

$$\mathbf{X}-\mathbf{Z} = (\mathbf{X}-\mathbf{Y}) + (\mathbf{Y}-\mathbf{Z}),$$

and apply triangle inequality with $\mathbf{X}$ and $\mathbf{Y}$ replaced by $\mathbf{X}-\mathbf{Y}$ and $\mathbf{Y}-\mathbf{Z}$.

Corollary: If $\mathbf{X}$ and $\mathbf{Y}$ are in $\mathbb{R}^n$, then

$$|\mathbf{X}-\mathbf{Y}| \geq \big||\mathbf{X}| - |\mathbf{Y}|\big|.$$

Proof: Since

$$\mathbf{X} = \mathbf{Y} + (\mathbf{X}-\mathbf{Y}),$$

Triangle inequality implies that

$$|\mathbf{X}| \leq |\mathbf{Y}| + |\mathbf{X}-\mathbf{Y}|,$$

which is equivalent to $|\mathbf{X}| - |\mathbf{Y}| \leq |\mathbf{X}-\mathbf{Y}|$.

Interchanging $\mathbf{X}$ and $\mathbf{Y}$ yields

$$|\mathbf{Y}| - |\mathbf{X}| \leq |\mathbf{Y}-\mathbf{X}|.$$

Since $|\mathbf{X}-\mathbf{Y}| = |\mathbf{Y}-\mathbf{X}|$, the last two inequalities imply the stated conclusion.

Theorem: If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are members of $\mathbb{R}^n$ and a is a scalar, then

- $|a\mathbf{X}| = |a|\,|\mathbf{X}|$.
- $|\mathbf{X}| \geq 0$, with equality if and only if $\mathbf{X} = \mathbf{0}$.
- $|\mathbf{X}-\mathbf{Y}| \geq 0$, with equality if and only if $\mathbf{X} = \mathbf{Y}$.
- $\mathbf{X}\cdot\mathbf{Y} = \mathbf{Y}\cdot\mathbf{X}$.
- $\mathbf{X}\cdot(\mathbf{Y}+\mathbf{Z}) = \mathbf{X}\cdot\mathbf{Y} + \mathbf{X}\cdot\mathbf{Z}$.
- $(c\mathbf{X})\cdot\mathbf{Y} = \mathbf{X}\cdot(c\mathbf{Y}) = c(\mathbf{X}\cdot\mathbf{Y})$.

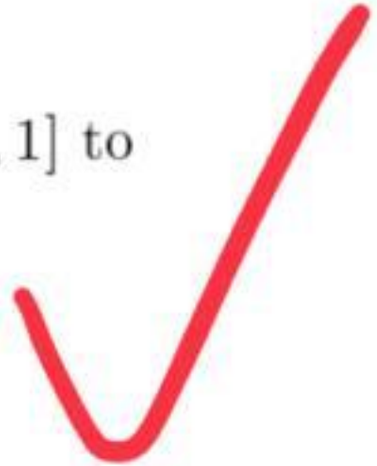
This proof was also in my paper- 5 marks - subjective

for all $n \geq 1$. Since ε can be arbitrarily small, it follows that

$$\|F_n - F\|_S = 1 \qquad \text{for all} \qquad n \geq 1.$$

Example: If $F_n(x) = x^n$, $n \geq 1$, then $\{F_n\}$ converges pointwise on $S = [0, 1]$ to

$$F(x) = \begin{cases} 1, & x = 1, \\ 0, & 0 \leq x < 1. \end{cases}$$



However, the convergence is uniform on $[0, \rho]$ if $0 < \rho < 1$, since then

$$\|F_n - F\|_{[0,\rho]} = \rho^n$$

and $\lim_{n\to\infty} \rho^n = 0$. Another way to say the same thing: $\{F_n\}$ converges uniformly on every closed subset of $[0, 1)$.

1.5 Cauchy's Uniform Convergence Criterion

We know that $\sum 1/n^2 < \infty$.

1.8.3 Weierstrass M-test/dominated Convergence Test

Theorem The series $\sum f_n$ converges uniformly on S if

$$\|f_n\|_S \leq M_n, \quad n \geq k, \tag{1.20}$$

where $\sum M_n < \infty$.

