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	MIDTERM EXAMINATION SEMESTER FALL 2004 MTH401- Differential Equations	Total Marks: 50 Duration: 60min

Instructions

1. Attempt all questions.
2. The Time allowed for this paper is 60 minutes.
3. This examination is closed book, closed notes, closed neighbors; any one found cheating will get zero grades in the course MTH401 Differential Equations.
4. You are not allowed to use any type of Table for Formulae of Differentiation and integration during your exam.
5. Each objective type question carries 2 marks and each Descriptive question carries 10 marks. So write every step of the solution of descriptive question to get maximum marks.
6. Do not ask any questions about the contents of this examination from anyone. If you think that there is something wrong with any of the questions, attempt it to the best of your understanding.

Question No: 1 **Marks: 2**

A differential equation said to be ordinary differential equation if it contains only ordinary derivatives with respect to single variable.

T

F

Question No: 2 **Marks: 2**

A solution of the differential equation of the form y= f (x) is called the implicit solution.

T

F

Question No: 3 **Marks: 2**

Logistic equations are applications of non-linear equations.

T

F

Question No: 4 **Marks: 2**

The given functions $f_1(x) = 5, f_2(x) = \cos^2 x, f_3(x) = \sin^2 x$ are linearly independent.

T

F

Question No: 5 **Marks: 2**

A set of functions whose wronskian is zero then set of functions may or may not be dependent.

T

F

Question No: 6 **Marks: 10**

(a) Define separable form. Just separate the variables of the given differential equation. $\sec y \frac{dy}{dx} + \sin(x-y) = \sin(x+y)$

Solution

The differential equation of the form d y / d x = f (x, y) is called **separable** if it can be written in the form

d y / d x = h(x)g(y)

$$\sec y \frac{dy}{dx} + \sin(x-y) = \sin(x+y)$$

$$\sec y \frac{dy}{dx} = \sin(x+y) - \sin(x-y)$$

$$\sec y \frac{dy}{dx} = \sin(x+y) - \sin(x-y)$$

$$\sec y \frac{dy}{dx} = \sin x \cos y + \cos x \sin y - \sin x \cos y + \cos x \sin y$$

$$\sec y \frac{dy}{dx} = 2\cos x \sin y$$

$$\frac{dy}{2\cos y \sin y} = \cos x dx$$

$$\frac{dy}{\sin 2y} = \cos x dx$$

$$\operatorname{cosec} 2y dy = \cos x dx$$

(b) Check whether the given differential equation is exact or not if not then make it exact also show that it is exact **(Just make the equation exact do not solve it further)**.

$$e^x dx + (e^x \cot y + 2y \operatorname{cosec} y) dy = 0$$

Solution

$e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)dy = 0$

$M(x, y) = e^x, \quad N(x, y) = e^x \cot y + 2y \operatorname{cosec} y$

$M_y = 0, \quad N_x = e^x \cot y$

$M_y \neq N_x$

Thus it is not exact now we apply techniques to make it exact

$e^x dx + (e^x \cot y + 2y \operatorname{cosec} y)dy = 0$

$$\frac{N_x - M_y}{M} = \frac{e^x \cot y - 0}{e^x} = \cot y$$

$$IF = e^{\int \cot y dy} = e^{\ln \sin y} = \sin y$$

$e^x \sin y dx + (e^x \cos y + 2y)dy = 0$

$M(x, y) = e^x \sin y, N(x, y) = e^x \cos y + 2y$

$M_y = e^x \cos y, \; N_x = e^x \cos y$

$M_y = N_x$

Which shows that equation is exact

Question No: 7 **Marks: 10**

(a) Solve Bernoulli equation $y + 2\frac{dy}{dx} = y^3(x-1)$ (**Just make the given equation linear in v, do not**)

Solution

$$xy - \frac{dy}{dx} = y^3(x-1)$$

$$xy^{-2} - \frac{dy}{dx}y^{-3} = (x-1)$$

$put\, y^{-2} = v$

$$-2\frac{dy}{dx}y^{-3} = \frac{dv}{dx}$$

$$-\frac{dy}{dx}y^{-3} = \frac{1}{2}\frac{dv}{dx}$$

Then

$$\frac{1}{2}\frac{dv}{dx} + vx = (x-1)$$

$$\frac{dv}{dx} + 2vx = 2(x-1)$$

(b) The radioactive isotope of the lead, Pb-209, decay at a rate proportional to the amount present at any time and has a half-life of 4 hours. If 2 grams of the lead is present initially, how long will it take for 80% of the lead to decay? **(Just make the model of the radioactive decay as well as describe the given conditions do not further solve)**

Solution

Suppose that A_0 is the initial amount of isotope, as given A_0 is 100 and A (t) be the amount present at time t governed by the differential equation. It

$$\frac{dA}{dt} \alpha A$$

$$\frac{dA}{dt} = kA$$

$$\frac{dA}{A} = kdt$$

Integrate both sides

$$\ln A = kt + c$$

$$Ae^{-} = e^{kt+c}$$

$$Ae^{-}e^{-} = e^{kt}e^{c}$$

$$A = P_0 e^{kt} \qquad say\ A_0 = e^{c}$$

Where A_0 is the initial amount of isotope

$A_0 = 2 = A(0)$

A (4)=2/2=1

Then we have to find time when radioactive isotope will take 80% decay. So as A initially given 2 and 80% of 2 is 8/5 so decay would be 1-8/5 = -3/5

P (t) = -3/5 , t=?

