Practice Questions of Lectures 1-22

Lecture 1

Q1: What is the difference between quadrant and octant?

Q2: Describe two applications of Differential Calculus and two applications of Integral Calculus.

Q3: In two dimensions, in which quadrant, are the following points located?

a) $(2, 5)$
b) $(-3, 7)$
c) $(9, -2)$
d) $(-8, -1)$

Q4: In three dimensions, in which octant, are the following points located?

a) $(1, 2, 3)$
b) $(-5, 7, 1)$
c) $(9, -2, 6)$
d) $(-9, -3, 9)$
e) $(-4, -2, -6)$
f) $(-1, 2, -3)$
g) $(2, 2, -2)$

Q5: Write down the equations of the following:

a) *x* – axis
b) *y* – axis
c) *z* – axis
d) *xy* – plane
e) *yz* – plane
f) *xz* – plane

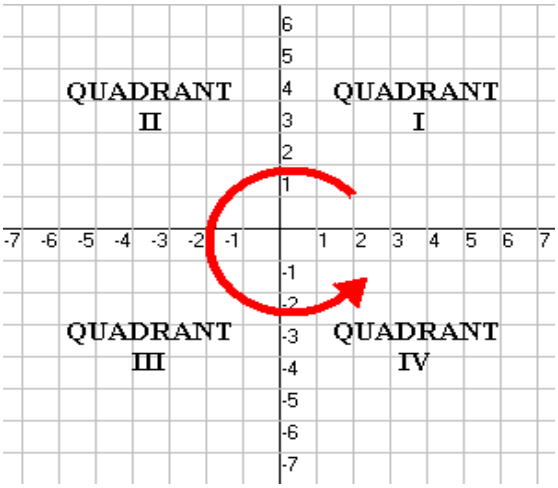
Q6: Identify the functions of single variable, two and three variables.

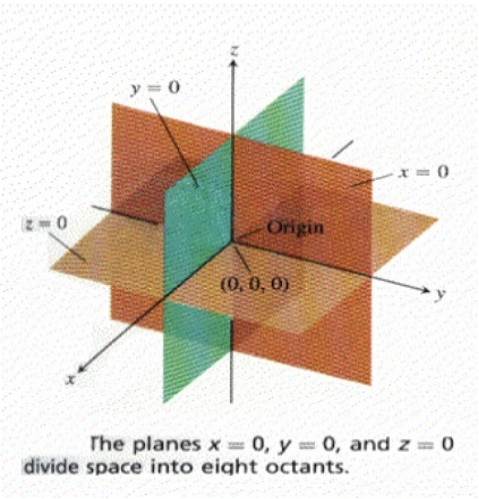
a) $y = 2x + 1$
b) $z = 5x + 6y + 10$
c) $w = 3xyz + 5$
d) $y = x^2 + 5x + 6$
e) $A = \pi r^2$
f) $P = 2(x + y)$

Answers

Q1: In two dimension plane, say xy- plane, we divide the plane into four parts. Each part is called quadrant.

In three dimension space, we divide the whole space in eight parts. Each part is called Octant.





The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

Q2: See handouts lecture No. 1. The answer is there.

Q3: (a) First quadrant (b) Second quadrant (c) 4^{th} quadrant (d) 3^{rd} quadrant

Q4: (a) First octant (b) 2^{nd} octant (c) 4^{th} octant

(d) 3^{rd} octant (e) 7^{th} octant (f) 6^{th} octant

(g) 5^{th} octant

Q5: See the page 14 Lecture No. 3

Q6: (a) single variable (b) two variables (c) three variables

(d) single variable (e) single variable (f) two variables

Lecture 2,3

Q#1: Let $f(x,y) = x^2 + 9x^3y - 2y$. Is the function defined at the point (4 , 2)? If yes, what is its value and if not, give the reason.

Sol:

$$f(x,y) = x^2 + 9x^3y - 2y$$
$$f(4,2) = (4)^2 + 9(4)^3(2) - 2(2)$$
$$= 16 + 1152 - 4 = 1164$$

Q#2: Let $f(x,\ y,\ z) = x^2 ye^{xyz}$. Is the function defined at point $(1,\ 3,\ 6)$? If yes, what is its value and if not, give the reason.

Sol:

$$f(x,\ y,\ z) = x^2 ye^{xyz}$$
$$f(1,3,6) = (1)^2(3)\mathrm{e}^{(1)(3)(6)} \ = \ 196979907.4$$

Q#3: Let $f(x,\ y,\ z)$ such that $f(\mathrm{x},\ \mathrm{y},\ \mathrm{z}) = \ln\sqrt{x^2 + y^2 + z^2}$. Is the function defined at origin? If yes, what is its value and if not, give the reason.

Sol: f is not defined at origin.

Q#4: Find the points where the parabola, represented by the equation $y = x^2 - 4x$, meets x-axis.

Sol: See solution of Q#5.

Q#5: Find at least two points that lie on the parabola $y = x^2 - 6x + 8$.

Sol: put $y = 0$; $x^2 - 6x + 8 = 0$ (after factorizing $x = 2$ and $x = 4$)

Q#6: Find the mid point between (6,2,4) and (2, 5, 8).

Sol: $\left(\frac{6+2}{2}\ ,\ \frac{2+5}{2}\ ,\ \frac{4+8}{2}\right) = \left(\frac{8}{2}\ ,\frac{7}{2}\ ,\frac{12}{2}\right) = (4,\frac{7}{2},6\ )$

Q#7: Find the distance between the points $A(4,\ 5,\ 9)$ and $B(1,\ 3,\ 7)$.

Solution: $\sqrt{(4-1)^2 + (5-3)^2 + (9-7)^2} \ = \ \sqrt{17}$

Q#8: For a line joining the points $A(3,\ 2,\ 6)$ and $B(0,\ 3,\ 4)$, find

a) Direction ratios of the line AB

0-3 , 3-2 , 4-6 = -3 , 1 , -2

b) Direction cosines of the line AB

$$\frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$$

Q#9:

Find the vertex point of the parabola $y = 2x^2 + 6x + 3$

Given equation of parabola:

$$y = 2x^2 + 6x + 3$$

$$x-\text{coordinate of vertex} = x_0 = \frac{-b}{2a} = \frac{-6}{2(2)} = \frac{-6}{4} = \frac{-3}{2}$$

$$y-\text{coordinate of vertex} = y_0 = 2\left(\frac{-3}{2}\right)^2 + 6\left(\frac{-3}{2}\right) + 3$$

$$= \frac{18}{4} - \frac{18}{2} + 3$$

$$= \frac{-6}{4} = \frac{-3}{2}$$

So the vertex point is $\left(\frac{-3}{2}, \frac{-3}{2}\right)$

Q#10: Find the points where the parabola $y = x^2 - 5x$ meets x-axis.