Proof: From Cauchy's convergence criterion for series of constants, there is for each $\varepsilon > 0$ an integer N such that

$$M_n + M_{n+1} + \cdots + M_m < \varepsilon \quad \text{if} \quad m \geq n \geq N.$$

which, because of (1.20), implies that

$$\|f_n\|_S + \|f_{n+1}\|_S + \cdots + \|f_m\|_S < \varepsilon \quad \text{if} \quad m, n \geq N.$$

Objective: gave definition and asked the name

$$\|f_n + f_{n+1} + \cdots + f_m\|_S < \varepsilon \quad \text{if} \quad m, n \geq N.$$

Due to Cauchy's criterion, we conclude that $\sum f_n$ converges uniformly on S.

Recall the following necessary condition for uniform convergence:

defined on K and $\{F_n\}$ converges uniformly then $\{F_n\}$ is equicontinuous on K.

For $n > N$ and $|x - y| < \delta$, we have

$$\begin{aligned} |F_n(x) - F_n(y)| &\leq |F_n(x) - F_N(x)| + |F_N(x) - F_N(y)| \\ &\quad + |F_N(y) - F_n(y)| \\ &< 3\varepsilon. \end{aligned}$$

Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set S and if $\{F_n\}$ is a pointwise bounded and equicontinuous on S, then

Objective

1. $\{F_n\}$ is uniformly bounded on S,
2. $\{F_n\}$ contains a uniformly convergent subsequence.

Proof: Since $\{F_n\}$ is equicontinuous then by definition for every $\varepsilon > 0$, we have

$$|F_n(x) - F_n(y)| < \varepsilon, \qquad |x - y| < \delta.$$

From Analysis I, we know that S is compact then there are finitely many points $p_1, p_2, ...p_r$ in S such that to every $x \in S$ corresponds at least one p_1 such that $|x - p_1| < \delta$.

Since $\{F_n\}$ is pointwise bounded, there exists $M_i < \infty$ such that

$$|F_n(p_i)| < M_i, n \in \mathbb{N}.$$

If we take

$$M = \max\{M_1, ..., M_r\},$$

then $|F_n(x)| < M + \varepsilon$ for every $x \in S$. This proves the first part of the theorem.

Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set S and if $\{F_n\}$ is a pointwise bounded and equicontinuous on S, then

1. $\{F_n\}$ contains a uniformly convergent subsequence.

Proof: Let E be a countable dense subset of S. Then from previous theorem we have a subsequence $\{F_{n_i}(x)\}$ such that the subsequence $\{F_{n_i}(x)\}$ converges for every $x \in E$.

Fix the notation $F_{n_i}(x) = g_i$, we shall prove that $\{g_i\}$ converges uniformly on S.

Let $\varepsilon > 0$, and choose δ as before. Let $V(x, \delta)$ be the set of all $y \in S$ such that $|x - y| < \delta$.

Since E is dense in S, and S is compact, there are finitely many points $x_1, ..., x_m$ in E such that

$$S \subset V(x_1, \delta) \cup ... \cup V(x_m, \delta) \qquad (*).$$

Since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer N such that

Therefore, the differentiated series is also uniformly convergent on $[-r, r]$ for every r, so the term by term differentiation is legitimate and

$$E'(x) = E(x), \quad -\infty < x < \infty.$$

This is not surprising if you recognize that $E(x) = e^x$.

Remark: Failure to verify that the given series converges at some point can lead to erroneous conclusions.

Example: For example, differentiating

$$\sum_{n=1}^{\infty} \cos \frac{x}{n} \tag{1.31}$$

term by term.

We have

$$-\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}.$$

Since

$$\left| \frac{1}{n} \sin \frac{x}{n} \right| \leq \frac{|x|}{n^2} \leq \frac{r}{n^2}, \quad |x| \leq r,$$

and $\sum 1/n^2 < \infty$. which converges uniformly on $[-r, r]$ for every r,

We cannot conclude from this that (1.31) converges uniformly on $[-r, r]$. In fact, it diverges for every x.