Question No: 8 **Marks:10**

(a) Find a second solution of following differential equations where the first solution is given **(also write the formulae)**.

$$x^2y, -4xy, +6y = 0; \quad y_1 = x^2$$

Solution

$x^2y'' - 4xy' + 6y = 0;\ y_1 = x^2$

differential equation can be written as

$$y'' - \frac{4}{x}xy' + \frac{6}{x^2} = 0$$

the 2nd solution is given by

$$y_2 = y_1 \int \frac{e^{-\int pdx}}{y_1} dx$$

$$y_2 = x^2 \int \frac{e^{-\int 4/x\, dx}}{(x^2)^2} dx$$

$$y_2 = x^3 \ln x \int \frac{e^{\int 4/x\, dx}}{(x^3 \ln x)^2} dx$$

$$y_2 = x^2 \int \frac{e^{4\ln x}}{x^4} dx$$

$$y_2 = x^3 \ln x \int \frac{e^{\ln x^4}}{x^4} dx$$

$$y_2 = x^2 \int \frac{x^4}{x^4} dx$$

$$y_2 = x^2 \int dx$$

$$y_2 = x^3$$

(b) Solve the differential equation by the undetermined coefficient **(superposition approach)**

$$y'' - 2y' - 3y = 4\sin\theta$$

If complimentary solution is given below

$$y_c = c_1 e^{3x} + c_2 e^{-x}$$

Then just find particular solution.

Solution

We find a particular solution of non-homogeneous differential equation.

Suppose input function

$$y_p = A\cos\theta + B\sin\theta$$

Then

$$y'_p = -A\sin\theta + B\cos\theta$$

$$y''_p = -A\cos\theta - B\sin\theta$$

Substituting in the given differential equation

$$-A\cos\theta - B\sin\theta - 2(-A\sin\theta + B\cos\theta) - 3(A\cos\theta + B\sin\theta) = 4\sin\theta$$

$$(-A \quad 2B - 3A)\cos\theta + (-B \quad 2A - 3B)\sin\theta = 4\sin\theta$$

From the resulting equations

$-A-2B-3A=0;$ $\quad -B+2A-3B=4$

$-4A-2B=0;$ $\quad 2A-4B=4$

$2A+B=0;$ $\quad A-2B=2 \rightarrow A=2+2B$

$\rightarrow 2(2+2B)+B=0$

$\rightarrow +4+5B=0$

$\rightarrow B=\frac{-4}{5}$

$\rightarrow A=2+2\left(\frac{-4}{5}\right)=\frac{2}{5}$

$y_p=\frac{2}{5}\cos\theta-\frac{4}{5}\sin\theta$

Question No: 9 **Marks: 10**

(a) Solve differential equation by the undetermined coefficient **(annihilator operator).**

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}=e^{2x}$$

If complimentary solution is given below

$$y_c=c_1+c_2e^{4x}$$

Then just find general solution.

Solution

In this case of input function is

g(x)= e^{2x}

further

$$(D-2)(g(x))=(D-2)(e^{2x})=0$$

Therefore the differential operator D^3 annihilates the function g. operating on both sides

$$(D-2)(D^2-4D)y=(D-2)(e^{2x})$$

$$(D-2)(D^2-4D)y=0$$

This is the homogeneous equation of order 3. Next we solve this higher order equation.

Thus auxiliary equation is

$$(m-2)(m^2-4m)=0$$

$$m(m-2)(m-4)=0$$

$$m=0,2,4$$

Thus its general solution of the differential equation must be

$$y=c_1+c_2e^{4x}+c_3e^{2x}$$

(b) Solve the differential equation by the variation of parameters

$y''-4y+3=\cos x$

If complimentary solution is given below

$y_c = c_1 e^{3x} + c_2 e^{x}$

Then just find particular solution **(do not integrate).**

Solution

$y''-4y+3=\cos x$

This equation is already in standard form

$y'' + P(x)y' + Q(x)y = f(x)$

Therefore, we identify the function f(x) as

$f(x) = \cos x$

We construct the determinants

Since $y_1 = e^{3x}$, $y_2 = e^{x}$ so

$$W(y_1, y_2) = \begin{vmatrix} e^{3x} & e^{x} \\ 3e^{3x} & e^{x} \end{vmatrix} = e^{3x}e^{x} - 3e^{3x}e^{x} = -2e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & e^{x} \\ \cos x & e^{x} \end{vmatrix} = \cos x e^{x}$$

$$W_2 = \begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & \cos x \end{vmatrix} = \cos x e^{3x}$$

We determine the derivatives of the function u_1 and u_2

$$u'_1 = \frac{W_1}{W} = \frac{\cos x e^{x}}{-2e^{4x}} \rightarrow u_1 = \int \frac{\cos x}{-2} e^{-3x} dx$$

$$u'_2 = \frac{W_2}{W} = \frac{\cos x e^{3x}}{-2e^{4x}} \rightarrow u_2 = \int \frac{\cos x}{-2} e^{-x} dx$$

$$y_p = u_1 e^{3x} + u_2 e^{x}$$

is required particular solution