Given equation of the parabola is

$$y = x^2 - 5x \quad ----(1)$$

At x-axis, $y = 0$

So eq (1) becomes

$$0 = x^2 - 5x$$

$$0 = x(x-5)$$

$$0 = x \quad or \quad x-5 = 0$$

$$0 = x \quad or \quad x = 5$$

So the parabola meets x-axis at $(0, 0)$ and $(5, 0)$.

Q#11: Find at least two points that lie on the parabola $y = x^2 - 7x + 12$.

put

$y = 0$

$\Rightarrow x^2 - 7x + 12 = 0$

$(x-4)(x-3) = 0$

$x = 4$ or x=3

so the two points on parabola be (4,0) and (3,0).

Lecture 4

Q1: Convert the Cartesian coordinates $(2, 3)$ into polar coordinates (r, θ)

Q2: Convert the Rectangular coordinates $(1, 2, 3)$ into Cylindrical coordinates (r, θ, z).

Q3: Convert the spherical coordinates $(10, \frac{\pi}{3}, \pi)$ into

(a) cylindrical coordinates (r, θ, z)
(b) rectangular coordinates (x, y, z)

Answers

Q1: $r = \sqrt{13}, \quad \theta = \tan^{-1}\left(\frac{3}{2}\right)$

Q2: $r = \sqrt{5}, \quad \theta = \tan^{-1}(2), \; z = 3$

Q3: (a) $r = 0, \quad \theta = \frac{\pi}{3}, \; z = -10$

(b) $x = 0, \quad y = 0, \quad z = -10$

Lecture No. 4 (with Solutions)

Q1: Convert the Cartesian coordinates $(3, 4)$ into polar coordinates (r, θ)

Sol: since $x = 3, y = 4.$

Here, we can find the polar coordinates (r, θ)

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5.$

$$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{4}{3}).$$

Q2: Convert the Rectangular coordinates $(-2,\ 2\sqrt{3},\ 4)$ into Spherical polar coordinates $(\rho,\ \theta\ ,\phi)$.

Sol Q2(a): since $x = -2, y = 2\sqrt{3}, and\ z = 4$

Here, we can find the spherical coordinates (ρ,θ,ϕ)

Since $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+12+16} = \sqrt{32} = 4\sqrt{2}.$

$$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{2\sqrt{3}}{-2}) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{4}{4\sqrt{2}}\right) = \frac{\pi}{4}$$

Thus, the point is $(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4})$ in spherical coordinates.

Question 2 (b): Convert the Rectangular coordinates $(-\sqrt{3},\ 1,\ -2)$ into Spherical polar coordinates $(\rho,\ \theta\ ,\phi)$.

Solution: Since $x = -\sqrt{3}, y = 1, and\ z = -2$

Here, we can find the spherical coordinates (ρ,θ,ϕ)

Since $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{3+1+4} = \sqrt{8} = 2\sqrt{2}.$

$$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{1}{\sqrt{-3}}) = \frac{5\pi}{6}$$

$$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-2}{2\sqrt{2}}\right) = \frac{3\pi}{4}$$

Thus, the point is $(2\sqrt{2}, \frac{5\pi}{6}, \frac{3\pi}{4})$ in spherical coordinates.

Q3: Express the given equation in cylindrical and spherical coordinates:

$$(x+y)^2 - z^2 + 4 = 0$$

Solution:

$(x+y)^2 - z^2 + 4 = 0$

$x^2 + y^2 + 2xy - z^2 + 4 = 0$ ..... (1)

For cylindrical coordinates, $x = r\cos\theta, y = r\sin\theta, and\ z = z$, so (1) becomes

$(r\cos\theta)^2 + (r\sin\theta)^2 + 2(r\cos\theta)(r\sin\theta) - z^2 + 4 = 0$

$r^2\cos^2\theta + r^2\sin^2\theta + 2r^2\cos\theta\sin\theta - z^2 + 4 = 0$

$r^2(\cos^2\theta + \sin^2\theta) + 2r^2\cos\theta\sin\theta - z^2 + 4 = 0$

$r^2(1) + 2r^2\cos\theta\sin\theta - z^2 + 4 = 0$

$r^2(1 + 2\cos\theta\sin\theta) - z^2 + 4 = 0$

$r^2(1 + \sin 2\theta) - z^2 + 4 = 0$

For spherical coordinates, $x = p\sin\phi\cos\theta, y = p\sin\phi\sin\theta, and\ z = p\cos\phi$, so (1) becomes

$(p\sin\phi\cos\theta)^2 + (p\sin\phi\sin\theta)^2 + 2(p\sin\phi\cos\theta)(p\sin\phi\sin\theta) - (p\cos\phi)^2 + 4 = 0$

$p^2\sin^2\phi(\cos^2 + \sin^2\theta) + 2p^2\sin^2\phi\cos\theta\sin\theta - p^2\cos^2\phi + 4 = 0$

$p^2\sin^2\phi(1) + 2p^2\sin^2\phi\cos\theta\sin\theta - p^2\cos^2\phi + 4 = 0$

$2p^2\sin^2\phi\cos\theta\sin\theta + 4 = p^2\cos^2\phi - p^2\sin^2\phi$

$2p^2\sin^2\phi\cos\theta\sin\theta + 4 = p^2(\cos^2\phi - \sin^2\phi)$

$p^2\cos 2\phi - 2p^2\sin^2\phi\cos\theta\sin\theta = 4$

$p^2(\cos 2\phi - 2\sin^2\phi\cos\theta\sin\theta) = 4$

$$p^2(\cos 2\phi - \sin^2\phi \sin 2\theta) = 4$$

$$2p^2 \sin^2\phi \cos\theta \sin\theta + 4 = p^2(\cos^2\phi - \sin^2\phi)$$

Lecture 5

Q.1 Find the limit of the following function:

$$\lim_{(x,y)\to(1,3)} \frac{x^2 - 3xy + y^2}{x^2 y + 4x + 2y^2}$$

Q.2: By considering different path approach, find whether

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x + y^3}$$ exists or not

Q.3 By considering different path approach show that the function

$$f(x,y) = \frac{x^3 - y}{y}$$ has not limit as $(x,y) \to (0,0)$

Answers

Q1: $\frac{1}{25}$

Q2: Limit exists.