1.9 Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{1.32}$$

where x_0 and $a_0, a_1, \ldots,$ are constants, is called a *power series in* $x - x_0$. If $x_0 = 0$ then power series becomes

$$\sum_{n=0}^{\infty} a_n x^n.$$

Theorem: The radius of convergence of $\sum a_n (x - x_0)^n$ is given by

$$\frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

if the limit exists in the extended real number system.

Example: Consider the sequence of functions

$$F_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, \qquad S = [0, 1].$$

I was asked to work out ans for this which is 1

Then $|F_n| \leq 1$, so that $\{F_n(x)\}$ is uniformly bounded on $[0, 1]$. Also

$$\lim_{n\to\infty} F_n(x) = 0, \qquad x \in [0, 1].$$

But

$$F_n(\frac{1}{n}) = 1,$$

so no subsequence can converge uniformly on $[0, 1]$.

so no subsequence can converge uniformly on $[0, 1]$.

1.13 Equicontinuous Functions on a Set

A family of functions $\mathcal{F}$ defined on the set S is *equicontinuous* if for all $f \in \mathcal{F}$ and for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| \leq \varepsilon \quad \text{if} \quad x_1, x_2 \in [a, b], \quad |x_1 - x_2| < \delta. \tag{1.58}$$

Remark: It is clear that every member of $\mathcal{F}$ is uniformly continuous.

Theorem: If $\{F_n\}$ is a pointwise bounded sequence of functions on a countable set S, then $\{F_n\}$ has a subsequence $\{F_{n_k}\}$ such that subsequence converges for all $x \in S$.

Proof: Let $\{x_i\}$, $i = 1, 2, 3, ...$ be the points of S arranged in a sequence.

Since $\{F_n(x_i)\}$ is bounded, there exists a subsequence, which we shall denote by $\{F_{i,k}\}$, such that $\{F_{i,k}(x_i)\}$ converges as $k \to \infty$.

Consider the sequences $S_1, S_2, ...$, defined by

$$S_1 : F_{1,1}\ F_{1,2}\ F_{1,3}\ F_{1,4}\ ...$$

$$S_2 : F_{2,1}\ F_{2,2}\ F_{2,3}\ F_{2,4}\ ...$$

$$S_3 : F_{3,1}\ F_{3,2}\ F_{3,3}\ F_{3,4}\ ...$$

.....

Consider the sequences $S_1, S_2, ...$, defined by

$$S_1 : F_{1,1}\ F_{1,2}\ F_{1,3}\ F_{1,4}\ ...$$

$$S_2 : F_{2,1}\ F_{2,2}\ F_{2,3}\ F_{2,4}\ ...$$

$$S_3 : F_{3,1}\ F_{3,2}\ F_{3,3}\ F_{3,4}\ ...$$

.....

Objective type main theorem ki statement thi

The sequence has the following properties

- S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, ...$

ensures that $\{F_{n,n}(x_i)\}$ converges as $n \to \infty$ for every $x \in S$.

Theorem: If K is a compact subset and if $\{F_n\}$ is a sequence of continuous functions defined on K and $\{F_n\}$ converges uniformly then $\{F_n\}$ is equicontinuous on K.

Proof: Since the sequence of functions $\{F_n\}$ is uniformly convergent, for every $\varepsilon > 0$, there is an integer N such that

$$\|F_n - F_N\|_K < \varepsilon, \qquad n > N.$$

We know that continuous functions on compact sets are uniformly continuous, there is a $\delta > 0$ such that

$$|F_i(x) - F_i(y)| < \varepsilon, \quad |x - y| < \delta, 1 \leq i \leq N.$$

Theorem: If K is a compact subset and if $\{F_n\}$ is a sequence of continuous functions defined on K and $\{F_n\}$ converges uniformly then $\{F_n\}$ is equicontinuous on K.

For $n > N$ and $|x - y| < \delta$, we have

$$\begin{aligned} |F_n(x) - F_n(y)| &\leq |F_n(x) - F_N(x)| + |F_N(x) - F_N(y)| \\ &\quad + |F_N(y) - F_n(y)| \\ &< 3\varepsilon. \end{aligned}$$

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Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set

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Use Maclaurin series to...
https://answers.yahoo.com

let f(x) = 2(sin x)(cos x)

and let g(x) = sin (2x)

the first 3 terms of the Maclaurin series for f are: f(0), f '(0)x and f "(0)x^2/2, and:

f(0) = 2 sin 0 cos 0 = 2*0*1 = 0

f '(x) = -2(sin^2 x) + 2(cos^2 x), so f '(0) = 2

f "(x) = -8(sin x)(cos x), so f "(0) = 0

Similarly, calculating derivatives of g(x):

g(0) = 0

g'(x) = 2cos(2x), so g'(0) = 2

g"(x) = -4sin(2x), so g"(0) = 0

you can see that since f "(x) = -4f(x), and g"(x) = -4g(x), we will have equal terms for all terms of their maclaurin series, so f = g.

Answer:

$$\sin 2x = \sum_{n=0}^{\infty} \frac{2 \cdot 4^n}{(2n+1)!} x^{2n+1}.$$

Explanation:

Any general Maclauren series for a function $f(x)$ may be expressed as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!} x^n$$

So in this case, if we write the first few derivatives of the function out we get

$$f = \sin 2x \Rightarrow f(0) = 0$$
$$f' = 2\cos 2x \Rightarrow f'(0) = 2$$
$$f'' = -4\sin 2x \Rightarrow f''(0) = 0$$
$$f''' = 8\cos 2x \Rightarrow f'''(0) = 8$$
$$f^4 = -16\sin 2x \Rightarrow f^4(0) = 0$$
$$f^5 = 32\cos 2x \Rightarrow f^5(0) = 32$$

Continuing in this way, we eventually get that every even term is zero and hence vanishes from the Maclaren series, and every odd term has value $(2 \cdot 4^n)$.

Therefore the Mclauen power series expansion for this function is :

$$\sin 2x = \sum_{n=0}^{\infty} \frac{2 \cdot 4^n}{(2n+1)!} x^{2n+1}.$$

Thus, the Maclaurin series for $\sin(x)$ is

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= 0 + \left(\frac{1}{1!}x\right) + 0 + \left(\frac{-1}{3!}x^3\right) + 0 + \left(\frac{1}{5!}x^5\right) + \dots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\end{aligned}$$

The Maclaurin Expansion of cos(x)

The Maclaurin series expansion for cos(*x*) is given by

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

This formula is valid for all real values of *x*.

MTH631 27 DECEMBER 2018
PAPER

Using the McLaurin series $log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ find the value of

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} \quad as\ n \to \infty$$

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The Grapes of Math. Vu 3rd Semester

MTH 631

① Express the expression $S_n(x) = \frac{1}{n+n^3x^2}$ (5)

differential term by term.

② Define Schwirz's theorem and proof $|x-z| \leq |x-y|+|y-z|$ (5)

③ Start from Maclauies series of $\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ and end with

~~Ad~~ Abel's theorem and evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)}$$

More... Photo reply Edit

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$$

Periodic 10 Find Fourier coefficients

Mark 10

Mth 631

(1) Prove the identity by maclurin series of

$2\sin x \cos x = \sin 2x$. (10 mark)

(2) Find 1st four terms by maclurin series

$\frac{e^{-x^2}}{1+x^2}$ (5 marks)

(3) $S_n = e^{-nx}$ given that

$\int_0^1 S_n dx = \int_0^1 \sum_{n=0}^{\infty} S_n dx$ — Find

(5 marks)

MTH 631

Q#1 Show

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

for $n = 1, 2, 3, 4 \cdots$

Q#2 Show limit of a continous funtion is discentinous on R

where $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$

Q#3. Starting from Maclaurian Series $\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$

$|x| < 1$

show by ables theorems

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$$



v. tough

Starting from Maclaurin Series.

$$\log(1-x) = -\sum_{n\geq 0}^{\infty} \frac{x^{n+1}}{n+1} \qquad |x|<1$$

Use Abel's theorem to evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$$

If $f(x) = x$, $-\pi < x < \pi$

$f(\pi) = -\pi$

Determine fourier sine series of the function $f(x)$.