Lecture No. 5 (Solutions)

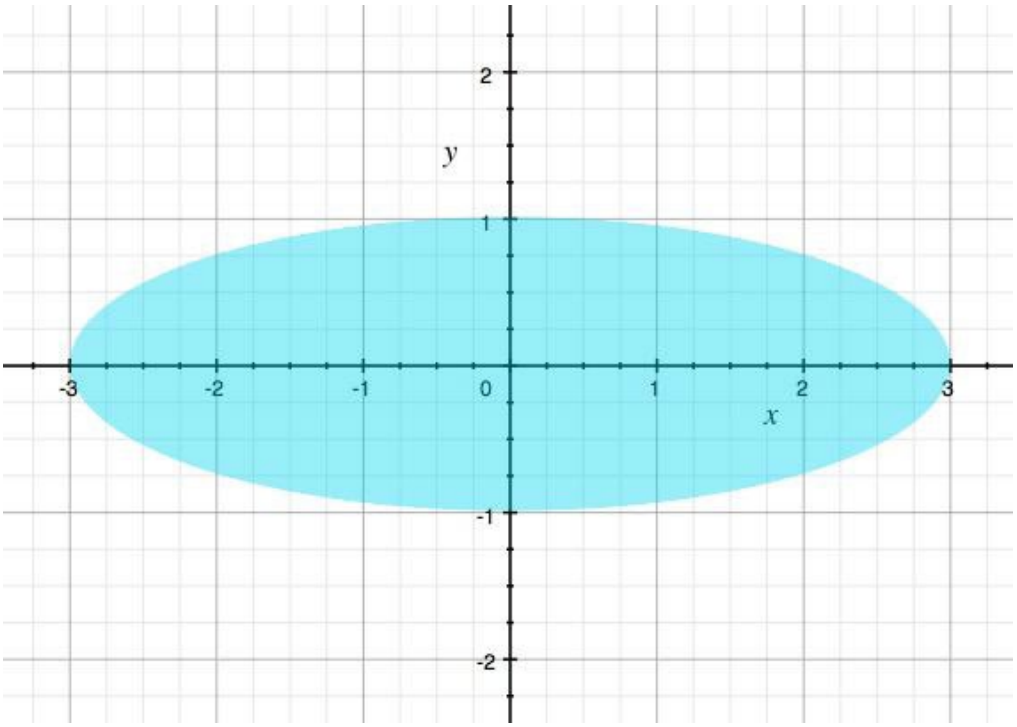
Q1: Find the domains of the following functions and explain it through graph.

$$f(x,y) = \ln(9 - x^2 - 9y^2)$$

We see that $9 - x^2 - 9y^2$ must be positive.

We have $9 - (x^2 + 9y^2) > 0 \Rightarrow x^2 + 9y^2 < 9 \Rightarrow \frac{x^2}{9} + y^2 < 1$

The domain is the set of all points interior to the ellipse centered at the origin with a=3, b=1.



Q2: Find the limit of the following function.

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2 \ln(xyz)}{e^{xz}}$$

$$\lim_{(x,y,z)\to(1,1,1)} \frac{x^2 In(xyz)}{e^{xz}}$$

By applying limit, we get

$$= \frac{(1)^2 In(1)}{e^1} = \frac{1\times 0}{e} = 0.$$

Q3: By using different path approach, find whether $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$ exist or not.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (x^2-y^2)$$

Along x-axis $y=0.$

$$\lim_{(x,y)\to(0,0)} x^2 = 0$$

Along y-axis $x=0$.

$$\lim_{(x,y)\to(0,0)} -y^2 = 0$$

Along the line $y=mx$.

$$\lim_{(x,y)\to(0,0)} x^2-(mx)^2 = \lim_{(x,y)\to(0,0)} x^2(1-m^2) = 0.$$

Along the parabola $y=x^2$.

$$\lim_{(x,y)\to(0,0)} x^2-(x^2)^2 = \lim_{(x,y)\to(0,0)} 1-x^2 = 1$$

Thus the limit does not exist

Q4: Find the limit, if (x, y)$\to (0,0)$ along the line $y = x^3$.

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^2}$$

If $(x,y)\to(0,0)$ along the line $y=x^3$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^3x^3}{x^2+(x^3)^2} = \lim_{(x,y)\to(0,0)} \frac{x^6}{x^2+x^6} = \lim_{(x,y)\to(0,0)} \frac{x^6}{x^2(1+x^4)} = \lim_{(x,y)\to(0,0)} \frac{x^4}{(1+x^4)} = \frac{0}{1} = 0.$$

Lecture 6

Q.1: Check whether the following function is continuous or not.

$$f(x,y)=\begin{cases}\dfrac{x^2y}{2x^2-y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Q.2: Find the region where the following function is continuous.

$$f(x,y)=2x\sin x-\frac{2y}{x-1}$$

Q.3 Given that $f(t)=\dfrac{2t}{t+1}$ and $g(x,y)=x+y$. Find $f\circ g(x,y)$

Answers

Q1: Continuous

Q2: on entire *xy*-plane where $x\neq 1$

Q3: $\dfrac{2(x+y)}{x+y+1}$

Lecture No. 6 (Solutions)

Q1: : Evaluate the given limits.

$$\lim_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)$$

Solution: $\lim\limits_{(x,y)\to(1,-1)} e^{-xy}\cos(x+y)=e^{-1(-1)}\cos(1+(-1))=e^{1}\cos(0)=e$

Q2: Find the region where the function $f(x,y)=\dfrac{x^2}{x^3-y^2}$ is continuous.