Show that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$$

is uniformly convergent for all real values of x and $p > 1$

Note: Submit this sheet to Superintendent, before leaving the Examination Center.

Math 631

Date:

Find four nonzero maclurian series.

7 $\frac{\cos x}{1+x^6}$ (5 marks)

8 $S_n(x) = \frac{nx}{1+n^2x^2}$ is differential on $[0, 1]$

(5 marks)

Q. Drive maclurian series $2\sin x \cos x = 2\sin 2x$ ~~and~~ and write the maclusin series of $\sin x$, $\cos x$, $\sin 2x$.

(10 marks)

Schwarz's Inequality:-

$$|X.Y| \leq |X||Y|$$

if x and Y are in R^n with equality iff and only iff vector is the scalar multiple of the other.

If $y \neq 0$, t is any real no.

$$0 \leq \sum_{i=1}^{n} (x_i - t y_i)^2$$

$$= \sum_{i=1}^{n} x_i^2 - 2t \sum_{i=1}^{n} x_i y_i + t^2 \sum_{i=1}^{n} y_i^2$$

$$= |x|^2 - 2(x.y)t + t^2 |y|^2$$

$$= t = \frac{(x.y) \pm \sqrt{(x.y)^2 - 4|x|^2|y|^2}}{|y|^2}$$

Hence $(x.y)^2 \leq |x|^2 |y|^2$

$\Rightarrow |x.y| \leq |x||y|$

We consider the sequence

$$E : F_{1,1}\ F_{2,2}\ F_{3,3}\ \cdots$$

By (3) property E is a subsequence of S_n, for $n = 1, 2, 3, \ldots$. The order in which t functions appear is the same in each sequence, i.e., if one function precedes anoth in S_1, they are in the same relation in every S_n, until one or the other is deleted

Hence, when going from one row in the above array to the next below, functi may move to the left but never to the right. The (2) property of the sequen ensures that $\{F_{n,n}(x_i)\}$ converges as $n \to \infty$ for every $x \in S$.

Theorem: If K is a compact subset and if $\{F_n\}$ is a sequence of continuous functi defined on K and $\{F_n\}$ converges uniformly then $\{F_n\}$ is equicontinuous on K.

Proof: Since the sequence of functions $\{F_n\}$ is uniformly convergent, for ev [redacted]re is an integer N such that

$$\|F_n - F_N\|_K < \varepsilon, \qquad n > N.$$

We know that continuous functions on compact sets are uniformly continuous, th is a $\delta > 0$ such that

$$|F_i(x) - F_i(y)| < \varepsilon, \quad |x - y| < \delta, 1 \le i \le N.$$

Theorem: If K is a compact subset and if $\{F_n\}$ is a sequence of continuous functio defined on K and $\{F_n\}$ converges uniformly then $\{F_n\}$ is equicontinuous on K.

For $n > N$ and $|x - y| < \delta$, we have

$$\begin{aligned} |F_n(x) - F_n(y)| &\le |F_n(x) - F_N(x)| + |F_N(x) - F_N(y)| \\ &\quad + |F_N(y) - F_n(y)| \\ &< 3\varepsilon. \end{aligned}$$

Theorem: If $\{F_n\}$ is a sequence of continuous functions defined on a compact set S and if $\{F_n\}$ is a pointwise bounded and equicontinuous on S, then

$$\frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\, dx.$$

$$\frac{1}{\pi}\int_0^{\pi} 2\sin x \cos nx\, dx \qquad 2\sin a\cos b = \sin(a+b)+\sin(a-b)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(1+n)x + \sin(1-n)x\, dx$$

$$= \frac{1}{\pi}\left[\frac{-1}{n+1}\cos(1+n)x - \frac{1}{1-n}\cos(1-n)x\right]_0^{\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1}\cos(1+n)\pi - \frac{1}{1-n}\cos(1-n)\pi + \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{1+n}(-1)^{1+n} - \frac{1}{1-n}\cos(1-n)\pi + \frac{1}{1+n} - \frac{1}{1-n}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{1+n}\left\{\left(1-(-1)^{(1+n)}\right)\right\} + \frac{1}{1-n}\left(1-(-1)^{1-n}\right)\right]$$

if n is odd.

$$a_n = 0$$

if n is even.

$$a_{2n} = \frac{1}{\pi}\left[\frac{2}{1+2n} + \frac{2}{1-2n}\right]$$

$$= \frac{1}{\pi}\left[\frac{4}{1-4\pi^2}\right] \qquad n \geq 1$$

$f(x) = \frac{a_0}{2} + \sum$

$f(x) = \frac{4}{2\pi} + \sum$

$f(x) = |x|$

$f(x) = \frac{a_0}{2} +$

$a_0 = \frac{1}{\pi}$

$a_n = \frac{1}{\pi}$

$b_n =$

$f(x) = |\sin x| \quad [-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$f(-x) = |-\sin x| = |\sin x|$ even function.

$b_n = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x|\, dx$$

$$\frac{2}{\pi} \int_{0}^{\pi} |\sin x|\, dx$$

$\sin x \geq 0$ in $[0, \pi]$

$$\frac{2}{\pi} \int_{0}^{\pi} \sin x\, dx$$

$$\frac{2}{\pi} [-\cos x]_0^{\pi} = \frac{2}{\pi} [-\cos \pi - \cos 0] = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |\sin x| \cos nx\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos nx\, dx$$

$f^{iv} = \frac{1}{(1+x)^4}(-10\sin\log(1+x))$

$f^{v} = \frac{-4}{(1+x)^5}[-10\sin\log(1+x)] + \frac{1}{(1+x)^4}\left[10\cos\log(1+x)\cdot\frac{1}{1+x}\right]$

$f^{iv} = \frac{1}{(1+x)^5}[40\sin\log(1+x) + 10\cos\log(1+x)]$

at $x=0$

$f(0) = \sin\log(1+0) = 0$

$f'(0) = \frac{1}{1+0}\cos\log(1+0) = 1$

$f''(0) = -\frac{1}{(1+0)^2}(\sin\log(1+0) + \cos\log(1+0))$

$= -(1) = -1$

$f'''(0) = \frac{1}{(1+0)^3}(3\sin\log(1+0) + \cos\log(1+0)$

$f^{iv}(0) = \frac{1}{(1+0)^4}[-10\sin\log(1+0)] = 0$

$f^{v}(0) = \frac{1}{(1+0)^5}[40\sin\log(1+0) + 10\cos\log(1+0)$

$f^{v}(0) = = 10$

Find Fourier Series for $f(x) = x^2$ $(0, 2\pi)$ $(-\pi, \pi)$

Fourier coefficient for $a_0, \ldots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Norm of function $fx = \frac{x}{1+nx}$ converge pointwise o to ... in ...

$f(-x) = (-x)^2$ limit point at $(0, \infty)$

$= x$

Same even.

a_0, a_n $b_n = x$

① $\Rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$

② $a_n = \frac{1}{\pi} \int_{\pi}^{\pi} f(x) \cos nx\,dx$

③ $b_n = \frac{1}{\pi} \int f(x) \sin nx\,dx$

Thus $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx$ $f(x)$ is even

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2\,dx$$

$$a_0 = \frac{2}{\pi} \left| \frac{x^3}{3} \right|_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\,dx$$ even so

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx$$

$$a_n = \frac{4(-1)^n}{n^2}$$

Similarly;

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx.$$

$\sin x$ is odd and x^2 is even.

by property of definite integral;

$$b_n = 0$$

a_0, b_n, a_n on Fourier sari.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{2}{3}\frac{\pi^2}{2} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}$$

$$= \pi^2/3 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx = \text{Ans}$$

$$\int_0^{\pi} x^2 \cos nx\,dx$$

$uv_1 - u'v_2 + u''v_3$

$$\int x^3 \sin nx$$

$$x^3\int \sin nx\,dx - \int 3x^2 \int \sin nx + 6x \int \sin nx$$

$$x^3\left(\frac{-\cos nx}{n}\right) - \int 3x^2 \frac{-\sin nx}{n^2} + 6x\frac{\cos nx}{n^3} - 6\frac{\sin nx}{n^4}$$