Solution: $f(x,y)$ is continuous in the entire two dimensional space except at the point

where $y=\sqrt{x^3}$

Q3: Examine the continuity of f at the point (0, 0).

$$f(x,y)=\begin{cases}\dfrac{2xy^2}{x^3+y^3} & if\,(x,y)\neq(0,0)\\ 0 & if\,(x,y)=(0,0)\end{cases}$$

Solution:

Let $(x,\mathrm{y})\to(0,0)$ along the line $y=mx$ so that $y\to 0\,as\,x\to 0$, then

$$f(x,y)=\frac{2x\ (m^2x^2)}{x^3+m^3x^3}$$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0}\frac{2m^2x^3}{x^3(1+m^3)}$$

$$=\lim_{x\to 0}\frac{2m^2}{(1+m^3)}\ ,\ \text{since}\ x\neq 0$$

$$=\frac{2m^2}{1+m^3}$$

So that limit value will vary as m changes. Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

so f is discontinuous at (0, 0).

Q4: Let $f(x,y) = e^{x-y}$ be a surface. Find the partial derivatives of f with respect to *x* and *y* .

Solution: $f_x = e^{x-y}, f_y = -e^{x-y}$

Q5: Find the first order partial derivatives of the given function.

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

Solution:

$$f_x = \frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

Lecture#7

Q1. If the function $f(x,\ y) = e^{2x}\cos y$, then find $\frac{\partial f}{\partial x}$ at point $(\ln 3,\ \pi)$.

Answer (-18)

Q2. If $f(x,\ y) = \sin(x^4y^3)$, then find $\frac{\partial f}{\partial x}$.

Answer ($4x^3y^3\cos(x^4y^3)$)

Q3. If $f(x,\ y) = x\ln(1+xy^{-1/4})$, find $\dfrac{\partial f}{\partial y}$.

Answer ($-\dfrac{x^2}{4(y^{5/4}+xy)}$)

Q4. Given that $\mathrm{h}(u,v) = \ln uv^2 + \cos v$. Find $\dfrac{\partial^2 h}{\partial v^2}$.

Answer ($-\dfrac{2}{v^2} - \cos v$)

Q5. If $\mathrm{g}(x,\ y) = (x+3)\ln(xy)$, then find both first order partial derivatives.

Answer $\left((x+3)\dfrac{x}{xy} = \dfrac{x+3}{y}\right)$

Q.6: Verify Euler's theorem if $f(x,y) = e^{xy}\sin y$

Q7 : Let $g = \ln(3x-3y) + \sin(x+y)$

a) Find $\dfrac{\partial^2 g}{\partial x^2}$

b) Find $\dfrac{\partial^2 g}{\partial y^2}$

c) Check whether $\dfrac{\partial^2 g}{\partial x^2} = \dfrac{\partial^2 g}{\partial y^2}$

Answer $\dfrac{\partial^2 g}{\partial x^2} = \dfrac{\partial^2 g}{\partial y^2}$ holds true.

Lecture#8

Q1. Suppose $f(x,\ y) = \dfrac{x}{y}$ *where* $x = e^u$ *and* $y = \ln u$. Use chain rule to find $\dfrac{df}{du}$ and write the answer in variable *u* only.

Answer ($\dfrac{[u(\ln u) - 1]e^u}{u(\ln u)^2}$)

Q2.

If $h = g(u, v, w)$

$u = u(t)$

$v = v(t)$

$w = w(t)$, then state the chain rule for derivative of h with respect to t.

Answer $\dfrac{dh}{dt} = \dfrac{\partial h}{\partial u}\dfrac{du}{dt} + \dfrac{\partial h}{\partial v}\dfrac{dv}{dt} + \dfrac{\partial h}{\partial w}\dfrac{dw}{dt}$

Q3. Let $u = f(x,\ y) = xy - \cos y$

$x(t) = e^t$

$y(t) = \sqrt{t^2 + 1}$

Use chain rule to find $\dfrac{du}{dt}$ and show it as a function of t.

Answer ($\dfrac{e^t(t^2+1) + t(e^t + \sin\sqrt{t^2+1})}{\sqrt{t^2+1}}$)

Q4. *Let* $w = g(u,\ v) = \ln(u + v)$

$u(t) = e^{2t}$

$v(t) = t^2 + t$

Use chain rule to find $\dfrac{dw}{dt}$ and express it as a function of t.

Answer ($\frac{dw}{dt} = \frac{2e^{2t} + 2t + 1}{e^{2t} + t^2 + t}$)

Q5: Find the partial derivatives with respect to *x* and *y* at point $(1, 2)$.

$$g(x, y) = x^3 y^2 \ln(xy)$$

Answer $g_x(1,2) = 12\ln 2 + 4, \quad g_y(1,2) = 4\ln 2 + 2$

Lecture#9

Q1.

Let $f(x, y) = \frac{\cos x}{y}$, $x = u^2 - v$ and $y = v + e^u$. Find $\frac{\partial f}{\partial u}$ and simplify it as a function of u and v.

Answer ($-\frac{2u\sin(u^2 - v)e^u}{(v + e^u)} - \frac{e^{2u}\cos(u^2 - v)}{(v + e^u)^2}$)

Q2. Let $z = f(x, y) = e^{xy}$, $x(s,t) = 3s\cos t$ and $y(s,t) = 4t^2 s$. Use chain rule to find $\frac{\partial z}{\partial t}$ and simplify it as a function of s and t.

Answer ($12s^2 te^{12s^2t^2\cos t}(-t\sin t + 2\cos t)$)

Q3.

If $h = g(x, y, z)$

$u = u(r, s, t)$

$v = v(r, s, t)$

$w = w(r, s, t)$, then state the chain rule for derivative of h with respect to r.

Answer

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial r}$$

Lecture 10

Q.1: Find a vector that is perpendicular to both of the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and

$2\hat{i} + 4\hat{j} - \hat{k}$

Q.2: Find the scalar product of the vectors $3\hat{i} + 2\hat{j} + \hat{k}$ and $4\hat{i} - \hat{j} + 2\hat{k}$.

Q.3: Find the value of '$\bar{a}$' if $a\hat{i} + \hat{j} + 5\hat{k}$ and $\hat{i} - 3\hat{j} + \hat{k}$ are orthogonal to each other.