$$x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\Big|_0^{\pi}$$

$$\pi^2\left(\frac{\sin n\pi}{n}\right) - 2\pi\left(\frac{-\cos n\pi}{n^2}\right) + 2\frac{\sin n\pi}{n^3}\Big|_0^{\pi}$$

$$0 + \frac{2\pi(\cos n\pi)}{n^2} - 0\Big|_0^{\pi}$$

$$\frac{2}{\pi}\left[\frac{2\pi(-1)^n}{n^2} - \frac{2\pi\cos n(0)}{n^2}\right]$$

$$\frac{4\pi}{n^3}(-1)^n \quad \frac{2\pi(-1)^n - 2\pi}{n^2}$$

$$\frac{2}{\pi}\cdot\frac{2\pi((-1)^n - 1)}{n^2} = \frac{4\pi}{n^3}$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f''''(0) + \frac{x^5}{5!} f'''''(0) + \cdots$$

$$= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(10)$$

$$= x - x^2/2 + x^3/6 + \frac{x^5}{120} \times 10$$

$5 \times 4 \times 3 \times 2 \times 1 = 120$

$$f(x) = x - x^2/2 + x^3/6 + x^5/12 + \cdots$$ Ans

$f(x) = \sin\log(1+x)$

$f'(x) = \cos\log(1+x)\cdot\frac{1}{1+x}$

$f''(x) = -\sin\log(1+x)\cdot\left(\frac{1}{1+x}\right)^2 + \cos\log(1+x)(-1)(1+x)^{-2}$

$= -\left(\frac{1}{1+x}\right)^2\left[\sin\log(1+x) + \cos\log(1+x)\right]$

$f'''(x) = 2(1+x)^{-3}\left[\sin\log(1+x) + \cos\log(1+x)\right] - \left(\frac{1}{1+x}\right)^2\left[\cos\log(1+x)\left(\frac{1}{1+x}\right) - \sin\log(1+x)\cdot\frac{1}{1+x}\right]$

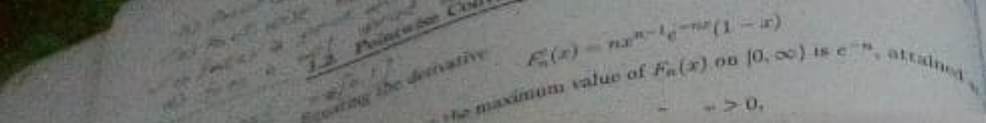
$= \frac{2}{(1+x)^3}\left[\sin\log(1+x) + \cos\log(1+x)\right] - \frac{1}{(1+x)^3}\left[\cos\log(1+x) - \sin\log(1+x)\right]$

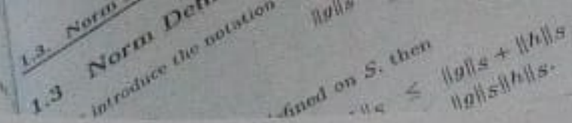
$= \frac{1}{(1+x)^3}\left[2\sin\log(1+x) + 2\cos\log(1+x)\right] - \cos\log(1+x) + \sin\log(1+x)$

$= \frac{1}{(1+x)^3}\left[3\sin\log(1+x) + \cos\log(1+x)\right]$

$f^{(iv)}(x) = \frac{-3}{(1+x)^4}\left[3\sin\log(1+x) + \cos\log(1+x)\right] + \frac{1}{(1+x)^3}\left\{3\cos\log(1+x)\frac{1}{1+x} - \sin\log(1+x)\frac{1}{1+x}\right\}$

$= \frac{1}{(1+x)^4}\left[-9\sin\log(1+x) - 3\cos\log(1+x)\right] + 3\cos\log(1+x) - \sin\log(1+x)\Big\}$





Norm of a function:

Norm is a function that assigns a strictly positive length or size to each vector in vector space except of zero vector

if $P : V \longrightarrow R$

$v \in V$ is

$\|v\| = P(v)$

$\|g\|_S = \sup_{x \in S} |g(x)|$ $\quad g : S \longrightarrow R$

$|g(x)|, \quad x \in S$