Answers

Q1: $-10\hat{i} + 7\hat{j} + 8\hat{k}$

Q2: 12

Q3: -2

Lecture 11

Q1: Let $\bar{a} = 2\hat{i} + 5\hat{j} + 6\hat{k},\ \bar{b} = -\hat{i} + 3\hat{j} - 7\hat{k},\ \bar{c} = 3\hat{i} + 5\hat{j} + 3\hat{k}.$ Find $\bar{a} \times \bar{b}.\bar{c}$

Q2: Find the gradient of $f(x, y) = ye^x + \cos(xy)$

Q3: Find the gradient of $g(x, y) = 5x^3 y^2$

Q4: Find the directional derivative of the function $f(x, y) = xy + \sin(xy)$ at $(2, 0)$ in the direction of $\bar{a} = 4\hat{i} + 3\hat{j}$.

Answers

Q1: -86

Q2: $\nabla f = \left(ye^x - y \sin(xy)\right)\hat{i} + \left(e^x - x \sin(xy)\right)\hat{j}$

Q3: $\nabla g = 15x^2 y^2\, \hat{i} + 10x^3 y\, \hat{j}$

Q4: $\nabla f(2,0).\hat{a} = \frac{12}{5}$

Lecture No. 12

Q.1: Find an equation of the tangent plane to the surface $z = 3y^2 - 2x^2 + x$ at the point

$(2, -1, -3)$.

Q.2 : Find an equation of the tangent plane to the surface $z = x\sin(x + y)$ at the point

$(-1, 1, 0)$.

Q.3: Find an equation of the plane passing through point $(2,1,3)$ and perpendicular to the

vector $\vec{n} = \hat{i} + \hat{j} + 2\hat{k}$.

Answers

Q1: $z = -7x - 6y + 5$

Q2: $x + y + z = 0$

Q3: $x + y + 2z = 9$

Lecture#13

Q1.

Find equation of normal line to the given surface $f(x, y, z) = x^2 - 2y^2 - z^2 - 4$ at the point $P(-6, 2, \sqrt{24})$

Solution

$f(x, y, z) = x^2 - 2y^2 - z^2 - 4$

$f_x = 2x$, $f_y = -4y$, $f_z = -2z$

$f_x(-6,2,\sqrt{24}) = -12$, $f_y(-6,2,\sqrt{24}) = -8$, $f_z(-6,2,\sqrt{24}) = -2\sqrt{24}$

Equation of normal line of the surface is

$$\frac{x+6}{-12} = \frac{y-2}{-8} = \frac{z-\sqrt{24}}{-2\sqrt{24}}$$

$$\frac{x+6}{3} = \frac{y-2}{2} = \frac{z-\sqrt{24}}{\sqrt{6}}$$

Q: Find the Equation of the tangent plane to the surface $f(x,y,z) = xz - 4$ at the point

P (-2, 2,-2)

Solution

$f(x,y,z) = xz - 4$

$f_x = z$, $f_y = 0$, $f_z = x$

$f_x(-2,2,-2) = -2$, $f_y(-2,2,-2) = 0$, $f_z(-2,2,-2) = -2$

Equation of tangent plane to the surface is

$$-2(x+2) - 2(z+2) = 0$$
$$2(x+2) + 2(z+2) = 0$$
$$2x + 2z + 8 = 0$$
$$2(x+z+4) = 0$$

Q3 Show that the following two surfaces intersect orthogonally.

$x^2 + y^2 + 2z^2 - 16 = 0$

$x^2 + y^2 - 4z = 0$

Solution

let

$f(x,\ y,\ z) = x^2 + y^2 + 2z^2 - 16 = 0 - - - - - - - - - - - - - - - (1)$

$g(x,\ y,\ z) = x^2 + y^2 - 4z = 0 - - - - - - - - - - - - - - - - - (2)$

$f_x(x,\ y,\ z) = 2x$

$f_y(x,\ y,\ z) = 2y$

$f_z(x,\ y,\ z) = 4z$

$g_x(x,\ y,\ z) = 2x$

$g_y(x,\ y,\ z) = 2y$

$g_z(x,\ y,\ z) = -4$ 1+1+1

$$f_x g_x + f_y g_y + f_z g_z = (2x)(2x) + (2y)(2y) + (4z)(-4)$$
$$= 4x^2 + 4y^2 - 16z$$
$$= 4(x^2 + y^2 - 4z)$$

Since from equation (2) , $x^2 + y^2 - 4z = 0$ *So*

$f_x g_x + f_y g_y + f_z g_z = 4(x^2 + y^2 - 4z) = 0$

Thus, the surfaces $x^2 + y^2 + 2z^2 - 16 = 0$ and $x^2 + y^2 - 4z = 0$ are orthogonal at every point of intersection.

Lecture#14

Q1.

Find the critical points for the function

$f(x, y) = 2x^2 - 4x + xy^2 - 1$

Solution

$f(x,y) = 2x^2 - 4x + xy^2 - 1$

$f_x = 4x - 4 + y^2$

$f_y = 2xy$

The critical points of f satisfy the equations

$4x - 4 + y^2 = 0 - - - - - - - - - - - - (1)$

$2xy \quad = 0 - - - - - - - - - - (2)$

$2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0 \; or \; y = 0$

If $x = 0$ *then from* $(1), y = \pm 2.$

If $y = 0$ *then from* $(1), \mathrm{x} = 1.$

*Thus the critical po*int *are* $(1, 0), (0, 2), (0, -2)$

Q2.

Find the critical points of the function.

$f(x,y) = \sin x + \sin y$

Solution

$f(x,y) = \sin x + \sin y$

$f_x = \cos x$

$f_y = \cos y$ 1+1

The critical points of f satisfy the equations

$\cos x = 0 - - - - - - - - - - - - - - (1)$

$\mathrm{cosy} = 0 - - - - - - - - - - - - - - - (2)$

From (1)

$x = \cos^{-1}(0) \Rightarrow \; x = \mathrm{m}\pi + \frac{\pi}{2}$

From (2)

$y = \cos^{-1}(0) \Rightarrow \; y = \mathrm{n}\pi + \frac{\pi}{2}$

*Thus the critical po*int*s of a given function are* $(\mathrm{m}\pi + \frac{\pi}{2}, \mathrm{n}\pi + \frac{\pi}{2})$

Q3 Find the extrema for the following function.

$f(x,y) = x^2 - xy + y^2 + 6x$

Solution

$f(x,\ y) = x^2 - xy + y^2 + 6x$

$f_x = 2x - y + 6$

$f_y = -x + 2y$

The critical points of f satisfy the equations

$f_x = 2x - y + 6 = 0 - - - - - - - - - - - - (1)$

$f_y = -x + 2y = 0 - - - - - - - - - - - - (2)$

Now put x=2y in (1), we have

$2(2y) - y + 6 = 0 \Rightarrow$ 4y-y+6=0 $\Rightarrow$ 3y=-6 $\Rightarrow$ y=-2

Put y=-2 in (2)

$-x + 2(-2) = 0 \Rightarrow x$=-4

Thus the only critical points is $(-4, -2)$

$f_{xx} = 2\ ,\ \ f_{yy} = 2,\quad f_{xy} = -1$

$D = f_{xx}\ f_{yy} - f_{xy}^{\ 2}$

$For\ \ (-4, -2),\ \ D = 3 > 0$ Re$lative$ min$imum$

Q4.

Find the critical points of the function.

$f(x,y) = 3x^2\ y + y^3 - 3x^2 - 3y^2 + 2$

Solution

$f(x,y) = 3x^2\ y + y^3 - 3x^2 - 3y^2 + 2$

$f_x = 6xy - 6x$

$f_y = 3x^2 + 3y^2 - 6y$

The critical points of f satisfy the equations

$6xy - 6x = 0 - - - - - - - - - - - - (1)$

$3x^2 + 3y^2 - 6y = 0 - - - - - - - - - (2)$

$From\,(1)$

$6x(y-1) = 0 \Rightarrow x = 0 \;and\; y = 1$

$For\; x = 0, we\; have$

$3y(y-2)=0 \Rightarrow y=0, y=2$

For y=1

$3x^2 + 3(1)^2 = 6(1) \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$Thus\;\; the\;\; critical\;\; poi\,\text{nt}\;\; are\;\; (0,0),(0,2),(1,1),(-1,1)$

Lecture#15

Q1. Find the critical points of the function $f(x,\ y) = x^2 + 4xy + 2y^2 + 4x - 8y + 7$ and then classify them into relative maxima, relative minima and saddle points.

Solution

$f(x,\ y) = x^2 + 4xy + 2y^2 + 4x - 8y + 7$

$f_x = 2x + 4y + 4$

$f_y = 4x + 4y - 8$

The critical points of f satisfy the equations

$f_x = 2x + 4y + 4 = 0 - - - - - - - - - - - - (1)$

$f_y = 4x + 4y - 8 = 0 - - - - - - - - - - - - (2)$

Subtracting (2) from (1), we have

$-2x+12=0 \Rightarrow 2x=12 \Rightarrow x=6$

Now put x=2 in (1), we have

$4y+16=0 \Rightarrow 4y=-16 \Rightarrow y=-4$

Thus the only critical points is $(6,\ -4)$

$f_{xx} = 2\ ,\ \ f_{yy} = 4,\ \ \ f_{xy} = 4$

$D = f_{xx}\ f_{yy} - f_{xy}^{\ 2} = 8 - 16 = -8$

$For\;\; (6,-4),\;\; D = -8 < 0$ saddle point

Q2. Consider a function $f(x,y) = x^3 + 6xy + 3y^2 - 9x$ .One of its critical point is (1, -1). Find, whether (1, -1) is relative maxima, relative minima or saddle point of $f(x,y)$

Solution

$f(x,y) = x^3 + 6xy + 3y^2 - 9x$

$f_x = 3x^2 + 6y - 9$

$f_y = 6x + 6y$

$f_{xx} = 6x$,

$f_{yy} = 6,$

$f_{xy} = 6$

$f_{xx}(1,-1) = 6$,

$f_{yy}(1,-1) = 6$,

$f_{xy}(1,-1) = 6$

$D = 36 - 36 = 0$

since $D > 0,\ f_{xx} > 0,$ therefore, $f(x,y)$ *has Saddle* point *at* $(1,-1).$

Q3. Find and classify the critical points of $f(x,y) = 2xye^{-x^2-y^2}$

Solution

$f(x,y) = 2xye^{-x^2-y^2}$

$f_x = 2ye^{-x^2-y^2} - 4x^2ye^{-x^2-y^2}$

$= 2ye^{-x^2-y^2}(1-2x^2)$

$f_y = 2xe^{-x^2-y^2} - 4y^2xe^{-x^2-y^2}$

$= 2xe^{-x^2-y^2}(1-2y^2)$

The critical points of f satisfy the equations

$f_x = 2ye^{-x^2-y^2}(1-2x^2) = 0 - - - - - - - - - - - - (1)$

$f_y = 2xe^{-x^2-y^2}(1-2y^2) = 0 - - - - - - - - - - - - (2)$

$(1) \Rightarrow ye^{-x^2-y^2}(1-2x^2) = 0 \Rightarrow y = 0 \ or \ 1 = 2x^2 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$(2) \Rightarrow xe^{-x^2-y^2}(1-2y^2) = 0 \Rightarrow x = 0 \ or \ 1 = 2y^2 \Rightarrow y = \pm\frac{1}{\sqrt{2}}$

critical points are $(0,\ 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}), (\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

$$fxx = 2\left(-4xye^{-x^2-y^2} - 2x\left(1 - 2x^2\right)ye^{-x^2-y^2}\right)$$

$$= -4xy\left(3 - x^2\right)e^{-x^2-y^2}$$

$$fxy = 2\left(1 - 2x^2\right)\left(1 - 2y^2\right)e^{-x^2-y^2}$$

$$fyy = -4xy\left(3 - y^2\right)e^{-x^2-y^2},$$

$D = f_{xx}\ f_{yy} - f_{xy}^{\ 2}$

$For \ (0,0), \ D = -4 < 0 \ \ Saddle \ \ point$

$For \ \left(1/\sqrt{2}, 1/\sqrt{2}\right) = \left(-1/\sqrt{2}, -1/\sqrt{2}\right), \ D = 49 > 0$

$fxx\left(1/\sqrt{2}, 1/\sqrt{2}\right) = fxx\left(-1/\sqrt{2}, -1/\sqrt{2}\right) < 0, \ \text{Re}lative \ maximum.$

$For \ \left(-1/\sqrt{2}, 1/\sqrt{2}\right) = \left(1/\sqrt{2}, -1/\sqrt{2}\right), \ D = 49 > 0$

$fxx\left(1/\sqrt{2}, 1/\sqrt{2}\right) = fxx\left(-1/\sqrt{2}, -1/\sqrt{2}\right) > 0, \ \text{Re}lative \ \min imum$

Q4. For a function $f(x,y) = x^2 + (y+1)^2$, $f_x = 2x$, $f_y = 2(y+1)$. One of the critical points is $(0,-1)$. Check whether it is relative maxima, relative minima or a saddle point.

Solution

$f_{xx} = 2, \ f_{yy} = 2, \ \ f_{xy} = 0$

$f_{xx}(0,-1) = 2 > 0, \ \ f_{yy}(0,-1) = 2$

$D = f_{xx}(0,-1)f_{yy}(0,-1) - f_{xy}^{\ 2}(0,-1)$

$= (2)(2) - 0 = 4 > 0$

since $D > 0, \ f_{xx} > 0,$ therefore, $f(x,y) \ has \ \text{Re}lative \ \min imum \ at \ (0,-1).$

Lecture 16

Q1. If $f(x,y)=2x-3y^2+xy$ and R is the triangular region with vertices O (0, 0), A (3, 0) and B (1, 4), then find the critical points along the line segment OB at which the absolute extreme values of $f(x,y)$ can occur.

Solution

As the equation of the line for two points is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(\mathrm{x}-\mathrm{x}_1)$$

So the equation of line segment OB from $(0,0)$ to $(1,4)$ is $y=4x$ as compared to general equation of line. By putting $y=4x$ in $f(x,y)=2x-3y^2+xy$, we have

$$\begin{aligned} f(x,x) &= 2x-3(4x)^2+x(4x) \\ &= 2x-48x^2+4x^2 \\ &= 2x-44x^2 \\ f_x(x,x) &= 2-88x \end{aligned}$$

Equating with zero, we get

$$2-88x=0 \Rightarrow x=\frac{1}{44}$$

$$y=4(\frac{1}{44})=\frac{1}{11}$$

Thus, the critical point at which the function may assume the absolute extrema is $\left(\frac{1}{44},\frac{1}{11}\right)$.

Q 2. If $f(x,y)=xy+x+1$ and R is the triangular region with vertices O (0, 0), A (6, 0) and B (0, 5), then find the critical points along the line segment AB at which the absolute extrema of $f(x,y)$ can occur.

Solution

Equation of the line segment from $(6,0)$ to $(0,5)$ is $y=-\frac{5}{6}x+5$ as compared to general equation of straight line. By putting $y=-\frac{5}{6}x+5$ in $f(x,y)=xy+x+1$, we have

$$f(x,-\frac{5}{6}x+5)=x(-\frac{5}{6}x+5)+x+1$$

$$=-\frac{5}{6}x^2+6x+1$$

$$f_x(x,-\frac{5}{6}x+5)=-\frac{5}{3}x+6$$

Equating with zero, we get

$$-\frac{5}{3}x+6=0\Rightarrow x=\frac{18}{5}$$

$$y=-\frac{5}{6}(\frac{18}{5})+5=2$$

Thus, the critical point at which $f(x,y)$ may assume the absolute extrema is $\left(\frac{18}{5},2\right)$.

Q3. Find the critical point for the given function $f(x,y)=x^2-2xy+y^2$ along the line $y=2x-3$ at which the absolute extrema of the function can occur.

Solution

As $f(x,y)=x^2-2xy+y^2$, simplifies it to single variable x

$$f(x,2x-3)=x^2-2x(2x-3)+(2x-3)^2$$

$$=x^2-6x+9$$

As critical point exist when $f'(x,2x-3)=0$

So here $2x-6=0$

$\Rightarrow x=3$

$y=2(3)-3=3$

Thus the critical point exists at (3, 3).

Q4. Let $f(x,y)=xy-x-y$ and R is the triangular region with vertices O (0, 0), A (0, 4) and B (5, 0). Find the critical points along the line segment OA at which the absolute extrema of $f(x,y)$ can occur.

Solution

Since along the line segment OA $x=0$, so the given function becomes

$f(0,y)=-y$

$f_y(0,y)=-1$

As the critical point exist where $f_y(0,y)=0$.

Since $-1\neq 0$, therefore, there does not exist any critical point along the line OA at which the function $f(x,y)$ may assume the extreme values.

Lecture#17

Q1. Consider three positive integers whose sum is 35 and the product of two integers with the cube of third integer will be maximum. Write down the function to be maximized.

Solution

The function to be maximized is
$f(x,y) = xy(35-x-y)^3$.

Q2. Show that $f(x,y) = xy(11-x-y)$ is maximum at $x=5, y=5$, by using second partial derivative test.

Solution

As $f(x,y) = 11xy - x^2y - xy^2$

$f_x = 11y - 2xy - y^2,\ f_y = 11x - x^2 - 2xy$

$f_{xx} = -2y,\ f_{yy} = -2x,\ f_{xy} = 11 - 2x - 2y$

at $x=5,\ y=5$

$f_{xx} = -10 < 0,\ f_{yy} = -10,\ f_{xy} = -9$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$= (-10)(-10) - (-9)^2 = 19 > 0$

Hence, given function is maximum at the given points.

Q3. Using second partial derivative test show that the function $f(x,y) = 2xy(5-x-y)$ is maximum at $x=1, y=1$.

Solution

As $f(x,y) = 10xy - 2x^2y - 2xy^2$

$f_x = 10y - 4xy - 2y^2,\ f_y = 10x - 2x^2 - 4xy$

$f_{xx} = -4y,\ f_{yy} = -4x,\ f_{xy} = 10 - 4x - 4y$

at $x=1, y=1$

$f_{xx} = -4 < 0,\ f_{yy} = -4,\ f_{xy} = 2$

$D = (f_{xx})(f_{yy}) - (f_{xy})^2$

$= (-4)(-4) - 4 = 12 > 0$

Hence, given function is maximum at given points.

Lecture#18

Q1. Evaluate the following double integral.

$$\iint \left(x^3 y - 6x^2\right)\, dy\, dx$$

Solution

$$\iint \left(x^3 y - 6x^2\right)\, dy\, dx = \int \left(\frac{x^3 y^2}{2} - 6x^2 y\right) dx$$
$$= \frac{x^4 y^2}{8} - 2x^3 y$$

Q2. Evaluate the double integral

$$\int_2^6 \int_{-1}^{3} \left(7xy^2 + 3x^3 y\right)\, dx\, dy.$$

Solution

$$\int_2^6 \int_{-1}^{3} \left(7xy^2 + 3x^3 y\right)\, dx\, dy = \int_2^6 \left| \frac{7x^2 y^2}{2} + \frac{3x^4 y}{4} \right|_{-1}^{3} dy$$
$$= \int_2^6 \left(\frac{7}{2} y^2 \left(3^2 - (-1)^2\right) + \frac{3}{4} y \left(3^4 - (-1)^4\right) \right) dy$$
$$= \int_2^6 \left(\frac{7}{2} y^2 (9-1) + \frac{3}{4} y (81-1) \right) dy$$
$$= \int_2^6 \left(\frac{7}{2} y^2 (8) + \frac{3}{4} y (80) \right) dy$$
$$= \int_2^6 \left(28y^2 + 60y\right) dy$$
$$= \left| \frac{28y^3}{3} + 30y^2 \right|_2^6$$
$$= \left| \frac{28}{3}\left(6^3 - 2^3\right) + 30\left(6^2 - 2^2\right) \right|_2^6$$
$$= \frac{28}{3}(216-8) + 30(36-4)$$
$$= \frac{8704}{3}$$

Q3. Evaluate $\int_{-1}^{3}\int_{1}^{4}(3x^3y+x^2)dx\ dy$.

Solution

$$\int_{-1}^{3}\int_{1}^{4}(3x^3y+x^2)dx\ dy = \int_{-1}^{3}\left|\frac{3x^4y}{4}+\frac{x^3}{3}\right|_{1}^{4} dy$$

$$= \int_{-1}^{3}(\frac{765y}{4}+21)dy$$

$$= \left|\frac{765y^2}{8}+21y\right|_{-1}^{3}$$

$$= \frac{765}{8}[(3)^2-(-1)^2]+21[(3)-(-1)]$$

$$= 765+84 = 849$$

Q4. Evaluate each of the following integrals over the given region *D*.

$$\iint_D e^{\frac{x}{y}}dA, D=\{\{x,y\}\,|\,1\le y\le 2, y\le x\le y^3\}$$

Solution

$$\iint_D \mathbf{e}^{\frac{x}{y}}dA = \int_1^2\int_y^{y^3}\mathbf{e}^{\frac{x}{y}}dx\,dy = \int_1^2 y\,\mathbf{e}^{\frac{x}{y}}\Big|_y^{y^3} dy$$

$$= \int_1^2 y\,\mathbf{e}^{y^2}-y\mathbf{e}^1\,dy$$

$$= \left(\frac{1}{2}\mathbf{e}^{y^2}-\frac{1}{2}y^2\mathbf{e}^1\right)\Big|_1^2 = \frac{1}{2}\mathbf{e}^4-2\mathbf{e}^1$$

Answer $(\frac{1}{2}e^4-2e^1)$

Practice Questions Lectures 19-22

Q#1: Evaluate $\int_0^3 \int_{x^2}^{3x} (3x+1)\, dy\, dx$

Q#2:

If the order of integration for the integral $\int_0^6 \int_{y/3}^{2} e^{x^2}\, dxdy$ is changed. Find the change in the limits of new integral.

Q#3: Evaluate the integral $\int_0^3 \int_0^{2x} e^{x^2}\, dydx$.

Q#4: Evaluate the integral $\int_0^3 \int_0^{y^2} \cos y^3\, dxdy$.

Solution of Practice Question Lecture 19-22

Q#1: Evaluate $\int_0^3 \int_{x^2}^{3x} (3x+1)\, dy\, dx$

Solution:

$$\int_0^3 \int_{x^2}^{3x} (3x+1)\, dy\, dx$$

$$= \int_0^3 [3x(\mathrm{y})_{x^2}^{3x} + (\mathrm{y})_{x^2}^{3x}]\, dx$$

$$= \int_0^3 [3x(3\mathrm{x} - x^2) + (3\mathrm{x} - x^2)]\, dx$$

$$== \int_0^3 [9x^2 - -3x^3 + 3\mathrm{x} - \mathrm{x}^2]\, dx = \frac{99}{4}$$

If the order of integration for the integral $\int_0^6 \int_{y/3}^{2} e^{x^2}\, dxdy$ is changed. Find the change in the limits of new integral.

Solution:

$$\frac{y}{3} \le x \le 2 \Rightarrow y \le 3x \le 6$$
$$0 \le y \le 3x \le 6$$
$$0 \le 3x \le 6$$
$$0 \le x \le 2$$

Q#3: Evaluate the integral $\int_0^3 \int_0^{2x} e^{x^2} dydx$.

Solution:

$$\int_0^3 \int_0^{2x} e^{x^2} dydx$$
$$= \int_0^3 e^{x^2} (\mathrm{y})_0^{2x} dx$$
$$= \int_0^3 e^{x^2} 2xdx = (e^{x^2})_0^3 = e^{3^2} - 1$$
$$= e^9 - 1$$

Q#4: Evaluate the integral $\int_0^3 \int_0^{y^2} \cos y^3 dxdy$.

Solution:

$$\int_0^3 \int_0^{y^2} \cos y^3 dxdy$$
$$= \int_0^3 (x)_0^{y^2} . \cos y^3 \ dy$$
$$= \int_0^3 y^2 . \cos y^3 \ dy$$
$$= \frac{1}{3} \int_0^3 3y^2 . \cos y^3 \ dy$$
$$= \frac{1}{3} \left(\sin y^3 \right)_0^3 = = \frac{1}{3} \sin 27$$